AN APPROXIMATION RESULT FOR FREE DISCONTINUITY FUNCTIONALS BY MEANS OF NON-LOCAL ENERGIES

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Abstract. We approximate, in the sense of Γ-convergence, free discontinuity functionals with linear growth by a sequence of non local integral functionals depending on the average of the gradient on small balls. The result extends to higher dimension what already proved in [15], where there is the proof of the general one-dimensional case, and in [16], where the n-dimensional case with ϕ =Id is treated. Moreover we investigate whether it is possible to approximate a given free discontinuity functional by means of non-local energies.

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1. Introduction

Many variational problems recently under consideration, like computer vision problems (image segmentation, signal processing and so on) involve "free discontinuity functionals" (according to a terminology introduced by E. DeGiorgi in [12]); a weak form of those functionals in BV space takes the form

$$F(u) = \int_\Omega \phi(|\nabla u|)dx + \int_{S_u} \vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + c|D^c u|(\Omega), \tag{1.1}$$

where $Du = \nabla u dx + (u^+ - u^-)d\mathcal{H}^{n-1} + D^c u$ is the decomposition of the distributional derivative of a BV-function u, ϕ is convex and ϑ is subadditive.

Several approximation methods of these functionals are known. For example the Ambrosio and Tortorelli approximation (see [3] and [4]) of the Mumford Shah functional

$$MS(u) = \int_\Omega |\nabla u|^2 dx + c\mathcal{H}^{n-1}(S_u), \quad u \in SBV(\Omega)$$

via elliptic functionals; see [17] for applications to computer vision problems. A variant of this method was studied in [1] for functional with linear growth in the gradient. Moreover a finite difference method scheme for free discontinuity functionals was studied by Gobbino and Mora in [13]. About the non-local approximation we refer the reader to [7], [9] and [8]; in this last paper Braides and Dal Maso approximate the Mumford-Shah functional. A variant of this method was studied in [16], where a convergence result is proved for the family

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u| dy\right)dx, \quad u \in W^{1,1}(\Omega),$$

to a functional F of type (1.1), with $\phi(t) = t$.

In this paper first, in section 4, we will extend result [16] to a family of functions f_ε, depending on ε, instead of f, in view to obtain a convergence result (already proved in the one-dimensional case in [15]) to a functional (1.1) with a more general growth in the gradient with respect to the case ϕ =Id considered in [16]. Then, in section 5, we will investigate whether it is possible to construct a family $(F_\varepsilon)_{\varepsilon>0}$ which Γ-converges to a given functional F of type (1.1) with linear growth in the gradient. The main difficulty is to find a family of functions f_ε which gives rise to a prescribed surface energy density, because this leads to consider integral equations for which the properties of the solutions can be determined only in few special cases. However, for example, we will obtain an explicit approximation for the functional given by

$$F(u) = \int_\Omega |\nabla u| dx + \int_{S_u} \frac{|u^+ - u^-|}{1 + |u^+ - u^-|} d\mathcal{H}^{n-1} + |D^c u|(\Omega),$$

proposed by Shah in [18] as a segmentation functional with linear growth.

2. Notation and preliminaries

Let $n \geq 1$ be a fixed integer. The open ball with center x and radius r is indicated by $B_r(x)$; the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure of a Borel set $B \subseteq \mathbb{R}^n$ are denoted by $|B|$ (or $\mathcal{L}^n(B)$) and $\mathcal{H}^{n-1}(B)$, respectively. We use standard notation for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$.

2.1. **Functions of bounded variation.** For a thorough treatment of BV functions we refer to [2]. For a function $u \in BV(\Omega)$ the distributional derivative of u is given by

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u + D^c u.$$

We say that $u \in SBV(\Omega)$ if $|D^c u|(\Omega) = 0$. A function $u \in L^1(\Omega)$ is a *generalized function of bounded variation* ($u \in GBV(\Omega)$) if $u^T := (-T) \vee u \wedge T$ belongs to $BV(\Omega)$ for every $T \geq 0$. If $u \in GBV(\Omega)$, then ∇u, S_u, $u^+(x)$ and $u^-(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$, and $|D^c u|$ turn out to be well-defined ([2], Theorem 4.34 and Definition 4.33).

