DAMAGE AS THE Γ-LIMIT OF MICROFRACTURES IN LINEARIZED ELASTICITY UNDER THE NON-INTERPENETRATION CONSTRAINT

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Abstract. A homogenization result is given for a material with brittle periodic inclusions, under the requirement that the interpenetration of matter is forbidden. According to the ratio between the softness of the inclusions and the size of the microstucture, three different limit models are deduced via Γ-convergence. In particular it is shown that in the limit the non-interpenetration constraint breaks the symmetry between states where the material is in extension and in compression.

Keywords: brittle fracture, damage, non-interpenetration, homogenization, Γ-convergence, integral representation

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1. Introduction

The subject of this paper is a homogenization result for a composite material given by a periodic fine mixture of an unbreakable material and a very brittle one. We consider the case in which the unbreakable material is arranged in a connected grid (reinforced fibers), while the brittle material forms a disconnected set of inclusions. An example of such a composite in the two-dimensional case is illustrated in Figure 1. One of the most interesting points of our analysis is the requirement that a non-interpenetration constraint be satisfied between the lips of the microfractures.

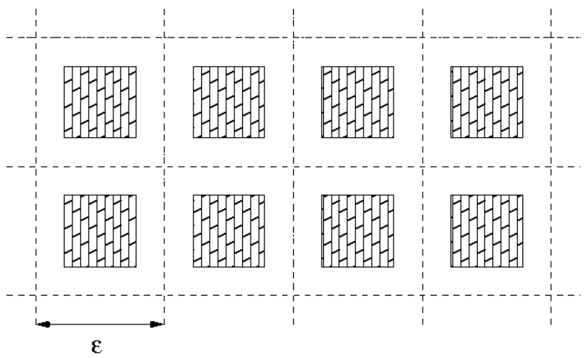

Figure 1. Fine periodic mixture at scale ε.

More precisely, let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, be the region occupied by the composite material and let $\varepsilon > 0$ be a small parameter representing the size of the periodic mixture. Let εQ be the periodicity cell, where $Q := (0,1)^n$. We denote by $\varepsilon I \subset \varepsilon Q$ the brittle inclusion in the periodicity cell εQ.

A displacement of Ω will be a vector valued function $u \in SBD_0(\Omega)$, the space of special functions of bounded deformation satisfying homogeneous Dirichlet boundary conditions on $\partial\Omega$. An admissible displacement has to fulfill also the infinitesimal non-interpenetration condition $[u]\cdot\nu_u \geq 0$ on the jump set J_u, where $[u]$ is the jump of u and ν_u is the normal to the jump set (see e.g. [12]). Physically, this constraint means that the two lips of a fracture cannot interpenetrate.

The energy associated to a given displacement u will be a Mumford-Shah-like functional $\mathcal{F}^\varepsilon$ consisting of a volume term, representing the elastic energy, and a surface term, penalizing the opening of a fracture in the material. More precisely

$$\mathcal{F}^\varepsilon(u) := \int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u\,dx + \int_{J_u} g_{\alpha_\varepsilon}\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big) d\mathcal{H}^{n-1}(x), \tag{1.1}$$

where $g_{\alpha_\varepsilon} : \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to [0, +\infty]$ is a Q-periodic function in the first variable, defined for $y \in Q$, $z \in \mathbb{R}^n$, and $\nu \in S^{n-1}$ by

$$g_{\alpha_\varepsilon}(y, z, \nu) := \begin{cases} \alpha_\varepsilon & \text{if } y \in I \text{ and } z \cdot \nu \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

and α_ε is a positive parameter depending on ε (see Section 3 for more details).

We remark that the density g_{α_ε} prevents interpenetration of the fracture lips and ensures that the discontinuity set of u can only be in the brittle part of the material. When u satisfies these constraints, then the surface term in the energy reduces to $\alpha_\varepsilon \mathcal{H}^{n-1}(J_u)$, i.e., our model describes a brittle fracture according to the Griffith criteria (see [12] for a similar model).

The overall properties of the composite material described by the functional $\mathcal{F}^\varepsilon$ can be approximated by a *homogenized* functional, which is given by the Γ-limit of $\mathcal{F}^\varepsilon$ as ε goes to zero. In our case we assume that $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$, and we show that the limit model depends on the behaviour of the ratio $\frac{\alpha_\varepsilon}{\varepsilon}$ as ε goes to zero. According to this limit, three different models are deduced via Γ-convergence. They do have, however, a common feature: They describe an unbreakable material. This means that, even if at scale ε the material has periodically distributed microscopic cracks, when ε goes to zero no macroscopic crack appears. This is due to the fact that in the periodicity cell εQ the brittle region εI is well separated from the boundary of εQ and this prevents small cracks from glueing together into a macroscopic fracture.

We recall that the case of generalised anti-planar shear has been treated in [21]. Also in that case the Γ-limit exhibited, in the three regimes illustrated above, a gain in regularity for the relevant displacements. The case of brittle inclusions $\varepsilon I \subset \varepsilon Q$ with vanishing distance δ_ε from the boundary of εQ and $\alpha_\varepsilon = 1$ has been treated in the recent papers [5] and [17].

In this paper we derive three different models corresponding to the limit $\frac{\alpha_\varepsilon}{\varepsilon}$ being zero (subcritical case), finite (critical case) or $+\infty$ (supercritical case).

In the subcritical case, $\alpha_\varepsilon << \varepsilon$, the limit functional is given, for $u \in H^1_0(\Omega; \mathbb{R}^n)$, by

$$\mathcal{F}^0(u) = \int_\Omega f_0(\mathcal{E}u)dx. \tag{1.3}$$

The limit energy density f_0 is defined for $\xi \in \mathbb{M}^{n\times n}$ by the cell formula

$$f_0(\xi) := \inf\bigg\{ \int_Q \mathbb{C}\mathcal{E}(\xi x + w) : \mathcal{E}(\xi x + w)dx : w \in SBD_\#(Q), J_w \subset I,\ [w]\cdot\nu_w \geq 0 \text{ a.e. on } J_w \bigg\}, \tag{1.4}$$

where $SBD_\#(Q) \subset SBD(Q)$ denotes the functions with periodic boundary conditions on ∂Q. We notice that f_0 is anisotropic, even assuming that $\mathbb{C}$ is isotropic (see Remark 5.5).

An interesting result is that, due the non-interpenetration constraint, f_0 fails to be a quadratic form. Indeed, taking $\mathbb{C}$ to be

$$\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\, Id \otimes Id,$$

where $\lambda, \mu > 0$, $(\mathbb{I})_{ijkl} = \delta_{ik}\delta_{jl}$, and $(Id \otimes Id)_{ijkl} = \delta_{ij}\delta_{kl}$, it turns out that $f_0(Id) \neq f_0(-Id)$ (see Lemma 5.3). On the contrary, when the non-interpenetration constraint in not imposed, one can prove in a similar way to [21] that the limit density is

$$\hat{f}_0(\xi) := \inf\bigg\{ \int_Q \mathbb{C}\mathcal{E}(\xi x + w) : \mathcal{E}(\xi x + w)dx : w \in SBD_\#(Q), J_w \subset I \bigg\}, \tag{1.5}$$

which is a quadratic form for every choice of the tensor $\mathbb{C}$. An interpretation of the fact that $f_0(Id) \neq f_0(-Id)$ is the following. For $\xi = Id$ the body is subject to a boundary displacement of pure extension in all directions. In this case, the solutions of (1.4) have discontinuities, since the non-interpenetration constraint is compatible with the boundary conditions and it is energetically convenient to have a nonempty jump set. On the contrary, when $\xi = -Id$, i.e., in a regime of pure compression, the optimal w in (1.4) is $w = 0$. This happens because the minimisers of the problem (1.5) corresponding to $\xi = -Id$ are not admissible for (1.4), since they do not satisfy the non-interpenetration constraint. Therefore the non-interpenetration constraints acts as a *selection mechanism* for the minimisers in (1.5).

Another important remark is that the limit energy describes a damaged material. Indeed, for a large class of matrices $\xi \in \mathbb{M}^{n\times n}$ it turns out that $f_0(\xi) \lneqq \mathbb{C}\xi : \xi$, and this means that the elastic moduli of the material are reduced by homogenization. Therefore the possible presence of microfractures at scale ε translates into a damage of the material at a macroscopic scale.

In the supercritical regime, $\alpha_\varepsilon >> \varepsilon$, the limit model, for $H^1_0(\Omega;\mathbb{R}^n)$, is given by the functional

$$\mathcal{F}^\infty(u) = \int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u \, dx. \tag{1.6}$$

Therefore, the (possible) presence of cracks in the approximating problems has no effect in the limit. Indeed, in this case the ε-energy highly penalises displacements having discontinuities, so that the limit material has the same elastic properties as the original one and no damage occurs.

We want to underline that in this regime the Γ-limit is the same as if the non-interpenetration constraint were not imposed. The feature which makes this case mathematically different from the corresponding one in [21] is the lack of a lower semicontinuity result in SBD when no a priori bound on the L^∞-norm of the displacements is given. Hence, in order to prove the Γ-convergence result for this scaling, we need a modified version of the proof of lower semicontinuity in SBD given in [6], where the assumption of the equiboundedness of the L^∞-norm of the displacements is replaced by the assumption that the measure of the jump sets of the displacements goes to zero (see Lemma 7.2).

In the critical regime, $\alpha_\varepsilon = \varepsilon$, the limit functional, for $u \in H^1_0(\Omega;\mathbb{R}^n)$, is

$$\mathcal{F}^{hom}(u) = \int_\Omega f_{hom}(\mathcal{E}u) dx, \tag{1.7}$$

where the density f_{hom} is defined for $\xi \in \mathbb{M}^{n\times n}$ by the asymptotic cell problem

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \ \inf\Big\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}(\xi x + w) : \mathcal{E}(\xi x + w) + \mathcal{H}^{n-1}(J_w) : w \in SBD_0\big((0,t)^n\big), \\ J_w \subset \tilde{I} \cap (0,t)^n,\ [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w \Big\},$$

and the set $\tilde{I}$ is the periodic set having I as periodicity cell, see (3.1). Notice that this is the only case where the cell formula involves both volume and surface terms. This is because, when $\alpha_\varepsilon = \varepsilon$, the volume and the surface terms of $\mathcal{F}^\varepsilon$ have the same order. Moreover, the limit functional describes a damaged material, as shown in Lemma 6.4.

The fact that the critical scaling for the parameter α_ε is ε is supported by several results in fracture mechanics (see, e.g., [23]). In particular, Braides and Truskinovsky [9] have recently proved that, starting from a purely atomistic model where ε is the lattice spacing and the interactions between neighbouring atoms are described by Lennard-Jones-like potentials, the limiting continuous model is given by the Mumford-Shah functional, where the measure of the discontinuity set is weighted by the parameter ε.

The plan of the paper is the following. In Sections 2 and 3 we define the mathematical setting of the problem and introduce the energy functional. In Section 4 we show that the limit functional obtained via Γ-convergence admits an integral representation, while Sections 5-7 are devoted to the description of the limit functionals in the subcritical, critical and supercritical cases.

2. Preliminaries

In this section we collect some definitions and results that will be widely used throughout the paper. In order to make precise the mathematical setting, we recall some properties of rectifiable sets and we include a brief presentation of the spaces SBV and SBD. We refer the reader to [3] and to [22] for further details.

