# SMOOTH APPROXIMATION OF WEAK FINSLER METRICS

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ABSTRACT. Smooth Finsler metrics are a natural generalization of Riemannian ones and have been widely studied in the framework of differential geometry. The definition can be weakened by allowing the metric to be only Borel measurable. This generalization is necessary in view of applications, such as, for instance, optimization problems. In this paper we show that smooth Finsler metrics are dense in Borel ones, generalizing the results obtained in [15]. The case of degenerate Finsler distances is also discussed.

## 1. INTRODUCTION

A Finsler metric on a differential manifold $M$ is a map $\varphi : TM \to [0, +\infty)$ such that $\varphi(x, \cdot)$ is convex and positively 1-homogeneous on $T_xM$ for every $x \in M$. A non-symmetric distance on $M$ can be associated to $\varphi$ as follows:

$$d_\varphi(x, y) := \inf \left\{ \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\, \mathrm{d}t \;\middle|\; \gamma \in \mathrm{Lip}([0, 1], M),\ \gamma(0) = x,\ \gamma(1) = y \right\} \quad (1)$$

for $x, y \in M$, provided the class of admissible paths $\gamma$ is non-void (for instance, assume $M$ connected). Distances of this kind are usually known in literature as *Finsler distances.* Finsler metrics are a natural generalization of Riemannian ones. The smooth case has been largely investigated in the last century in the framework of differential geometry. The literature on the subject is wide; an introduction is supplied, for instance, by [3].

The above definitions still make sense under weaker regularity assumptions on the metric. In fact, we may allow $\varphi$ to be only Borel-measurable. This generality is necessary in view of applications. In optimization problems, for instance, one is often interested in minimizing a cost functional which takes the form $\mathcal{C}(d_\varphi)$, where $\varphi$ is to be chosen in some suitable class of Finsler metrics. Then, if one restricts himself to consider smooth or continuous metrics only, the problem may not have a solution: indeed, when attacking the problem via the direct method of the Calculus of Variations, it might happen that the minimizing sequences converge to distances deriving from Finsler metrics that are shown to be Borel-measurable only (see [1, 5, 8]). Note that problems of this kind fall into the framework of shape optimization: in fact, $\varphi$ describes the geometric properties of the metric space $(M, d_\varphi)$, and may possibly depend on some quantities one is allowed to vary. For example, if $a(\cdot)$ is the density of a viscous material that fills a region $M$ of the space, the distance between points is expected to be proportional to the resistance opposed by the medium and

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may be described by $d_\varphi$ with $\varphi(x,\xi) := a(x)|\xi|$. If now we let $a(\cdot)$ vary in a suitable family of functions satisfying integral and pointwise constraints, the corresponding shape optimization problem is that of finding a "best" way to distribute a certain amount of this material on $M$. A problem of this kind has been considered, for instance, in [7].

Measurable Finsler metrics have been also considered in [16, 17, 18], where this point of view was adopted to suitably generalize the notions of Riemannian and Finsler metrics for Lipschitz manifolds, namely topological manifolds with a countable basis whose changes of coordinates are Lipschitz maps. Lipschitz manifolds are a generalization of polyhedra, and were introduced to treat the case of manifolds with singularities, such as vertices, edges, conical points, even not isolated.

The perspective can be also reversed: instead of dealing with singularities carried by the manifold, one could treat the case of a sufficiently smooth manifold equipped with an irregular metric. For example, the problem concerning geodesics with obstacles can be studied as a problem about metrics with singularities. This topic is also of physical interest, since the irregularities of the metric may actually represent irregularities of the materials where the physical phenomena take place: think, for instance, to geometrical optics in non-homogeneous media.

With this regard, we underline the relation with the theory of Hamilton-Jacobi equations. In the study of equations of eikonal type, a central role is in fact played by a Finsler distance associated to the equation, the so called *optical length function*. The metric character of equations of this kind has been recognized and explored by several authors [20, 22, 23, 25]. The study of Finsler metrics in the weak measurable setting acquires therefore further interest in view of generalizations of the theory of viscosity solutions for Hamilton-Jacobi equations with discontinuous ingredients, a topic which is the object of growing attention [4, 10, 11, 12, 26].

In this paper we consider the space $\mathcal{D}_\alpha$ of non-symmetric distances defined through (1), where $M$ is now replaced by the closure of a connected open subset $\Omega$ of $\mathbb{R}^N$, and $\varphi$ varies in the family $\mathcal{M}_\alpha$ of Borel-measurable Finsler metrics that satisfy the following bounds for two fixed positive constants $\alpha$ and $\beta$:

$$\alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \quad \text{on } \overline{\Omega}\times\mathbb{R}^N.$$

The aim is to show that continuous (smooth) Finsler metrics are dense in Borel ones. More precisely, we will show that any element of $\mathcal{D}_\alpha$ is the uniform limit of a suitable sequence of distances derived through (1) from continuous (smooth) metrics belonging to $\mathcal{M}_\alpha$ (Theorems 4.1 and 4.5). The hard side of the job corresponds to Theorem 4.1: indeed, once the density result is proved for continuous metrics, the analogous result for smooth ones is obtained through a standard mollification argument.

These results can be read as the counterpart of those obtained in [15], where the case of symmetric distances was considered, and analogous density theorems for continuous and smooth Riemannian metrics were obtained. As a matter of fact, the proofs exploit similar ideas; in particular, the key observation still corresponds to Lemma 3.1 (*cf.* [15, Lemma 3.4]), which allows to replace the uniform convergence of distances with a pointwise convergence on a fixed, countable subset of $\overline{\Omega}\times\overline{\Omega}$. On the other hand, new arguments have to be introduced to overcome the difficulties produced by the non-symmetric character of distances here considered.

We also wish to underline the content of Theorem 4.2: it amounts to saying that any geodesic distance $d$, locally equivalent to the Euclidean one, can be obtained from a convex Finsler metric through (1), a fact which is not trivial at all: notice indeed that the Finsler metric $\varphi_d(x,\xi)$, obtained from $d$ by derivation (*cf.* formula (7) below), is proved to be convex in $\xi$ for almost every $x$ only.

The density results obtained for $\alpha > 0$ are then extended to the case $\alpha = 0$. The main difference between the two cases relies on the fact that, while $\mathcal{D}_\alpha$ is closed as a metric space when $\alpha > 0$, this is no longer true when $\alpha = 0$. This fact is investigated in more detail in Section 5 through suitable, explicit examples. We remark that the family $\mathcal{D}_0$ includes distances for which the local equivalence with the Euclidean one fails to hold somewhere. The interest for this class of degenerate distances is motivated by the study of Hamilton-Jacobi equations of eikonal type in the critical case (see [12, 20]).

Finally, we remark that our results also holds if $\mathbb{R}^N$ is replaced by a $N$-dimensional, differential manifold without boundary and of class $C^1$. Since all arguments exploit local properties, proofs can actually be rephrased by using local coordinates. We have preferred, however, to consider this more special case not to add further technicalities.

Our paper is organized as follows: Section 2 is devoted to notation and definitions. In particular, some results holding in classical metric spaces are generalized to cover the case of non-symmetric distances. In Section 3 we state some preliminary propositions, in Section 4 we prove our main results for the class of non-symmetric distances $\mathcal{D}_\alpha$ when $\alpha > 0$, and in Section 5 we examine the case $\alpha = 0$.