2.2. **Γ-convergence.** For the general theory see [6] and [11]. Let (X, d) be a metric space. Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that $(F_j)_{j\in\mathbb{N}}$ Γ-*converges*, as $j \to +\infty$, to $F\colon X \to \overline{\mathbb{R}}$, if for all $u \in X$ we have:

i) for every sequence $(u_j)_{j\in\mathbb{N}}$ converging to u

$$F(u) \leq \liminf_{j\to+\infty} F_j(u_j);$$

ii) there exists a sequence $(u_j)_{j\in\mathbb{N}}$ converging to u such that

$$F(u) \geq \limsup_{j\to+\infty} F_j(u_j).$$

We extend this definition of convergence to families depending on a real parameter. Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-*converges*, as $\varepsilon \to 0$, to $F\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ the sequence $(F_{\varepsilon_j})_{j\in\mathbb{N}}$ Γ-converges to F.

2.3. **Supremum of measures.** We recall the following useful tool, which can be found in [2].

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let $\mathcal{A}(\Omega)$ be the the family of its open subsets. Let λ be a positive Borel measure on Ω, and $\mu\colon \mathcal{A}(\Omega) \to [0, +\infty)$ a set function which is superadditive on open sets with disjoint compact closures (i.e. if $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$, then $\mu(A \cup B) \geq \mu(A) + \mu(B)$). Let $(\psi_i)_{i\in I}$ be a family of positive Borel functions on Ω. Assume that*

$$\mu(A) \geq \int_A \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I\,.$$

Then

$$\mu(A) \geq \int_A \sup_{i\in I} \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega)\,.$$

The following Corollary is an easy consequence of Lemma 2.1.

Corollary 2.2. *Let μ be as in Lemma 2.1. Let $\lambda_1, \ldots, \lambda_N$ be mutually singular Borel measures, and $\psi_1, \ldots, \psi_N$ positive Borel functions on Ω. Suppose that*

$$\mu(A) \geq \int_A \psi_i \, d\lambda_i \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i = 1, \ldots, N.$$

Then

$$\mu(A) \geq \sum_{i=1}^{N} \int_A \psi_i \, d\lambda_i \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

3. Setting of the problem and main results

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. In this paper we consider the same framework of [15], which we briefly recall.

Let $c \in (0,+\infty)$ and let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a convex function with $\phi(0) = 0$ and with

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c.$$

Let $(f_\varepsilon)_{\varepsilon>0}$ be a family of functions satisfying the following conditions.

- **(A1)** For every $\varepsilon > 0$, $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ is a non-decreasing continuous function with $f_\varepsilon(0) = 0$; moreover, there exists $a_\varepsilon > 0$ such that $a_\varepsilon \to 0$ as $\varepsilon \to 0$ and f_ε is concave in $(a_\varepsilon, +\infty)$.
- **(A2)** $$\lim_{(\varepsilon,t)\to(0,0)} \frac{f_\varepsilon(t)}{\varepsilon\phi\left(\frac{t}{\varepsilon}\right)} = 1\,.$$
- **(A3)** f_ε converges uniformly on the compact subsets of $[0,+\infty)$; we denote the limit of f_ε by f, which is a continuous, concave and non-decreasing function.
- **(A4)** There exists $L > 0$ such that $$|f_\varepsilon(s) - f_\varepsilon(t)| \leq L|s-t|, \quad \forall s,t > 0.$$

Notice that we obtain the case considered in [16] by taking $c = 1$, $\phi(t) = t$, and the constant sequence $f_\varepsilon \equiv f$ with

$$\lim_{t\to 0} \frac{f(t)}{t} = 1.$$

Consider the family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $L^1(\Omega) \to [0,+\infty]$ of the form

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right)dx & \text{if } u \in W^{1,1}(\Omega)\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

For every open subset A of Ω we set

$$F_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right)dx & \text{if } u \in W^{1,1}(\Omega)\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Clearly, $F_\varepsilon\big(\cdot,\Omega\big)$ coincides with the functional F_ε defined in (3.1). The lower and upper Γ-limits of $\big(F_\varepsilon(\cdot,A)\big)$ will be denoted by $F'(\cdot,A)$ and $F''(\cdot,A)$, respectively.

3.1. The convergence result.

Let us now state the main convergence Theorem.

