On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$

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Abstract

It has been recently conjectured that the isoperimetric sets in the Heisenberg group $\mathbb{H}^n$ could coincide with the solutions to a "restricted" isoperimetric problem within the class of sets having finite perimeter and smooth boundary with cylindrical symmetry. In this paper, we derive new properties of those restricted isoperimetric sets, that we call $\mathbb{H}$-*bubbles*. In particular, we show that their boundary has constant mean $\mathbb{H}$-curvature and, surprisingly, that it is the union of all Carnot-Carathéodory geodesics connecting two special points. We believe that this latter property could help for proving that $\mathbb{H}$-bubbles are the general isoperimetric sets of $\mathbb{H}^n$ and for better understanding the geometric structure of the group. Besides, we show that the classical method based on the Brunn-Minkowski inequality, which serves to prove that Euclidean balls are isoperimetric sets in $\mathbb{R}^n$, has no direct and easy counterpart in $\mathbb{H}^n$.

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1 Introduction

It is well-known that Euclidean balls in $\mathbb{R}^n$ are, up to negligible sets, the unique solutions to the *isoperimetric problem* in $\mathbb{R}^n$, that is, the unique minimizers of the *perimeter* among all measurable sets with same n-dimensional Lebesgue measure.

This paper is devoted to the isoperimetric problem in the *Heisenberg group* $\mathbb{H}^n$, where the Euclidean geometry of $\mathbb{R}^{2n+1}$ is replaced by a *sub-Riemannian* geometry induced by a suitable family of *horizontal* vector fields. Before giving the definition and some properties of this group, let us point out the relationship between the isoperimetric problem and the *isoperimetric inequalities*. Recall that both $\mathbb{R}^n$ and $\mathbb{H}^n$ belong to the wider class of *Carnot groups*, i.e., structures of the form $(\mathbb{G}, ., \delta_\lambda, d_c)$, where $(\mathbb{G}, .)$ is a connected and simply connected Lie group, δ_λ is a (family of) dilation(s) and d_c is the *Carnot-Carathéodory metric* (see [1, 5, 6, 9] for more details and section 2 for more precise definitions). It is known that isoperimetric inequalities of the type

$$C\,|F|^{\frac{Q-1}{Q}} \leq P_{\mathbb{G}}(F) \tag{1.1}$$

hold for all measurable $F \subset \mathbb{G}$ with $|F| < \infty$, and for some positive C depending only on $\mathbb{G}$ (see [8, 15]); here, $|\cdot|$ denotes the n-dimensional Lebesgue measure (the *Haar measure* of the group $\mathbb{G} \simeq \mathbb{R}^m$), Q is the *homogeneous dimension* of $\mathbb{G}$ and $P_{\mathbb{G}}(F)$ denotes the $\mathbb{G}$-perimeter defined with

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respect to the family of horizontal vector fields (see section 2). Since $|\cdot|$ and $P_{\mathbb{G}}$ are, respectively, Q and $(Q-1)$-homogeneous with respect to dilations δ_λ, one can write (1.1) for $|F|=1$ and easily obtain that the best (the largest) constant C that can be plugged into (1.1) is exactly the infimum of $P_{\mathbb{G}}(F)$ under the volume constraint $|F|=1$, that is, the perimeter of any possibly existing solution to the isoperimetric problem, scaled to have unit volume. For more details on sets of finite $\mathbb{H}$-perimeter, the reader may refer to [7].

The existence of isoperimetric sets (i.e., solutions to the isoperimetric problem) in Carnot groups has been recently proved in [10], where some general properties of those sets are also carried out: more precisely, one can essentially prove that these sets are bounded, with Alhfors-regular boundary verifying a condition of "good" geometric separation (the so-called *condition B*). Moreover, at least for Carnot groups of *step* 2 and in particular for the Heisenberg group $\mathbb{H}^n$, the connectedness can also be proved as a consequence of being a *domain of isoperimetry*. Yet a more precise characterization of isoperimetric sets in a general Carnot group is still an open (and difficult) problem.

The natural candidates to solve the isoperimetric problem are the balls associated with the Carnot-Carathéodory metric because they are the counterparts in $\mathbb{H}^n$ of the Euclidean balls in $\mathbb{R}^n$. However, as shown in a recent work by Monti ([12]), these balls are *not* isoperimetric. Now, at least in the particular case of the first Heisenberg group $\mathbb{H}^1$, a hopefully good approximation of an isoperimetric set can be obtained as the result of numerical simulations that are mathematically supported by an approximation result of sets of finite $\mathbb{H}$-perimeter with polyhedral sets ([11]). Starting from different polyhedra and using Brakke's Surface Evolver [2], the minimization of the $\mathbb{H}$-perimeter at constant volume leads, up to left-translations, to a unique, apparently smooth and convex (in the Euclidean sense) body with an evident cylindrical symmetry (see figure 1) plus a symmetry with respect to the z-plane (recall that the points of $\mathbb{H}^n$ can be seen as the pairs $[z,t]\in\mathbb{C}^n\times\mathbb{R}\simeq\mathbb{R}^{2n+1}$). These symmetry properties are fully coherent with the fact that all rotations around the t-axis (which are symplectic maps of $\mathbb{H}^1$), as well as maps of the form $(x,y,t)\mapsto(y,x,-t)$, are automorphisms of $\mathbb{H}^1$ (see [5]).

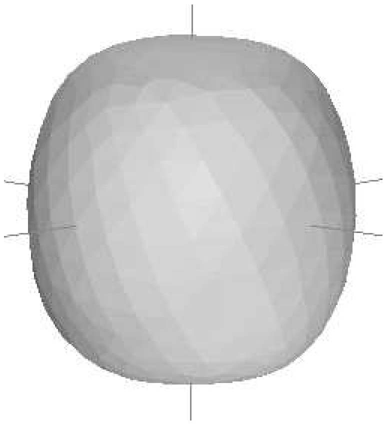

Figure 1: Using Brakke's Surface Evolver, the minimization of the $\mathbb{H}$-perimeter at constant volume in the Heisenberg group $\mathbb{H}^1$, taking a polyhedron as a starting configuration, produces this approximate "isoperimetric" set that notably differs from the Carnot-Carathéodory ball having same volume (which, for instance, is not convex - see the figure in [12]).