A set $\Gamma \subset \mathbb{R}^n$ is rectifiable if there exists $N_0 \subset \Gamma$ with $\mathcal{H}^{n-1}(N_0) = 0$, and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^n$ such that

$$\Gamma \setminus N_0 \subset \bigcup_{i\in\mathbb{N}} M_i.$$

For every $x \in \Gamma \setminus N_0$ we denote the normal to Γ at x by $\nu_{M_i}(x)$. It turns out that the normal is well defined (up to the sign) for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

SBV functions. Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $SBV(U)$ as the set of functions $u \in L^1(U)$ such that the distributional derivative Du is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Du(A) = D^a u(A) + D^j u(A) = \int_A \nabla u\, dx + \int_{S_u\cap A} [u](x)\, \nu_u(x)\, d\mathcal{H}^{n-1}(x),$$

where ∇u is the approximate differential of u, S_u is the jump set of u (which is a rectifiable set), $\nu_u(x)$ is the normal to S_u at x, and $[u](x)$ is the jump of u at x.
For every $p \in]1, +\infty[$ we set

$$SBV^p(U) = \big\{u \in SBV(U) : \nabla u \in L^p(U;\mathbb{R}^n), \mathcal{H}^{n-1}(S_u) < +\infty\big\}.$$

If $u \in SBV(U)$ and $\Gamma \subset U$ is rectifiable and oriented by a normal vector field ν, then we can define the traces u^+ and u^- of u on Γ, which are characterized by the relations

$$\lim_{r\to 0} \frac{1}{r^n} \int_{\Omega\cap B_r^\pm(x)} |u(y) - u^\pm(x)|\, dy = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma,$$

where $B_r^\pm(x) := \{y \in B_r(x) : (y-x)\cdot\nu \gtrless 0\}$ and $B_r(x)$ is the open ball with radius r and center x.

BD functions. Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $BD(U)$ as the set of functions $u \in L^1(U;\mathbb{R}^n)$ such that the symmetric part of the distributional derivative Du is a bounded Radon measure. We denote with Eu the symmetric part of Du, i.e.,

$$Eu := \{(Eu)_{ij}\}, \quad (Eu)_{ij} := \frac{1}{2}\,(D_i u_j + D_j u_i).$$

We can split the symmetric gradient into its absolutely continuous, jump and Cantor parts with respect to the Lebesgue measure, as

$$Eu = E^a u + E^j u + E^c u = \mathcal{E}u\, dx + E^j u + E^c u.$$

Sections of BD functions. Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, $u \in BD(U)$, and let $\xi \in S^{n-1}$. We denote by π_ξ the hyperplane orthogonal to ξ passing through the origin and by U^ξ the orthogonal projection of U on π_ξ. Let $y \in \mathbb{R}^n$; the section of U corresponding to y is denoted by U_y^ξ, that is, $U_y^\xi := \{t \in \mathbb{R} : y + t\,\xi \in U\}$. We can define the section $u_y^\xi : U_y^\xi \to \mathbb{R}$ as $u_y^\xi(t) := u(y + t\,\xi)\cdot\xi$, for every $t \in U_y^\xi$. Then, it holds:
(i) for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$ the function u_y^ξ belongs to $BV(U_y^\xi)$;
(ii) $(\mathcal{E}u(y + t\,\xi)\xi, \xi) = \nabla u_y^\xi(t)$;
(iii) $(\mathcal{E}u\xi, \xi) = \int_{U^\xi} \nabla u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(\mathcal{E}u\xi, \xi)| = \int_{U^\xi} |\nabla u_y^\xi|\, d\mathcal{H}^{n-1}(y)$;
(iv) $(E^j u\xi, \xi) = \int_{U^\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(E^j u\xi, \xi)| = \int_{U^\xi} |D^j u_y^\xi|\, d\mathcal{H}^{n-1}(y)$;
(v) $(E^c u\xi, \xi) = \int_{U^\xi} D^c u_y^\xi d\mathcal{H}^{n-1}(y)$, $|(E^c u\xi, \xi)| = \int_{U^\xi} |D^c u_y^\xi|\, d\mathcal{H}^{n-1}(y)$.

SBD(U) functions. We say that a function $u \in BD(U)$ belongs to $SBD(U)$ if Eu is a Radon measure that for every open set $A \subset U$ can be represented as

$$Eu(A) = E^a u(A) + E^j u(A) = \int_A \mathcal{E}u\, dx + \int_{J_u\cap A} [u](x) \odot \nu_u(x) d\mathcal{H}^{n-1}(x),$$

where J_u is the jump set of u (which is a rectifiable set), $\nu_u(x)$ is the normal to J_u at x, and $[u](x)$ is the jump of u at x. We have that if $u \in SBD(U)$, then its sections are in $SBV(U^\xi_y)$ for every $\xi \neq 0$ and for $\mathcal{H}^{n-1}$-a.e. $y \in U^\xi$. We set, for every $p \in]1, +\infty[$,

$$SBD^p(U) = \big\{u \in SBD(U) : \mathcal{E}u \in L^p(U; \mathbb{M}^{n\times n}_{sym}), \mathcal{H}^{n-1}(J_u) < +\infty\big\}.$$

Finally, we denote by $SBD^p_0(U)$ the space

$$SBD^p_0(U) = \big\{u \in SBD^p(U) : tr(u) = 0 \text{ on } \partial U\big\}.$$

3. Formulation of the problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We assume for simplicity that $\partial\Omega$ is C^2, although this condition may be weakened. Let $\varepsilon > 0$; we consider the periodic structure in $\mathbb{R}^n$ generated by an ε-homothetic of the basic cell $Q := (0,1)^n$. For notational brevity we will use the superscript ε to denote the ε-homothetic of any domain so that, in particular, $Q^\varepsilon := \varepsilon Q$. For every $0 < \varrho < \frac{1}{2}$ we denote with Q_ϱ the cube concentric with Q and with side $1 - 2\varrho$, i.e., $Q_\varrho := (\varrho, 1-\varrho)^n$. Let $0 < \delta < \frac{1}{2}$ be fixed; we assume that every periodicity cell Q^ε has a brittle inclusion of the form εI, where $I \subseteq \overline{Q}_\delta$ is a finite union of disjoint sets given by the closure of domains with Lipschitz boundary. To make the computations more explicit, in some of the results presented in the paper we will choose $I = \overline{Q}_\delta$. We define the periodic set $\tilde{I}$ generated by the inclusion I, i.e.,

$$\tilde{I} := \bigcup_{h\in\mathbb{Z}^n} (I + h), \tag{3.1}$$

and the subsets $I(\varepsilon), \Omega(\varepsilon) \subset \Omega$, representing the brittle inclusions in Ω and the unbreakable part of the material, respectively, i.e.,

$$I(\varepsilon) := \Omega \cap \varepsilon\tilde{I}, \qquad \Omega(\varepsilon) := \Omega \setminus I(\varepsilon). \tag{3.2}$$

Notice that we can split the boundary of $\Omega(\varepsilon)$ as $\partial\Omega(\varepsilon) = \Gamma(\varepsilon) \cup S(\varepsilon)$, where

$$\Gamma(\varepsilon) := \partial\Omega(\varepsilon) \cap \partial\Omega \quad \text{and} \quad S(\varepsilon) := \partial\Omega(\varepsilon) \cap \Omega. \tag{3.3}$$

Let $\mathbb{C} = (\mathbb{C}_{ijkl})$ be the elasticity tensor, considered as a symmetric positive definite linear operator from $\mathbb{M}^{n\times n}_{sym}$ into itself. It turns out that there exists two constants $0 < \vartheta_m \leq \vartheta_M$ such that for any $\xi \in \mathbb{M}^{n\times n}_{sym}$, it holds

$$\vartheta_m |\xi|^2 \leq \mathbb{C}\xi : \xi \leq \vartheta_M |\xi|^2, \tag{3.4}$$

where $\xi : \eta = \text{trace}(\xi\eta^T) = \sum_{ij} \xi_{ij}\eta_{ij}$ and $|\xi|^2 = \xi : \xi$ is the standard Euclidean norm. Clearly, the tensor $\mathbb{C}$ is symmetric with respect to any interchange of indices, that is,

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}. \tag{3.5}$$

The analysis developed in the present paper can be extended to more general measurable and Q-periodic functions $\mathbb{C}$. Therefore, in particular, it covers the case of $\mathbb{C}$ being constant in I and in $Q \setminus I$, but with different constant values.

To every displacement $u \in SBD^2_0(\Omega)$ we associate the energy

$$\mathcal{F}^\varepsilon(u) = \int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u \, dx + \int_{J_u} g_\alpha\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big) \, d\mathcal{H}^{n-1}(x),$$

where $g_\alpha : \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to [0, +\infty]$ is defined as

$$g_\alpha(y, z, \nu) = \begin{cases} \alpha & \text{if } y \in \tilde{I} \text{ and } z \cdot \nu \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and α is a positive parameter. Owing to the Q-periodicity of g_α in the first variable, the function

$$x \mapsto g_\alpha\Big(\frac{x}{\varepsilon}, z, \nu\Big)$$

is Q^ε-periodic. The volume term in the expression of $\mathcal{F}^\varepsilon$ represents the elastic energy, while the surface integral describes the energy needed to open a crack. More precisely, the density g_α forces the deformation u to have a jump set contained in the fragile part of the material and the lips of the fracture to avoid interpenetration.

We assume as in [21] that $\alpha = \alpha_\varepsilon$ depends on ε and goes to zero as $\varepsilon \to 0$ and we analyse the asymptotic behaviour of the functional $\mathcal{F}^\varepsilon$ in the cases $\alpha_\varepsilon \ll \varepsilon$ (subcritical regime), $\alpha_\varepsilon \approx \varepsilon$ (critical regime), and $\alpha_\varepsilon \gg \varepsilon$ (supercritical regime).

For the purposes of our analysis, it is convenient to rewrite the functional as follows

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u \, dx + \alpha_\varepsilon \, \mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2_0(\Omega), J_u \subset I(\varepsilon), \\ & [u]\cdot\nu_u \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{3.6}$$

Before treating the different cases we have just described, we state a Korn Inequality for perforated domains, together with an extension result that will be often used in the following. For the proof we refer to [20, Theorem 4.5, Theorem 4.2], respectively.

Definition 3.1. Let ω be an unbounded domain of $\mathbb{R}^n$ with a Q-periodic structure, where $Q := (0,1)^n$. Assume that the cell of periodicity $\omega \cap Q$ is a domain with a Lipschitz boundary. Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a positive parameter $\varepsilon > 0$, we set $\Omega(\varepsilon) := \Omega \cap \varepsilon\,\omega$. Moreover, we set $\Gamma(\varepsilon) := \partial\Omega \cap \varepsilon\,\omega$. We define the space $H^1(\Omega(\varepsilon), \Gamma(\varepsilon); \mathbb{R}^n)$ as

$$H^1(\Omega(\varepsilon), \Gamma(\varepsilon); \mathbb{R}^n) := \{v \in H^1(\Omega(\varepsilon);\mathbb{R}^n) : tr(v) = 0 \text{ on } \Gamma(\varepsilon)\}.$$

Theorem 3.2. *For any vector-valued function $u \in H^1(\Omega(\varepsilon), \Gamma(\varepsilon); \mathbb{R}^n)$ the inequality*

$$||u||_{(H^1(\Omega(\varepsilon)))^n} \leq k||Eu||_{(L^2(\Omega(\varepsilon)))^{n\times n}}$$

is valid, where $k > 0$ is a constant independent of u and ε.