## 2. Notation and definitions

We write here a list of symbols used throughout this paper.

| | |
|---|---|
| $\Omega$ | open connected subset of $\mathbb{R}^N$ |
| $\mathbb{S}^{N-1}$ | $(N-1)$ – dimensional unitary sphere of $\mathbb{R}^N$ |
| $B_r(x), B_r$ | open ball in $\mathbb{R}^N$ of radius $r$ centred in $x$ and 0 respectively |
| $I$ | closed interval $[0,1]$ |
| $\mathcal{L}^k$ | $k$-dimensional Lebesgue measure |
| $\mathcal{H}^k$ | $k$-dimensional Hausdorff measure |
| $\|x\|$ | Euclidean norm of the vector $x \in \mathbb{R}^N$ |
| $\mathbb{R}_+$ | non-negative real numbers |
| $\mathscr{P}(\mathbb{R}^N)$ | family of all subsets of $\mathbb{R}^N$. |

We will denote by $N$ an integer number. A subset of $\mathbb{R}^N$ is said to be *negligible* if its $N$-dimensional Lebesgue measure is null. When not otherwise specified, the word *curve* or *path* will always denote a Lipschitz function from the interval $I := [0,1]$ to $\overline{\Omega}$. The family of all such curves will be denoted by $\mathrm{Lip}(I,\overline{\Omega})$, and is equipped with the metric of uniform convergence; in particular, we will say that the sequence $(\gamma_n)_n$ converges to $\gamma$ to mean that $\sup_{t\in I} |\gamma_n(t) - \gamma(t)|$ tends to zero as $n$ goes to infinity. Unless otherwise specified, any curve $\gamma$ is always supposed to be parametrized by constant speed, *i.e.* in such a way that $|\dot\gamma(t)|$ is constant for $\mathcal{L}^1$-a.e. $t \in I$. We will denote by $\mathrm{Lip}_{x,y}$ the family of curves $\gamma$ which join $x$ to $y$, *i.e.* such that $\gamma(0) = x$ and $\gamma(1) = y$. The domain $\Omega$ will be always assumed to satisfy the following condition:

$$\forall\, r > 0\ \exists\, C_r \geq 1 \ \text{ s.t. } \quad d_\Omega(x,y) \leq C_r|x-y| \qquad \forall\, x,y \in \overline{\Omega} \cap B_r, \tag{$\Omega$}$$

where $d_\Omega$ is the Euclidean geodesic distance in $\overline{\Omega}$, *i.e.* the distance defined through formula (4) below with $\varphi(x,\xi) = |\xi|$ identically in $\overline{\Omega} \times \mathbb{R}^N$. Note that $(\Omega)$ depends on the regularity of $\partial\Omega$; when the latter can be (locally) expressed as a graph of a function $h$, $(\Omega)$ is equivalent to require that $h$ is Lipschitz continuous.

For a measurable function $f : I \to \mathbb{R}^N$, $\|f\|_\infty$ stands for $\sqrt{\sum_{i=0}^N \|f_i\|^2_{L^\infty(I)}}$, where $f_i$ and $\|f_i\|_{L^\infty(I)}$ denote the $i$-th component of $f$ and the $\mathrm{L}^\infty$-norm of $f_i$ respectively.

In the sequel, a function $d$ defined on $\overline{\Omega} \times \overline{\Omega}$ will be said to be a *distance on* $\overline{\Omega}$ if the following conditions hold:

(i) $d(x,x) = 0 \quad$ for every $x \in \overline{\Omega}$;

(ii) $d(x,y) \le d(x,z) + d(z,y) \quad$ for every $x, y, z \in \overline{\Omega}$.

With respect to the classical definition of distance, two conditions are not required: first, $d$ may be *non-symmetric*, namely the identity $d(x,y) = d(y,x)$ may fail to hold in $\overline{\Omega} \times \overline{\Omega}$; second, $d$ can possibly be *degenerate*, namely it might happen that $d(x,y) = 0$ for some $x \neq y$. Degenerate distances, however, are only considered at the end of the paper, in Section 5.

We begin by extending to this setting some well known definitions and classical results for usual distances. We will omit the proofs of the next theorems, which can be derived by minor changes from those proposed, for instance, in [2] in the classical case.

First, let us define the $d$-length of $\gamma$, obtained as the supremum of the $d$-lengths of inscribed polygonal curves:

$$\mathrm{L}_d(\gamma) := \sup \left\{ \sum_{i=0}^{m-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big) \;\Big|\; 0 = t_0 < t_1 < .. < t_m = 1,\ m \in \mathbb{N} \right\}. \tag{2}$$

**Definition 2.1** (Geodesic distance)**.** We will say that $d$ is a *geodesic distance* if it satisfies the following identity:

$$d(x,y) = \inf \left\{ \mathrm{L}_d(\gamma) \mid \gamma \in \mathrm{Lip}_{x,y} \right\} \qquad \text{for every } (x,y) \in \overline{\Omega} \times \overline{\Omega}.$$

All distances considered in the sequel will fulfill the following hypotheses:

(d1) $d$ is geodesic;

(d2) there exist two non-negative constants $\alpha$ and $\beta$ such that

$$\alpha|x-y| \le d(x,y) \le \beta|x-y| \quad \text{locally in } \Omega$$

(*i.e.* for every $x_0 \in \Omega$ there exists an open ball $B_r(x_0) \subset \Omega$ such that the above inequality holds for every $x, y \in B_r(x_0)$).

Any distance $d$ which satisfies the above hypotheses induces on $\overline{\Omega}$ a topology which is equivalent to the Euclidean one when $\alpha > 0$, and weaker, in general, if $\alpha = 0$. Notice that $\alpha > 0$ implies that $d$ is non-degenerate. By applying to our framework a classical theorem due to Busemann (*cf.* [2, Theorem 4.3.1]), we obtain what follows.

**Proposition 2.2.** *The length functional* $\mathrm{L}_d$ *is lower semicontinuous with respect to the uniform convergence of paths, namely if* $(\gamma_n)_n$ *converges to* $\gamma$*, then*

$$\mathrm{L}_d(\gamma) \le \liminf_{n \to +\infty} \mathrm{L}_d(\gamma_n).$$

*Moreover, when* $\alpha > 0$*, for every couple of points* $x, y$ *in* $\overline{\Omega}$ *there exists a curve* $\gamma \in \mathrm{Lip}_{x,y}$ *which is a path of minimal d-length, i.e. such that* $\mathrm{L}_d(\gamma) = d(x,y)$*.*

As in the symmetric case, we can define the metric derivative of a curve.

**Definition 2.3** (Metric derivative)**.** Given a curve $\gamma$, we define the *metric derivative* $|\dot{\gamma}|_d(t)$ of $\gamma$ at the point $t \in (0,1)$ as

$$|\dot{\gamma}|_d(t) := \limsup_{h\to 0^+} \frac{d(\gamma(t),\gamma(t+h))}{h}. \tag{3}$$

The $d$-length of a curve $\gamma$ admits an integral representation in terms of its metric derivative.

**Theorem 2.4.** *For every curve $\gamma$ the* lim sup *at the right-hand side of (3) is actually a* limit *for $\mathcal{L}^1$-a.e. $t \in I$. Moreover we have*

$$\mathrm{L}_d(\gamma) = \int_0^1 |\dot{\gamma}|_d(t)\,\mathrm{d}t.$$

We now make precise what we mean by weak Finsler metric.

**Definition 2.5.** A Borel-measurable function $\varphi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$ is said to be a *(weak) Finsler metric* on $\overline{\Omega}$ if

(i) $\varphi(x,\cdot)$ is positively 1-homogeneous for every $x \in \overline{\Omega}$;
(ii) $\varphi(x,\cdot)$ is convex for $\mathcal{L}^N$-a.e. $x \in \overline{\Omega}$.

We will say that the metric $\varphi$ is *convex* if (ii) holds for every $x \in \overline{\Omega}$. We will say that $\varphi$ is a *continuous* (resp. *smooth*) *Finsler metric* if $\varphi(\cdot,\xi)$ is continuous (resp. smooth) on $\overline{\Omega}$ for every $\xi \in \mathbb{R}^N$.

We now fix two non-negative constants $\alpha$ and $\beta$ and we consider the following family of functions:

$$\mathcal{M}_\alpha := \left\{\varphi \text{ Finsler metrics on } \overline{\Omega} \;\middle|\; \alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \text{ on } \overline{\Omega}\times\mathbb{R}^N\right\}.$$

For each $\varphi \in \mathcal{M}_\alpha$, we can define a function $d_\varphi$ on $\overline{\Omega}\times\overline{\Omega}$ through the formula

$$d_\varphi(x,y) := \inf\left\{\mathbb{L}_\varphi(\gamma) \mid \gamma \in \mathrm{Lip}_{x,y}\right\}, \tag{4}$$

where the length functional $\mathbb{L}_\varphi$ is defined by

$$\mathbb{L}_\varphi(\gamma) := \int_0^1 \varphi(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t. \tag{5}$$

The main properties of $d_\varphi$ are summarized below.