Motivated by the results of our simulations and generalizing to $\mathbb{H}^n$, $n \geq 1$, we are naturally led to consider a "restricted" isoperimetric problem, that is the minimization of the ratio $P_{\mathbb{H}}(F)/|F|^{\frac{Q}{Q-1}}$ on the subclass $\mathcal{F}$ of sets whose boundary S can be written as a union $S^+ \cup S^-$ where $S^+ = S \cap \{t \geq 0\}$ is the graph of some radial, smooth and non identically zero function $g(z) = f(|z|)$ whereas S^- is the symmetric of S^+ with respect to the z-plane.

It can be proved that this restricted isoperimetric problem admits solutions (see Theorem 3.3) that we call $\mathbb{H}$-*bubbles*. We believe that those are the right candidates to solve the (general) isoperimetric problem in $\mathbb{H}^n$ as suggested by our numerical results and, above all, because their intrinsic mean curvature turns out to be constant (see Theorem 3.3), as happens for Euclidean balls in $\mathbb{R}^n$. Unfortunately, this belief remains conjectural because it is still unknown whether isoperimetric sets are cylindrically symmetric. In addition, if such symmetry seems natural in $\mathbb{H}^1$ for the reasons mentioned before, it is less evident in $\mathbb{H}^n$ when $n \geq 2$ because the rotations around the t-axis are no more necessarily group automorphisms.

There are many techniques for proving that Euclidean balls are isoperimetric sets in $\mathbb{R}^n$. Among them, the most general are probably those involving a symmetrization procedure (Steiner symmetrization, Schwartz symmetrization, etc.) and the one based on the Brunn-Minkowski inequality (see [3, 4]). However, the symmetrization techniques seem to have no direct counterpart in $\mathbb{H}^n$; it is indeed an open question whether there exists in $\mathbb{H}^n$ a symmetrization procedure that does not increase the $\mathbb{H}$-perimeter. The other general technique we have in mind is based on the Brunn-Minkowski inequality in $\mathbb{R}^n$

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \qquad \forall A, B \subset \mathbb{R}^n,$$

whose natural counterpart in $\mathbb{H}^n$ could be

$$|A.B|^{1/Q} \geq |A|^{1/Q} + |B|^{1/Q}, \qquad \forall A, B \subset \mathbb{H}^n, \tag{1.2}$$

where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$ (see Section 2). However, Monti ([13]) has recently observed that (1.2) cannot hold for *any* pair of measurable sets $A, B \subset \mathbb{H}^n$ since, otherwise, Carnot-Carathéodory balls would be isoperimetric sets, and we recall that this is known to be false. Anyway, the Brunn-Minkowski based proof does not require that (1.2) be true for all pairs of sets. Actually, we could prove that a $\mathbb{H}$-bubble E is isoperimetric in $\mathbb{H}^n$ if the following conditions are fulfilled:

(C1) $|A.\delta_\epsilon E|^{1/Q} \geq |A|^{1/Q} + \epsilon |E|^{1/Q}$ (at least) for every A such that $|A| = |E|$ and for all $\epsilon > 0$.

(C2) $|E.\delta_\epsilon E| = |\delta_{1+\epsilon} E| + \mathrm{o}(\epsilon)$.

(C3) the (upper) Minkowski content associated to E, i.e.,

$$\mathcal{M}_E(A) := \limsup_{\epsilon \to 0} \frac{|A.\delta_\epsilon(E)| - |A|}{\epsilon}, \qquad \forall A \text{ measurable } \subset \mathbb{H}^n \text{ with } |A| < \infty$$

verifies $\mathcal{M}_E(A) = C P_{\mathbb{H}}(A)$ for all A with smooth boundary and for some $C > 0$ depending only on $|E|$.

Under these properties, one could prove that $\mathcal{M}_E(A) \geq \mathcal{M}_E(E)$ for all measurable A with smooth boundary and measure $|A| = |E|$, which would imply that E is isoperimetric. Indeed, we deduce from (C1) that

$$\begin{aligned}\frac{|A.\delta_\epsilon(E)|-|A|}{\epsilon} &= \frac{(|A.\delta_\epsilon(E)|^{1/Q})^Q-|A|}{\epsilon}\\ &\geq \frac{|A|(1+|\delta_\epsilon E|^{1/Q}|A|^{-1/Q})^Q-|A|}{\epsilon}\\ &\geq \frac{Q|A|^{(Q-1)/Q}|E|^{1/Q}\epsilon+\mathrm{o}(\epsilon)}{\epsilon},\end{aligned}$$

thus

$$\mathcal{M}_E(A)\geq Q|A|^{(Q-1)/Q}|E|^{1/Q} \tag{1.3}$$

Besides, (C2) implies that

$$\frac{|E.\delta_\epsilon(E)|-|E|}{\epsilon}=\frac{(1+\epsilon)^Q|E|-|E|+\mathrm{o}(\epsilon)}{\epsilon}$$

and therefore,

$$\mathcal{M}_E(E)=Q|E|. \tag{1.4}$$

We deduce from (1.3) and (1.4) above that for every A such that $|A|=|E|$ there holds:

$$\mathcal{M}_E(A)\geq\mathcal{M}_E(E).$$

By (C3) together with the smoothness of ∂E and a density argument, we would finally get that E is isoperimetric.

(C1) is obviously weaker than (1.2) and (C2) is a weak counterpart in $\mathbb{H}^n$ of the equality $B+\epsilon B=(1+\epsilon)B$ that holds for any Euclidean ball $B\subset\mathbb{R}^n$ and for all $\epsilon>0$. The argument above becomes clearly unusable if (C1) or (C2) are further weakened, for example if we assume that $|E.\delta_\epsilon E|=|\delta_{1+\epsilon}E|+\mathrm{O}(\epsilon)$. Thus, (C1), (C2) and (C3) seem to be minimal requirements for applying the Brunn-Minkowski based strategy. The results we show in this paper are unfortunately negative. We prove indeed in Proposition 4.1 that (C2) is false, at least in the group $\mathbb{H}^1$, and that the relation must rather be written as

$$|\delta_{1+\epsilon}E|-|E.\delta_\epsilon E|\geq C\epsilon \tag{1.5}$$

for ϵ small enough and for some $C>0$ depending only on $|E|$. This implies that (C1) is false, too, because it would give for $A=E$

$$|E.\delta_\epsilon E|\geq(1+\epsilon)^Q|E|=|\delta_{1+\epsilon}E|,$$

which is obviously incompatible with (1.5). One already knew that the Brunn-Minkowski inequality could not be true for every pair of sets and this last result reveals that the inequality is actually false even for the very particular pair $(E,\delta_\epsilon E)$ of presumably isoperimetric sets.