Theorem 3.3. *Let Ω_0 be a bounded domain such that $\overline{\Omega} \subset \Omega_0$ and $dist(\partial\Omega_0, \Omega) > 1$. Then for every sufficiently small ε there exists a linear extension operator $T^\varepsilon : H^1(\Omega(\varepsilon), \Gamma(\varepsilon); \mathbb{R}^n) \to H^1_0(\Omega_0; \mathbb{R}^n)$ and three constants $k_0, k_1, k_2 > 0$ such that*

$$\begin{aligned} ||T^\varepsilon u||_{(H^1(\Omega_0))^n} &\leq k_1||u||_{(H^1(\Omega(\varepsilon)))^n}, \\ ||D(T^\varepsilon u)||_{(L^2(\Omega_0))^{n\times n}} &\leq k_2||Du||_{(L^2(\Omega(\varepsilon)))^{n\times n}}, \\ ||E(T^\varepsilon u)||_{(L^2(\Omega_0))^{n\times n}} &\leq k_3||Eu||_{(L^2(\Omega(\varepsilon)))^{n\times n}}, \end{aligned}$$

for any $u \in H^1(\Omega(\varepsilon), \Gamma(\varepsilon); \mathbb{R}^n)$, where the constants k_0, k_1, k_2 do not depend on ε.

Moreover, $(T^\varepsilon u)_{|A} = 0$ for any open set A such that $\bar{A} \subset \Omega_0 \setminus \Omega$, if ε is sufficiently small.

4. Integral representation of the Γ-limit

In this section we will prove a Γ-convergence result for the functionals $\mathcal{F}^\varepsilon$, together with a characterisation of the Γ-limit via an integral representation. The arguments used in the proof are independent of the rate of convergence to zero of α_ε with respect to ε, so in this section we do not need to treat the three cases separately. Moreover, we prove that the limit energy is finite only on the space of H^1-functions, meaning that in the limit discontinuous displacements are no longer admissible. Nevertheless, a careful analysis of the limit energy density will show that, in some regimes, the material is *damaged* (see Sections 5-7).

We first show that the functional $\mathcal{F}' := \Gamma - \liminf_\varepsilon \mathcal{F}^\varepsilon$ is finite only on $H^1_0(\Omega;\mathbb{R}^n)$.

Theorem 4.1. *Let $\mathcal{G} : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ be the functional defined as*

$$\mathcal{G}(u) = \begin{cases} \displaystyle\int_\Omega A^0\mathcal{E}u : \mathcal{E}u \, dx & \text{if } u \in H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases} \tag{4.1}$$

where $A^0 = (A^0_{ijkh})$ is the fourth order tensor with constant coefficients given by the solution of the cell problem

$$A^0\xi : \xi = \min\left\{ \int_{Q\setminus I} \mathbb{C}\mathcal{E} : \mathcal{E}w \, dy : w - \xi\, y \in H^1_\#(Q;\mathbb{R}^n)\right\},$$

for $\xi \in \mathbb{M}^{n\times n}_{sym}$. *Then it holds*

$$\mathcal{F}'(u) := \Gamma - \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u) \geq \mathcal{G}(u) \quad \textit{for every } u \in L^2(\Omega;\mathbb{R}^n). \tag{4.2}$$

Proof. Let $u \in L^2(\Omega;\mathbb{R}^n)$ and let (u^ε) be a sequence converging to u strongly in L^2 and such that $\mathcal{F}^\varepsilon(u^\varepsilon) \leq c < +\infty$. Let us define the auxiliary functional $\mathcal{G}^\varepsilon : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ as

$$\mathcal{G}^\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)\mathbb{C}\mathcal{E}v : \mathcal{E}v dx & \text{if } v \in H^1(\Omega(\varepsilon),\Gamma(\varepsilon);\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n), \end{cases} \tag{4.3}$$

where $\Omega(\varepsilon)$ is defined as in (3.2), and a is a Q-periodic function given by

$$a(y) = \begin{cases} 0 & \text{for } y \in I, \\ 1 & \text{for } y \in Q \setminus I. \end{cases}$$

It is well known that the sequence $(\mathcal{G}^\varepsilon)$ Γ-converges (with respect to the strong topology of L^2) to the functional $\mathcal{G}$ defined in (4.1). For further details we refer to [13] and [20]. We are going to prove that $\mathcal{G}^\varepsilon$ evaluated on a suitable extension of u^ε provides a lower bound for $\mathcal{F}^\varepsilon(u^\varepsilon)$, from which the claim follows.

As $\mathcal{F}^\varepsilon(u^\varepsilon) \leq +\infty$ we have that the sequence $(\mathcal{E}u^\varepsilon)$ is equibounded in $L^2(\Omega(\varepsilon);\mathbb{M}^{n\times n})$. Moreover, as $J_{u^\varepsilon} \cap \Omega(\varepsilon) = \emptyset$, by Theorem 3.2 $u^\varepsilon \in H^1(\Omega(\varepsilon);\mathbb{R}^n$. Now, let $\Omega_0 \supset \Omega$ with $\text{dist}(\Omega,\partial\Omega_0) > 1$ and let us denote with $\hat{u}^\varepsilon \in H^1_0(\Omega_0;\mathbb{R}^n)$ the extension of u^ε, whose existence is guaranteed by Theorem 3.3. The quoted theorem also ensures that the sequence $(\mathcal{E}\hat{u}^\varepsilon)$ is equibounded in $L^2(\Omega_0;\mathbb{M}^{n\times n})$. Hence, by the Korn Inequality we deduce that $\hat{u}^\varepsilon$ is equibounded in $H^1_0(\Omega_0;\mathbb{R}^n)$. We denote by $\hat{u}$ its weak limit in H^1. We claim that $u = \hat{u}$ a.e. in Ω. This follows by the Riemann-Lebesgue Lemma, as

$$0 = \lim_{\varepsilon\to 0} \int_\Omega a\left(\frac{x}{\varepsilon}\right)|u^\varepsilon - \hat{u}^\varepsilon|^2 dx = \vartheta \int_\Omega |u - \hat{u}|^2 dx,$$

where $\vartheta > 0$ is the $*$-weak limit of $a(\frac{\cdot}{\varepsilon})$ in $L^\infty(\Omega)$. Therefore, from the previous expression we conclude immediately that $u = \hat{u}$ a.e. on Ω. Moreover, since by the properties of the extension $\hat{u}^\varepsilon \in H^1_0(\Omega';\mathbb{R}^n)$ for every $\overline{\Omega} \subset \Omega' \subset \Omega_0$ (at least for small ε), then $u \in H^1_0(\Omega;\mathbb{R}^n)$. Finally we have

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{G}^\varepsilon(\hat{u}^\varepsilon),$$

from which we deduce the bound (4.2). □

We now prove that the sequence $(\mathcal{F}^\varepsilon)$ admits a Γ-convergence subsequence. This will be done by proving that the functionals $\mathcal{F}^\varepsilon$ satisfy a technical estimate (see 4.7). We first need to introduce some definitions and results that will be used in the following. For further references see [14].

Definition 4.2. Let $(G^\varepsilon) : L^2(\Omega;\mathbb{R}^n) \to \overline{\mathbb{R}}$ be a sequence of functionals, where the space $L^2(\Omega;\mathbb{R}^n)$ is endowed with the distance induced by the norm. Define the functionals G' and G'' as follows:

$$G' := \Gamma - \liminf_{\varepsilon\to 0} G^\varepsilon \quad \text{and} \quad G'' := \Gamma - \limsup_{\varepsilon\to 0} G^\varepsilon.$$

Definition 4.3. Let $\mathcal{A}(\Omega)$ denote the family of the open subsets of Ω. We say that a functional $G : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ is increasing (on $\mathcal{A}(\Omega)$) if for every $u \in L^2(\Omega;\mathbb{R}^n)$ the set function $G(u,\cdot)$ is increasing on $\mathcal{A}(\Omega)$.

Definition 4.4. Given a functional $G : L^2(\Omega;\mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0,+\infty]$, we define its inner regularisation as

$$G_-(u,A) := \sup\big\{G(u,B) : B \in \mathcal{A}(\Omega), B \subset\subset A\big\}.$$

Observe that if G is increasing, then also G_- is increasing.

Definition 4.5. We say that a sequence (G^ε) is $\overline{\Gamma}$-convergent to a functional G whenever

$$G = (G')_- = (G'')_-.$$

We have the following general compactness theorem.

Theorem 4.6. *Every sequence of increasing functionals has a $\overline{\Gamma}$-convergent subsequence.*

Since for every $\varepsilon > 0$ the functional $\mathcal{F}^\varepsilon$ is increasing, we deduce by Theorem 4.6 that there exists a $\overline{\Gamma}$-convergent subsequence in L^2. In order to pass from $\overline{\Gamma}$- to Γ-convergence a crucial step is to show that the functionals $\mathcal{F}^\varepsilon$ satisfy the so-called *fundamental estimate*. The latter can be seen as an approximated subadditivity of $\mathcal{F}^\varepsilon$, and it is essential in proving that the limit functional is a measure. As a first step, we localise the sequence $(\mathcal{F}^\varepsilon)$; that is, for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every open set $A \in \mathcal{A}(\Omega)$ we define

$$\mathcal{F}^\varepsilon(u,A) = \begin{cases} \displaystyle\int_A \mathbb{C}\mathcal{E}u : \mathcal{E}u\,dx + \alpha_\varepsilon\,\mathcal{H}^{n-1}(J_u \cap A) & \text{if } u \in SBD^2(A), J_u \subset I(\varepsilon)\cap A, \\ & [u]\cdot\nu_u \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{4.4}$$

For a fixed $u \in L^2(\Omega;\mathbb{R}^n)$ we can extend $(\mathcal{F}^\varepsilon)(u,\cdot)$ to a measure $(\mathcal{F}^\varepsilon)^*(u,\cdot)$ on the class of Borel sets $\mathcal{B}(\Omega)$ in the usual way:

$$(\mathcal{F}^\varepsilon)^*(u,B) := \inf\big\{\mathcal{F}^\varepsilon(u,A) : A \in \mathcal{A}(\Omega), B \subseteq A\big\}.$$

Next theorem provides an extension of the fundamental estimate to the space SBD^2. The proof ca be simply obtained by adapting the proof of [8, Proposition 3.1], valid for SBV functions, to the present case.

Theorem 4.7 (Fundamental estimate in SBD^2)**.** *For every $\eta > 0$ and for every A', A'' and $B \in \mathcal{A}(\Omega)$, with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: for every $\varepsilon > 0$ and for every $u \in SBD^2(A'')$ such that $J_u \subset I(\varepsilon)\cap A''$ and $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_u, and for every $v \in SBD^2(B)$ such that $J_v \subset I(\varepsilon)\cap B$ and $[v]\cdot\nu_v \geq 0$ $\mathcal{H}^{n-1}$-a.e. on J_v, there exists a function $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in a neighborhood of $\bar{A}'$, spt $\varphi \subset A''$ and $0 \leq \varphi \leq 1$ such that*

$$\mathcal{F}^\varepsilon(\varphi\,u + (1-\varphi)\,v, A'\cup B) \leq (1+\eta)\,\mathcal{F}^\varepsilon(u,A'') + (1+\eta)\,\mathcal{F}^\varepsilon(v,B) + M\int_T |u-v|^2dx,$$

where $T := (A''\setminus A')\cap B$.

We can finally state our Γ-convergence result for a subsequence of $(\mathcal{F}^\varepsilon)$.