**Proposition 2.6.** *The function $d_\varphi$ given by (4) is well defined on $\overline{\Omega}\times\overline{\Omega}$ and satisfies the following properties:*

(i) $0 \le d_\varphi(x,y) \le d_\varphi(x,z) + d_\varphi(z,y)$ *for all* $x,y,z \in \overline{\Omega}$*;*
(ii) $\alpha|x-y| \le d_\varphi(x,y) \le \beta|x-y|$ *locally in* $\Omega$*.*

**Proof.** Let $\gamma$ be a curve. Since the map $t \mapsto \big(\gamma(t),\dot{\gamma}(t)\big)$ is Lebesgue measurable on $I$, and $\varphi$ is Borel measurable on $\overline{\Omega}\times\mathbb{R}^N$, their composition $\varphi(\gamma(t),\dot{\gamma}(t))$ is Lebesgue measurable on $I$. Therefore the integral in (5) is well defined and so is $d_\varphi$. The remainder of the assertion is a simple consequence of the definitions. $\square$

Next proposition clarifies the relation between the functional (5) and the (intrinsic) metric length functional. The proof can be derived by minor changes from that of Theorem 4.3 in [16].

**Proposition 2.7.** *Let $d := d_\varphi$ with $\varphi \in \mathcal{M}_\alpha$. Then, for any $\gamma \in \mathrm{Lip}(I, \overline{\Omega})$, we have that $\mathrm{L}_d(\gamma) \leq \mathbb{L}_\varphi(\gamma)$. In particular, $d$ is a geodesic distance according to Definition 2.1. If $\alpha > 0$, we have moreover:*

$$\mathrm{L}_d(\gamma) = \inf \left\{ \liminf_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) \;\middle|\; (\gamma_n)_n \text{ converges to } \gamma \text{ in } \mathrm{Lip}(I, \overline{\Omega}) \right\}, \tag{6}$$

*namely $\mathrm{L}_d$ is the relaxed functional of $\mathbb{L}_\varphi$ on $\mathrm{Lip}(I, \overline{\Omega})$.*

**Remark 2.8.** By Proposition 2.7, $\mathbb{L}_\varphi$ will coincide with $\mathrm{L}_d$ whenever $\mathbb{L}_\varphi$ is lower semicontinuous on $\mathrm{Lip}(I, \overline{\Omega})$. This happens, for instance, when $\varphi$ is lower semicontinuous on $\overline{\Omega} \times \mathbb{R}^N$ and $\varphi(x, \cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \overline{\Omega}$ (*cf.* [6, Theorem 4.1.1]).

Let us denote by $\mathcal{D}_\alpha$ the family of distances on $\overline{\Omega}$ generated by the metrics $\mathcal{M}_\alpha$, namely

$$\mathcal{D}_\alpha := \{d_\varphi \text{ distance on } \overline{\Omega} \text{ given by (4)} \mid \varphi \in \mathcal{M}_\alpha\}.$$

We endow $\mathcal{D}_\alpha$ with the metric given by the uniform convergence on compact subsets of $\overline{\Omega} \times \overline{\Omega}$. The convergence of a sequence of distances $(d_n)_n$ to $d$ with respect to this metric will be denoted by $d_n \xrightarrow{\mathcal{D}_\alpha} d$. When $\alpha > 0$, this convergence is equivalent to the $\Gamma$-convergence of the relative length functionals $\mathrm{L}_{d_n}$ to $\mathrm{L}_d$ with respect to the uniform convergence of paths. This result has been proved in [8, Theorem 3.1] considering usual symmetric distances, but actually it still holds in the non-symmetric case too and the proof is the same. Moreover, we have (*cf.* [8, Theorem 3.1]):

**Theorem 2.9.** *Assume $\alpha > 0$. Then the set $\mathcal{D}_\alpha$, endowed with the metric given by the uniform convergence of distances on $\overline{\Omega} \times \overline{\Omega}$, is a metrizable compact space.*

Let us stress that the interesting part of the result provided by Theorem 2.9 corresponds to the closed character of the space $\mathcal{D}_\alpha$, since the compactness trivially follows from Ascoli-Arzelà Theorem. The requirement that $\alpha$ is strictly positive is necessary, as we will see in Section 5.

In general, as condition $(\Omega)$ is satisfied, $\mathcal{D}_\alpha$ can be seen as a subspace of Finsler distances on $\mathbb{R}^N$: indeed, given $d \in \mathcal{D}_\alpha$, a suitable Finsler distance $\overline{d}$ on $\mathbb{R}^N$ which extends $d$ can be constructed as shown in [15, Remark 2.7]. Therefore, by possibly replacing $d$ with $\overline{d}$, the following definition always makes sense:

$$\varphi_d(x,\xi) := \limsup_{h\to 0^+} \left( \frac{d(x, x+h\xi)}{h} \wedge \beta|\xi| \right) \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{7}$$

The function $\varphi_d$ is a Finsler metric, as proved in [18]; moreover, for each $n \in \mathbb{N}$, there holds

$$|\varphi_d(x,\xi) - \varphi_d(x,\eta)| \leq \beta C_n |\xi - \eta| \quad \text{for every } x \in B_n \cap \overline{\Omega},\ \xi, \eta \in \mathbb{R}^N,$$

where $C_n$ are the constants appearing in $(\Omega)$. We will refer to $\varphi_d$ as the metric associated to $d$ by derivation. In view of Proposition 2.6 and [15, Remark 2.10], for every curve $\gamma$ the following holds:

$$|\dot\gamma|_d(t) = \limsup_{h\to 0^+} \frac{d(\gamma(t), \gamma(t)+h\dot\gamma(t))}{h} \leq \beta|\dot\gamma(t)| \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I,$$

therefore by Theorem 2.4

$$\mathrm{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t \quad \text{for all } \gamma \in \mathrm{Lip}(I, \overline{\Omega}), \tag{8}$$

*i.e.* $\mathrm{L}_d = \mathbb{L}_{\varphi_d}$ on $\mathrm{Lip}(I, \overline{\Omega})$ (*cf.* [18, Theorem 2.5]). In particular, $d = d_{\varphi_d}$ according to (4).

We recall the following definitions for set-valued maps (*cf.* [13]).

**Definition 2.10.** Let $X$ be a subset of $\mathbb{R}^N$ and $C : X \to \mathscr{P}(\mathbb{R}^N)$ a set-valued map such that $C(x)$ is compact for every $x \in X$. We will say that

(i) $C(\cdot)$ is *Borel measurable* on $X$ if the set $\{x \in X \,:\, C(x) \cap A \neq \emptyset\}$ is Borel-measurable in $\mathbb{R}^N$ for every open subset $A$ of $\mathbb{R}^N$.

(ii) $C(\cdot)$ is *upper semicontinuous* at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that
$$C(z) \subset C(x) + B_\varepsilon \qquad \text{for all } z \in B_\delta(x) \cap X.$$

(iii) $C(\cdot)$ is *lower semicontinuous* at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that
$$C(x) \subset C(z) + B_\varepsilon \qquad \text{for all } z \in B_\delta(x) \cap X.$$

(iv) $C(\cdot)$ is *continuous* at $x \in X$ if it is both upper and lower semicontinuous at $x$.

The support function of a subset $C$ of $\mathbb{R}^N$ is the function $\sigma_C$ defined by

$$\sigma_C(\xi) := \sup\{\langle \xi, p\rangle \mid p \in C\} \qquad \text{for every } \xi \in \mathbb{R}^N,$$

which is convex and positively 1-homogeneous on $\mathbb{R}^N$.

Given a Borel measurable set-valued map $C : \overline{\Omega} \to \mathscr{P}(\mathbb{R}^N)$ such that $C(x)$ is compact and contains 0 for every $x \in \overline{\Omega}$, a Finsler metric $\varphi$ can be defined on $\overline{\Omega} \times \mathbb{R}^N$ as follows:

$$\varphi(x, \xi) := \sigma_{C(x)}(\xi) \qquad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

Moreover, if $C(\cdot)$ is upper semicontinuous (resp. lower semicontinuous) on $\overline{\Omega}$, then $\varphi(\cdot, \xi)$ is upper semicontinuous (resp. lower semicontinuous) on $\overline{\Omega}$ as well, for every $\xi \in \mathbb{R}^N$.

