Upper bound bifurcation time in exponent related dynamic evolution for the average distance functional

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July 27, 2011

Abstract

In this paper we consider the dynamic irreversible evolution of a connected network related to an average distance functional minimization problem, with associated dissipation term. Our goal is to determine whether and when new branches may appear. Tools belonging to minimizing movements and optimal transportation theory with free Dirichlet regions will be used extensively. Then we will show an application of conditions found to a particular class of configurations, and give an upper bound estimate for the branching time for them.

Keywords: optimal transport, Euler scheme, minimizing movements, average distance

2010 Mathematics Subject Classification: 49Q10, 49J40, 49L15

1 Introduction

De Giorgi in [11] presented the so-called “minimizing movements theory” to study evolution problems endowed with a variational structure. In this paper we will consider the general dynamic, rate independent, evolution for connected networks related to the average distance functional, and our main goal is to determine whether optimal sets will exhibit a bifurcation at some time.

Let \( \Omega \) be a compact connected subset of \( \mathbb{R}^2 \), let us define

\[
A_l(\Omega) := \{ \mathcal{X} \subseteq \Omega : \mathcal{X} \text{ compact, connected}, \dim_{\mathcal{H}} \mathcal{X} = 1 \text{ and } \mathcal{H}^1(\mathcal{X}) \leq l \}, \quad A(\Omega) := \bigcup_{j \geq 0} A_j.
\] (1.1)

As we can see both \( A_l \) and \( A \) depend on the domain \( \Omega \), but in order to simplify notations, when the domain is clear and there is no risk of confusion, we will omit this dependence.

Now we introduce the main functionals of this paper: the first is

\[
F : A \rightarrow (0, \infty), \quad F(S) := \int_{\Omega} \text{dist}(x, S)dx,
\]
which will be referred as “energy” in the following. Then, as we are studying dynamic evolutions, another functional, the “dissipation” is required:

**Definition 1.1.** Given a domain $\Omega$, a dissipation is a functional

$$D : A^2 \rightarrow [0, \infty]$$

which verifies:

- $D(S, S) = 0$ for any $S \in A$,
- $D(S_1, S_2) > 0$ for any $S_1, S_2 \in A$ with $\mathcal{H}^1(S_1 \Delta S_2) > 0$

Notice that the dissipation is generally not symmetric. For the measure choice in the definition of $F$, as we will see in the following, the Lebesgue measure copes superbly in relating geometric extension of a set with its measure.

Let us introduce a first result on the monotonicity of $F$:

**Proposition 1.2.** Given a domain $\Omega$, for any $S_1, S_2 \in A$, with $S_1 \subseteq S_2$, we have $F(S_1) \geq F(S_2)$.

**Proof.** The proof is easy: as $S_1 \subseteq S_2$, for any $x \in \Omega$ we have

$$\text{dist}(x, S_1) \geq \text{dist}(x, S_2)$$

and integrating on $\Omega$

$$\int_{\Omega} \text{dist}(x, S_1)dx \geq \int_{\Omega} \text{dist}(x, S_2)dx$$

which concludes the proof.

Moreover we see from the proof of Proposition 1.2 that if there exists $\Omega' \subseteq \Omega$ with $L^2(\Omega') > 0$ and $\text{dist}(x, S_1) > \text{dist}(x, S_2)$ for any $x \in \Omega'$, then $F(S_1) > F(S_2)$.

This result says that prescribing the maximum length is the same as prescribing the length, i.e. for any $h > 0$

$$\min_{X \in A_h} F(X) = \min_{H^1(X') = h} F(X')$$

and

$$\argmin_{A_h} F = \argmin_{A_h \setminus \bigcup_{0 \leq h' < h} A_{h'}} F.$$

So in the paper, we can use these two constraints indifferently.

This paper will be structured as follows:

- in Section 2 we will present some sharp estimates for the energy functional, and introduce the minimizing movements;
- in Section 3 we will determine the “suitable” evolution, and then give sufficient conditions to force a branching behavior;
- in Section 4 we try to build a particular class of configurations for which results in Section 3 can be easily applied, and we will give an upper bound for their branching time.
Notations

The most used in this paper will be:

- $\Omega$ to denote the domain,
- $\Sigma$ to denote the minimizing movement function,
- $\varepsilon, \xi, \eta, \delta, r, \rho$ to denote small positive number,
- $l, a$ to denote generic positive number,
- $S$ to denote generic connected compact sets in the domain,
- $S_0$ to denote the initial datum of an Euler scheme/minimizing movement,
- $w(k, \cdot), w(k) \ (k \in \mathbb{N})$ to denote the $(k + 1)$-th set of an Euler scheme.

To avoid using excessive number of different notations, some symbols will be used in more situations: unless explicitly specified, if a notation is used in two different Definitions/Propositions/Lemma/Theorems, there is no connection between them, so there is no risk of confusion.

The only notable exceptions are

- $A_l (\text{with } l \geq 0)$, and $A$: if there is a given domain $\Omega$, they always denote the sets defined in (1.1),
- $F$ which always stands for the average distance functional
- $V(\cdot)$ which stands for the Voronoi cell of the point.

Moreover, we will work only with domains in $\mathbb{R}^2$ which are closure of an open connected bounded and sufficiently regular set, and the word “domain” will always refer to a similar domain.

2 Preliminaries

In this section we first present some estimates for the energy functional; then we will introduce the minimizing movements, and determine the “suitable” ones.