Theorem 3.1. *Let $(F_\varepsilon)_{\varepsilon>0}$ be as in (3.1), with f_ε satisfying conditions* **(A1)-(A4)**. *Then (F_ε) Γ-converges, in the L^1-topology, as $\varepsilon \to 0$, to $\mathcal{F}\colon L^1(\Omega) \to [0,+\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega \phi(|\nabla u|)dx + \int_{S_u} \vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + c|D^c u|(\Omega) & \text{if } u \in GBV(\Omega)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

where

$$\vartheta(s) = 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1}\right) dt. \tag{3.3}$$

Moreover we have the following compactness property which turns out to be useful in view to obtain convergence of minima and minimizers.

Theorem 3.2. **(Compactness)** *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a positive infinitesimal sequence and let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $L^1(\Omega)$ such that $||u_j||_\infty \le M$, and $F_{\varepsilon_j}(u_j) \le M$ for a suitable constant M independent of j. Then there exists a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

3.2. **The approximation result.** Let $\phi, \vartheta\colon [0,+\infty) \to [0,+\infty)$ be such that:

i) $\phi(0) = \vartheta(0) = 0$.
ii) ϕ is convex and ϑ is concave.
iii) There exists $c \in (0,+\infty)$ such that

$$\lim_{t\to+\infty}\frac{\phi(t)}{t} = \lim_{t\to 0}\frac{\vartheta(t)}{t} = c.$$

Let $F\colon L^1(\Omega) \to [0,+\infty]$ be the functional given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega \phi(|\nabla u|)dx + \int_{S_u}\vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + c|D^c u|(\Omega) & \text{if } u \in GBV(\Omega)\,, \\ +\infty & \text{otherwise.}\end{cases}$$

The following approximation result for F holds.

Theorem 3.3. *Let us suppose that $\vartheta \in C^1([0,+\infty))$; moreover we assume that the function*

$$t \mapsto \vartheta(t) + \frac{n-1}{2}t\vartheta'(t)$$

is non-decreasing and concave. Then for any $\varepsilon > 0$ there exists $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ satisfying conditions **(A1)-(A4)** *and be such that the family $(F_\varepsilon)_{\varepsilon>0}$ given by (3.1) Γ-converges to F as $\varepsilon \to 0$ with respect to the strong topology of L^1.*

4. Proof of the convergence result

In this section we will prove Theorem 3.1 combining Proposition 4.5 with Proposition 4.9.

4.1. **Compactness and lower bound.** In this subsection we prove the lower bound for the functional $F'(\cdot, A)$. First we need the following Lemma which is, up to some modifications, Theorem 6.1 in [16].

Lemma 4.1. *Let $g\colon [0,+\infty) \to [0,+\infty)$ be a continuous, concave and non-decreasing function with*

$$\lim_{t\to 0^+}\frac{g(t)}{t} = \alpha \in (0,+\infty).$$

For any $\varepsilon > 0$ and A open subset in Ω let $G_\varepsilon\colon L^1(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ be the functional given by

$$G_\varepsilon(u,A) = \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_A g\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|dy\right)dx & \text{if } u \in W^{1,1}(\Omega)\,, \\ +\infty & \textit{otherwise.}\end{cases}$$

Then for any $u \in BV(\Omega)$ and for any A open subset of Ω we have

$$G'(u,A) \ge \alpha\int_A |\nabla u|dx + \int_{S_u\cap A}\vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + \alpha|D^c u|(A) \tag{4.1}$$

where

$$\vartheta(s) = 2\int_0^1 g\left(\frac{\omega_{n-1}}{\omega_n}s(\sqrt{1-t^2})^{n-1}\right)dt, \quad s \ge 0.$$

Proof. Theorem 6.1 in [16] says that inequality (4.1) holds if $g \in C^2$ and if g is strictly concave. For every $h \in \mathbb{N}$ let $g_h\colon [0,+\infty) \to [0,+\infty)$ be a strictly concave, non decreasing and C^2-function with $(1-1/h)g(t) \le g_h(t) \le g(t)$; then

$$G'(u,A) \ge \alpha(1-1/h)\int_A |\nabla u|dx + \int_{S_u\cap A} \vartheta_h(|u^+ - u^-|)d\mathcal{H}^{n-1} + \alpha(1-1/h)|D^c u|(A),$$

where

$$\vartheta_h(s) = 2\int_0^1 g_h\left(\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1}\right)dt, \quad s \ge 0.$$