Theorem 4.8. *Let ε be a sequence converging to zero. Then there exists a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^\sigma : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that, for every $A \in \mathcal{A}(\Omega)$,*

$$\mathcal{F}^\sigma(\cdot,A) = \Gamma - \lim_{\varepsilon\to 0}\mathcal{F}^{\sigma(\varepsilon)}(\cdot,A)$$

in the strong L^2-topology. Moreover, for every $u \in L^2(\Omega;\mathbb{R}^n)$, the set function $\mathcal{F}^\sigma(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω.

Proof. Since for every $\varepsilon > 0$ the functional $\mathcal{F}^\varepsilon$ is increasing, we deduce by Theorem 4.6 that there exists a subsequence $(\sigma(\varepsilon))$ and a functional $\mathcal{F}^\sigma : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that $\mathcal{F}^\sigma = \overline{\Gamma}(L^2) - \lim_{\varepsilon\to 0}\mathcal{F}^{\sigma(\varepsilon)}$. We put a superscipt σ in order to underline that the limit functional may depend on the subsequence. Now we define the nonnegative increasing functional $K : L^2(\Omega;\mathbb{R}^n)\times\mathcal{A}(\Omega) \to [0,+\infty]$ as

$$K(u,A) := \begin{cases} \displaystyle\int_A |\mathcal{E}u|^2dx & \text{if } u_{|A} \in H^1(A;\mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, K is a measure with respect to A. Moreover, by (3.4) we have that $0 \leq \mathcal{F}^{\sigma(\varepsilon)} \leq \vartheta_M\,K$ for every $\varepsilon > 0$ and by Theorem 4.7 the fundamental estimate holds uniformly for the subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$. Therefore, we can proceed as in [14, Proposition 18.6] and we obtain that

$$\mathcal{F}^\sigma(u,A) = (\mathcal{F}^\sigma)'(u,A) = (\mathcal{F}^\sigma)''(u,A)$$

for every $u \in L^2(\Omega;\mathbb{R}^n)$ and for every $A \in \mathcal{A}(\Omega)$ such that $K(u,A) < +\infty$.

Fix $A \in \mathcal{A}(\Omega)$. We observe that from Theorem 4.1 we have the bound $\mathcal{F}'(\cdot, A) \geq \vartheta_m \mathcal{G}(\cdot, A)$, where we have localised the functional $\mathcal{G}$ defined in (4.1) as in (4.4). Notice that, by definition,

$$\mathcal{F}^\sigma(\cdot, A) = (\mathcal{F}^\sigma)'(\cdot, A) \geq \mathcal{F}'(\cdot, A).$$

Hence we deduce that $\mathcal{F}^\sigma(\cdot, A) \geq \vartheta_m \mathcal{G}(\cdot, A)$. This entails in particular that the Γ-limit of $\mathcal{F}^{\sigma(\varepsilon)}(\cdot, A)$ is finite only on $H^1(A; \mathbb{R}^n)$, which is the same domain where $K(\cdot, A)$ is finite, and is given by $\mathcal{F}^\sigma(\cdot, A)$. This proves the stated convergence for the subsequence $\big(\mathcal{F}^{\sigma(\varepsilon)}\big)$.

Finally, $\mathcal{F}^\varepsilon(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. Then, by Theorem 4.7 and [14, Theorem 18.5] we have that for every $u \in L^2(\Omega; \mathbb{R}^n)$ the set function $\mathcal{F}^\sigma(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω. □

We now show general properties for the Γ-limit of $\mathcal{F}^\varepsilon$, even if, so far, we have only proved the convergence of a subsequence. The fact that the whole sequence $(\mathcal{F}^\varepsilon)$ converges will follow from the characterization of the Γ-limit, which will depend only on the symmetric gradient of the deformation and not on the subsequence $\sigma(\varepsilon)$. This will be done separately for the different regimes in Theorems 5.1, 6.2, 7.5, respectively. In the remaining part of this section we therefore assume that the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges to a functional that we call $\mathcal{F}$, and we omit the superscript σ.

Lemma 4.9. *The restriction of the functional* $\mathcal{F} : L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty]$ *to* $H^1_0(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega)$ *satisfies the following properties: for every* $u, v \in H^1_0(\Omega; \mathbb{R}^n)$ *and for every* $A \in \mathcal{A}(\Omega)$

(a) $\mathcal{F}$ *is local, i.e.,* $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ *whenever* $u_{|A} = v_{|A}$*;*

(b) *the set function* $\mathcal{F}(u, \cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure on* Ω*;*

(c) $\mathcal{F}(\cdot, A)$ *is sequentially weakly lower semicontinuous on* $H^1_0(\Omega; \mathbb{R}^n)$*;*

(d) *for every* $a \in \mathbb{R}^n$ *we have* $\mathcal{F}(u, A) = \mathcal{F}(u + a, A)$*;*

(e) $\mathcal{F}$ *satisfies the bound*

$$0 \leq \mathcal{F}(u, A) \leq \vartheta_M \int_A |\mathcal{E}u|^2 dx.$$

Proof. Properties (a) and (c) follow from the fact that $\mathcal{F}(\cdot, A)$ is the Γ-limit of the sequence $\mathcal{F}^\varepsilon(\cdot, A)$, while (b) comes from Theorem 4.8. For property (d) we can proceed as follows. Let $u \in H^1_0(\Omega; \mathbb{R}^n)$, $A \in \mathcal{A}(\Omega)$ and consider a recovery sequence $(u^\varepsilon) \subset L^2(\Omega; \mathbb{R}^n) \cap SBD^2(A)$ satisfying the usual constraints for the jump set, converging to u strongly in $L^2(\Omega; \mathbb{R}^n)$ and such that $\big(\mathcal{F}^\varepsilon(u^\varepsilon, A)\big)$ converges to $\mathcal{F}(u, A)$. Then $(u^\varepsilon + a)$ converges to $u + a$ in $L^2(\Omega; \mathbb{R}^n)$ and

$$\mathcal{F}(u + a, A) \leq \liminf_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon + a, A) = \liminf_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u^\varepsilon, A) = \mathcal{F}(u, A).$$

On the other hand, $\mathcal{F}(u, A) = \mathcal{F}((u + a) + (-a), A) \leq \mathcal{F}(u + a, A)$, hence (d) is proved. Property (e) follows by the uniform bound (3.4) and by the lim inf-inequality, since

$$\mathcal{F}(u, A) \leq \liminf_{\varepsilon \to 0} \mathcal{F}^\varepsilon(u, A) \leq \vartheta_M \int_A |\mathcal{E}u|^2 dx.$$

□

Next theorem shows that the functional $\mathcal{F}$ admits an integral representation.

Theorem 4.10. *There exists a unique quasi-convex function* $f : \mathbb{M}^{n \times n} \to [0, +\infty[$ *with the following properties:*

(i) $0 \leq f(\xi) \leq \vartheta_M |\xi|^2$ *for every* $\xi \in \mathbb{M}^{n \times n}$*;*

(ii) $\mathcal{F}(u, A) = \displaystyle\int_A f(\nabla u) dx$ *for every* $A \in \mathcal{A}(\Omega)$ *and for every* $u \in H^1(A; \mathbb{R}^n)$*.*

Proof. Notice that the functional $\mathcal{F}$ satisfies all the assumptions of [14, Theorem 20.1], so by Lemma 4.9 the Carathéodory function $f : \Omega \times \mathbb{M}^{n \times n} \to \mathbb{R}$ defined as

$$f(y, \xi) := \limsup_{\varrho \to 0} \frac{\mathcal{F}(\xi\, x, B_\varrho(y))}{\mathcal{L}^n(B_\varrho(y))}$$

provides the integral representation

$$\mathcal{F}(u,A)=\int_A f(x,\nabla u)dx$$

for every $A\in\mathcal{A}(\Omega)$ and for every $u\in L^2(\Omega;\mathbb{R}^n)$ such that $u_{|A}\in H^1(A;\mathbb{R}^n)$. Moreover the same theorem ensures that for a.e. $x\in\Omega$ the function $f(x,\cdot)$ is quasi-convex on $\mathbb{M}^{n\times n}$ and that

$$0\le f(x,\xi)\le\vartheta_M|\xi|^2\quad\text{for a.e. } x\in\mathbb{R}^n \text{ and for every } \xi\in\mathbb{M}^{n\times n}.$$

The fact that f is independent of the first variable can be proved in the usual way (see for instance [21, Theorem 5.4]). □

In the next sections we will use a slightly different notation for the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$. More precisely, it will be denoted by $\mathcal{F}^0$ in the subcritical case, by $\mathcal{F}^{hom}$ in the critical regime, and by $\mathcal{F}^\infty$ in the supercritical case.

5. Subcritical regime: very brittle inclusions

In this section we shall analyse the subcritical case, where the fragility coefficient of the brittle inclusions in the material is much smaller than the size ε of the periodic structure, i.e., $\frac{\alpha_\varepsilon}{\varepsilon}\to 0$.

5.1. Cell formula.

We localise the sequence $(\mathcal{F}^\varepsilon)$ given in (3.6) as in (4.4). Theorem 4.10 ensures that it admits a Γ-convergence subsequence to a limit functional $\mathcal{F}^0$. We shall prove that the limit density can be characterized in terms of an asymptotic cell problem and that it is independent of the subsequence. More precisely, the limit energy density is the function $f_0:\mathbb{M}^{n\times n}\to[0,+\infty)$ defined as follows:

$$f_0(\xi):=\inf\left\{\int_Q \mathbb{C}\mathcal{E}(\xi\,x+w):\mathcal{E}(\xi\,x+w)dx : w\in SBD^2_\#(Q), J_w\subset I,\ [w]\cdot\nu_w\ge 0 \text{ a.e. on } J_w\right\}. \tag{5.1}$$

Next theorem shows that the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ can be expressed in terms of the homogenization formula (5.1). The proof is a simple adaptation of the proof of [21][Theorem 5.6] and therefore will be omitted.

Theorem 5.1. *The density f of the limit functional $\mathcal{F}$ (see Theorem 4.10) coincides with the function f_0 defined by the cell formula (5.1), i.e., for every $\xi\in\mathbb{M}^{n\times n}$*

$$f(\xi)=f_0(\xi).$$

Remark 5.2. The previous theorem implies in particular that in the subcritical regime the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence. Moreover, from the cell formula we deduce that the limit density function depends only on the symmetric part of its argument.

When the elasticity tensor $\mathbb{C}$ is isotropic and $I=\overline{Q}_\delta:=[\delta,1-\delta]^n$, $0<\delta<\frac{1}{2}$, we can give a more explicit description of the density f_0, as shown in the following lemma.

Lemma 5.3. *Let $\mathbb{C}$ be of the special form $\mathbb{C}=2\,\mu\,\mathbb{I}+\lambda\,Id\otimes Id$, $\mu,\lambda>0$, and let f_0 be the corresponding limit density defined as in (5.1). Then it turns out that $f_0(Id)\neq f_0(-Id)$.*

Proof. By the assumption on $\mathbb{C}$ we have that, for every $w\in SBD^2(Q)$

$$\mathbb{C}\mathcal{E}u=2\,\mu\,\mathcal{E}w+\lambda(\mathcal{E}w:Id)\,Id=2\,\mu\,\mathcal{E}w+\lambda(\operatorname{div}w)\,Id\in\mathbb{M}^{n\times n}_{sym}.$$

First step: $f_0(Id)\lneqq 2\,\mu\,n+\lambda\,n^2$.