2.1 Energy estimates

The first is an easy upper bound:

**Lemma 2.1.** Given a domain $\Omega$, let be $S_1, S_2 \in A, S_1 \subset S_2$, then

$$F(S_1) - F(S_2) \leq O(\max_{y \in S_2} \text{dist}(y, S_1)).$$
Proof. Writing the thesis explicitly we have

\[ F(S_1) - F(S_2) = \int_{\Omega} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx \]

and as \( S_1 \subset S_2 \)

\[ \int_{\{x \in \Omega: \text{dist}(x, S_1) = \text{dist}(x, S_2)\}} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx = 0. \]

So the thesis is equivalent to estimate

\[ \int_{\{x \in \Omega: \text{dist}(x, S_1) > \text{dist}(x, S_2)\}} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx. \]

As for an point \( z \in \Omega \) we have

\[ \text{dist}(z, S_1) \leq \text{dist}(z, S_2) + \max_{y \in S_2} \text{dist}(y, S_1) \]

so integrating we have

\[ \int_{\Omega} \text{dist}(z, S_1) \leq \int_{\Omega} \text{dist}(z, S_2) + |\Omega| \max_{y \in S_2} \text{dist}(y, S_1) \]

or equivalently

\[ F(S_1) - F(S_2) \leq |\Omega| \max_{y \in S_2} \text{dist}(y, S_1), \]

and the proof is complete. \( \square \)

Now we want a lower bound:

**Definition 2.2.** Given a domain \( \Omega \), \( S \in A \) a generic element, a non endpoint \( P \in S \) is “smooth” if there exists \( r > 0 \) such that:

1. there exists an homeomorphism \( f : B(P, r) \cap S \rightarrow (0, 1) \);
2. there exists an unique direction \( \theta \) such that for any sequence \( P_n \rightarrow P \) in \( B(P, r) \) the directions of the line \( L(P_n, P) \) converge to \( \theta \).

A subset of \( S \) is smooth is all its non endpoints are smooth.

**Proposition 2.3.** Given a domain \( \Omega \), let be \( S \subset \Omega \) be a connected set, if we add a segment \( \lambda_{\varepsilon} \) to a smooth non endpoint of \( S \) (with \( H^1(\lambda_{\varepsilon}) = \varepsilon \)), then the “gain” \( F(S) - F(S_{\varepsilon}) \) is comparable with \( \varepsilon^{3/2} \), where \( S_{\varepsilon} := S \cup \lambda_{\varepsilon} \).

Fig. 1: All the shaded area, whose area is comparable with \( \varepsilon^{1/2} \), gains something in path.
Proof. Upon rescaling, the configuration can be brought to the following in figure, so all the computations can be done here.

Fig. 2: For graphical purposes the borders are a bit larger, but the considered domain is \([-1,1] \times [0,1]\); notice that \(X, Y, W, Z\) are not on the border, but they are the midpoints between the \(y\) axis and the intersections of the border with \(y = \varepsilon\) and \(y = 1\) respectively.

Adding such a segment to \(S\), the gain is on the shaded region; if a point \((x, y)\) can choose a shorter path, then it must satisfy

\[
\text{dist}((x, y), S \cup \lambda_\varepsilon) < \text{dist}((x, y), S)
\]

thus

\[
(x^2 + (y - \varepsilon)^2)^{1/2} < |y|,
\]

which leads to

\[
y > \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2}.
\]

Now we have to estimate its area: as we are doing our computation in the rectangle \([-1,1] \times [0,1] \subset \mathbb{R}^2\), the parabola has boundaries defined by the last inequality and \([-1,1] \times \{1\}\). The intersections between \(\{(x, y) : y = \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2}\}\) and \([-1,1] \times \{1\}\) are

\[
x^\pm := \pm \sqrt{2\varepsilon - \varepsilon^2}.
\]

So the area of the shaded region is

\[
2\sqrt{2\varepsilon - \varepsilon^2} - \int_{x^-}^{x^+} \left(\frac{t^2}{2\varepsilon} + \frac{\varepsilon}{2}\right)dt = \frac{2}{3}\sqrt{2\varepsilon - \varepsilon^2}(2 - \varepsilon).
\]

The parabola contains the trapezium \(XYWZ\), and \(\mathcal{H}^1(XY) = \varepsilon, \mathcal{H}^1(WZ) = \sqrt{2\varepsilon - \varepsilon^2}\) and the height is \(1 - \varepsilon\). The gain in path here is at least \(\varepsilon/2\) (this minimum is attained on points \(X\) and \(Y\)), so the gain for the energy functional is at least

\[
\frac{\varepsilon}{2}(1 - \varepsilon - \sqrt{2\varepsilon - \varepsilon^2}) \geq \frac{\varepsilon^{3/2}}{8},
\]

so the choice \(K = 1/8\) is acceptable. \(\square\)
From the proof we can see that given any $h > 0$, if we rescale all the configuration with the transformation
$$x \mapsto xh, \quad y \mapsto yh$$
then the same argument gives that the lower bound for the gain in energy scales by the factor $h^2$.

Here is another lower bound for the energy functional:

**Proposition 2.4.** Given a domain $\Omega$, let $S \in A$ be a smooth set, and let it have an endpoint $O$ which satisfies:

(*) there exist $\rho, \theta > 0$ and a triangle $T' \subset V(O)$ with a vertex in $O$ and sides $\rho, \rho, \rho \sqrt{2}(1 - 2 \cos \theta)$ (the order is not relevant) that does not intersect $S$.

Then there exists $\varepsilon_0$ such for any $\varepsilon < \varepsilon_0$ adding a segment $\lambda_\varepsilon$ at $O$, with $H^1(\lambda_\varepsilon) = \varepsilon$ in $O$ is more convenient that adding any connected set with same length at any non endpoint.