## 3. Some preliminary results

We collect in this section some results we will need in the proofs of our main theorems. To begin, we state a simple but important consequence of Ascoli-Arzelà Theorem.

**Lemma 3.1.** *Let $(d_n)_n$ be a sequence contained in $\mathcal{D}_\alpha$ which converges pointwise to some $d \in \mathcal{D}_\alpha$ on a dense subset of $\overline{\Omega} \times \overline{\Omega}$. Then $d_n \xrightarrow{\mathcal{D}_\alpha} d$.*

A first application of Lemma 3.1 is provided in the proof of the following proposition.

**Proposition 3.2.** *Assume $\alpha > 0$. Let $\varphi$, $\varphi_n \in \mathcal{M}_\alpha$ and $d$ and $d_n$ be the distances associated respectively to $\varphi$ and $\varphi_n$ through (4). Then $d_n \xrightarrow{\mathcal{D}_\alpha} d$ in the following cases:*

(i) $(\varphi_n)_n$ *converges uniformly to* $\varphi$ *on compact subsets of* $\overline{\Omega} \times \mathbb{R}^N$*;*
(ii) $\varphi_n$ *are lower semicontinuous in* $x$*, convex in* $\xi$ *and converge increasingly to* $\varphi$ *pointwise on* $\overline{\Omega} \times \mathbb{R}^N$*;*
(iii) $(\varphi_n)_n$ *converges decreasingly to* $\varphi$ *pointwise on* $\overline{\Omega} \times \mathbb{R}^N$*.*

**Proof.** By Lemma 3.1 it is sufficient to prove that $(d_n)_n$ converges pointwise to $d$. For any fixed $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ set $X := \mathrm{Lip}_{x,y}$. We want to prove that $\lim_n d_n(x, y) = d(x, y)$.

(i) For $\gamma \in X$ and the uniform convergence of $\varphi_n$ we have

$$\mathbb{L}_\varphi(\gamma) = \lim_{n\to+\infty} \mathbb{L}_{\varphi_n}(\gamma) \geq \limsup_{n\to+\infty} d_n(x, y),$$

which entails $d(x, y) \geq \limsup_n d_n(x, y)$ by taking the infimum over all possible curves $\gamma \in X$. Let us now choose a sequence of curves $(\gamma_n)_n \subset X$ such that $\mathbb{L}_{\varphi_n}(\gamma_n) \leq d_n(x, y) + 1/n$. Since the curves $\gamma_n$ are equi-Lipschitz continuous (as $\mathbb{L}_{\varphi_n}(\gamma_n) \geq \alpha \int_0^1 |\dot\gamma_n(t)|\, dt$ and all curves are parametrized by constant speed), the uniform convergence of the metrics implies that $\limsup_n |\mathbb{L}_{\varphi_n}(\gamma_n) - \mathbb{L}_\varphi(\gamma_n)| = 0$, therefore

$$d(x, y) \leq \liminf_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) = \liminf_{n\to+\infty} \mathbb{L}_{\varphi_n}(\gamma_n) = \liminf_{n\to+\infty} d_n(x, y),$$

hence the claim.

(ii) $(\mathbb{L}_{\varphi_n})_n$ is an increasing sequence of lower semicontinuous functionals on $X$ (*cf.* Remark 2.8), which converges pointwise to $\mathbb{L}_\varphi$ (*i.e.* $\lim_n \mathbb{L}_{\varphi_n}(\gamma) = \mathbb{L}_\varphi(\gamma)$ for each $\gamma \in X$). By [14, Remark 5.5] the functionals $\mathbb{L}_{\varphi_n}$ $\Gamma$-converge to $\mathbb{L}_\varphi$. In particular, since the functionals $\mathbb{L}_{\varphi_n}$ are equi-coercive (as $\mathbb{L}_{\varphi_n}(\gamma) \geq \alpha \int_0^1 |\dot\gamma| dt$ for every $n$), we have, by the crucial result of $\Gamma$-convergence [14, Theorem 7.4], that the sequence $(\inf_X \mathbb{L}_{\varphi_n})_n$ converges to $\inf_X \mathbb{L}_\varphi$, that is

$$\lim_{n\to+\infty} d_n(x, y) = \lim_{n\to+\infty} \inf_{\gamma\in X} \mathbb{L}_{\varphi_n}(\gamma) = \inf_{\gamma\in X} \mathbb{L}_\varphi(\gamma) = d(x, y).$$

(iii) By monotonicity we get $d(x, y) \leq \inf_n d_n(x, y)$. To show the reverse inequality, take a curve $\gamma \in X$. By the monotone convergence theorem and by the definition of $d_n(x, y)$ we have

$$\mathbb{L}_\varphi(\gamma) = \inf_n \mathbb{L}_{\varphi_n}(\gamma) \geq \inf_n d_n(x, y),$$

and the claim easily follows by taking the infimum over all curves in $\mathrm{Lip}_{x,y}$. $\square$

**Remark 3.3.** Notice that Proposition 3.2 (iii) holds for $\alpha = 0$ too.

We end this brief section with two results which will be needed in the sequel. The first one is just a restatement of Theorem 1 in [19, Section 6.6].

**Theorem 3.4.** *Suppose* $\gamma : I \to \mathbb{R}^N$ *is a Lipschitz continuous curve. Then for each* $\varepsilon > 0$*, there exists a* $C^1$ *curve* $\overline{\gamma} : I \to \mathbb{R}^N$ *such that:*

$$\mathcal{L}^1\left(\{t \in I \,|\, \overline{\gamma}(t) \neq \gamma(t) \text{ or } \dot{\overline{\gamma}}(t) \neq \dot\gamma(t)\,\}\right) \leq \varepsilon.$$

*In addition*

$$\left\|\dot{\overline{\gamma}}\right\|_\infty \leq c\, \|\dot\gamma\|_\infty$$

*for some constant* $c$ *depending only on* $N$*.*

The second result is an easy application of Severini-Egoroff's theorem. A proof is provided in [15].

**Lemma 3.5.** *Let $\gamma$ be an injective Lipschitz continuous curve, $\Gamma := \gamma((0,1)) \subset \overline{\Omega}$ and $a : \overline{\Omega} \to [\alpha, \beta]$ a Borel function. Then there exists a sequence of continuous functions $a_k : \Gamma \to [\alpha, \beta]$ such that $a_k(x)$ converge to $a(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$. Moreover, for every $\varepsilon > 0$ there exists a Borel subset $B_\varepsilon \subset \Gamma$ such that $\mathcal{H}^1(\Gamma \backslash B_\varepsilon) < \varepsilon$ and $a_k$ converge uniformly to $a$ on $B_\varepsilon$.*

## 4. The density results for non-degenerate distances

Throughout this section, $\alpha$ is always assumed to be strictly positive. Our first density result is then stated as follows:

**Theorem 4.1.** *Let $d \in \mathcal{D}_\alpha$. Then there exists a sequence $(\varphi_n)_n$ of continuous and convex Finsler metrics in $\mathcal{M}_\alpha$ such that $d_{\varphi_n} \xrightarrow{\mathcal{D}_\alpha} d$, where $d_{\varphi_n}$ is the distance associated to $\varphi_n$ through (4).*

When $d$ is symmetric, Theorem 4.1 has been already proved in [15]. In this case, the approximating metrics $\varphi_n(x, \xi)$ can be chosen of the form $a_n(x)|\xi|$. This accounts to saying that, for any $x \in \overline{\Omega}$, each metric $\varphi_n(x, \cdot)$ is the support function of the closed convex set $C^n(x) := \{p \in \mathbb{R}^N : |p| \leq a_n(x)\}$. This remark suggests how to generalize the proof to non-symmetric distances: for each $n \in \mathbb{N}$, we set

$$\varphi_n(x, \xi) := \sigma_{C^n(x)}(\xi) \qquad \text{for every } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N,$$

where $\sigma_K$ denotes the support function of the set $K \subset \mathbb{R}^N$, and $C^n : \overline{\Omega} \to \mathscr{P}(\mathbb{R}^N)$ is a set-valued map which has to be suitably defined. Of course, since we want to approximate a non-symmetric distance $d$, the sets $C^n(x)$ will be chosen non-symmetric with respect to 0, in general. The proof of Theorem 4.1 then proceeds by following the same idea used in [15]: we define the approximating sequence of set-valued maps $C^n(\cdot)$ in such a way to control the convergence of the induced distances $d_{\varphi_n}$ to $d$ pointwise on a dense subset of $\overline{\Omega} \times \overline{\Omega}$ (in view of Lemma 3.1). To this aim, we set $S := \mathbb{Q}^N \cap \overline{\Omega}$. Obviously $S \times S$ is countable and dense in $\overline{\Omega} \times \overline{\Omega}$, so we write

$$S \times S := \{(x_i, y_i) \,|\, i \in \mathbb{N}\}.$$

This notation is adopted in the remainder of the paper with no further mention.