First of all, we can notice that $f_0(Id)$ can be rewritten as

$$f_0(Id):=\inf\left\{\int_Q \mathbb{C}\mathcal{E}w:\mathcal{E}w\,dx : w-x\in SBD^2_\#(Q), J_w\subset\overline{Q}_\delta,\ [w]\cdot\nu_w\ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\right\}. \tag{5.2}$$

For $i = 1, \dots, n$, let us denote with $\{\partial Q^i_{+\delta}, \partial Q^i_{-\delta}\}$ the opposite *hyperfaces* of ∂Q_δ which are orthogonal to the vector e_i. More precisely,

$$\partial Q^i_{\pm\delta} := \{x \in \partial Q_\delta : x \cdot e_i \gtrless 0\}.$$

We claim that the function w defined as

$$w(x) = \begin{cases} x & \text{if } x \in Q \setminus Q_\delta, \\ 0 & \text{if } x \in Q_\delta, \end{cases}$$

is a competitor for the minimisation problem in (5.2). Indeed, $w - x \in SBD^2_0(Q) \subset SBD^2_\#(Q)$ and $J_w \subset \partial Q_\delta$. It remains to check the non-interpenetration condition for almost every $x \in J_w$. Notice that if $\hat{x} \in J_w \cap \partial Q^i_{+\delta}$ for some i, then

$$[w](\hat{x}) \cdot \nu_w(\hat{x}) = \hat{x} \cdot e_i \geq \min_{x \in \partial Q^i_{+\delta}} (x \cdot e_i) > 0.$$

On the other hand, if $\hat{x} \in J_w \cap \partial Q^i_{-\delta}$ for some i, then

$$[w](\hat{x}) \cdot \nu_w(\hat{x}) = \hat{x} \cdot (-e_i) \geq - \max_{x \in \partial Q^i_{-\delta}} (x \cdot e_i) > 0.$$

Therefore w is a competitor in (5.2), and we obtain by comparison that

$$f_0(Id) \leq \int_Q \mathbb{C}\mathcal{E}w : \mathcal{E}w \, dx = \mathcal{L}^n(Q \setminus Q_\delta)(2\,\mu\, n + \lambda n^2) \lneqq 2\,\mu\, n + \lambda n^2.$$

Second step: $f_0(-Id) = 2\,\mu\, n + \lambda\, n^2$.

In order to prove this relation it is more convenient to use the characterization of the density f_0 in the form (5.1). We are going to prove that $w = 0$ is a minimiser of (5.1), for $\xi = -Id$. To this aim, let $v \in SBD^2_\#(Q)$ be such that $J_v \subset \overline{Q}_\delta$ and $[v] \cdot \nu_v \geq 0$ $\mathcal{H}^{n-1}$- a.e. on J_v. For $\eta \geq 0$ we define the function

$$I(\eta) := \frac{1}{2} \int_Q \mathbb{C}\mathcal{E}(-x + \eta\, v) : \mathcal{E}(-x + \eta\, v) dx.$$

We claim that

$$\left(\frac{d}{d\eta} I(\eta)\right)_{|\eta=0} \geq 0 \tag{5.3}$$

for every admissible v. By convexity, (5.3) is a necessary and sufficient condition for the minimality of $w = 0$ in (5.1), for $\xi = -Id$. We notice that condition (5.3) is equivalent to

$$\int_Q (2\mu + \lambda n) Id : \mathcal{E}v \, dx \leq 0 \tag{5.4}$$

for every admissible v. Let now v be an admissible competitor in the minimisation problem. Integrating by parts and using the periodicity assumption on v, we have

$$\int_Q (2\mu + \lambda n) Id : \mathcal{E}v \, dx = -(2\mu + \lambda n) \sum_{i=1}^n \int_{J_v} [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1}. \tag{5.5}$$

As v satisfies the non-interpenetration condition

$$\sum_{i=1}^n [v_i](x)\, \nu_{v_i}(x) \geq 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_v,$$

and is arbitrary, from (5.5) follows (5.4), and the claim. □

Remark 5.4. As immediate corollary from the previous lemma we deduce that, in general, the limit density f_0 is not a quadratic form.

Remark 5.5 (Anisotropy of the limit energy)**.** We are going to show that the limit energy is not isotropic, in the case of an isotropic elasticity tensor $\mathbb{C}$. Therefore the isotropy of the elastic energy is not preserved by homogenization. We recall that for the energy density f_0 being isotropic means that, for every $\xi \in \mathbb{M}^{n\times n}$

$$f_0(\xi) = f_0(\xi R) \quad \text{for every } R \in SO(n). \tag{5.6}$$

We will prove that, for the choice $\xi = -Id$, there exists a rotation $R_0 \in SO(n)$ such that the equality (5.6) is violated, i.e., such that

$$f_0(-Id) \neq f_0(-R_0). \tag{5.7}$$

A fundamental step in this direction is the explicit expression of $f_0(-Id)$ provided by Lemma 5.3. We will prove (5.7) by showing that $f_0(-R_0) \lneqq \mathbb{C}(-R_0) : (-R_0) = f_0(-Id)$.

Let $\xi \in \mathbb{M}^{n\times n}_{sym}$ be a diagonal matrix. We denote with $(\nu_1, \dots, \nu_n)$ its eigenvalues. We will prove that the minimality of $w = 0$ in (5.1) forces $\nu_1 = \dots = \nu_n < 0$.

Let $v \in SBD^2_\#(Q)$ such that $J_v \subset Q_\delta$ and $[v]\cdot\nu_v \geq 0$ $\mathcal{H}^{n-1}$- a.e. on J_v, and let $\eta \geq 0$. We define

$$I(\eta) := \frac{1}{2}\int_Q \mathbb{C}\mathcal{E}(\xi\, x + \eta\, v) : \mathcal{E}(\xi\, x + \eta\, v)dx.$$

Let us suppose that $w = 0$ is a minimiser in (5.1). Since the functional in (5.1) is convex, the minimality of $w = 0$ is equivalent to

$$\left(\frac{d}{d\eta}I(\eta)\right)_{|\eta=0} = \frac{1}{2}\left(\frac{d}{d\eta}\int_Q \mathbb{C}\mathcal{E}(\xi\, x + \eta\, v) : \mathcal{E}(\xi\, x + \eta\, v)dx\right)_{|\eta=0} \geq 0 \tag{5.8}$$

for every admissible v, which is in turn equivalent to

$$\int_Q \mathbb{C}\xi : \mathcal{E}v\, dx \geq 0 \tag{5.9}$$

for every admissible v. Integrating by parts, the left hand side in the previous expression becomes

$$\int_Q \mathbb{C}\xi : \mathcal{E}v\, dx = -\sum_{i,j=1}^n \int_{J_v} (\mathbb{C}\xi)_{ij}\, [v_j]\, \nu_{v_i} d\mathcal{H}^{n-1}.$$

Therefore, as $(\mathbb{C}\xi)_{ij} = (2\mu\nu_i + \lambda\sum_{k=1}^n \nu_k)\delta_{ij}$, (5.4) reduces to

$$-\sum_{i=1}^n \int_{J_v} (2\mu\nu_i + \lambda\sum_{k=1}^n \nu_k)\, [v_i]\, \nu_{v_i} d\mathcal{H}^{n-1} \geq 0 \tag{5.10}$$

for every admissible v. As v satisfies the non-interpenetration condition

$$\sum_{i=1}^n [v_i](x)\, \nu_{v_i}(x) \geq 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_v, \tag{5.11}$$

and is arbitrary, we conclude that the quantities $(2\mu\nu_i + \lambda\sum_{k=1}^n \nu_k)$ are forced to be equal for every i and negative. This clearly implies that alle the eigenvalues ν_i of ξ are equal and negative, i.e., ξ is a negative multiple of the identity.

Therefore, choosing R_0 to be a diagonal matrix with eigenvalues $\nu_i = \pm 1$ (with at least a positive eigenvalue) and $\det R_0 = 1$, (5.7) follows.

Remark 5.6. Another important consequence of Lemma 5.3 is that the limit functional $\mathcal{F}_0$ describes a material where damage occurs. Indeed, at least in the isotropic case and for $I = \overline{Q}_\delta$, we proved that $f_0(Id) \lneqq 2\,\mu\, n + \lambda\, n^2 = \mathbb{C}Id : Id$. More in general, similar computations show that $f_0(\xi) \lneqq \mathbb{C}\xi : \xi$ for all the tensors ξ with the property that there exists a constant $c_\xi = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that

$$\max_{x\in\partial Q^i_{-\delta}} \big((\xi\, x)\cdot e_i\big) < c_i < \min_{x\in\partial Q^i_{+\delta}} \big((\xi\, x)\cdot e_i\big) \quad \text{for every } i = 1, \dots, n. \tag{5.12}$$

6. Critical regime: $\alpha_\varepsilon \approx \varepsilon$.

In this section we shall analyse the case in which the fragility coefficient of the inclusions in the material α_ε is of the same order of the size ε of the periodic structure. We can assume, without loss of generality, that $\alpha_\varepsilon = \varepsilon$. The energy of the material is thus given by

$$\mathcal{F}^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u\,dx + \varepsilon\,\mathcal{H}^{n-1}(J_u) & \text{if } u \in SBD^2_0(\Omega), J_u \subset I(\varepsilon), \\ & [u]\cdot\nu_u \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases}$$

6.1. **Homogenization formula.** We localise the sequence $(\mathcal{F}^\varepsilon)$ as in (4.4). Theorem 4.10 ensures that it admits a Γ-convergence subsequence to a limit functional $\mathcal{F}^{hom}$. We shall prove that the limit density can be characterized in terms of an asymptotic cell problem and that it is independent of the subsequence. More precisely, the limit energy density is the function $f_{hom} : \mathbb{M}^{n\times n} \to [0,+\infty)$ defined as

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}(\xi\,x + w) : \mathcal{E}(\xi\,x+w)dx + \mathcal{H}^{n-1}(J_w) : \\ w \in SBD^2_0\big((0,t)^n\big), J_w \subset \tilde{I}\cap(0,t)^n, [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\Big\}, \tag{6.1}$$

where $\tilde{I}$ is defined as in (3.1). We state rigorously this results in the next two theorems, whose proofs follow easily from the proofs of [21][Theorem 5.5, Theorem 5.6], respectively.

Theorem 6.1. *The function f_{hom} in (6.1) is well defined, that is the function*

$$g(t) := \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}(\xi\,x + w) : \mathcal{E}(\xi\,x+w)dx + \mathcal{H}^{n-1}(J_w) : \\ w \in SBD^2_0\big((0,t)^n\big), J_w \subset \tilde{I}\cap(0,t)^n, [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\Big\},$$

admits a limit as $t \to +\infty$.

Theorem 6.2. *The density f of the limit functional $\mathcal{F}^{hom}$ (see Theorem 4.10) coincides with the function f_{hom} defined by the cell formula (6.1), i.e., for every $\xi \in \mathbb{M}^{n\times n}$*

$$f(\xi) = f_{hom}(\xi).$$

Remark 6.3. Notice that from this theorem we deduce that also in the critical case the whole sequence $(\mathcal{F}^\varepsilon)$ Γ-converges, since the formula for the limit energy density does not depend on the subsequence. Moreover, we deduce that the limit density function depends only on the symmetric part of its argument.

Next lemma shows that the limit functional in the critical regime describes a damaged material. We restrict our attention to the isotropic case, i.e., $\mathbb{C} = 2\,\mu\,\mathbb{I} + \lambda\,Id\otimes Id$ with $\mu, \lambda > 0$ and to $I = \overline{Q}_\delta = [\delta, 1-\delta]^n$.