![Fig. 4: the presence of the shaded triangle $T'$ makes adding at an endpoint more convenient than at a non endpoint at least when the added portion has sufficient small length.](image)

*Proof.* Adding $\lambda_\varepsilon$ at a smooth non endpoint, as stated in Proposition 2.3, will decrease the energy by a quantity comparable with $\varepsilon^{3/2}$, and from the proof in [6], bounded by $\varepsilon^{3/2}\text{diam } \Omega$. But adding it at $O$ and in the shaded triangle, with $\varepsilon$ small enough, will cause:

$$F(S_\varepsilon) - F(S) \leq \int_{\Omega} \text{dist}(x, S_\varepsilon) dx - \int_{\Omega} \text{dist}(x, S) dx \leq -C\varepsilon L^2(T).$$

Now we estimate a lower bound value for $C$: if we add the segment $\lambda_\varepsilon$ at $O$, along the bisector of the marked angle in Figure 4 (whose value is $\theta$), then all points on $JKK'J'$ (where $J', K'$ are midpoints of segment $OJ$ and $OK$) will have a gain in path to $S$ at least

$$\frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - \frac{\varepsilon \rho}{2} \cos \frac{\theta}{2}} \approx \varepsilon \cos \frac{\theta}{2} - O(\varepsilon^2) \quad (2.1)$$

as this is the gain of points on $OJ$ and $OK$, and points inside gain even more. Notice that

$$\varepsilon \cos \frac{\theta}{2} - O(\varepsilon^2) > \frac{\varepsilon}{2} \cos \frac{\theta}{2}$$
for any \( \varepsilon < \frac{\rho}{2} \cos \frac{\theta}{2} \), the total gain in energy is not less than \( \frac{\varepsilon}{2} \cos \frac{\theta}{2} \) multiplied by the area of trapezium \( JK'KJ' \), i.e.

\[
\frac{3}{8} \varepsilon \cos \frac{\theta}{2} \mathcal{L}^2(T').
\]

So for \( \varepsilon \) such that \( \frac{3}{8} \varepsilon \cos \frac{\theta}{2} \mathcal{L}^2(T') > \varepsilon^{3/2} \text{diam } \Omega \), i.e.

\[
\varepsilon < \left( \frac{3 \cos \frac{\theta}{2} \mathcal{L}^2(T')}{8 \text{diam } \Omega} \right)^2 \wedge \left( \frac{\rho \cos \frac{\theta}{2}}{2} \right)
\]

we have that adding \( \lambda \varepsilon \) to \( O \) is more convenient than adding it at an non endpoint.

This result can be slightly generalized to another class of points:

**Definition 2.5.** Given a domain \( \Omega, \) \( S \in A \) a generic element, a non endpoint \( P \in S \) is “angular” if there exists \( r > 0 \) such that:

1. there exists an homeomorphism \( f : B(P, r) \cap S \rightarrow (0,1) \), and without loss of generality, \( f(P) = 1/2 \);
2. there exist \( 0 < \theta < \pi \) such that for any sequences \( \{x_n\} \rightarrow 1/2, \{y_n\} \rightarrow 1/2 \) the value of angles \( f^{-1}(x_n)Pf^{-1}(y_n) \) accumulates in \( \{0, \theta\} \).

Geometrically, one may imagine that a point \( P \) is angular when the tangent vectors form an angle (or a cuspid) here. The following result holds:

**Lemma 2.6.** Given a domain \( \Omega \), let \( S \in A \) be an arbitrary element, and suppose there exists \( Q \in S \) angular and let be \( \delta > 0 \) such that \( B(Q, \delta) \cap S \) is homeomorphic to \((0,1)\). Then the Voronoi cell \( V(Q) \) contains a triangle \( T_Q \) with sides \( \rho_Q > 0 \) and angle \( \hat{Q} > 0 \).

Fig. 5: All the points in the shaded area belong to \( V(Q) \), and it contains a triangle.
For the proof we refer to [12].

So we can apply the same argument found in the proof of Proposition 2.4 to these points, essentially obtaining the same conclusion, i.e. adding a small segment of length $\varepsilon$ the gain for $F$ has order $O(\varepsilon)$. Thus the following result holds:

**Proposition 2.7.** Given a domain $\Omega$, let $S \in A$, and suppose that at least one of the following assumptions hold:

(i) there exists an angular non endpoint $Q$ and $\xi > 0$, such that $B(Q, \xi) \cap S$ is homeomorphic to $(0, 1)$;
(ii) there exists an endpoint $U$ and $\xi > 0$, such that $B(U, \xi) \cap S$ is homeomorphic to $(0, 1]$.

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$

$$\min_{J \in A_{\varepsilon} \setminus A_0} F(S \cup J) \leq F(S) - K\varepsilon$$

where $K$ is a positive constant dependent only on geometric quantities.

### 2.2 Minimizing movements

Let us recall here some notions about minimizing movements, first in the abstract case.

Let be $X$ a set, endowed with a convergence structure. Given a time $T > 0$, let be $F$ a functional

$$F : [0, T] \times X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}.$$ 

We present first the Euler scheme in the abstract case: given $\varepsilon > 0$ and $X_0 \in X$ an initial datum, let be

$$\begin{cases}
  w(0) = X_0 \\
  w(n+1) \in \text{argmin} F((n+1)\varepsilon, \cdot, w(n))
\end{cases}$$

and let us consider the function $u_\varepsilon : [0, T] \rightarrow X$ obtained by setting

$$u_\varepsilon(t) = w\left(\left\lfloor \frac{t}{\varepsilon} \right\rfloor\right),$$

with $\lfloor \cdot \rfloor$ denoting the integer part mapping.