Since $g_h \to g$ uniformly on compact subsets of $[0,+\infty)$, we have $\vartheta_h \to \vartheta$ pointwise, so that we conclude by Fatou's Lemma. □

Remark 4.2. *Let g and G_ε as in Lemma 4.1; then the compactness property given by Theorem 3.2 holds in this case. (Theorem 3.2 in* [16]*).*

In the previous Lemma, as in [16], the function g does not depend on ε and it has linear behaviour near 0. In this case the energy density of the absolutely continuous part of the Γ-limit is simply given by $\alpha|\nabla u|$. Condition **(A2)** says that the function f_ε tends to be a rescaled of a convex function ϕ near 0:

$$f_\varepsilon(t) \sim \varepsilon\phi\left(\frac{t}{\varepsilon}\right), \quad t \sim 0.$$

In this setting the energy density of the volume part of the Γ-limit will be given by $\phi(|\nabla u|)$.

Proposition 4.3. *Let $u \in BV(\Omega)$ and A be an open subset of Ω; then*

$$F'(u,A) \ge \int_A \phi(|\nabla u|)dx + c|D^c u|(A).$$

Proof. Let $h \in N$, $\delta \in (0,1)$ and $\ell_h^\delta(t) = c_h^\delta t + d_h^\delta$ be a family of affine functions with

$$\sup_{h\in\mathbb{N}} \ell_h^\delta(t) = (1-\delta)\phi(t), \quad \forall t \ge 0.$$

We can suppose $(1-\delta)c = \sup_h c_h^\delta$; according to **(A2)** let $t_\delta > 0$ with

$$f_\varepsilon(t) \ge (1-\delta)\varepsilon\phi(t/\varepsilon)$$

for every $t \in [0,t_\delta]$ whenever $0 < \varepsilon < \varepsilon_\delta$. Consider the function

$$g_{\varepsilon,h}^\delta(t) := \begin{cases} c_h^\delta t + \varepsilon d_h^\delta & 0 \le t \le t_\delta \\ c_h^\delta t_\delta + \varepsilon d_h^\delta & t > t_\delta. \end{cases}$$

By construction $f_\varepsilon \ge g_{\varepsilon,h}^\delta$; moreover $g_{\varepsilon,h}^\delta - \varepsilon d_h^\delta$ does not depend on ε. Setting $f_h^\delta(t) := g_{\varepsilon,h}^\delta(t) - \varepsilon d_h^\delta$, we can write

$$F_\varepsilon(u,A) \ge \frac{1}{\varepsilon}\int_A f_h^\delta\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right)dx + d_h^\delta|A|. \tag{4.2}$$

Taking into account (4.1)

$$F'(u,A) \ge \int_A c_h^\delta|\nabla u|dx + c_h^\delta|D^c u|(A) + d_h^\delta|A| = \int_A \ell_h^\delta(|\nabla u|)dx + c_h^\delta|D^c u|(A).$$

Since $F'(u,\cdot)$ is a superadditive function on open sets of Ω with disjoint compact closures, by Lemma 2.1 we obtain

$$F'(u,A) \ge (1-\delta)\int_A \phi(|\nabla u|)dx + (1-\delta)c|D^c u|(A)$$

and by arbitrariness of $\delta \in (0,1)$

$$F'(u,A) \ge \int_A \phi(|\nabla u|)dx + c|D^c u|(A).$$

□

Proof of Theorem 3.2. It follows from (4.2) and from Remark (4.2). □

As in [15] the surface energy density of the Γ-limit depends on the limit of the family $(f_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$, i.e. it depends on the function f. The correct lower bound can be easily deduced by a direct application of Lemma 4.1.

Proposition 4.4. *For every $u \in BV(\Omega)$ and A open subset of Ω it turns out that*

$$F'(u,A) \geq \int_{S_u\cap A} \vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1}.$$

Proof. Let $0 < T < \sup f$, $\delta \in (0,1)$ and

$$f^T(t) := \begin{cases} f(t) & 0 \leq t \leq T \\ f(T) & t > T. \end{cases}$$

Since f_ε is non-decreasing and since f_ε converges uniformly to f on compact subsets of $[0,+\infty)$, for ε sufficiently small $f_\varepsilon(t) \geq (1-\delta)f^T(t)$ for every $t \in [0,+\infty)$. Then for any $v \in W^{1,1}(\Omega)$ we get

$$F_\varepsilon(v,A) \geq \frac{1-\delta}{\varepsilon}\int_A f^T\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla v(y)|dy\right)dx.$$