For the sake of completeness, we also prove in Proposition 4.2 that (C2) and (C3) are actually *incompatible*. Though it is redundant with Proposition 4.1, this result illustrates well the close connection between perimeter, Minkowski content and volume.

Besides those negative results, we give new properties of $\mathbb{H}$-bubbles that could be of help to find (hopefully) a rigorous proof of their optimality and also to better understand the geometry of $\mathbb{H}^n$ in general. The first property is the previously mentioned fact that the mean $\mathbb{H}$-curvature of the boundary of an $\mathbb{H}$-bubble (following the quite natural definition proposed in [16]) is constant, and this agrees with the Euclidean case, where balls verify precisely the same property. The second one, which is indeed the most interesting and surprising, is the fact that the boundary of any $\mathbb{H}$-bubble is the union of the (infinitely many) geodesics connecting the *north pole* and the *south pole* of the bubble. This quite unexpected property, now without any Euclidean counterpart (recall that Euclidean geodesics connecting two given points are reduced to a single segment!), can be checked very easily, once one knows the equations of geodesics in $\mathbb{H}^n$ and the explicit solution of the Euler equation in the radial case.

In conclusion, even though the solution to the isoperimetric problem in $\mathbb{H}^n$ has not been found yet, we believe that our observations might shed some light and suggest new approaches to this problem, even in the context of more general Carnot groups.

The paper is organized as follows: after introducing the notations in section 2, we give in section 3 the announced properties of $\mathbb{H}$-bubbles and we finally show in section 4 why the Brunn-Minkowski based strategy fails.

2 Notations and main facts about $\mathbb{H}^n$

The Heisenberg group $\mathbb{H}^n$ coincides with $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ and we shall frequently denote its elements by $P = [z,t]$, where $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$. We will also sometimes identify $z = x+iy$ with the $2n$-tuple (x,y), where $x, y \in \mathbb{R}^n$. Like for any Carnot group, the algebraic and metric structure of $\mathbb{H}^n$ can be completely derived via exponential map from its tangent, stratified Lie algebra $\mathcal{G}$ generated by the following family of *horizontal* vector fields: for $i = 1, \ldots, n$, define

$$\begin{aligned} X_i(P) &= \partial_{x_i} + 2y_i\partial_t \\ Y_i(P) &= \partial_{y_i} - 2x_i\partial_t, \end{aligned}$$

where $P = [x+iy,t]$. Note that the stratification is non-trivial, since $\mathcal{G} = H \oplus V$, where $H = \text{span}\,\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ is the so-called horizontal subbundle and $V = \text{span}\,\{[X_i, Y_i]\} = \text{span}\,\{\partial_t\}$ is the center of the algebra (as usual, $[X,Y] = XY - YX$ denotes the commutator of the two fields X and Y). The resulting group operation on $\mathbb{H}^n$ is

$$P.P' = [z,t].[z',t'] = [z+z', t+t'+2\,\text{Im}(\sum_{i=1}^n z_i \cdot \bar{z}_i')]. \tag{2.1}$$

Thanks to (2.1), one defines a family of *left translations* on $\mathbb{H}^n$ as the group automorphisms $\tau_P : \mathbb{H}^n \to \mathbb{H}^n$ which associate to any $Q \in \mathbb{H}^n$ the point $\tau_P(Q) = P.Q$. There is also a family of intrinsic *dilations* on $\mathbb{H}^n$, given by $\delta_\lambda([z,t]) = [\lambda z, \lambda^2 t]$, with $\lambda > 0$.

To complete the Carnot structure, we define the *Carnot-Carathéodory metric* as follows. We say that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{H}^n$ is a sub-unit curve if there exist $2n$ measurable functions $h_1, \ldots, h_{2n} : [0,T] \to \mathbb{R}$ such that $\sum_{j=1}^{2n} h_j^2(s) \le 1$ and

$$\dot{\gamma}(s) = \sum_{i=1}^n h_i(s)X_i(\gamma(s)) + h_{i+n}(s)Y_i(\gamma(s))$$

for a.e. $s \in [0,T]$. By Chow's Theorem, any two points P and Q in $\mathbb{H}^n$ can be joined by a sub-unit curve. Then the Carnot-Carathéodory distance between P and Q is

$$\begin{aligned} d_c(P,Q) = \inf\{T > 0\,;\ &\text{there exists a sub-unit curve } \gamma : [0,T] \to \mathbb{H}^n \\ &\text{such that } \gamma(0) = P,\ \gamma(T) = Q\}. \end{aligned}$$

It is worth noting that the distance d_c is coherent with the group structure and the dilations: indeed,

$$d_c(\tau_P(Q), \tau_P(W)) = d_c(Q, W) \quad \text{and} \quad d_c(\delta_\lambda(P), \delta_\lambda(Q) = \lambda d_c(P, Q)$$

for all $P, Q, W \in \mathbb{H}^n$ and $\lambda > 0$. Given P and Q as above, there always exists a sub-unit curve joining P and Q of length $d_c(P, Q)$ (i.e., a minimal geodesic). We recall here the equations for geodesics of unit length starting from $[0, 0]$, since all other geodesics can be recovered by left-translations and dilations (see [12] and the references therein). Let $s \in [0, 1]$ be the time-length parameter and $\phi \in [-2\pi, 2\pi]$, and let $A_i, B_i \in \mathbb{R}$ such that $\sum_{i=1}^n A_i^2 + B_i^2 = 1$: then the set of equations

$$\begin{cases} x_i(s) = \dfrac{A_i(1-\cos(\phi s)) + B_i \sin(\phi s)}{\phi} & i = 1, \dots, n \\ y_i(s) = \dfrac{-B_i(1-\cos(\phi s)) + A_i \sin(\phi s)}{\phi} & i = 1, \dots, n \\ t(s) = 2\,\dfrac{\phi s - \sin(\phi s)}{\phi^2} \end{cases} \tag{2.2}$$

defines a geodesic $\gamma(s)$ connecting $[0, 0]$ with the point $[x + iy, t]$, whose coordinates are

$$\begin{cases} x_i = x_i(1) = \dfrac{A_i(1-\cos\phi) + B_i \sin\phi}{\phi} & i = 1, \dots, n \\ y_i = y_i(1) = \dfrac{-B_i(1-\cos\phi) + A_i \sin\phi}{\phi} & i = 1, \dots, n \\ t = t(1) = 2\dfrac{\phi - \sin\phi}{\phi^2} \end{cases} \tag{2.3}$$

(of course, this gives a parameterization of the boundary of the Carnot-Carathéodory ball with unit radius). Finally, the structure $(\mathbb{H}^n, \cdot, \delta_\lambda, d_c)$ provides an example of *Carnot group*, as said in the introduction.