The minimizing movement is someway a limit of Euler schemes with progressively smaller time step:

**Definition 2.8.** Given $T > 0$, the function $u : [0, T] \rightarrow X$ is a minimizing movement associated with initial datum $u_0$ and kinetic term $F$, and we will write $u \in MM(F, X, u_0)$ if there exists a sequence $\varepsilon_n \downarrow 0$ for which

$$\forall t \in [0, T] \quad u_{\varepsilon_n}(t) \rightarrow u(t).$$
In our case we are working on the space \( A \) endowed with the convergence in the Hausdorff distance, and we have to choose our functional \( F \); recall that we are working in irreversible context, thus a sort of irreversibility condition must be put in the dissipation term.

We will work with the following class of dissipation terms: given parameters \( \alpha > 1, \varepsilon > 0 \), we define

\[
D_{\alpha, \varepsilon} : A^2 \rightarrow [0, \infty], \quad D_{\alpha, \varepsilon}(X_1, X_2) := \begin{cases} \frac{H^1(X_1 \setminus X_2)^\alpha}{\varepsilon} & \text{if } X_2 \subset X_1 \\ \infty & \text{otherwise} \end{cases}.
\]

The functionals we are going to consider in this paper have all the form

\[
F_{\alpha, \varepsilon} : A^2 \rightarrow [0, \infty], \quad F_{\alpha, \varepsilon}(X_1, X_2) := F(X_1) + D_{\alpha, \varepsilon}(X_1, X_2)
\]

where \( \alpha, \varepsilon \) are parameters.

So, given parameters \( \alpha > 1, \varepsilon > 0 \), time \( T > 0 \), positive time step \( \xi > 0 \) and an initial datum \( S_0 \in A \), our Euler scheme is

\[
\begin{cases}
w(0) = S_0 \\
w(n + 1) \in \arg\min F_{\alpha, \varepsilon}(\cdot, w(n))
\end{cases}
\]

and \( \Sigma_\xi(t) := w \left( \frac{t}{v(\xi)} \right) \) can be defined, where \( v(\xi) \) is a suitable time scaling to be determined in the following.

Now fix parameters \( \alpha, \varepsilon, S \in A \), and let us analyze what happens if we add a set \( J_{\varepsilon'} \) with length \( \varepsilon' \) to \( S \): if at least one condition of Proposition 2.7 is verified, then the gain for \( F \) is of order \( O(\varepsilon) \), thus for \( F_{\alpha, \varepsilon} \) we have

\[
F_{\alpha, \varepsilon}(S, S) - F_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = F(S) - F(S \cup J_{\varepsilon'}) - \frac{H^1(J_{\varepsilon'})^\alpha}{\varepsilon}
\]

and analyzing its orders we have

\[
F_{\alpha, \varepsilon}(S, S) - F_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon') - \frac{O(\varepsilon'^\alpha)}{\varepsilon}
\]

If no condition of Proposition 2.7 is satisfied, then the gain has order \( O(\varepsilon^{3/2}) \), thus the orders read

\[
F_{\alpha, \varepsilon}(S, S) - F_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon^{3/2}) - \frac{O(\varepsilon'^\alpha)}{\varepsilon}.
\]

Without loss of generality we put \( \varepsilon' := \varepsilon^c \), with \( c > 0 \), thus the order equations read

\[
F_{\alpha, \varepsilon}(S, S) - F_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon^c) - O(\varepsilon^{ca-1})
\]

and

\[
F_{\alpha, \varepsilon}(S, S) - F_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon^{3c}) - O(\varepsilon^{ca-1})
\]

respectively.
Now we have to compute the optimal $c$: let us consider the first case. Maximizing
\[
F_{\alpha,\varepsilon}(S, S) - F_{\alpha,\varepsilon}(S \cup J', S) = O(\varepsilon^c) - O(\varepsilon^{c\alpha - 1})
\]
is equivalent to maximize
\[
H\varepsilon^c - \varepsilon^{c\alpha - 1}
\]
where the nature of $H$ will be discussed later. By direct differentiation we find that the optimal $c$ is
\[
c_{opt} := \frac{1}{\alpha - 1}(1 + \log \varepsilon H - \log \varepsilon \alpha).
\]

We have to estimate the value of $H$: it is clear that $H$ is upper bounded, the key point is to determine a lower bound for it:

- if we are under the hypothesis of Proposition 2.7, then $H$ is a positive constant, thus passing to the limit for $\varepsilon \downarrow 0$ we have
  \[
  \lim_{\varepsilon \downarrow 0} c_{opt} = \frac{1}{\alpha - 1};
  \]

- if not, Proposition 2.3 states that we can always have a gain comparable with $\varepsilon^{3/2}$. Moreover, as we are considering evolutions in a limited period of time, and the final set has limited length, from the proof of Proposition 2.3 we conclude that there always exists a constant $B > 0$ such that there exists a point for which we can scale to the configuration of Proposition 2.3, i.e. $H \geq C\varepsilon^{1/2}$ with some $C > 0$. Then passing to the limit we have
  \[
  \lim_{\varepsilon \downarrow 0} c_{opt} = \frac{3}{2\alpha - 1}.
  \]

The following result is very useful:

**Proposition 2.9.** Given a domain $\Omega$, an arbitrary initial datum $S_0 \in A$, and consider the Euler scheme (in a time interval $[0, T]$)
\[
\begin{align*}
  w(0) &:= S_0 \\
  w(k + 1) &\in \text{argmin}_{w(n) \subset X, H^1(X) \leq H^1(w(k)) + o(1)} F(X)
\end{align*}
\]
and let be:
\[
W_\varepsilon := \{ d' : w(d') is obtained from w(d' - 1) by adding some length at endpoints \}.
\]

Then, if $\lim_{\varepsilon \to 0} F(w \left( \left\lfloor \frac{T}{\varepsilon} \right\rfloor \right)) < F(S_0)$. Then we have $\limsup_{\varepsilon \to 0} \frac{|W_\varepsilon|}{|T/\varepsilon|} = D > 0$.