By (4.1) and by arbitrariness of $\delta \in (0,1)$

$$F'(u,A) \geq \int_{S_u\cap A}\vartheta^T(|u^+ - u^-|)d\mathcal{H}^{n-1},$$

where ϑ^T is defined as in (3.3) replacing f by f^T. We can conclude by taking the limit for $T \to \sup f$, noticing that $\vartheta^T \nearrow \vartheta$. □

Proposition 4.5. *Let $u \in GBV(\Omega)$ and A be an open subset of Ω; then*

$$F'(u,A) \geq \int_A \phi(|\nabla u|)dx + \int_{S_u\cap A}\vartheta|u^+ - u^-|d\mathcal{H}^{n-1} + c|D^c u|(A).$$

Proof. Let $u \in BV(\Omega)$; from Proposition 4.3 we have

$$F'(u,A) \geq \int_A \phi(|\nabla u|)dx\,, \quad F'(u,A) \geq c|D^c u|(A);$$

moreover Proposition 4.4 gives

$$F'(u,A) \geq \int_{S_u\cap A}\vartheta\big(|u^+ - u^-|\big)d\mathcal{H}^{n-1}\,.$$

Apply now Corollary 2.2 with $\lambda_1 = \mathcal{L}^n$, $\lambda_2 = \mathcal{H}^{n-1} \llcorner S_u$ and $\lambda_3 = |D^c u|$. Therefore $F'(u) \geq \mathcal{F}(u)$.

Let us now consider the case $u \in GBV(\Omega)$. We can find a positive infinitesimal sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ and a sequence $(u_j)_{j\in\mathbb{N}}$ in $W^{1,1}(\Omega)$ converging to u in $L^1(\Omega)$ and such that $F_{\varepsilon_j}(u_j) \xrightarrow{j} F'(u)$. Define $u_j^T = (-T)\vee u_j \wedge T$, and $u^T = (-T)\vee u \wedge T$. Since $u_j^T \to u^T$ in $L^1(\Omega)$, and $u^T \in BV(\Omega)$, we have

$$F'(u) = \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) \geq \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j^T) \geq$$

$$\geq \int_\Omega \phi(|\nabla u^T|)dx + \int_{S_u\cap\Omega}\vartheta\big(|(u^T)^+ - (u^T)^-|\big)\,d\mathcal{H}^{n-1} + c|D^c u^T|(\Omega).$$

We conclude by taking the limit as $T \to +\infty$. □

4.2. **Estimate from above of the upper Γ-limit.** In this section we only sketch the proof of the upper estimate for the Γ-limit (for more completeness see §7 in [16]). Thanks to the density result in [10] (Theorem 3.1) and to the relaxation result in [5] (Theorem 3.1) and in [1] (Theorem 3.2) it is sufficient to take into account the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

i) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;
ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite number of $(n-1)$-dimensional simplices;
iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$.

In the computation of the upper Γ-limit we will use a generalization of Proposition 3.6 in [16]: the same proof works since the relaxation of F_ε depends only on the continuity and on the monotonicity of the function f_ε.

Proposition 4.6. *For every $\varepsilon > 0$, the relaxed functional of F_ε in the L^1-topology is given by*

$$\overline{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)dx \tag{4.3}$$

for every $u \in BV(\Omega)$.

Remark 4.7. *From the preceding proposition we easily deduce that $F''(u) < +\infty$ if $u \in BV(\Omega)$. Indeed, the upper Γ-limit of (F_ε) coincides with the upper Γ-limit of $(\overline{F}_\varepsilon)$; therefore, since $f_\varepsilon(t) \le Lt$, if we set $\mu = |Du|$, we have:*

$$F''(u) \le L\limsup_{\varepsilon\to0}\overline{F}_\varepsilon(u) \le L\limsup_{\varepsilon\to0}\int_\Omega \frac{1}{|B_\varepsilon(x)\cap\Omega|}\int_\Omega 1_{B_\varepsilon(x)}\,d\mu\,dx\,,$$

where $1_{B_\varepsilon(x)}$ is the characteristic function of $B_\varepsilon(x)$. There exists a constant $\gamma > 0$ such that:

$$|B_\varepsilon(x)\cap\Omega| \ge \gamma\varepsilon^n \tag{4.4}$$

for any $x \in \Omega$ and $\varepsilon < diam(\Omega)$; therefore

$$F''(u) \le L\limsup_{\varepsilon\to0}\frac{1}{\gamma\varepsilon^n}\int_\Omega d\mu(y)\int_\Omega 1_{B_\varepsilon(x)}(y)\,dx \le L\frac{\omega_n}{\gamma}\mu(\Omega) < +\infty\,.$$