Lemma 6.4. *There exists $\xi \in \mathbb{M}^{n\times n}$ such that $f_{hom}(\xi) \lneqq \mathbb{C}\xi : \xi$.*

Proof. Let us rewrite the limit energy density in the following way:

$$f_{hom}(\xi) := \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}w : \mathcal{E}w dx + \mathcal{H}^{n-1}(J_w) : w - \xi x \in SBD^2_\#\big((0,t)^n\big), \\ J_w \subset \overline{Q}_\delta, [w]\cdot\nu_w \geq 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\Big\}, \tag{6.2}$$

for every $\xi \in \mathbb{M}^{n\times n}$. Let $\xi \in \mathbb{M}^{n\times n}$ and assume that there exists a constant $c_\xi = (c_1,\dots,c_n) \in \mathbb{R}^n$ with the property (5.12) as in Lemma 5.3. Let us restrict our attention to the case when in (6.2) $t \in \mathbb{N}$. The general case can be deduced in the same way. Then, it is easy to check that the function w_ξ defined as

$$w_\xi(x) = \begin{cases} \xi\,x & \text{if } x \in Q\setminus Q_\delta, \\ c_\xi & \text{if } x \in Q_\delta, \end{cases}$$

and extended by periodicity in $(0,t)^n$ is a competitor in (6.2). Therefore, for the class of matrices ξ defined by the condition (5.12) we have

$$f_{hom}(\xi) \leq \lim_{t\to+\infty} \frac{1}{t^n}\left\{ \int_{(0,t)^n} \mathbb{C}\mathcal{E}w_\xi : \mathcal{E}w_\xi dx + \mathcal{H}^{n-1}(J_{w_\xi})\right\} \leq \mathcal{L}^n(Q_\delta)\mathbb{C}\xi : \xi + P(Q_\delta, Q).$$

Then, in order to prove the theorem it is sufficient to choose a matrix $\xi \in \mathbb{M}^{n\times n}$ satisfying the property (5.12) and such that

$$\mathcal{L}^n(Q_\delta)\mathbb{C}\xi : \xi + P(Q_\delta, Q) \lneqq \mathbb{C}\xi : \xi.$$

In particular $\xi = \kappa Id$ with $\kappa >> 1$ provides a possible choice. □

7. Supercritical regime: stiffer inclusions

In this section we shall analyse the supercritical case, where the fragility coefficient α_ε of the inclusions in the material is bigger than the size ε of the periodic structure.

Before studying this case, we state a technical lemma which will be used in the following. For the proof we refer to [21].

Lemma 7.1. *Let $a_h : \Omega \to \mathbb{R}_+$ be a sequence of measurable functions such that*

$$a_h \to a \quad \textit{in measure.}$$

Then, for every $v \in L^2(\Omega;\mathbb{R}^m)$ and for every sequence $(v_h) \subset L^2(\Omega;\mathbb{R}^m)$ such that

$$v_h \rightharpoonup v \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^m),$$

it turns out that

$$\int_\Omega a|v|^2 dx \leq \liminf_{h\to+\infty} \int_\Omega a_h|v_h|^2 dx.$$

In the following we present a proper modification of the argument used in [2] and in [6] to prove compactness and lower semicontinuity in SBD. We refer also to [10, Lemma 5.1] for a similar result.

Lemma 7.2. *Let $U \subset \mathbb{R}^n$ be an open set, let $w \in L^2(U;\mathbb{R}^n)$ and let (w_h) be a sequence converging to w strongly in L^2. Assume that $||\mathcal{E}w_h||_{L^2(U;\mathbb{M}^{n\times n})} \leq c$ and that $\mathcal{H}^{n-1}(J_{w_h}) \to 0$ as $h \to +\infty$. Then $w \in H^1(U;\mathbb{R}^n)$ and*

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \textit{weakly in } L^2(U;\mathbb{M}^{n\times n}).$$

Proof. We can assume up to a subsequence that

$$\mathcal{H}^{n-1}(J_{w_h}) \leq \frac{1}{h^2}.$$

First step: $w \in H^1(U;\mathbb{R}^n)$.
Let $\xi \in S^{n-1}$, $y \in \Pi^\xi$ and let us define for every $h \in \mathbb{N}$ the sections $(w_h)^\xi_y(t) := w_h(y + t\,\xi)\cdot\xi$. It is well known that $(w_h)^\xi_y \in SBV^2\big(U^\xi_y\big)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi$. Moreover, since $w_h \to w$ strongly in L^2, it follows that, up to subsequences,

$$(w_h)^\xi_y \to w^\xi_y \quad \text{strongly in } L^2\big(U^\xi_y\big) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi.$$

Let us denote with N_1 the set such that $(w_h)^\xi_y \in SBV^2\big(U^\xi_y\big)$ and $(w_h)^\xi_y \to w^\xi_y$ strongly in L^2 for every $y \in \Pi^\xi \setminus N_1$. As we already noticed, $\mathcal{H}^{n-1}(N_1) = 0$. Let us define the set E_h as

$$E_h := \bigcup_{j\geq h} J_{w_j}.$$

From the inequality $\mathcal{H}^{n-1}(J_{w_h}) \leq \frac{1}{h^2}$, it turns out that $\mathcal{H}^{n-1}(E_h) \to 0$ as $h \to +\infty$. Hence for every $\vartheta > 0$ there exists $h(\vartheta)$ such that $\mathcal{H}^{n-1}(E_{h(\vartheta)}) < \vartheta$. Clearly, $J_{w_h} \subset E_{h(\vartheta)}$ for every $h \geq h(\vartheta)$.

Let us denote with $(E_{h(\vartheta)})^\xi$ the projection of the set $E_{h(\vartheta)}$ on Π^ξ. From the definition it follows that $(w_h)^\xi_y \in H^1\big(U^\xi_y\big)$ for every $y \in (\Pi^\xi \setminus (E_{h(\vartheta)})^\xi) \setminus N_1$ and for $h \geq h(\vartheta)$. Moreover, the H^1 norm of $(w_h)^\xi_y$ is equibounded. Indeed, using Fubini's Theorem we can write

$$\int_U |\mathcal{E} w_h \xi \cdot \xi|^2 dx = \int_U |\nabla w_h \xi \cdot \xi|^2 dx = \int_{\Pi^\xi} \left[\int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt \right] d\mathcal{H}^{n-1}(y), \tag{7.1}$$

and, as $\xi \in S^{n-1}$, we have

$$\int_U |\mathcal{E} w_h \xi \cdot \xi|^2 dx \leq \int_U |\mathcal{E} w_h|^2 dx, \tag{7.2}$$

where the right-hand side is equibounded by assumption. Hence from (7.1) we obtain

$$\int_{\Pi^\xi} \left[\int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt \right] d\mathcal{H}^{n-1}(y) \leq c. \tag{7.3}$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h \to +\infty} \int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt = \lim_{k(y) \to +\infty} \int_{U^\xi_y} |\nabla (w_{k(y)})^\xi_y|^2 \, dt. \tag{7.4}$$

The bound (7.3) guarantees that there exists a function v such that, up to a further subsequence $w_{j(y)} \subset w_{k(y)}$, we have

$$(w_{j(y)})^\xi_y \rightharpoonup v \quad \text{weakly in } H^1\big(U^\xi_y\big), \tag{7.5}$$

for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi \setminus (E_{h(\vartheta)})^\xi$. Since for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi$ the whole sequence $(w_h)^\xi_y$ converges to w^ξ_y strongly in L^2, (7.5) implies that

$$(w_{j(y)})^\xi_y \rightharpoonup w^\xi_y \quad \text{weakly in } H^1\big(U^\xi_y\big). \tag{7.6}$$

By the lower semicontinuity in H^1 and by (7.4) we obtain the inequality

$$\int_{U^\xi_y} |\nabla (w^\xi_y)|^2 \, dt \leq \liminf_{j(y) \to +\infty} \int_{U^\xi_y} |\nabla (w_{j(y)})^\xi_y|^2 \, dt = \liminf_{h \to +\infty} \int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt, \tag{7.7}$$

valid for $\mathcal{H}^{n-1}$-almost every $y \in (\Pi^\xi \setminus (E_{h(\vartheta)})^\xi)$. Integrating (7.7) with respect to y and using the Fatou Lemma we get

$$\int_{\Pi^\xi \setminus (E_{h(\vartheta)})^\xi} \left[\int_{U^\xi_y} |\nabla (w^\xi_y)|^2 \, dt \right] d\mathcal{H}^{n-1}(y) \leq \liminf_{h \to +\infty} \int_{\Pi^\xi \setminus (E_{h(\vartheta)})^\xi} \left[\int_{U^\xi_y} |\nabla (w_h)^\xi_y|^2 \, dt \right] d\mathcal{H}^{n-1}(y). \tag{7.8}$$

Hence, by (7.3) we obtain

$$\int_{\Pi^\xi \setminus (E_{h(\vartheta)})^\xi} \left[\int_{U^\xi_y} |\nabla (w^\xi_y)|^2 \, dt \right] d\mathcal{H}^{n-1}(y) \leq c, \tag{7.9}$$

where the constant c is independent of ϑ. Using the estimate (7.9), that $w \in L^2(U; \mathbb{R}^n)$, and that $w^\xi_y \in H^1\big(U^\xi_y\big)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi \setminus (E_{h(\vartheta)})^\xi$, we conclude that $w \in H^1(U; \mathbb{R}^n)$. Indeed, let us define the sets E_∞ and E_0 as

$$E_\infty := \cap_h E_h \quad \text{and} \quad E_0 := \lim_h E_h,$$

where the convergence in the definition of E_0 is almost everywhere with respect to the Hausdorff measure. From $\mathcal{H}^{n-1}(E_h) \leq \frac{1}{h^2}$ and $E_{h+1} \subset E_h$, it turns out that

$$\mathcal{H}^{n-1}(E_\infty) = 0 = \mathcal{H}^{n-1}(E_0).$$

Now, since $\Pi^\xi \setminus (E_\infty)^\xi$ is contained in $\Pi^\xi \setminus (E_h)^\xi$ for h large enough, we have that $w^\xi_y \in H^1\big(U^\xi_y\big)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi \setminus (E_\infty)^\xi$. Hence, as $\mathcal{H}^{n-1}(E_\infty) = 0$, we conclude that for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi$ the section $w^\xi_y \in H^1\big(U^\xi_y\big)$. On the other hand, using the Monotone Convergence Theorem in (7.9), we have

$$\lim_{h(\vartheta)\to\infty} \int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi} \left[\int_{U^\xi_y} |\nabla(w^\xi_y)|^2\, dt\right] d\mathcal{H}^{n-1}(y) = \int_{\Pi^\xi\setminus(E_0)^\xi} \left[\int_{U^\xi_y} |\nabla(w^\xi_y)|^2\, dt\right] d\mathcal{H}^{n-1}(y) \le c. \tag{7.10}$$

Again, the fact that $\mathcal{H}^{n-1}(E_0) = 0$ implies that

$$\int_{\Pi^\xi} \left[\int_{U^\xi_y} |\nabla(w^\xi_y)|^2\, dt\right] d\mathcal{H}^{n-1}(y) \le c. \tag{7.11}$$

At this point we can apply [3, Proposition 3.105] to conclude that

$$\nabla(w^\xi_y) = D_t[w(y+t\xi)\cdot\xi] = Dw\xi\cdot\xi = Ew\xi\cdot\xi \in L^2(U),$$

and this holds for every ξ. Using the identity

$$Ew\xi\cdot\eta = \frac{1}{2}[Ew(\xi+\eta)\cdot(\xi+\eta) - Ew\xi\cdot\xi - Ew\eta\cdot\eta] \quad \forall\, \xi,\eta,$$

we conclude that $Ew \in L^2(U;\mathbb{M}^{n\times n})$. Therefore, since $w \in L^2(U;\mathbb{R}^n)$, the Korn Inequality ensures that $w \in H^1(U;\mathbb{R}^n)$.