**Proof.** The proof is easy: the gain in energy at step $d$ is comparable with $O(\varepsilon)$ only if $d \in W_\varepsilon$, and not larger than $|\Omega|\varepsilon$ thus (upon in first order approximation):
\[
F(w(d - 1)) - F(d) \leq |\Omega|\varepsilon \chi_{W_\varepsilon}(d);
\]
repeating this argument we have

\[ F(S_0) - F(d) \leq |\Omega| \varepsilon \sum_{j=1}^{d} \chi_{W_\varepsilon}(j) \]

and then

\[ F(S_0) - F(w\left(\left\lceil \frac{T}{\varepsilon}\right\rceil\right)) \leq |\Omega| \varepsilon \sum_{j=1}^{\lfloor T/\varepsilon \rfloor} \chi_{W_\varepsilon}(j) \]

and clearly \( \sum_{j=1}^{\lfloor T/\varepsilon \rfloor} \chi_{W_\varepsilon}(j) = |W_\varepsilon| \). So if \( \frac{|W|}{T/\varepsilon} \to 0 \) means \( \frac{|W|}{\varepsilon + 1} \to 0 \), which implies \( \varepsilon |W_\varepsilon| \to 0 \) and ultimately

\[ \lim_{\varepsilon \to 0} F(S_0) - F(w\left(\left\lceil \frac{T}{\varepsilon}\right\rceil\right)) = 0 \]

which is a contradiction. \( \square \)

So we will assume that our kinetic has the form

\[ F_{\alpha,\varepsilon}(X_1, X_2) := F(X_1) + \frac{1}{2\varepsilon} \frac{H^1(X_1 \setminus X_2) \varepsilon^{\alpha - 1}}{\varepsilon}. \]

Moreover, as we have seen that the optimal value for the above mentioned \( c \) converges to \( \frac{1}{\alpha - 1} \), we have to take account of this, i.e. the evolution we are going to consider will have the form

\[ \begin{cases} w(0) := S_0 \\ w(k + 1) \in \arg\min_{w} F_{\alpha,\varepsilon}(\cdot, w(k)) \end{cases} \]

with the time scaling

\( \Sigma_\varepsilon : [0, T] \to A, \quad \Sigma_\varepsilon(t) := w\left(\left\lceil \frac{t}{\varepsilon^{\alpha - 1}}\right\rceil\right) \).

In the following we will consider the evolution with \( \alpha = 2 \) first, as the results for the case \( \alpha > 1 \) can be deduced with similar arguments.

### 3 Sufficient conditions

In this section we present the main results of our paper, i.e. conditions sufficient to force a branching behavior. Moreover, in this dynamic case the evolution may have locally “stable” points, i.e. configurations from which any further evolution (at least when adding sets with small length) is not optimal (see [14] for instance).
3.1 Changing topology

Now we investigate all the situations that may appear during the evolution. Given an initial datum \( S_0 \in A, \Sigma : [0, T] \rightarrow A \) a minimizing movement function, a time \( T_0 \in (0, T] \), the following behaviors are possible:

1. \( \Sigma \) evolves by adding length at endpoints, i.e. there exists \( \delta > 0 \) such that given \( t \in (T_0, T_0 + \delta) \), any simple point of \( \Sigma(T_0) \) is simple in \( \Sigma(t) \) too, any triple point of \( \Sigma(T_0) \) is triple in \( \Sigma(t) \) too, etc...;

2. \( \Sigma \) evolves by adding length at a non endpoint, i.e. there exists a point of \( \Sigma(T_0) \) which does not verify the condition stated in choice (1) for any \( \Sigma(t) \), \( t > T_0 \).

We will analyze the energy and the dissipation separately.

**Lemma 3.1.** Given a domain \( \Omega \), parameter \( \varepsilon > 0 \), \( S \in A \), suppose that there is an angular point \( X \in S \) which verifies the hypothesis of Lemma 2.6, and the triangle \( T_X \subset V(X) \) is large enough to have a gain \( K\varepsilon \) \((K > 1/2)\) using the procedure described in the proof of Proposition 2.4. Then \( S \) cannot be a local minimum for \( F_{2,\varepsilon} \).

**Proof.** As \( X \) verifies conditions of Lemma 2.6, its Voronoi cell \( V(X) \) contains a triangle \( T_X \) with positive measure. Then if we add a segment \( I_{\varepsilon'} \in \Lambda_{\varepsilon}' \setminus \Lambda_0 \) (with \( \varepsilon' \) to be defined later) in a way described in the proof of Proposition 2.4, using the same argument we have that there exists \( \varepsilon'' > 0 \) such that for any \( \varepsilon'' < \varepsilon' \) the gain \( F(S) - F(S \cup I_{\varepsilon'}) \geq K\varepsilon'' \). So choosing \( \varepsilon' \in (0, \varepsilon_0) \), we have for \( F \):

\[
F(S) - F(S \cup I_{\varepsilon'}) = O(\varepsilon') \geq K\varepsilon',
\]

with \( K > 1/2 \).