Lemma 4.8. *Let $g \in L^1(\Omega)$ and let A be an open subset of Ω. Then*

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_A f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} g(y)\,dy\right)dx = \int_A \phi(g(x))\,dx.$$

Proof. Let $g_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap\Omega} g(y)\,dy$. Since $g \in L^1(\Omega)$, from Lebesgue's Differentiation Theorem g_ε converges to g in $L^1(\Omega)$ and a.e. in Ω. We can now conclude by the Dominated Convergence Theorem, recalling **(A2)** and $f_\varepsilon(t) \le Lt$ (by **(A4)**). □

Proposition 4.9. *Let $u \in GBV(\Omega)$; then*

$$F''(u) \le \int_\Omega \phi(|\nabla u|)dx + \int_{S_u}\vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + c|D^c u|(\Omega).$$

Sketch of the proof. We only consider the case u satisfying i), ii) and iii); thus S_u is contained in the union of a finite collection $K_1, \dots, K_m$ of $(n-1)$-dimensional simplexes. Since the upper Γ-limit of (F_ε) coincides with the upper Γ-limit of $(\overline{F}_\varepsilon)$, we have $F''(u) \le \limsup_\varepsilon \overline{F}_\varepsilon(u)$.

Fix $\sigma > 0$ and let S be the union of the relative boundaries of K_i (observe that $\mathcal{H}^{n-1}(S) = 0$). Let $\Omega_\sigma = \{x \in \Omega :\ d(x,\partial\Omega) > \sigma,\ d(x,S) > \sigma\}$; by Proposition 4.6 and by the argument used in Remark 4.7 we have:

$$\overline{F}_\varepsilon(u) \le \overline{F}_\varepsilon(u,\Omega_\sigma) + L\frac{\omega_n}{\gamma}|Du|(A_{\varepsilon,\sigma})\,,$$

where $A_{\varepsilon,\sigma} = \{x \in \Omega :\ d(x,\Omega\setminus\Omega_\sigma) < \varepsilon\}$. The second term goes to 0 as $\varepsilon,\sigma \to 0$. Without loss of generality we can suppose $m = 1$ and $K \subseteq \{x \in \mathbb{R}^n :\ x_1 = 0\}$. Let

$$K_\varepsilon = \{x \in \mathbb{R}^n :\ d(x,K) < \varepsilon\}.$$

Then for ε small we get

$$\overline{F}_\varepsilon(u,\Omega_\sigma) \le \overline{F}_\varepsilon(u,\Omega_\sigma\setminus \overline{K}_\varepsilon) + \overline{F}_\varepsilon(u,\Omega_\sigma\cap K_\varepsilon).$$

Since $u \in W^{k,\infty}(\Omega\setminus\overline{K})$ by Lemma 4.8 we deduce that

$$\limsup_{\varepsilon\to 0} \overline{F}_\varepsilon(u,\Omega_\sigma\setminus\overline{K}_\varepsilon) \le \int_\Omega \phi(|\nabla u|)dx.$$

Let us now estimate the second term $\overline{F}_\varepsilon(u,\Omega_\sigma\cap K_\varepsilon)$. By condition **(A4)** we have

$$\overline{F}_\varepsilon(u,\Omega_\sigma\cap K_\varepsilon) \le \frac{1}{\varepsilon}\int_{\Omega_\sigma\cap K_\varepsilon} f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u(y)|\,dy + \frac{1}{\omega_n\varepsilon^{n-1}}|D^s u|\big(B_\varepsilon(x)\big)\right)dx \le$$

$$\le L\int_{\Omega_\sigma\cap K_\varepsilon}\fint_{B_\varepsilon(x)} |\nabla u(y)|\,dy\,dx + \frac{1}{\varepsilon}\int_{\Omega_\sigma\cap K_\varepsilon} f_\varepsilon\left(\frac{|D^s u|\big(B_\varepsilon(x)\big)}{\omega_n\varepsilon^{n-1}}\right)dx\,.$$