It is not difficult to check that the $(2n+1)$-dimensional Lebesgue measure on $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ is the Haar measure of the group, invariant under left-translations and $(2n+2)$-homogeneous with respect to dilations (this degree of homogeneity coincides with the so-called *homogeneous dimension* of $\mathbb{H}^n$, from now on denoted as Q). In particular, the topological dimension of $\mathbb{H}^n$ $(= 2n + 1)$ is strictly less than its Hausdorff dimension $(= Q = 2n + 2)$.

We now define the sets with finite $\mathbb{H}$-perimeter. If Ω is an open subset of $\mathbb{H}^n$ and $F \subset \mathbb{H}^n$ is measurable, we set

$$P_{\mathbb{H}}(F, \Omega) = \sup \left(\int_{\Omega \cap F} \operatorname{div}_{\mathbb{H}} \phi \, d\mathcal{L}^{2n+1} \, ; \; \phi \in C_0^1(\Omega, \mathbb{R}^{2n}), \; \|\phi\|_\infty \le 1 \right),$$

where

$$\operatorname{div}_{\mathbb{H}} \phi = \sum_{i=1}^n X_i \phi_i + Y_i \phi_{i+n}$$

for any $\phi = (\phi_1, \dots, \phi_{2n}) \in C_0^1(\Omega, \mathbb{R}^{2n})$. Here $C_0^1(\Omega, \mathbb{R}^{2n})$ denotes the space of $\mathbb{R}^{2n}$-valued functions of class C^1 (in the Euclidean sense) with compact support in Ω. Of course, F will be said to have finite $\mathbb{H}$-perimeter in Ω if and only if $P_{\mathbb{H}}(F, \Omega) < \infty$. As for the notation, we will write $P_{\mathbb{H}}(F)$ instead of $P_{\mathbb{H}}(F, \mathbb{H}^n)$. Among the various properties of the $\mathbb{H}$-perimeter, we just recall the invariance with respect to left-translations and the $(Q-1)$-homogeneity with respect to dilations.

It is also worth recalling that the definition of $\mathbb{H}$-perimeter is closely related to that of $BV_{\mathbb{H}}$ space, hence it can be useful to define the horizontal (distributional) gradient of a function $f : \mathbb{H}^n \to \mathbb{R}$:

$$\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f).$$

If F has Lipschitz boundary in Ω (in the Euclidean sense) we have the following *integral representation* of the $\mathbb{H}$-perimeter, as a consequence of Green's formulae:

$$P_{\mathbb{H}}(F, \Omega) = \int_{w \in \partial F \cap \Omega} |C(w) \cdot \nu_F(w)| \, \partial \mathcal{H}^{2n}(w), \tag{2.4}$$

where $\mathcal{H}^{2n}$ is the Euclidean $(2n)$-dimensional Hausdorff measure, $\nu_F(w)$ is the Euclidean normal vector to ∂F at w, and $C(w)$ is the $(2n \times 2n+1)$-matrix whose rows are given by the components of the vector fields $X_i(w)$ and $Y_i(w)$, i.e.,

$$C(w) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 2y_1 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 2y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 2y_n \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2x_1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & -2x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -2x_n \end{bmatrix}, \qquad w = [x+iy, t]. \tag{2.5}$$

Another way of computing the $\mathbb{H}$-perimeter, at least for a suitable subclass of measurable sets, is given by the *Minkowski content*, defined as the following limit (if it exists):

$$\mathcal{M}_B(F, \Omega) = \lim_{\epsilon \to 0^+} \frac{|(F.B_\epsilon) \cap \Omega| - |F \cap \Omega|}{\epsilon}, \tag{2.6}$$

where $B(P, r)$ denotes the open ball of center P and radius r with respect to d_c (we shall keep this notation in the sequel and simply write B_r when $P = [0, 0]$). Indeed, as a particular case of a more general result of [14], we have the following

Theorem 2.1 (Monti-Serra Cassano) *Let $E \subset \mathbb{H}^n$ be a bounded, open set of class C^∞. Then*

$$\mathcal{M}_B(F) = P_{\mathbb{H}}(E).$$

Going back to the isoperimetric problem, we now recall the theorem proved in [10] about the existence of isoperimetric sets in any Carnot group, here specialized to the context of $\mathbb{H}^n$:

Theorem 2.2 (Leonardi-Rigot) *For all $v > 0$, there exists a measurable set $I \subset \mathbb{H}^n$ which is isoperimetric, that is, it verifies $|I| = v$ and minimizes the $\mathbb{H}$-perimeter among all measurable $F \subset \mathbb{H}^n$ such that $|F| = v$. Moreover, I is open, bounded and connected, its boundary is Alhfors-regular and verifies the condition B (a kind of good geometric separation property) in an intrinsic sense, and it is also a domain of isoperimetry (i.e., a relative isoperimetric inequality holds on I).*

Again, we remark that this result is not enough to completely identify isoperimetric sets, since the recovered properties are too generic.

3 $\mathbb{H}$-bubbles

The purpose of this section is to give new properties of $\mathbb{H}$-bubbles. We first show that they are isoperimetric sets within a particular class of sets of finite perimeter, that we now define:

Definition 3.1 *We call $\mathcal{F}$ the class of sets F of finite perimeter whose boundary ∂F can be written, up to left-translations, as $\partial^+ F \cup \partial^- F$, with $\partial^+ F$ and $\partial^- F$ being the graphs of, respectively, $g(z)$ and $-g(z)$, where $g(z) = f(|z|)$ is a smooth, nonnegative, radial function defined on some $2n$-ball $D_r \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ of radius r centered at 0, and such that $g = 0$ on ∂D_r.*

As mentioned in the introduction, we shall also prove that, as for Euclidean balls in $\mathbb{R}^n$, our $\mathbb{H}$-bubbles have constant mean $\mathbb{H}$-curvature, following a definition that has been proposed by Pauls in [16] and that we recall:

Definition 3.2 *Let Ω be an open subset of $\mathbb{R}^2$ and $g : \Omega \to \mathbb{R}$ a smooth function. Define the function $F(x, y, t) = g(x, y) - t$, whose zero level-set is precisely the graph of g in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$. Then, the quantity*

$$H_{cc}(g) = -\operatorname{div}_{\mathbb{H}} \frac{\nabla_{\mathbb{H}} F}{|\nabla_{\mathbb{H}} F|}$$

is called the mean $\mathbb{H}$-curvature of the graph of g.