Second step: convergence of the symmetric gradient. Let us define, for a given scalar function $v \in L^2(U)$, the functional

$$L^\xi_y(w_h, v) := \int_{U^\xi_y} |\nabla(w_h)^\xi_y - v(t,y)|^2\, dt.$$

Using (7.2) and the fact that $v \in L^2(U)$, we obtain the bound

$$\int_{\Pi^\xi} L^\xi_y(w_h, v) d\mathcal{H}^{n-1}(y) \le \int_U |\mathcal{E}w_h\xi\cdot\xi - v|^2 dx \le c.$$

Now, let $w_{k(y)}$ be a subsequence (depending on y) of w_h such that

$$\liminf_{h\to+\infty} L^\xi_y(w_h, v) = \lim_{k(y)\to+\infty} L^\xi_y(w_{k(y)}, v). \tag{7.12}$$

The bound (7.3) guarantees that, up to a further subsequence $w_{j(y)} \subset w_{k(y)}$,

$$(w_{j(y)})^\xi_y \rightharpoonup w^\xi_y \quad \text{weakly in } H^1\big(U^\xi_y\big)$$

for $\mathcal{H}^{n-1}$-almost every $y \in \Pi^\xi \setminus (E_{h(\vartheta)})^\xi$, and in particular

$$\nabla(w_{j(y)})^\xi_y - v \rightharpoonup \nabla w^\xi_y - v \quad \text{weakly in } L^2\big(U^\xi_y\big).$$

Hence, by the lower semicontinuity of the functional L^ξ_y and by (7.12), we obtain

$$L^\xi_y(w, v) \le \liminf_{j(y)\to+\infty} L^\xi_y(w_{j(y)}, v) = \liminf_{h\to+\infty} L^\xi_y(w_h, v).$$

Integrating the previous expression with respect to y leads to

$$\int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi} L^\xi_y(w, v)\, d\mathcal{H}^{n-1}(y) \le \liminf_{h\to+\infty} \int_{\Pi^\xi\setminus(E_{h(\vartheta)})^\xi} L^\xi_y(w_h, v) d\mathcal{H}^{n-1}(y).$$

As $w \in H^1(U;\mathbb{R}^n)$ we can pass to the limit as $\vartheta \to 0$ in the previous expression and we get

$$\int_U |\mathcal{E}w\xi\cdot\xi - v|^2 dx \le \liminf_{h\to+\infty} \int_U |\mathcal{E}w_h\xi\cdot\xi - v|^2 dx. \tag{7.13}$$

Since (7.13) holds true for every $v \in L^2(U)$ we have that, for every $\xi \in S^{n-1}$,

$$\mathcal{E}w_h\xi\cdot\xi \rightharpoonup \mathcal{E}w\xi\cdot\xi \quad \text{weakly in } L^2(U). \tag{7.14}$$

Now we consider a basis $\{\xi_1,\dots,\xi_n\}$ of $\mathbb{R}^n$ such that $\xi_i + \xi_j \in S^{n-1}$ for every $i \ne j$, and specify $\xi = \xi_i + \xi_j$ in (7.14). Then we have

$$\mathcal{E}w_h \rightharpoonup \mathcal{E}w \quad \text{weakly in } L^2(U;\mathbb{M}^{n\times n}),$$

and this concludes the proof. □

In the next two lemmas we state some Γ-convergence results that will be used in the proof of the main result of this section.

Lemma 7.3. *Let us fix $0 < \bar\delta < \delta < \frac{1}{2}$ so that $I \subseteq \overline{Q}_\delta \subset\subset Q_{\bar\delta}$. For every $h \in \mathbb{N}$, let $\mathcal{G}^h : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ be the functional defined as*

$$\mathcal{G}^h(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \mathbb{C}\mathcal{E}w : \mathcal{E}w\,dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}),\ J_w \subset I, \mathcal{H}^{n-1}(J_w) \le \frac{1}{h^2}, \\ & \quad [w]\cdot\nu_w \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Then the sequence $(\mathcal{G}^h)$ Γ-converges with respect to the strong topology of L^2 to the functional $\mathcal{G} : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ given by

$$\mathcal{G}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \mathbb{C}\mathcal{E}w : \mathcal{E}w\,dx & \text{if } w \in H^1(Q_{\bar\delta};\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Proof. The proof of the liminf inequality follows by applying the previous lemma with $U = Q_{\bar\delta}$ and using the lower semicontinuity of the functionals, while the existence of the recovery sequence is immediate. □

Lemma 7.4. *Let $(\varphi_h), \varphi \in H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$ be such that $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$. For every $h \in \mathbb{N}$, let $\mathcal{G}^h_{\varphi_h} : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ be the functionals defined by*

$$\mathcal{G}^h_{\varphi_h}(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \mathbb{C}\mathcal{E}w : \mathcal{E}w\,dx + \mathcal{H}^{n-1}(J_w) & \text{if } w \in SBD^2(Q_{\bar\delta}), J_w \subset Q_\delta, \mathcal{H}^{n-1}(J_w) \le \frac{1}{h^2}, \\ & \quad [w]\cdot\nu_w \ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w,\ tr(w) = \varphi_h \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases} \tag{7.15}$$

Then the sequence $(\mathcal{G}^h_{\varphi_h})$ Γ-converges with respect to the strong topology of L^2 to the functional $\mathcal{G}_\varphi : L^2(Q_{\bar\delta};\mathbb{R}^n) \to [0,+\infty]$ given by

$$\mathcal{G}_\varphi(w) := \begin{cases} \displaystyle\int_{Q_{\bar\delta}} \mathbb{C}\mathcal{E}w : \mathcal{E}w\,dx & \text{if } w \in H^1(Q_{\bar\delta};\mathbb{R}^n),\ tr(w) = \varphi \text{ on } \partial Q_{\bar\delta}, \\ +\infty & \text{otherwise in } L^2(Q_{\bar\delta};\mathbb{R}^n). \end{cases}$$

Proof. First step: proof of compactness and liminf. Let $(w_h), w \in L^2(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_h \to w$ strongly in L^2 and $\mathcal{G}^h_{\varphi_h}(w_h) \le c < +\infty$. From the equality $\mathcal{G}^h_{\varphi_h}(w_h) = \mathcal{G}^h(w_h)$ and Lemma 7.3 we have that $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$; moreover

$$\liminf_{h\to+\infty} \mathcal{G}^h_{\varphi_h}(w_h) = \liminf_{h\to+\infty} \mathcal{G}^h(w_h) \ge \mathcal{G}(w).$$

It remains to show that $tr(w) = \varphi$ on $\partial Q_{\bar\delta}$. From the bound $\mathcal{G}^h_{\varphi_h}(w_h) \le c$ it follows that the sequence (w_h) is equibounded in $H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n)$, and hence

$$w_h \rightharpoonup w \quad \text{weakly in } H^1(Q_{\bar\delta} \setminus Q_\delta;\mathbb{R}^n).$$

The compactness of the trace operator gives

$$\varphi_h = (w_h)_{|\partial Q_{\bar\delta}} \to w_{|\partial Q_{\bar\delta}} \quad \text{strongly in } L^2(\partial Q_{\bar\delta};\mathbb{R}^n).$$

On the other hand, by assumption, $\varphi_h \to \varphi$ strongly in $H^{1/2}(\partial Q_{\bar\delta};\mathbb{R}^n)$. Therefore, $w_{|\partial Q_{\bar\delta}} = tr(w) = \varphi$.

Second step: limsup. Let $w \in H^1(Q_{\bar\delta};\mathbb{R}^n)$ be such that $w_{|\partial Q_{\bar\delta}} = \varphi$. Let us consider the sequence $(v_h) \subset H^1(Q_{\bar\delta};\mathbb{R}^n)$ such that $(v_h)_{|\partial Q_{\bar\delta}} = \varphi_h - \varphi$; it turns out that $v_h \to 0$ strongly in H^1. We claim that $w_h := v_h + w$ is a recovery sequence. Indeed, $(w_h)_{|\partial Q_{\bar\delta}} = \varphi_h$ and $w_h \to w$ strongly in H^1, hence $\mathcal{E}w_h \to \mathcal{E}w$ strongly in L^2. Since the functional $\mathcal{G}^h_{\varphi_h}$ gives a norm equivalent to the standard L^2-norm, we have the desired convergence. □

Finally we are ready to state and prove the convergence result for the functional $\mathcal{F}^\varepsilon$ in (3.6), in the supercritical regime $\frac{\alpha_\varepsilon}{\varepsilon} \to \infty$. We define the functional $\mathcal{F}^\infty : L^2(\Omega;\mathbb{R}^n) \to [0,+\infty]$ as

$$\mathcal{F}^\infty(u) = \begin{cases} \displaystyle\int_\Omega \mathbb{C}\mathcal{E}u : \mathcal{E}u\, dx & \text{in } H^1_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^n). \end{cases}$$

Next theorem shows that $\mathcal{F}^\infty$ is the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ in the case $\alpha_\varepsilon \gg \varepsilon$.

Theorem 7.5 (Γ-convergence)**.** *(i) Let $u \in L^2(\Omega;\mathbb{R}^n)$ and let (u^ε) be a sequence converging to u strongly in L^2 and having equibounded energy $\big(\mathcal{F}^\varepsilon(u^\varepsilon)\big)$. Then $u \in H^1_0(\Omega;\mathbb{R}^n)$ and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{F}^\infty(u). \tag{7.16}$$

(ii) For every $u \in H^1_0(\Omega;\mathbb{R}^n)$ there exists a sequence (u^ε) such that $u^\varepsilon \to u$ in $L^2(\Omega;\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \mathcal{F}^\infty(u). \tag{7.17}$$

Proof. Notice that (ii) trivially follows by taking $u^\varepsilon = u$ for every $\varepsilon > 0$, so that only (i) needs a proof.

(i) Let us write the domain Ω as union of cubes of side ε:

$$\Omega = \Big(\bigcup_{h\in\mathbb{Z}^n} \varepsilon\overline{(Q+h)}\Big) \cap \Omega.$$

We denote by $\{Q^\varepsilon_k\}_{k=1,\dots,N(\varepsilon)+N_r(\varepsilon)}$ an enumeration of the family of cubes $\varepsilon(Q+h)$ intersecting Ω, so that $Q^\varepsilon_k \subset \Omega$ for $k \in \{1,\dots,N(\varepsilon)\}$, and $Q^\varepsilon_k \cap \partial\Omega \neq \emptyset$ for $k \in \{N(\varepsilon)+1,\dots,N(\varepsilon)+N_r(\varepsilon)\}$. In the same way we can define the sets $\{I^\varepsilon_k\}_{k=1,\dots,N(\varepsilon)+N_r(\varepsilon)}$. Notice that $N(\varepsilon)$ is of order $1/\varepsilon^n$, while $N_r(\varepsilon)$ is of order $1/\varepsilon^{n-1}$.