Instead of providing a direct proof to Theorem 4.1, we prefer to break it into intermediate propositions, which will be proved separately. The first one can be read as a "dual" formulation of [15, Theorem 4.7].

**Theorem 4.2.** *Let $d \in \mathcal{D}_\alpha$. Then there exists a decreasing sequence of convex metrics $\psi_n \in \mathcal{M}_\alpha$ such that*

$$d_{\psi_n}(x_i, y_i) = d(x_i, y_i) \quad \textit{for each } i \leq n. \tag{9}$$

*Moreover, if we set $\psi(x, \xi) := \inf_n \psi_n(x, \xi)$ for every $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$, then $d = d_\psi$, that is every Finsler distance is induced by a convex Finsler metric.*

**Proof.** For each $(x_i, y_i) \in S \times S$, let $\gamma_i \in \mathrm{Lip}_{x_i, y_i}$ be a path of minimal $d$-length, *i.e.* $\mathrm{L}_d(\gamma_i) = d(x_i, y_i)$, and set $\Gamma_i := \gamma_i((0,1))$.
For every $x \in \Gamma_i$, let $\{\xi_1^i(x), \xi_2^i(x), \ldots, \xi_N^i(x)\}$ be an orthonormal basis of $\mathbb{R}^N$ such that $\xi_1^i(\gamma_i(t)) = \dot{\gamma}_i(t)/\|\dot{\gamma}_i\|_\infty$ for $\mathcal{L}^1$-a.e $t \in I$ (the curves $\gamma_i$ are assumed to be parametrized by constant speed), and set $a^i(x) := \varphi_d(x, \xi_1^i(x))$. Such vectors can

be chosen in such a way that the map $x \to \xi^i_j(x)$ is Borel-measurable on $\Gamma_i$ for each $1 \le j \le N$. For every $x \in \Gamma_i$, we denote by $Q^i(x)$ the rectangle in $\mathbb{R}^N$ defined by

$$Q^i(x) := \left\{ \mu_1 \xi^i_1(x) + \cdots + \mu_N \xi^i_N(x) \mid \mu_1 \in [-\beta, a^i(x)],\, \mu_2, \ldots, \mu_N \in [-\beta, \beta] \right\}.$$

We define the set-valued map $K^i(\cdot)$ on $\overline{\Omega}$ as follows:

$$K^i(x) := \begin{cases} Q^i(x) \cap B_\beta & \text{if } x \in \Gamma_i \\ B_\alpha & \text{if } x \in \{x_i, y_i\} \\ B_\beta & \text{otherwise,} \end{cases}$$

and set $C^n(x) := \cap_{i=1}^n K^i(x)$ for every $x \in \overline{\Omega}$. Clearly, $C^n(x)$ is a closed, convex set satisfying $B_\alpha \subset C_n(x) \subset B_\beta$ for every $x \in \overline{\Omega}$. Moreover the set-valued map $C^n(\cdot)$ is Borel measurable on $\overline{\Omega}$, as can be checked. The required metrics $\psi_n$ are then defined as follows:

$$\psi_n(x, \xi) := \sigma_{C^n(x)}(\xi) \quad \text{for every } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N,$$

for each $n \in \mathbb{N}$. Such functions $\psi_n$ are convex Finsler metrics belonging to $\mathcal{M}_\alpha$, as easily seen. In order to conclude, we need to show that (9) holds. This will be proved by showing that the curves $\gamma_i$ are still optimal for $d_{\psi_n}(x_i, y_i)$ for each $1 \le i \le n$.

To this end, first recall that the tangent spaces of two 1-rectifiable sets $M_1, M_2$ coincide at $\mathcal{H}^1$-a.e. point belonging to their intersection (*cf.* [21]). In particular, we get

$$\xi^1_i(x) = \pm \xi^1_h(x) \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma_i \cap \Gamma_h,$$

for every $1 \le i < h \le n$. This in turn implies that, for any fixed $1 \le i \le n$, the vector $a^i(x)\xi^i_1(x)$ belongs to $\partial C^n(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma_i$, and so that $\psi_n(\gamma_i(t), \dot\gamma_i(t)) = \varphi_d(\gamma_i(t), \dot\gamma_i(t))$ for $\mathcal{L}^1$-a.e. $t \in I$. Consequently we get

$$d_{\psi_n}(x_i, y_i) \le \int_0^1 \psi_n(\gamma_i, \dot\gamma_i)\, \mathrm{d}t = \int_0^1 \varphi_d(\gamma_i, \dot\gamma_i)\, \mathrm{d}t = d(x_i, y_i).$$

To prove the reverse inequality, choose a curve $\gamma \in \mathrm{Lip}_{x_i, y_i}$ and, for every $1 \le j < n$, set $I_{j+1} := \{t \in I \setminus \cup_{h \le j} I_h \mid \gamma(t) \in \gamma_{j+1}(I)\}$, $I_1 := \{t \in I \mid \gamma(t) \in \gamma_1(I)\}$ and $I_0 := I \setminus \cup_{j \le n} I_j$. By arguing as above, we deduce that, for each $1 \le j \le n$, the vector $\dot\gamma(t)$ is parallel to $\xi^j_1(\gamma(t))$ for $\mathcal{L}^1$-a.e. $t \in I_j$, hence $\psi_n(\gamma, \dot\gamma) \ge \varphi_d(\gamma, \dot\gamma)$ $\mathcal{L}^1$-a.e. on $I_j$ (depending on whether the two vectors are equally oriented or not). Note that this trivially holds for $j = 0$ too. Therefore we have

$$\mathbb{L}_{\psi_n}(\gamma) = \int_0^1 \psi_n(\gamma, \dot\gamma)\, \mathrm{d}t = \sum_{j=0}^n \int_{I_j} \psi_n(\gamma, \dot\gamma)\, \mathrm{d}t \ge \sum_{j=0}^n \int_{I_j} \varphi_d(\gamma, \dot\gamma)\, \mathrm{d}t \ge d(x_i, y_i).$$

By passing to the infimum over all possible curves $\gamma \in \mathrm{Lip}_{x_i, y_i}$ we get the claim.
To conclude the proof of the statement, set $\psi(x, \xi) := \inf_n \psi_n(x, \xi)$ for every $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$. By applying Proposition 3.2 (iii), we obtain

$$d_\psi(x_i, y_i) = \lim_{n \to +\infty} d_{\psi_n}(x_i, y_i) = d(x_i, y_i) \quad \text{for every } i \in \mathbb{N},$$

namely $d_\psi$ coincide with $d$ on $S \times S$, hence everywhere in $\overline{\Omega} \times \overline{\Omega}$ by density. □

Next, we prove the following

**Proposition 4.3.** *Let $d \in \mathcal{D}_\alpha$. Then, for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a lower semicontinuous, convex metric $\phi_\varepsilon \in \mathcal{M}_\alpha$ such that*

$$|d_{\phi_\varepsilon}(x_i, y_i) - d(x_i, y_i)| < \varepsilon \quad \text{for all } i \le n.$$