We now can state our main result about $\mathbb{H}$-bubbles.

Theorem 3.3 *There exists a unique (up to dilations and left-translations) solution E to the isoperimetric problem within the restricted class $\mathcal{F}$. This solution is called a $\mathbb{H}$-bubble and has the following properties*

1. *the mean $\mathbb{H}$-curvature of ∂E is constant;*
2. *∂E coincides with the union of all infinite geodesics connecting the north pole N and the south pole S of E (see figure 2).*

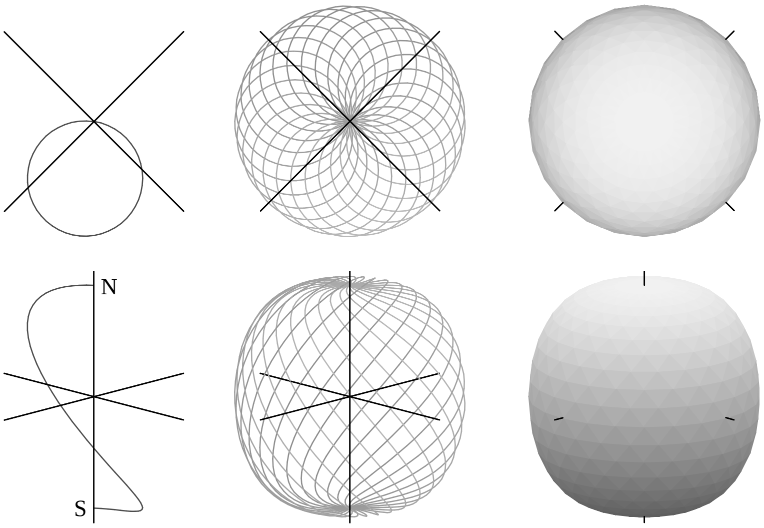

Figure 2: From left to right: one of the geodesics connecting S and N, few geodesics and, finally, the total geodesic envelope corresponding to the boundary of a $\mathbb{H}$-bubble in $\mathbb{H}^1 \simeq \mathbb{R}^3$. Figures on the top are views from above.

PROOF. The first part of the statement, that is the existence of a solution to the isoperimetric problem in $\mathcal{F}$, is somehow a known fact (see for instance [13]), but we give the proof for the sake of completeness. Using the integral representation (2.4), we can compute the isoperimetric ratio of any $F \in \mathcal{F}$ from its associated function $f(\rho)$, with $\rho \in [0, r]$. Indeed, we first compute the $\mathbb{H}$-perimeter of F by using (2.4) and the fact that $\partial^+ F$ and $\partial^- F$ give the same contribution to the whole perimeter (a consequence of the radial symmetry of f):

$$P_{\mathbb{H}}(F) = 2\int_{D_r} |C \cdot \nu| \sqrt{1 + |\nabla g|^2}\, dx\, dy = 4n\omega_{2n} \int_0^r \sqrt{4\rho^2 + f'(\rho)^2}\rho^{2n-1}\, d\rho,$$

where ω_{2n} denotes the volume of the unit ball in $\mathbb{R}^{2n}$. Then, the volume of F in terms of f is

$$|F| = 4n\omega_{2n} \int_0^r f(\rho)\rho^{2n-1}\, d\rho.$$

At this point, by computing the Euler equation of the functional $C_I(f) = P_{\mathbb{H}}(F)/|F|^{\frac{Q-1}{Q}}$ one obtains

$$\left(\frac{\rho^{2n-1} f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}}\right)' = -\lambda \rho^{2n-1},$$

and hence, by a first integration,

$$\frac{f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}} = -\frac{\lambda_n}{2}\rho, \tag{3.1}$$

with $\lambda_n = \frac{(Q-1)P_{\mathbb{H}}(F)}{nQ|F|}$. The assumed smoothness of ∂F implies that $f'(0) = 0$ and we obtain by integration the following solution to (3.1):

$$f(\rho) = f(0) + \frac{2}{\lambda_n^2}\left(\lambda_n\rho\sqrt{1 - \left(\frac{\lambda_n\rho}{2}\right)^2} + 2\arccos\left(\frac{\lambda_n\rho}{2}\right) - \pi\right),$$

whence we infer that ρ must be less than or equal to $r = \frac{2}{\lambda_n}$; moreover, if we ask that $f(r) = 0$ then we obtain $f(0) = \frac{2\pi}{\lambda^2}$ and the unique solution to (3.1) having this property can be written as

$$f(\rho) = \frac{2}{\lambda_n^2}\left(\lambda_n\rho\sqrt{1 - \left(\frac{\lambda_n\rho}{2}\right)^2} + 2\arccos\left(\frac{\lambda_n\rho}{2}\right)\right). \tag{3.2}$$

This function uniquely determines a solution to the isoperimetric problem in $\mathcal{F}$, that we call E.

Property 1 can be checked by direct computation, using (3.2) and the definition of the mean $\mathbb{H}$-curvature. Indeed, taking $\lambda_n = 2$ and $\rho^2 = \sum_{i=1}^n x_i^2 + y_i^2$ as before, one obtains

$$\begin{aligned}
H_{cc}(g) &= \sum_{i=1}^n \partial_{x_i}\left(\frac{-\frac{f'(\rho)}{\rho}x_i + 2y_i}{\sqrt{4\rho^2 + f'^2(\rho)}}\right) + \partial_{y_i}\left(\frac{-\frac{f'(\rho)}{\rho}y_i - 2x_i}{\sqrt{4\rho^2 + f'^2(\rho)}}\right)\\
&= \sum_{i=1}^n \frac{2\rho^2(1-\rho^2) + x_i(\partial_{y_i}\rho)\sqrt{1-\rho^2} - y_i(\partial_{x_i}\rho)\sqrt{1-\rho^2}}{\rho^2(1-\rho^2)}\\
&= \sum_{i=1}^n \frac{2\rho^2(1-\rho^2)}{\rho^2(1-\rho^2)}\\
&= 2n.
\end{aligned}$$

In particular, one sees that $H_{cc}(g)$ is given exactly by the multiplier $n\lambda_n = \frac{(Q-1)P_{\mathbb{H}}(F)}{Q|F|}$, as happens in the Euclidean case.