Because of the regularity of u outside $\{x_1=0\}$ the first of these last two terms tends to 0. Denote by R_ε the last integral term. We get

$$R_\varepsilon \le \frac{1}{\varepsilon}\int_{\Omega_\sigma\cap K_\varepsilon} f_\varepsilon\left(\frac{1}{\omega_n\varepsilon^{n-1}}|D^s u|\big(B_\varepsilon(x)\big)\right)dx\,.$$

If $K_{\varepsilon,\sigma}\subseteq\mathbb{R}^{n-1}$ denotes the projection of $\Omega_\sigma\cap K_\varepsilon$ onto $\{x_1=0\}$, it turns out that $\Omega_\sigma\cap K_\varepsilon$ is contained in $(-\varepsilon,\varepsilon)\times K_{\varepsilon,\sigma}$, which is a subset of $\{x\in\Omega:\ d(x,\partial\Omega)>\varepsilon\}$ for ε sufficiently small. Then

$$R_\varepsilon \le \frac{1}{\varepsilon}\int_{K_{\varepsilon,\sigma}} dy\int_{-\varepsilon}^{\varepsilon} f_\varepsilon\left(\frac{1}{\omega_n\varepsilon^{n-1}}|D^s u|\big(B_\varepsilon(s,y)\big)\right)ds\,.$$

Set $h(x)=u^+(x)-u^-(x)$ for $x\in\Omega\cap\{x_1=0\}$. The function h is continuous, hence uniformly continuous on any compact subset of $\Omega\cap\{x_1=0\}$; thus for ε sufficiently small

$$|D^s u|\big(B_\varepsilon(s,y)\big) = \int_{B_\varepsilon(s,y)\cap\{x_1=0\}} |h|\,d\mathcal{H}^{n-1} \le \omega_{n-1}(\sqrt{\varepsilon^2-s^2})^{n-1}(|h(0,y)|+\sigma).$$

Therefore

$$R_\varepsilon \le \frac{1}{\varepsilon}\int_{K_{\varepsilon,\sigma}}\left(\int_{-\varepsilon}^{\varepsilon} f_\varepsilon\Big(\frac{\omega_{n-1}}{\omega_n}\big(\sqrt{1-(s/\varepsilon)^2}\big)^{n-1}(|h(0,y)|+\sigma)\Big)ds\right)dy.$$

By the change of variable $t=s/\varepsilon$ we obtain

$$R_\varepsilon \le \int_{\Omega\cap\{x_1=0\}}\left(\int_{-1}^{1} f_\varepsilon\Big(\frac{\omega_{n-1}}{\omega_n}(\sqrt{1-t^2})^{n-1}(|h(z)|+\sigma)\Big)dt\right)d\mathcal{H}^{n-1}(z).$$

Since f_ε converges uniformly on compact subsets of $[0,+\infty)$ we have

$$\limsup_{\varepsilon\to 0} R_\varepsilon \le \int_{\Omega\cap\{x_1=0\}}\left(\int_{-1}^{1} f\Big(\frac{\omega_{n-1}}{\omega_n}(\sqrt{1-t^2})^{n-1}(|h(z)|+\sigma)\Big)dt\right)d\mathcal{H}^{n-1}(z).$$

Now we conclude by taking the limit as $\sigma\to 0$, and noting that $h(z)$ vanishes $\mathcal{H}^{n-1}$-a.e. outside S_u. □

5. Proof of the approximation result

In this section we want to proof Theorem 3.3. To do this we have to construct a family of functions f_ε satisfying **(A1)-(A4)** which converges uniformly on compact subsets of $[0,+\infty)$ to a solution f of the integral equation

$$\vartheta(s) = 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n}s(\sqrt{1-t^2})^{n-1}\right)dt,\quad s>0. \tag{5.1}$$

In the following Proposition we investigate the existence of solutions of equation (5.1).

Proposition 5.1. *Let $\vartheta \in C^1([0,+\infty))$; then the function $f\colon [0,+\infty) \to [0,+\infty)$ given by*

$$f(s) = \frac{1}{\pi}\int_0^1 \frac{\sqrt{u}}{\sqrt{1-u}}\left[\vartheta\left(\frac{\omega_n}{\omega_{n-1}} s u^{\frac{n-1}{2}}\right) + \frac{n-1}{2}\frac{\omega_n}{\omega_{n-1}} s u^{\frac{n-1}{2}} \vartheta'\left(\frac{\omega_n}{\omega_{n-1}} s u^{\frac{n-1}{2}}\right)\right] du \tag{5.2}$$

is a solution of (5.1).