**Proof.** First, note that, if the set-valued map $K^i(\cdot)$ defined in the proof of Theorem 4.2 were continuous on $\Gamma_i$, the set-valued map $C^n(\cdot)$ would be lower semicontinuous on $\overline{\Omega}$ and a possible choice for $\phi_\varepsilon$ would simply be $\psi_n$. Hence, the idea is that of modifying the definition of $K^i(\cdot)$ in order to get a continuous map on $\Gamma_i$ for each $i$. We adopt the notation used in the proof of Theorem 4.2.
Let $n \in \mathbb{N}$ be fixed, let $\gamma_i$, $\Gamma_i$ and $a^i(x)$ be defined as above for each $i \in \mathbb{N}$, and set $N_n = \cup_{i=1}^n \Gamma_i$. Let us choose $\lambda > 0$ and let $1 \le i \le n$. Note that $\gamma_i$ is injective by minimality, hence we can apply Lemma 3.5 to obtain a sequence of continuous functions $a^i_k : \Gamma_i \to [\alpha, \beta]$, $k \in \mathbb{N}$, converging pointwise to $a^i$ $\mathcal{H}^1$-a.e. on $\Gamma_i$ as $k$ goes to infinity. Let $\overline{\gamma}_i$ be the $C^1$-continuous curve obtained by applying Theorem 3.4 with $\gamma := \gamma_i$ and $\varepsilon := \lambda$. For every $x \in \Gamma_i$, let $\{\overline{\xi}^i_1(x), \overline{\xi}^i_2(x), \dots, \overline{\xi}^i_N(x)\}$ be an orthogonal basis of $\mathbb{R}^N$ such that:

(i) $\overline{\xi}^i_1(\gamma_i(t)) = \dot{\overline{\gamma}}_i(t)/\|\dot{\gamma}_i\|_\infty$ for $\mathcal{L}^1$-a.e $t \in I$;
(ii) $|\overline{\xi}^i_j(x)| = 1$ for every $2 \le j \le N$.

Such vectors can be chosen in such a way that the map $x \to \overline{\xi}^i_j(x)$ is continuous on $\Gamma_i$ for each $1 \le j \le N$. For each $k \in \mathbb{N}$ and every $x \in \Gamma_i$, we denote by $\overline{Q}^i_k(x)$ the rectangle in $\mathbb{R}^N$ defined by

$$\overline{Q}^i_k(x) := \Big\{ \mu_1 \overline{\xi}^i_1(x) + \cdots + \mu_N \overline{\xi}^i_N(x) \mid \mu_1 \in [-\beta, a^i_k(x)],\ \mu_2, \dots, \mu_N \in [-\beta, \beta] \Big\}.$$

We define the set-valued map $\overline{K}^i_k(\cdot)$ on $\overline{\Omega}$ as follows:

$$\overline{K}^i_k(x) := \begin{cases} \overline{Q}^i_k(x) \cap B_\beta & \text{if } x \in \Gamma_i \\ B_\alpha & \text{if } x \in \{x_i, y_i\} \\ B_\beta & \text{otherwise.} \end{cases}$$

It is easy to check by definition that $\overline{K}^i_k(\cdot)$ is lower semicontinuous for each $1 \le i \le n$, so the same is true for the set-valued map $\overline{C}^n_k(\cdot)$ defined by setting $\overline{C}^n_k(x) := \cap_{i=1}^n \overline{K}^i_k(x)$ for every $x \in \overline{\Omega}$. For each $k \in \mathbb{N}$, a convex, lower semicontinuous Finsler metric $\phi_k$ belonging to $\mathcal{M}_\alpha$ may be then defined as follows:

$$\phi_k(x, \xi) := \sigma_{\overline{C}^n_k(x)}(\xi) \vee \alpha|\xi| \quad \text{for every } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

We claim that

$$\limsup_{k \to +\infty} |d_{\phi_k}(x_i, y_i) - d(x_i, y_i)| \le (n\beta L)\,\lambda \quad \text{for all } 1 \le i \le n, \tag{10}$$

where $L := \sup_{1 \le i \le n} \|\dot{\gamma}_i\|_\infty$ This will be enough to conclude: indeed, it is sufficient to take $\lambda < \varepsilon/(n\beta L)$ and $\phi_\varepsilon := \phi_k$ with $k$ suitably large.

Let us then prove (10). For $1 \le i \le n$ we have by definition

$$d_{\phi_k}(x_i, y_i) \le \mathbb{L}_{\phi_k}(\gamma_i) = \int_0^1 \phi_k(\gamma_i, \dot{\gamma}_i)\,\mathrm{d}t.$$

Let $J_i := \{\, t \in I \,|\, \dot{\overline{\gamma}}_i(t) = \dot{\gamma}_i(t) \,\}$ and recall that, by Theorem 3.4, $\mathcal{L}^1(I \setminus J_i) \le \lambda$. An argument analogous to the one used in the proof of Theorem 4.2 shows that $\phi_k(\gamma_i, \dot{\gamma}_i) = a^i_k(\gamma_i)|\dot{\gamma}_i|$ $\mathcal{L}^1$-a.e. on $J_i$. By the dominated convergence theorem we then have

$$
\begin{aligned}
\limsup_{k\to+\infty} d_{\phi_k}(x_i,y_i) &\le \limsup_{k\to+\infty}\left(\int_{J_i} a_k^i(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t + \int_{I\setminus J_i}\beta\,|\dot\gamma_i(t)|\,\mathrm{d}t\right)\\
&\le \int_0^1 a^i(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t + \beta L\lambda = d(x_i,y_i)+\beta L\lambda. \qquad (11)
\end{aligned}
$$

Now, set $\Gamma_0 := \cup_{i=1}^n \gamma_i(I\setminus J_i)$, and remark that $\mathcal{H}^1(\Gamma_0)\le nL\lambda$. Fix an index $1\le i\le n$. For each $k\in\mathbb{N}$, pick up a curve $\tilde\gamma_k\in \mathrm{Lip}_{x_i,y_i}$ such that

$$
\mathbb{L}_{\phi_k}(\tilde\gamma_k) = d_{\phi_k}(x_i,y_i). \qquad (12)
$$

Note that such curves are injective by minimality. Since $\alpha\int_I|\dot{\tilde\gamma}_k|\,\mathrm{d}t\le \mathbb{L}_{\phi_k}(\tilde\gamma_k)$, by (12) and (11) we get that $\limsup_k\int_I|\dot{\tilde\gamma}_k|\,\mathrm{d}t<+\infty$. Let us choose an $\tilde\varepsilon>0$. By applying Lemma 3.5 to each open arc $\Gamma_i$, we can find a Borel set $B_{\tilde\varepsilon}\subset N_n$ and an infinitesimal sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\mathcal{H}^1(N_n\setminus B_{\tilde\varepsilon})<\tilde\varepsilon$ and $|a_k^i(x)-a^i(x)|<\delta_k$ for every $x\in B_{\tilde\varepsilon}$, $1\le i\le n$ and $k\in\mathbb{N}$. Let us set $I_k:=\{t\in I\,|\,\tilde\gamma_k(t)\in\Gamma_0\cup(N_n\setminus B_{\tilde\varepsilon})\}$. Then we have

$$
\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\ge \psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k)-\delta_k|\dot{\tilde\gamma}_k| \qquad \mathcal{L}^1\text{-a.e. on } I\setminus I_k. \qquad (13)
$$

Let us write

$$
\mathbb{L}_{\phi_k}(\tilde\gamma_k)=\int_{I_k}\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t+\int_{I\setminus I_k}\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t.
$$

As $\tilde\gamma_k(I_k)\subset\Gamma_0\cup(N_n\setminus B_{\tilde\varepsilon})$ for every $k\in\mathbb{N}$, by the Area-formula we have

$$
\int_{I_k}|\dot{\tilde\gamma}_k|\,\mathrm{d}t=\mathcal{H}^1(\tilde\gamma_k(I_k))\le\mathcal{H}^1(N_n\setminus B_{\tilde\varepsilon})+\mathcal{H}^1(\Gamma_0)<\tilde\varepsilon+nL\lambda.
$$