Property 2 is verified, for example, by computing the equation of the surface given by the union of all geodesics between $[0,0]$ and $[0,\frac{1}{\pi}]$ (the latter point lying on the t-axis at distance $d_c = 1$ from the origin), since the general case can be recovered by scaling:

$$\begin{cases} x_i(s) = \dfrac{1}{2\pi}\left(A_i(1-\cos(2\pi s)) + B_i \sin(2\pi s)\right) \\ y_i(s) = \dfrac{1}{2\pi}\left(-B_i(1-\cos(2\pi s)) + A_i \sin(2\pi s)\right) \\ t(s) = \dfrac{1}{\pi}\left(s - \dfrac{\sin(2\pi s)}{2\pi}\right) \end{cases} \qquad s \in [0,1],\ \sum_i A_i^2 + B_i^2 = 1. \tag{3.3}$$

Then, by left-translating the envelope with the vector $[0, -\frac{1}{2\pi}]$ (in this case, the left-translation coincides with the Euclidean one) and by expressing t as a function of $\rho = \sqrt{\sum_i x_i^2 + y_i^2}$, one gets

$$t(\rho) = \frac{\rho}{\pi}\sqrt{1-\pi^2\rho^2} + \frac{1}{2\pi^2}\arccos(2\pi^2\rho^2 - 1),$$

which, thanks to the identity

$$\arccos(2\pi^2\rho^2 - 1) = 2\arccos(\pi\rho),$$

becomes

$$t(\rho) = \frac{\rho}{\pi}\sqrt{1-\pi^2\rho^2} + \frac{1}{\pi^2}\arccos(\pi\rho). \tag{3.4}$$

It is now easy to check that (3.4) corresponds to (3.2) when $\lambda_n = 2\pi$. $\square$

Remark 3.4 The boundary of any $\mathbb{H}$-bubble E is, *up to left translations*, the union of ∂^+E and ∂^-E (see Definition 3.1). By scaling, any point of ∂^+E is of the general form $(ax, ay, a^2 f(\sqrt{x^2+y^2})$ where $\sum_i x_i^2 + y_i^2 \leq 1$ and f is the function defined on $[0,1]$ by

$$f(\rho) = \rho\sqrt{1-\rho^2} + \arccos(\rho) \tag{3.5}$$

The parameter a will be called the *horizontal radius* of E.

4 Failure of the Brunn-Minkowski based strategy

As already said in the introduction, the Brunn-Minkowski based strategy for proving that $\mathbb{H}$-bubbles are isoperimetric in a general sense fails, as implied by the following proposition (here specialized to $\mathbb{H}^1$ for more convenience):

Proposition 4.1 *Let E be a $\mathbb{H}$-bubble in $\mathbb{H}^1$. There exists $C > 0$, depending only on the horizontal radius of E (see Remark 3.4), such that, for ϵ small enough,*

$$|\delta_{1+\epsilon}E| - |E.\delta_\epsilon E| \geq C\epsilon \tag{4.1}$$

The proof of Proposition 4.1 will be given at the end of this paper. Now, for the reader's convenience, we prove a slightly different fact, i.e. that conditions (C2) and (C3) given in the introduction are actually incompatible. It is of course redundant in view of Proposition 4.1, but its proof is quite simpler and illustrates better the close connection between perimeter, Minkowski content and volume.

Proposition 4.2 *Let E be a $\mathbb{H}$-bubble in $\mathbb{H}^1$. Then, assuming the following properties*

(i) the upper Minkowski content associated with E, i.e.

$$\mathcal{M}_E(F) := \limsup_{\epsilon\to 0} \frac{|F.\delta_\epsilon(E)| - |F|}{\epsilon}$$

satisfies $\mathcal{M}_E(F) = \alpha P_{\mathbb{H}}(F)$ for all F with smooth boundary, α being a positive constant depending only on $|E|$,

(ii) $|E.\delta_\epsilon(E)| = |\delta_{1+\epsilon}(E)| + o(\epsilon)$,

leads to a contradiction.

Proof. From (i) and (ii) we get that

$$\alpha P_{\mathbb{H}}(E) = \lim_{\epsilon\to 0} \frac{[(1+\epsilon)^4 - 1]|E|}{\epsilon} = 4|E|. \tag{4.2}$$

On the other hand, it is not difficult to see that, by computing $\mathcal{M}_E(F_\delta)$ for

$$F_\delta := \{(x+iy,t) :\ x\in[0,\delta],\ y\in[0,1],\ t\in[0,1]\}$$

and owing to (i), one should obtain that $\mathcal{M}_E(F_\delta)\to 2\alpha$ as $\delta\to 0$, but this is possible only if the horizontal radius of E is exactly equal to α. Now, we compute the multiplier $\lambda_1 = \lambda_1(E)$ associated to E, by recalling its definition (see (3.1) for $n=1$) and thanks to (4.2):

$$\lambda_1(E) = \frac{3P_{\mathbb{H}}(E)}{4|E|} = \frac{3}{\alpha}. \tag{4.3}$$

Then, observe that, by rescaling E with factor $\frac{1}{\alpha\pi}$, one obtains $E_{\frac{1}{\pi}} = \delta_{\frac{1}{\alpha\pi}}(E)$ with horizontal radius $\frac{1}{\pi}$ and $\lambda_1(E_{\frac{1}{\pi}}) = 2\pi$ (as computed by taking the particular geodesic envelope in (3.3) with $n=1$). By the first equation in (4.3) and by the scaling properties of $\mathbb{H}$-perimeter and volume, we deduce that

$$\lambda_1(E) = \lambda_1(\delta_{\alpha\pi}(E_{\frac{1}{\pi}})) = \frac{\lambda_1(E_{\frac{1}{\pi}})}{\alpha\pi} = \frac{2}{\alpha},$$

which contradicts (4.3). $\square$

As mentioned in the introduction (see the paragraph below equation (1.5)), the following corollary is an easy consequence of Proposition 4.1.