From Section 2 we see that putting \( \varepsilon' := \varepsilon^c \), for \( \varepsilon \downarrow 0 \), the optimal exponent \( c \) converges to 1, thus \( \varepsilon' \downarrow \varepsilon \) and

\[
F(S) - F(S \cup I_{\varepsilon}) = O(\varepsilon) \geq K\varepsilon,
\]

The dissipation under these conditions, is \( \frac{\varepsilon}{2} \), thus there is a net gain for the functional \( F_{2,\varepsilon} \) definitively for \( \varepsilon \) small enough, and the proof is complete.

Now we can present a condition required to avoid branching behaviors for Euler schemes.

**Proposition 3.2.** Given a domain \( \Omega \), let \( S^{(1)}_0 \in A \) be a generic element, \( T \) a positive time and let us consider the Euler scheme

\[
\begin{cases}
  w(0) := S^{(1)}_0 \\
  w(k) \in \text{argmin}_{\varepsilon} \mathcal{F}_{2,\varepsilon}(\cdot, w(k - 1))
\end{cases}
\]

in the time interval \([0, T]\).

Suppose that there exist \( P_0 \in S^{(1)}_0 \) angular (with \( \eta \) large enough to fall under the hypothesis of Lemma 3.1) such that \( B(P_0, \eta) \cap (w(k) \setminus w(0)) = \emptyset \) for any \( k \). Then for \( \varepsilon \) sufficiently small, there is an upper bound \( T_{\text{max}} = T_{\text{max}}(\varepsilon, \xi, \Omega, \eta, P_0) \) such that \( T > T_{\text{max}} \) forces a branching behavior.
Proof. As $P_0$ is angular, from the proof of Lemma 3.1 gives that $S_0^{(1)}$ we can see that there exists a constant $K(P_0) > 1/2$ (thus larger than the dissipation) such that adding a segment $J_{\varepsilon'}$ (with $\varepsilon'$ small enough, in a manner similar to that found in the proof of Lemma 3.1 which creates a branching behavior) in $P_0$ yields

$$F(S) - F(S \cup J_{\varepsilon'}) \geq K(P_0)\varepsilon'.$$

Recalling that for $\varepsilon \downarrow 0$ the optimal choice is (upon higher order terms, which vanish relatively quickly) $\varepsilon' \to \varepsilon$, combining this with the dissipation term, the net gain for $F_{2,\varepsilon}$ becomes (putting $\varepsilon' = \varepsilon$)

$$F_{2,\varepsilon}(S, S) - F_{2,\varepsilon}(S \cup J_{\varepsilon}) \geq K^*(P_0)\varepsilon$$

with $K^*(P_0) > 0$ (as $K(P_0) > 1/2$ is larger than the dissipation).

In order to avoid this (which would create a branching behavior), for any $d \ w(d)$ must be obtained from $w(d - 1)$ by adding length at points of $\text{ext}(w(d - 1))$, and the gain in energy must be more than $K^*(P_0)$ (obviously the gain for $F_{2,\varepsilon}$ is always smaller than the gain for $F$), i.e.

$$F(w(d)) \leq F(w(d - 1)) - K^*(P_0)\varepsilon \quad \forall d = 1, \ldots$$

which leads to

$$F(w(d)) \leq F(w(0)) - dK^*(P_0)\varepsilon \quad \forall d = 1, \ldots.$$  

But clearly $0 \leq F(w(d)) \leq F(w(0)) - dK^*(P_0)\varepsilon$, so we must have

$$d \leq \frac{F(S_0^{(1)})}{K^*(P_0)\varepsilon},$$

and recalling that

$$\Sigma_\varepsilon(t) := w\left(\left\lfloor \frac{t}{\varepsilon} \right\rfloor\right),$$

this forces

$$\left\lfloor \frac{t}{\varepsilon} \right\rfloor \leq \frac{F(S_0^{(1)})}{K^*(P_0)\varepsilon}$$

thus the upper bound

$$T_{\max} := \frac{F(S_0^{(1)})}{K^*(P_0)}$$

In the entire proof we have not considered the contributions of higher order terms (the choice $\varepsilon' = \varepsilon$ is optimal only in the limit case), but these vanish quickly as $\varepsilon \downarrow 0$. Notice that this result holds definitely for $\varepsilon > 0$ small enough, and it is uniform, so it can be applied to the rate independent case:
Theorem 3.3. Given a domain Ω, let \( S_0^{(2)} \in A \) be a generic element, \( T \) a positive time, and \( \Sigma : [0, T] \rightarrow A \) a minimizing movement obtained as limit of the Euler schemes with time step \( \{ \varepsilon_n \} \downarrow 0 \)

\[
\begin{cases}
  w(0, n) = w(0) = S_0^{(2)} \\
  w(k, n) \in \operatorname{argmin} F_{2, \varepsilon_n}(X')
\end{cases}
\]

Suppose that there exist \( P_1 \in S_0^{(2)} \) angular and \( \eta'/2 \) (large enough to make the configuration fall under the hypothesis of Lemma 3.1) such that \( B(P_1, \eta') \cap (\Sigma(T) \setminus S_0^{(2)}) = \emptyset \). Then there is an upper bound \( T_{\text{max}} \) (depending only on geometrical quantities) such that \( T > T_{\text{max}} \) forces branching behavior.