Taking this remark into account we get

$$
\begin{aligned}
\int_{I_k}\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\mathrm{d}t &= \int_{I_k}\psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t+\int_{I_k}\left(\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)-\psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k)\right)\,\mathrm{d}t\\
&\ge \int_{I_k}\psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t-\beta\,(\tilde\varepsilon+nL\lambda)\,. \qquad (14)
\end{aligned}
$$

Inequalities (13) and (14) then yields

$$
\begin{aligned}
\mathbb{L}_{\phi_k}(\tilde\gamma_k) &\ge \int_0^1\psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t-\delta_k\int_{I\setminus I_k}|\dot{\tilde\gamma}_k|\,\mathrm{d}t-\beta\,(\tilde\varepsilon+nL\lambda)\\
&\ge d_{\psi_n}(x_i,y_i)-\delta_k\int_0^1|\dot{\tilde\gamma}_k|\,\mathrm{d}t-\beta\,(\tilde\varepsilon+nL\lambda)
\end{aligned}
$$

and therefore, as $\delta_k\int_0^1|\dot{\tilde\gamma}_k|\,\mathrm{d}t$ goes to zero when $k\to+\infty$, we obtain

$$
\liminf_{k\to+\infty} d_{\phi_k}(x_i,y_i)=\liminf_{k\to+\infty}\mathbb{L}_{\phi_k}(\tilde\gamma_k)\ge d_{\psi_n}(x_i,y_i)-\beta\,(\tilde\varepsilon+nL\lambda)\,.
$$

Since $\tilde\varepsilon$ was arbitrary and $d_{\psi_n}(x_i,y_i)=d(x_i,y_i)$, the above inequality coupled with (11) gives (10), as claimed. $\square$

We are now ready to give the

**Proof of Theorem 4.1.** First, we claim that, for every $\varepsilon>0$, there exists a continuous, convex metric $\varphi_\varepsilon\in\mathcal{M}_\alpha$ such that

$$
|d_{\varphi_\varepsilon}(x_i,y_i)-d(x_i,y_i)|<\varepsilon \quad \text{for all } i\le n. \qquad (15)
$$

Indeed, Proposition 4.3 provides a convex, lower semicontinuous metric $\phi_\varepsilon$ satisfying (15). By setting $\phi_\varepsilon(x,\xi) := \beta|\xi|$ when $x \in \mathbb{R}^N \setminus \overline{\Omega}$, $\phi_\varepsilon$ can be extended to $\mathbb{R}^N \times \mathbb{R}^N$. Then Lemma 2.2.3 of [6] easily implies the existence of an increasing sequence $(\Phi_n)_n$ of continuous, convex Finsler metrics on $\mathbb{R}^N$ such that $\Phi_n|_{\overline{\Omega}\times\mathbb{R}^N} \in \mathcal{M}_\alpha$ for each $n \in \mathbb{N}$ and $\phi_\varepsilon(x,\xi) = \sup_{n\in\mathbb{N}} \Phi_n(x,\xi)$ in $\overline{\Omega} \times \mathbb{R}^N$ (*cf.* Proof of [5, Theorem 3.1]). By Proposition 3.2 (ii), the sequence of distances $(d_{\Phi_n})_n$ converges to $d_{\phi_\varepsilon}$ in $\mathcal{D}_\alpha$, hence (15) is proved by setting $\varphi_\varepsilon := \Phi_n$ for $n$ sufficiently large.
It is now clear how to conclude: for each $n \in \mathbb{N}$, define $\varphi_n := \varphi_{\varepsilon_n}$ with $\varepsilon_n := 1/n$. The distances $d_{\varphi_n}$ converge to $d$ pointwise on $S \times S$, which is dense in $\overline{\Omega} \times \overline{\Omega}$, so the statement follows in view of Lemma 3.1. □

**Remark 4.4.** Note that the metrics $\phi_n$ obtained in the previous proof are actually defined and continuous in all $\mathbb{R}^N \times \mathbb{R}^N$.

The result established in Theorem 4.1 can now be easily improved by requiring the approximating metrics to be smooth.

**Theorem 4.5.** *Let $d \in \mathcal{D}_\alpha$. Then there exists a sequence $(\tilde{\varphi}_n)_n$ of smooth and convex Finlser metrics in $\mathcal{M}_\alpha$ such that $d_{\tilde{\varphi}_n} \xrightarrow{\mathcal{D}_\alpha} d$.*

**Proof.** Theorem 4.1 provides a sequence $(\varphi_n)_n$ of continuous, convex metrics in $\mathcal{M}_\alpha$ satisfying the claim. Such metrics are actually defined and continuous in $\mathbb{R}^N \times \mathbb{R}^N$ (*cf.* Remark 4.4). Take a sequence $(\rho_k)_k$ of convolution kernels in $\mathbb{R}^N$ and, for each $n \in \mathbb{N}$, let $(\rho_k * \varphi_n)(x,\xi) := \int_{\mathbb{R}^N} \rho_k(x-y)\varphi_n(y,\xi)\,\mathrm{d}y$. Clearly $(\rho_k * \varphi_n)_k$ is a sequence of convex, smooth metrics in $\mathcal{M}_\alpha$, uniformly converging to $\varphi_n$ on compact subsets of $\overline{\Omega} \times \mathbb{R}^N$. In view of Proposition 3.2 (i), the claim follows by setting $\tilde{\varphi}_n := \rho_k * \varphi_n$ for each $n \in \mathbb{N}$, with $k := k(n)$ suitably large. □

We end this section by providing an example of Finsler metric which gives rise to a non-symmetric distance. The latter plays a central role in the study of Hamilton-Jacobi equations of eikonal type.

**Example 4.6.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let us consider the following problem

$$\begin{cases} H(Du) = n(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{16}$$

where $n(x)$ is a continuous positive function defined on $\overline{\Omega}$, and $H$ is a non-negative convex function with $H(0) = 0$. Solutions and subsolutions to (16) are usually meant in the viscosity sense, since classical ones do not exist in general, as well known (see e.g. [23]). In particular, viscosity subsolutions of (16) are characterized by the property of being 1-Lipschitz continuous with respect to the distance $D$ defined by

$$D(x,y) := \inf\left\{ \int_0^1 \varphi(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t \;\middle|\; \gamma \in \mathrm{Lip}_{x,y}\left([0,1],\overline{\Omega}\right) \right\} \quad \text{for all } x,y \in \overline{\Omega},$$

where $\varphi$ is the Finsler metric associated to the equation as follows:

$$\varphi(x,\xi) := \sup\{\langle \xi, p\rangle \mid H(p) \le n(x)\} \qquad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

Moreover, the unique viscosity solution $v$ to (16) is given by the following Lax formula

$$v(x) := \min_{y\in\partial\Omega} D(y,x).$$

Note that, when the function $H$ is non-symmetric, the distance $D$ is, in general, non-symmetric as well. The distance $D$ is called *optical length function*. This name originates from geometrical optics. Indeed, we may regard the differential equation in (16) as the equation which rules the propagation of light in a non-isotropic medium, where $u(x)$ represents the time at which the light reaches the location $x$, and $n(x)$ is the refractive index of the medium at $x$. The presence of the function $H$ accounts for the fact that the speed of light depends on its direction of travel. According to Fermat's principle, a ray of light moves from one source $y$ to another point $x$ along the path which is the shortest in time, *i.e.* in such a way to realize $D(y, x)$.

When $n$ is not continuous, the metric $\varphi$ is not continuous either. This case is of clear interest: think, for instance, to propagation of light in a medium composed by two materials with different indexes of refraction. This motivates the study of first order Hamilton-Jacobi equations with measurable ingredients. A metric approach to this problem, relying on the analysis of the properties of the Finsler metric $\varphi$ and of the related optical length function, can be found in [4, 11, 24].

## 5. The degenerate case

We want to extend the density results obtained in the previous section to the degenerate case, namely when $\alpha = 0$. In fact, the analogous of Theorem 4.5 holds.