Corollary 4.3 *The Brunn-Minkowski inequality in $\mathbb{H}^1$ is false for pairs of the type $(E,\delta_\epsilon E)$ where E is an $\mathbb{H}$-bubble and ϵ is small enough.*

Appendix

Proof of Proposition 4.1. By a scaling and left-translation argument, there is no loss of generality if we assume that E has unit horizontal radius and is centered at the origin. Thus, in view of (3.5), the boundary of E can be totally described by the function f defined on $[0,1]$ by

$$f(\rho) = \rho\sqrt{1-\rho^2} + \arccos(\rho).$$

Thanks to the symmetries of E, we can restrict our study to the upper half-space $\{t \geq 0\}$. It is easily seen that the coordinates of any point of $\partial^+(E.\delta_\epsilon E)$ can be written as

$$(a+\epsilon x, b+\epsilon y, f(\sqrt{a^2+b^2}) + \epsilon^2 f(\sqrt{x^2+y^2}) + 2\epsilon(bx - ay))$$

for a suitable choice of $(a,b) \in D_1$ and $(x,y) \in D_1$, with D_1 denoting the unit disk in $\mathbb{R}^2$. The corresponding point of $\partial^+(\delta_{1+\epsilon}E)$ with same z-component has coordinates:

$$(a+\epsilon x, b+\epsilon y, (1+\epsilon)^2 f(\frac{\sqrt{(a+\epsilon x)^2 + (b+\epsilon y)^2}}{1+\epsilon})).$$

In order to show that $E.\delta_\epsilon E \subset \delta_{1+\epsilon}E$ and to estimate $|\delta_{1+\epsilon}E \setminus E.\delta_\epsilon E|$, we shall minimize with respect to a, b, x, y the following function:

$$F(a,b,x,y,\epsilon) = (1+\epsilon)^2 f(\frac{\sqrt{(a+\epsilon x)^2 + (b+\epsilon y)^2}}{1+\epsilon}) - f(\sqrt{a^2+b^2}) - \epsilon^2 f(\sqrt{x^2+y^2}) - 2\epsilon(bx - ay).$$

Indeed, let $A = (\alpha + i\beta, t_A)$ a point on $\partial^+(\delta_{1+\epsilon}E)$ and $B = (\alpha + i\beta, t_B)$ a corresponding point on $\partial^+(E.\delta_\epsilon E)$. It is easily seen that

$$t_A - t_B = \min_{\substack{a,b,x,y \\ a+\epsilon x = \alpha \\ b+\epsilon y = \beta}} F(a,b,x,y,\epsilon) \tag{4.4}$$

Since the function F is invariant with respect to any rotation applied simultaneously to (a,b) and (x,y), we can restrict our study to the case $a \in [0,1]$, $b = 0$ and the new function to study is

$$\phi_{a,\epsilon}(x,y) := F(a,0,x,y,\epsilon) = (1+\epsilon)^2 f(\frac{\sqrt{(a+\epsilon x)^2 + \epsilon^2 y^2}}{1+\epsilon}) - f(a) - \epsilon^2 f(\sqrt{x^2+y^2}) + 2\epsilon a y.$$

To overcome the difficulties of a direct minimization of F, we shall rather look for a lower bound of $\min\limits_{(x,y)\in B_1} \phi_{a,\epsilon}(x,y)$, which is uniform with respect to a. It is easily seen that, given $a \in [0,1]$, the minimal value of $f(\frac{\sqrt{(a+\epsilon x)^2+\epsilon^2 y^2}}{1+\epsilon})$ is attained for $x \geq 0$ (because of the monotonicity of f) whereas there is no dependence of the other terms on the sign of x. Analogously, $2\epsilon a y$ is minimal for $y \leq 0$ whereas the sign of y plays no role in the other terms. To sum up, our aim is to provide a suitable estimate of $\phi_{a,\epsilon}(x,y)$ from below, under the conditions

$$0 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq 0,$$

at each $a \in [0,1]$ and $\epsilon > 0$. We now prove the following facts: there exists $a_0 \in (0,1)$ and $\epsilon_0 > 0$ such that

(i) for any $a \in [0, a_0]$, the corresponding minimum of $\phi_{a,\epsilon}(x,y)$ behaves like $C\epsilon$ for all $0 < \epsilon < \epsilon_0$, where $C > 0$ depends only on a_0;

(ii) the minimum of $\phi_{a,\epsilon}(x,y)$, when $a \in (a_0, 1]$, is non-negative for all $0 < \epsilon < \epsilon_0$.

The two properties above are enough to conclude that $\delta_{1+\epsilon}E$ contains strictly $(E.\delta_\epsilon E)$ and moreover that the volume of $\delta_{1+\epsilon}E \setminus (E.\delta_\epsilon E)$ grows like $C\epsilon$, which implies that (4.1) is satisfied, at least for ϵ sufficiently small.

Let us begin with the proof of something stronger than (i), that is, we take any $a_0 \in (0,1)$ and will prove the existence of $C > 0$ and $\epsilon_0 > 0$ such that $\min_{x,y} \phi_{a,\epsilon}(x,y) \geq C\epsilon$ for all $a \in [0, a_0]$ and $\epsilon \in [0, \epsilon_0]$. Indeed, for any $\delta \in (0, 1/2)$, the function $(a, x, y) \mapsto F(a, 0, x, y, \epsilon)$ is of class C^2 on D_δ, where

$$D_\delta = \{(a, x, y, \epsilon) : \; a \in [\delta, 1-\delta], \; x \in [0,1], \; y \in [-\sqrt{1-x^2}, 0], \; \epsilon \in [0, \delta/2]\}.$$

It means in particular that $M_\delta = \max_{D_\delta} |\frac{\partial^2 F}{\partial \epsilon^2}(a, 0, x, y, \epsilon)|$ is finite and, by Taylor's formula, one can show that

$$\phi_{a,\epsilon}(x,y) \geq [\frac{\partial F}{\partial \epsilon}(a, 0, x, y, 0)]\,\epsilon - M_\delta \epsilon^2 \geq [2\arccos(1-\delta)]\,\epsilon - M_\delta \epsilon^2. \tag{4.5}$$

To prove this, it suffices to observe that

$$\begin{aligned}
\frac{\partial F}{\partial \epsilon}(a, 0, x, y, 0) &= 2f(a) + \frac{2a^3}{\sqrt{1-a^2}} - \frac{2a^2 x}{\sqrt{1-a^2}} + 2ay \\
&\geq 2f(a) + \frac{2a^3}{\sqrt{1-a^2}} - \frac{2a^2 x}{\sqrt{1-a^2}} - 2a\sqrt{1-x^2} \\
&= 2\arccos(a) + \frac{2a}{\sqrt{1-a^2}}(1 - ax - \sqrt{1-x^2}\sqrt{1-a^2}) \\
&\geq 2\arccos(a) + \frac{2a}{\sqrt{1-a^2}}(\frac{x^2+a^2}{2} - ax) \\
&\geq 2\arccos(1-\delta).
\end{aligned}$$