Proof. Making the change of variable

$$\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1} = y^{\frac{n-1}{2}}$$

it is easy to see, by direct computation, that equation (5.1) is equivalent to

$$\vartheta(s) = \left(\frac{\omega_n}{s\omega_{n-1}}\right)^{\frac{1}{n-1}} \int_0^{\left(\frac{s\omega_{n-1}}{\omega_n}\right)^{\frac{2}{n-1}}} \frac{f(y^{\frac{n-1}{2}})}{\sqrt{\left(\frac{s\omega_{n-1}}{\omega_n}\right)^{\frac{2}{n-1}} - y}} dy$$

which is an Abel integral equation of the form

$$\sqrt{x}\psi(x) = \int_0^x \frac{h(y)}{\sqrt{x-y}} dy$$

where

$$x = \left(\frac{s\omega_{n-1}}{\omega_n}\right)^{\frac{2}{n-1}}, \quad \psi(x) = \vartheta\left(\frac{x^{\frac{n-1}{2}}\omega_n}{\omega_{n-1}}\right), \quad g(y) = f(y^{\frac{n-1}{2}}).$$

An explicit solution is known for this kind of equations (see e.g. [14] using the Laplace transform methods):

$$h(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\sqrt{z}\psi(z)}{\sqrt{x-z}} dz. \tag{5.3}$$

By $z = xu$ and by derivation under the integral sign we have

$$h(x) = \frac{1}{\pi}\int_0^1 \frac{\sqrt{u}}{\sqrt{1-u}}(\psi(xu) + xu\psi'(xu))du$$

which is (5.2). □

Finally we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let ϕ and ϑ as in assumptions i), ii), iii) of subsection 3.2, with $\vartheta \in C^1([0,+\infty))$ and such that the map

$$t \mapsto \vartheta(t) + \frac{n-1}{2} t\vartheta'(t)$$

is non-decreasing and concave. For any $\varepsilon > 0$ consider the function $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ given by

$$f_\varepsilon(t) = \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \le t \le \varepsilon^\beta, \\ f(t-\varepsilon^\beta) + \varepsilon\phi(\varepsilon^{\beta-1}) & \text{if } t \ge \varepsilon^\beta, \end{cases} \tag{5.4}$$

where $\beta \in (0,1)$ and f is the solution of

$$\vartheta(s) = 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} s(\sqrt{1-t^2})^{n-1}\right) dt, \quad s > 0$$

given by (5.2). The assumption on the map $t \mapsto \vartheta(t) + [(n-1)/2]\, t\vartheta'(t)$ implies **(A1)** with $a_\varepsilon = \varepsilon^\beta$; it is easy to check **(A2)**, **(A3)** and **(A4)**. The conclusion follows from Theorem 3.1. □

5.1. **An explicit example.** Consider $\Omega \subseteq \mathbb{R}^2$, and the functionals with linear growth given, on $GBV(\Omega)$, by

$$F(u) = \int_\Omega |\nabla u| dx + \int_{S_u} \frac{|u^+ - u^-|}{1 + |u^+ - u^-|} d\mathcal{H}^{n-1} + |D^c u|(\Omega),$$

proposed in [18] as a segmentation functional. In order to obtain the approximation of this functionals we can apply Theorem 3.3 choosing $\phi(t) = t$ and f as in the formula (5.2) with

$$\vartheta(t) = \frac{t}{1+t}.$$

Indeed one can easily see that the function

$$\vartheta(t) + \frac{t}{2}\vartheta'(t) = \frac{2t^2 + 3t}{2(1+t)^2}$$

is non-decreasing and concave.

Let us compute, in this case, the solution of the integral equation (5.1). Since

$$n = 2, \quad \frac{\omega_2}{\omega_1} = \frac{\pi}{2}, \quad \vartheta'(t) = \frac{1}{(1+t)^2}$$

by formula (5.2) we easily get

$$f(s) = s \int_0^1 \frac{t(3 + s\pi\sqrt{t})}{\sqrt{1-t}(2 + \pi s\sqrt{t})^2} dt. \tag{5.5}$$

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