Proof. As \( (\Sigma(T) \setminus S_0^{(2)}) \cap B(P_1, \eta') = \emptyset \), \( \lim_{n \to \infty} (w\left(\left\lceil \frac{T}{\varepsilon_n} \right\rceil, n\right) \setminus S_0^{(2)}) \cap B(P_1, \eta') = \emptyset \) so there exist \( \bar{n} \) such that for any \( n \geq \bar{n} \)

\[
\lim_{n \to \infty} (w\left(\left\lceil \frac{T}{\varepsilon_n} \right\rceil, n\right) \setminus S_0^{(2)}) \cap B(P_1, \frac{\eta'}{2}) = \emptyset.
\]

Using the same argument found in the proof of Theorem 3.2, we can conclude that under this configuration any Euler scheme (with sufficiently small time step \( \varepsilon_n \)) can always add a set of length \( \varepsilon_n \) and obtain a net gain for \( F_{2, \varepsilon_n} \) not less than \( K'(P_1)\varepsilon_n \). Thus, if we want to avoid this, we must have

\[
F(w(d, n)) \leq F(w(d - 1, n)) - K'(P_1)\varepsilon_n \quad \forall d = 1, \ldots, \left\lceil \frac{T}{\varepsilon_n} \right\rceil
\]

holds for any \( n \), thus

\[
0 \leq F\left(\frac{T}{\varepsilon_n}\right) \leq F(w(0)) - (T - \varepsilon_n)K'(P_1)
\]

holds for any \( n \), and the upper bound (given by Proposition 3.2)

\[
T_{\varepsilon_n} = \varepsilon_n + \frac{F(S_0^{(2)})}{K'(P_1)}
\]

holds for any \( n \); passing to the limit \( n \to \infty \), it reads

\[
T_{\text{max}} = \frac{F(S_0^{(2)})}{K'(P_1)}
\]

for the rate independent case, which concludes the proof. \( \square \)

4 An application

In this section we present first a more stringent upper bound estimate for the branching time, based on geometrical arguments and energy considerations from Theorem 3.3.
Lemma 4.1. If there exist \( k, n \) such that \( (w(k, n) \setminus S_{0}^{\text{ini}}) \cap \Omega^{+} \neq \emptyset \), but \( (w(k - 1, n) \setminus S_{0}^{\text{ini}}) \cap \Omega^{+} = \emptyset \), this means \( w(k, n) \) is not homeomorphic to \( w(k - 1, n) \).

The proof is easy, and done in [12]: the key argument is that in order to pass to the other region, the set must cross the border, thus creating a branching and changing the topology.

Now we present an estimate relating the energy and the “free space”:

Lemma 4.2. Given a domain \( \Omega \), an element \( S_{1} \in A \), and suppose that there exists \( Q \in \Omega \) and \( R > 0 \) such that the ball \( B(Q, R) \cap S = \emptyset \). Then

\[
F(S_{1}) \geq \frac{4\pi R^{3}}{27}.
\]

Proof. The proof is easy: as \( B(Q, R) \cap S_{1} = \emptyset \), for any \( r < R \) all points \( x \in B(Q, r) \) verify \( \text{dist}(x, S_{1}) \geq R - r \), so

\[
F(S_{1}) = \int_{\Omega} \text{dist}(x, S_{1})dx \geq \int_{B(Q, r)} \text{dist}(x, S_{1})dx \geq (R - r)r^{2}.
\]

Differentiating the expression \( (R - r)r^{2} \), its maximum value is attained by \( r = \frac{2}{3}R \), which corresponds to

\[
F(S_{1}) \geq \frac{4\pi}{27}R^{3}
\]

and the proof is complete.

Lemma 4.3. Given a domain \( \Omega \), an element \( S_{2} \in A \), a point \( Q' \in S_{2} \) and suppose that its Voronoi cell \( V(Q') \) has \( |V(Q')| > 0 \). Then there exists \( Q \in \Omega \) and \( R > 0 \) such that \( B(Q, R) \cap S_{2} = \emptyset \).

Proof. For \( V(Q') \) we have \( |V(Q')| \leq \frac{\pi}{4} \text{diam}(V(Q'))^{2} \). Let be \( X_{1}, X_{2} \in V(Q') \) points such that \( \text{dist}(X_{1}, X_{2}) = \text{diam}(V(Q')) \):

\[
\text{dist}(X_{1}, X_{2}) = \text{diam}(V(Q')) \leq \text{dist}(X_{1}, Q') + \text{dist}(Q', X_{2})
\]

so \( \min\{\text{dist}(X_{1}, Q'), \text{dist}(Q', X_{2})\} \geq \frac{1}{2} \text{diam}(V(Q')) \).

Assume that \( \text{dist}(X_{1}, Q') > \frac{1}{2} \text{diam}(V(Q')) \): \( X_{1} \in V(Q') \) means for any \( i < \frac{1}{2} \text{diam}(V(Q')) \), \( B(X_{1}, i) \cap S_{2} = \emptyset \) to avoid \( \text{dist}(X_{1}, B(X_{1}, i) \cap S_{2}) \leq i < \frac{1}{2} \text{diam}(V(Q')) \).

So we can choose \( \bar{Q} := X_{1}, \bar{R} = \frac{1}{2} \text{diam}(V(Q')) \), and considering that \( \text{diam}(V(Q')) \geq \sqrt{\frac{4}{\pi}|V(Q')|} \), the proof is complete.