Now, take $0 < \eta < \frac{\pi}{8}$ such small that

$$f(\eta) \geq \frac{\pi}{4} \qquad \text{and} \qquad \min_{t \in [0,\eta]} f'(t) \geq -\frac{\pi}{8}.$$

Then, for $a, \epsilon \in [0, \frac{\eta}{2}]$ one easily obtains

$$\begin{aligned}
\phi_{a,\epsilon}(x,y) &\geq (1+\epsilon)^2 f(a+\epsilon) - f(a) - \epsilon^2 f(0) - 2\epsilon a \\
&\geq [f(a+\epsilon) - f(a)] + 2f(\eta)\epsilon - \epsilon^2 f(0) - \epsilon\eta \\
&\geq [\min_{t\in[0,\eta]} f'(t) + 2f(\eta) - \eta]\epsilon - f(0)\epsilon^2 \\
&\geq \frac{\pi}{4}\epsilon - \frac{\pi}{2}\epsilon^2
\end{aligned}$$

We therefore conclude that, whenever $a \in [0, 1-\delta]$ for $\delta < \eta/2$ (or, in other words, $a \in [0, a_0]$ with $a_0 > 1 - \eta/2$), the minimum of $\phi_{a,\epsilon}(x,y)$ grows at least like $C\epsilon$ for all $0 < \epsilon < \epsilon_0$, for some $C, \epsilon_0 > 0$ depending only on δ (or a_0), which proves (i).

To prove (ii), we need to estimate $\phi_{a,\epsilon}(x,y)$ more carefully. First, we write $\phi_{a,\epsilon}(x,y) = A + B + C$, where

$$\begin{aligned}
A &= \sqrt{(a+\epsilon x)^2 + \epsilon^2 y^2}\sqrt{(1+\epsilon)^2 - [(a+\epsilon x)^2 + \epsilon^2 y^2]} - a\sqrt{1-a^2} + \epsilon\arccos(a) + 2ay\epsilon, \\
B &= (1+\epsilon)^2 \arccos\left(\frac{\sqrt{(a+\epsilon x)^2 + \epsilon^2 y^2}}{1+\epsilon}\right) - (1+\epsilon)\arccos(a), \\
C &= -[\arccos(\sqrt{x^2+y^2}) + \sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}]\epsilon^2.
\end{aligned}$$

Then, thanks to the monotonicity and concavity of $\arccos(r)$ when $r \in [0,1]$, we get that:

$$B \geq (1+\epsilon)[(1+\epsilon)\arccos(\frac{a+\epsilon}{1+\epsilon}) - \arccos(a)] \geq 0,$$

$$C \geq -\pi\epsilon^2.$$

An estimate of A is more delicate, since a uniform control on the second derivative of F with respect to ϵ is not possible when a tends to 1. To simplify, we first drop the dependence on the variable y by observing that the function $r\sqrt{1-r^2}$ is decreasing when $r \in [1/\sqrt{2}, 1]$, hence for a sufficiently close to 1 and ϵ small, we can minimize with respect to y by taking its largest absolute value, i.e. $y = -\sqrt{1-x^2}$. Therefore, we are led to minimize the function

$$h_{a,\epsilon}(x) = \phi_{a,\epsilon}(x, -\sqrt{1-x^2}) = \sqrt{a^2+\epsilon^2+2ax\epsilon}\sqrt{(1+\epsilon)^2 - [a^2+\epsilon^2+2ax\epsilon]} - 2a\epsilon\sqrt{1-x^2}$$
$$- a\sqrt{1-a^2} + \epsilon\arccos(a).$$

Its first derivative is

$$h_{a,\epsilon}{}'(x) = a\epsilon\left[\frac{2x}{\sqrt{1-x^2}} + \frac{1-2a^2+2\epsilon-\epsilon^2-4a\epsilon x}{\sqrt{1-a^2+2\epsilon-2a\epsilon x}\sqrt{a^2+\epsilon^2+2a\epsilon x}}\right]$$

and one finds that

$$h_{a,\epsilon}{}'(x_0) = 0 \qquad \text{if and only if} \qquad x_0 = \frac{2a^2-1-2\epsilon+\epsilon^2}{1+\epsilon(2+\epsilon-4a)}.$$

Then we calculate that

$$h_{a,\epsilon}(x_0) = a\sqrt{1-a}[\sqrt{1+a+2\epsilon} - \sqrt{1+a}] + \epsilon[\arccos(a) - \sqrt{(1-a)(1+a+2\epsilon)}]$$

under the condition $\epsilon < \min(a, 1-\frac{1}{\sqrt{2}})$. Using the fact that $\arccos(a) \geq \sqrt{2(1-a)}$ and $\epsilon < a$, we obtain the following inequalities:

$$h_{a,\epsilon}(x_0) \geq \frac{\epsilon\sqrt{1-a}(1+a-2\epsilon)}{\sqrt{2}+\sqrt{1+a+2\epsilon}} \geq 0.$$

It remains to check the positivity of $h_{a,\epsilon}(x)$ at $x = 0, 1$. It is easily seen that $h_{a,\epsilon}(1) \geq 0$ for any $a \in [0,1]$ and $\epsilon > 0$. On the other hand, we get

$$\begin{aligned} h_{a,\epsilon}(0) &= \sqrt{(a^2+\epsilon^2)(1-a^2+2\epsilon)} - a\sqrt{1-a^2} - \epsilon(2a - \sqrt{2(1-a)}) \\ &\geq 2\epsilon\left[\frac{a^2}{\sqrt{(1+\epsilon^2)(1-a^2+2\epsilon)} + a\sqrt{1-a^2}} - \frac{2a^2+a-1}{\sqrt{2(1-a)}+2a}\right] \\ &= \epsilon K(a,\epsilon), \end{aligned}$$

and it is clear that $\lim_{(a,\epsilon)\to(1,0)} K(a,\epsilon) = +\infty$. In particular, there exists (a_0, ϵ_0) such that $K(a,\epsilon) \geq 0$ and all the previous estimates hold for any $a > a_0$ and $\epsilon < \epsilon_0$. This concludes the proof of (ii) and of the proposition. □

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