**Theorem 5.1.** *Let $d \in \mathcal{D}_0$. Then there exists a sequence $(\tilde{\varphi}_n)_n$ of smooth and convex Finsler metrics in $\mathcal{M}_0$ such that $d_{\tilde{\varphi}_n} \xrightarrow{\mathcal{D}_0} d$.*

**Proof.** Let $\varphi \in \mathcal{M}_0$ such that $d = d_\varphi$. For each $k \in \mathbb{N}$, set $\varphi_k(x, \xi) := \varphi(x, \xi) \vee \frac{1}{k}|\xi|$ in $\overline{\Omega} \times \mathbb{R}^N$. As $\varphi_k \in \mathcal{M}_{1/k}$, we can apply Theorem 4.5 to $d_{\varphi_k}$ for each $k \in \mathbb{N}$, to obtain a smooth Finsler metric $\tilde{\varphi}_k \in \mathcal{M}_{1/k}$ such that

$$|d_{\varphi_k}(x, y) - d_{\tilde{\varphi}_k}(x, y)| \leq \frac{1}{k} \quad \text{for every } x, y \in \overline{\Omega}.$$

Now the claim easily follows, since the sequence $(d_{\varphi_k})_k$ converges to $d$ in $\mathcal{D}_0$ in view of Remark 3.3. $\square$

As already remarked in Section 2, when $\alpha > 0$, the space $\mathcal{D}_\alpha$ is closed, that is, if $(d_n)_n$ is a sequence in $\mathcal{D}_\alpha$ converging to $d$, then $d$ belongs to $\mathcal{D}_\alpha$. Does the same property hold for $\alpha = 0$ too?

The point is to show that $d$ is still of geodesic type. When $\alpha > 0$, this is basically due to the fact that the corresponding length functionals are equi-coercive, namely $\mathrm{L}_{d_n}(\gamma) \geq \alpha \int_0^1 |\dot{\gamma}(t)|\, \mathrm{d}t$ for any curve $\gamma$ and for each $n \in \mathbb{N}$, and this means that any sequence of curves $(\gamma_n)_n \subset \mathrm{Lip}_{x,y}$ such that $\limsup_n \mathrm{L}_{d_n}(\gamma_n) < +\infty$ admits (at least) a cluster point $\gamma \in \mathrm{Lip}_{x,y}$. In particular, for a suitable choice of $(\gamma_n)_n$ and of $\gamma$, that yields

$$\mathrm{L}_d(\gamma) \leq \liminf_{n \to +\infty} \mathrm{L}_{d_n}(\gamma_n) = \lim_{n \to +\infty} d_n(x, y) = d(x, y),$$

which obviously means that $d$ is a geodesic distance. When $\alpha = 0$, instead, it may happen that the Euclidean lengths of the curves $(\gamma_n)_n$ diverge, and two critical phenomena may basically occur in this case: either the curves go to infinity and disappear in the limit, or they stay bounded and converge to a non-rectifiable curve (hence, no longer Lipschitz continuous). In both cases, the limit distance might be not of geodesic type; hence, the answer to the question previously raised is no: $\mathcal{D}_0$ is not closed. Explicit examples of these possible situations are provided below.

**Example 5.2.** Let $\Omega := \mathbb{R}^2$, and let us set $\Gamma_n := \{(x,y) \mid x \in \{0,1\},\, y \in [0,n]\} \cup [0,1] \times \{n\}$ for each $n \in \mathbb{N}$, $\Gamma_\infty := \{(x,y) \mid x \in \{0,1\},\, y \in [0,+\infty)\}$. For each $n \in \mathbb{N}$, we define the metric $\varphi_n$ on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows:

$$\varphi_n(x,\xi) := \begin{cases} \frac{1}{4n^2}|\xi| & \text{if } x \in \Gamma_n \text{ and } \xi \in \mathbb{R}^2 \\ \\ 2|\xi| & \text{if } x \in \mathbb{R}^2 \setminus \Gamma_n \text{ and } \xi \in \mathbb{R}^2 \end{cases}$$

Let $x_0 := (0,0)$, $y_0 := (1,0)$, and let $d_n := d_{\varphi_n}$ be the distance defined on $\mathbb{R}^2 \times \mathbb{R}^2$ via (4) for each $n \in \mathbb{N}$. Let us notice that $d_n(x_0,y_0) = (2n+1)/4n^2$: in fact, the $d_n$-minimizing path connecting $x_0$ to $y_0$ is given by the polygonal arc $\Gamma_n$. Up to subsequences, $(d_n)_n$ converges (uniformly on compact subsets of $\mathbb{R}^2 \times \mathbb{R}^2$) to some distance $d$, by Ascoli-Arzelà Theorem. Now we have that $d(x_0,y_0) = 0$, while the $d$-metric length of any curve connecting $x_0$ to $y_0$ is at least 2. To see this, simply notice that $\varphi_d(x,\xi) = 2|\xi|$ for every $x \in \mathbb{R}^2 \setminus \Gamma_\infty$. Hence $d$ is not of geodesic type, though uniform limit of geodesic distances.

**Example 5.3.** Let $\Omega := (0,1) \times (0,1)$ and let $q = (1/2,0), q' = (1,1/2)$. Our example relies upon the construction provided by Whitney in [27]. In this remarkable paper, Whitney recursively defines an arc $A$ joining $q$ to $q'$ of infinite length, and a function $f$, defined on $\overline{\Omega}$ and of class $C^1$, whose gradient is null on $A$, but $f(q) = 0$ and $f(q') = 1$ (in particular, $f$ is not constant on $A$).

Such an arc can be represented as the image of a continuous function $\gamma : I \to \overline{\Omega}$ with $\gamma(0) = q$, $\gamma(1) = q'$, as explained by the author. Our goal is to define a sequence of Lipschitz curves uniformly converging to $\gamma$. This can be easily done by exploiting Whitney's construction. Let us keep the same notation of [27] and assume we are at the $n$-th step of the iterative procedure leading to the definition of $A$, *i.e.* we have already defined squares $Q_{i_1\cdots i_t}$, points $q_{i_1\cdots i_t}$, $q'_{i_1\cdots i_t}$ and lines $A_{j_1\cdots j_t}$ (each $i_k = 0,1,2,3$, each $j_k = 0,1,2,3,4$) for $t \le n$. Then, an arc $A_n$ of finite length, joining $q$ to $q'$, can be obtained by connecting each point $q'_{i_1\cdots i_s}$ to $q_{i_1\cdots i_s+1}$, if $i_s \le 2$, and $q'_{i_1\cdots i_{s-1},3}$ to $q_{i_1\cdots i'_{s-1}}$ by means of a segment, and by gluing all these segments with all the lines $A_{j_1\cdots j_t}$, $t \le n$. Each arc $A_n$ may be represented as the image of a Lipschitz curve $\gamma_n$. Up to a reparametrization (not by constant speed, in particular), the curves $\gamma_n$ uniformly converge to $\gamma$, as easily understood.

Let us now denote by $\ell_n$ the Euclidean length of the curve $\gamma_n$, and define the following sequence $(a_n)_n$ of Riemannian metrics:

$$a_n(x) := \begin{cases} \frac{1}{n\ell_n} & \text{if } x \in A_n \\ \\ |Df(x)| & \text{if } x \in \overline{\Omega} \setminus A_n. \end{cases}$$

Let $d_n$ the distances on $\overline{\Omega} \times \overline{\Omega}$ associated to $a_n(x)|\xi|$ through (4). Obviously, $d_n(q,q') \le 1/n$. Up to subsequences, $(d_n)_n$ converges to some distance $d$, by Ascoli-Arzelà Theorem, and obviously $d(q,q') = 0$. Notice also that $\varphi_d(x,\xi) = |Df(x)||\xi|$ for every $x \in \overline{\Omega} \setminus A$. Now, let us take a curve $\xi \in \mathrm{Lip}_{q,q'}$. We have

$$\mathrm{L}_d(\xi) = \int_0^1 \varphi_d(\xi(t),\dot\xi(t))\,\mathrm{d}t \ge \int_0^1 |Df(\xi(t))|\,|\dot\xi(t)|\,\mathrm{d}t \ge f(q') - f(q) = 1,$$

so $d$ is not of geodesic type.

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