We recall here that given \( S' \in A \) (in a given domain \( \Omega \)), \( \varepsilon > 0 \) and \( H_{\varepsilon} \in A_{\varepsilon} \setminus \bigcup_{0 \leq \varepsilon' < \varepsilon} A_{\varepsilon'} \), adding \( H_{\varepsilon} \) to a generic point \( U \in S' \) the gain in energy is (upon higher order terms)

\[
F(S' \cup H_{\varepsilon}) \geq F(S') - \varepsilon|V(U)|. \tag{4.1}
\]

Consider a such configuration:

given a domain \( \Omega \), let \( S_{0}^{\text{ini}} \) be the initial datum, and there exist

\[15\]
• $P_0' \in S_{0,dat}^\epsilon$ angular and let be $\xi' > 0$ large enough to fall under the hypothesis of Lemma 3.1, such that $B(\xi') \cap S_{0,dat}^\epsilon$ is homeomorphic to $(0, 1)$; \\
• a closed injective path $\gamma^* : [0, 1] \rightarrow \Omega$ such that $\gamma^*([0, 1]) \subseteq S_{0,dat}^\epsilon$; the domain $\Omega$ is now divided in two regions, $\Omega^+$ and $\Omega^-$ with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \setminus \gamma^*([0, 1])$, and they correspond to the “interior” and the “exterior” part of $\gamma^*([0, 1])$ – the order is not relevant – given by the Jordan Curve Theorem; \\
• triangle $T_{P_0'} \subset V(P_0') \cap B(P_0', \xi')$ (its existence is given by Lemma 2.6) verifies $|T_{P_0'} \cap \Omega^+| > 0$, and $\text{ext}(S_{0,dat}^\epsilon) \subset \Omega^-$. \\

Notice that $S_{0,dat}^\epsilon$ is very similar to $S_{ini}^\epsilon$, and results as Lemma 4.1 holds.

In the rest of this subsection we will suppose that $\Omega^-$ is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\text{diam}(\Omega^-)$, $|\Omega^-|$.

Consider now a minimizing movement $\Sigma : [0, T] \rightarrow A$, limit of Euler schemes $\{\Sigma_{\epsilon'_n}\}_{n=0}^\infty$ (with time steps $\{\epsilon'_n\}_{n=0}^\infty \downarrow 0$):

\[
\begin{align*}
 w(0, n) &= w(0) := S_{0,dat}^\epsilon \\
 w(k, n) &\in \arg\min \mathcal{F}_{2,\epsilon_n}(\cdot, w(k-1, n)) \\
 \Sigma_{\epsilon'_n}(t) &:= w\left(\frac{t}{\epsilon'_n}, n\right), \quad \Sigma(t) = \lim_{n \rightarrow \infty} \Sigma_{\epsilon'_n}(t) \forall t \in [0, T].
\end{align*}
\]

The main estimate here is Theorem 4.4.

The notations introduced (except mute counters like $k$ and $n$) will have the same meaning in the following of this subsection.

Using the same arguments found in Theorems 3.2 and 3.3, we see that we can add $J_{\epsilon'_n}$ in $P_0'$ and have a net gain for $\mathcal{F}_{2,\epsilon_n}$ at least $K(P_0')\epsilon'_n$. As we have see that for $\epsilon_n \downarrow 0$ the optimal choice is $\epsilon'_n \rightarrow \epsilon_n$, as the gain in energy is larger than the gain for $\mathcal{F}_{2,\epsilon}$, we have that to avoid a branching behavior we must have

\[ F(w(k, n)) \leq F(w(0)) - kK(P_0')\epsilon'_n \quad \text{(4.2)} \]

i.e. $\forall t \in [0, T]$

\[
F(\Sigma_{\epsilon'_n}(t)) := F\left(\frac{t}{\epsilon_n}, n\right) \leq F(\Sigma_{0,dat}^\epsilon) - \frac{t}{\epsilon'_n} K(P_0')\epsilon'_n \leq F(\Sigma_{0,dat}^\epsilon) - \left(\frac{t}{\epsilon'_n} + 1\right) K(P_0')\epsilon'_n.
\]

Passing to the limit $n \rightarrow \infty$, this reads

\[
F(\Sigma(t)) := F(\Sigma_{0,dat}^\epsilon) - \lim_{n \rightarrow \infty} F\left(\frac{t}{\epsilon'_n}, n\right) \leq F(\Sigma_{0,dat}^\epsilon) - \lim_{n \rightarrow \infty} \frac{t}{\epsilon'_n} K(P_0')\epsilon'_n
\]

\[
\leq F(\Sigma_{0,dat}^\epsilon) - \lim_{n \rightarrow \infty} (\frac{t}{\epsilon'_n} + 1) K(P_0')\epsilon'_n = F(\Sigma_{0,dat}^\epsilon) - t K(P_0')
\]
To avoid a branching behavior, there exists an endpoint $P^*$ of $\Sigma(t)$ with $|V(P^*)| \geq K(P_0')$, then for Lemma 4.3 there exists a point $X \in \Omega^-$ such that the ball

$$B(X, v) \cap \Sigma(t) = \emptyset, \quad v = \sqrt{\frac{K(P_0')}{\pi}},$$

and Lemma 4.2 gives

$$F(\Sigma(t)) \geq \frac{4}{27} \sqrt{\frac{K(P_0')^3}{\pi}}.$$

But we must have

$$F(\Sigma(t)) \leq F(w(0)) - tK(P_0')$$

and combining the above inequalities,

$$F(w(0)) - tK(P_0') \geq \frac{4}{27} \sqrt{\frac{K(P_0')^3}{\pi}}$$

which gives $t \leq \frac{F(w(0))}{K(P_0')} - \frac{4}{27} \sqrt{\frac{K(P_0')^3}{\pi}}$. So we have proved the following result:

**Theorem 4.4.** For this configuration, with the above notations, an upper bound for the branching time is given by

$$T_{\text{max}} := \frac{F(S_0^{\text{flat}})}{K(P_0')} - \frac{4}{27} \sqrt{\frac{K(P_0')^3}{\pi}}.$$

Notice that the partition $\Omega^+ \cup \Omega^-$ is crucial as Lemma 4.1 makes impossible passing from one region to another without changing topology, so it prevents $\Sigma(t)$ from ever visit $T(P_0') \cap \Omega^+$ without exhibiting branching behaviors.

### References