We now classify the cubes Q^ε_k, with $k = 1,\dots,N(\varepsilon)$, according to the measure of the jump set that they contain. More precisely, let $\beta > 0$ be a parameter that will be specified later; we say that a cube Q^ε_k is *bad* whenever $\mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big) > \beta\,\varepsilon^{n-1}$, and *good* otherwise. Then, if $N_b(\varepsilon)$ denotes the number of bad cubes, we can assume without loss of generality that Q^ε_k is bad for $k \in \{1,\dots,N_b(\varepsilon)\}$ and good for $k \in \{N_b(\varepsilon)+1,\dots,N(\varepsilon)\}$.

First step: energy estimate on bad cubes. Let Ω_0, $\mathcal{G}^\varepsilon$ and $(\hat u^\varepsilon)$ be defined as in Theorem 4.1. Then

$$\mathcal{F}^\varepsilon(u^\varepsilon) \geq \mathcal{G}^\varepsilon(\hat u^\varepsilon) \geq \sum_{k=1}^{N_b(\varepsilon)} \int_{Q^\varepsilon_k\setminus I^\varepsilon_k} \mathbb{C}\mathcal{E}\hat u^\varepsilon : \mathcal{E}\hat u^\varepsilon dx.$$

We also notice that, from the energy bound relative to the sequence (u^ε), since in particular $\alpha_\varepsilon\mathcal{H}^{n-1}(J_{u^\varepsilon}) \leq c$, it follows that $N_b(\varepsilon) \leq c/(\alpha_\varepsilon\varepsilon^{n-1})$.

Second step: energy estimate on good cubes. Let us fix $k \in \{N_b(\varepsilon)+1,\dots,N(\varepsilon)\}$ and let us consider the localisation of the functional $\mathcal{F}^\varepsilon$, relative to the set Q^ε_k, i.e.,

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \int_{Q^\varepsilon_k} \mathbb{C}\mathcal{E}u^\varepsilon : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon\mathcal{H}^{n-1}\big(J_{u^\varepsilon} \cap Q^\varepsilon_k\big). \tag{7.18}$$

Define the function v^ε in the unit cube Q_k as $u^\varepsilon(\varepsilon y) =: \sqrt{\alpha_\varepsilon\varepsilon}\, v^\varepsilon(y)$. In terms of v^ε, the energy (7.18) can be written as

$$\mathcal{F}^\varepsilon\big(u^\varepsilon, Q^\varepsilon_k\big) = \alpha_\varepsilon\varepsilon^{n-1}\Big\{\int_{Q_k} \mathbb{C}\mathcal{E}v^\varepsilon : \mathcal{E}v^\varepsilon dx + \mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_k)\Big\}, \tag{7.19}$$

with $\mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_k) \leq \beta$. Therefore, by means of a change of variables we reduced to the study of a Mumford-Shah like functional over a fixed domain, with some constraints on the jump set. From now on we will omit the subscript k. Let $\bar\delta, \hat\delta$ be such that $I \subset\subset Q_{\bar\delta} \subset\subset Q_{\hat\delta} \subset\subset Q$.

We are going to replace the function v^ε with a new function $\hat v^\varepsilon$ solving a suitable minimisation problem in the inner square $Q_{\bar\delta}$ and agreeing with v^ε in $Q \setminus Q_{\bar\delta}$.

As first step we find local minimisers of the Mumford-Shah functional in $Q_{\bar\delta}$ under the previously introduced constraints for the jump set, where, according to the definition given in [15], local minimality is intended with respect to all perturbations with compact support. More precisely, we analyse the following problem:

$$\text{(LMin)}\ \operatorname{loc\,min}\left\{\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}w:\mathcal{E}w\,dx+\mathcal{H}^{n-1}(J_w):\quad w\in SBD^2(Q_{\hat\delta}),J_w\subset I,\mathcal{H}^{n-1}(J_w)\le\beta,\ [w]\cdot\nu_w\ge 0\ \mathcal{H}^{n-1}\text{-a.e. on } J_w\right\}.$$

Let us denote by $\mathcal{M}_\beta$ the class of solutions of (LMin). For a given $\hat v\in\mathcal{M}_\beta$, let us consider the function $\tilde v$ solving

$$\text{(Eul)}\quad\begin{cases}\operatorname{div}\mathbb{C}\mathcal{E}\tilde v=0 & \text{in } Q_{\bar\delta},\\ \tilde v=\hat v & \text{in } Q_{\hat\delta}\setminus Q_{\bar\delta}.\end{cases}$$

We want to prove that for every $\eta>0$ there exists $\beta>0$ such that for every $\hat v\in\mathcal{M}_\beta$ and for the corresponding $\tilde v$ we have

$$\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\tilde v:\mathcal{E}\tilde v\,dx\le(1+\eta)\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\hat v:\mathcal{E}\hat v\,dx. \tag{7.20}$$

We will prove (7.20) by contradiction. Suppose, for contradiction, that there exists $\eta>0$ such that for every $\beta>0$ we can find $\hat v\in\mathcal{M}_\beta$ and a corresponding $\tilde v$ for which

$$\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\tilde v:\mathcal{E}\tilde v\,dx>(1+\eta)\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\hat v:\mathcal{E}\hat v\,dx. \tag{7.21}$$

In particular, for $\beta=\frac{1}{h^2}$, (7.21) implies that for every $h>0$ there exist $\hat v_h$ and $\tilde v_h$ defined as above for which

$$\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\tilde v_h:\mathcal{E}\tilde v_h dx>(1+\eta)\int_{Q_{\hat\delta}}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx. \tag{7.22}$$

Since $Q_{\hat\delta}=\big(Q_{\hat\delta}\setminus Q_{\bar\delta}\big)\cup Q_{\bar\delta}$, we can split the previous integrals and, using the fact that $\tilde v_h=\hat v_h$ in $Q_{\hat\delta}\setminus Q_{\bar\delta}$, we obtain from (7.22)

$$\int_{Q_{\bar\delta}}\mathbb{C}\mathcal{E}\tilde v_h:\mathcal{E}\tilde v_h dx>(1+\eta)\int_{Q_{\bar\delta}}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx+\eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx. \tag{7.23}$$

Since the problem defining $\tilde v_h$ is linear, we can normalize the left-hand side of (7.23), so that

$$1=\int_{Q_{\bar\delta}}\mathbb{C}\mathcal{E}\tilde v_h:\mathcal{E}\tilde v_h dx>(1+\eta)\int_{Q_{\bar\delta}}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx+\eta\int_{Q_{\hat\delta}\setminus Q_{\bar\delta}}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx. \tag{7.24}$$

This means in particular that

$$\int_{Q_{\hat\delta}}|\mathcal{E}\hat v_h|^2dx\le\frac{1}{\eta}<+\infty. \tag{7.25}$$

Without loss of generality we can assume that $\int_{Q_{\hat\delta}\setminus I}\hat v_h dx=0$; therefore, since $J_{\hat v_h}\subset I$, (7.25) and the Korn Inequality imply that $||\hat v_h||_{H^1(Q_{\hat\delta}\setminus I)}\le c$. Hence, there exists $\hat v\in H^1(Q_{\hat\delta}\setminus I;\mathbb{R}^n)$ such that $\hat v_h\rightharpoonup\hat v$ weakly in H^1 and, in particular, strongly in L^2, in $Q_{\hat\delta}\setminus I$.

Now we claim that, for every $B\subset\subset Q_{\hat\delta}\setminus I$, we have the following convergence result:

$$\mathcal{E}\hat v_h\to\mathcal{E}\hat v\quad\text{strongly in } L^2(B;\mathbb{M}^{n\times n}_{sym}). \tag{7.26}$$

Indeed, the local minimality of $\hat v_h$ in $Q_{\hat\delta}$ implies that

$$\int_{Q_{\hat\delta}\setminus I}\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\phi\,dx=0\quad\text{for every }\phi\in H^1_0(Q_{\hat\delta}\setminus I;\mathbb{R}^n). \tag{7.27}$$

Then, choosing as test function $\phi=\psi\,(\hat v_h-\hat v)$, with $\psi\in C^1_0(Q_{\hat\delta}\setminus I)$, we obtain

$$\int_{Q_{\hat\delta}\setminus I}\psi\,\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v_h dx=\int_{Q_{\hat\delta}\setminus I}\psi\,\mathbb{C}\mathcal{E}\hat v_h:\mathcal{E}\hat v\,dx-\int_{Q_{\hat\delta}\setminus I}\mathbb{C}\mathcal{E}\hat v_h:\big((\hat v_h-\hat v)\nabla\psi\big)dx.$$

Since $\hat{v}_h \rightharpoonup \hat{v}$ weakly in $H^1(Q_{\hat{\delta}} \setminus I;\mathbb{R}^n)$, if we let $h \to +\infty$ in the previous equation we get

$$\lim_{h\to+\infty} \int_{Q_{\hat{\delta}}\setminus I} \psi\, \mathbb{C}\mathcal{E}\hat{v}_h : \mathcal{E}\hat{v}_h dx = \int_{Q_{\hat{\delta}}\setminus I} \psi\, \mathbb{C}\mathcal{E}\hat{v} : \mathcal{E}\hat{v}\, dx. \tag{7.28}$$

Finally, (7.28) together with the weak convergence of the sequence $\hat{v}_h$ in $H^1(Q_{\hat{\delta}} \setminus I;\mathbb{R}^n)$ imply that $\mathcal{E}\hat{v}_h$ converges strongly to $\mathcal{E}\hat{v}$ with respect to the norm induced on L^2 by the tensor $\mathbb{C}$ introduced in (3.4) and (3.5). The equivalence of this norm to the standard L^2 norm gives (7.26).

By the strong convergence of $\hat{v}_h$ to $\hat{v}$ in L^2, (7.26) and the Korn Inequality, we deduce

$$\hat{v}_h \to \hat{v} \quad \text{strongly in } H^1(B;\mathbb{R}^n).$$

This entails the convergence of the traces of $\hat{v}_h$ on $\partial Q_{\bar{\delta}}$, that is,

$$\varphi_h := (\hat{v}_h)_{|\partial Q_{\bar{\delta}}} \to \varphi := (\hat{v})_{|\partial Q_{\bar{\delta}}} \quad \text{strongly in } H^{1/2}(\partial Q_{\bar{\delta}};\mathbb{R}^n). \tag{7.29}$$

At this point, let us consider the following problems:

$$(\mathrm{Eul})_{\varphi_h} \quad \begin{cases} \operatorname{div} \mathbb{C}\mathcal{E}w = 0 & \text{in } Q_{\bar{\delta}} \\ w = \varphi_h & \text{on } \partial Q_{\bar{\delta}}, \end{cases} \qquad (\mathrm{Eul})_{\varphi} \quad \begin{cases} \operatorname{div} \mathbb{C}\mathcal{E}w = 0 & \text{in } Q_{\bar{\delta}} \\ w = \varphi & \text{on } \partial Q_{\bar{\delta}}. \end{cases}$$

Clearly, $\tilde{v}_h$ is the solution to $(\mathrm{Eul})_{\varphi_h}$ for every h. Let us call $\tilde{v}$ the solution to $(\mathrm{Eul})_{\varphi}$. From (7.29) it turns out that $\tilde{v}_h \to \tilde{v}$ strongly in $H^1(Q_{\bar{\delta}};\mathbb{R}^n)$, hence,

$$1 = \lim_{h\to+\infty} \int_{Q_{\bar{\delta}}} \mathbb{C}\mathcal{E}\tilde{v}_h : \mathcal{E}\tilde{v}_h dx = \int_{Q_{\bar{\delta}}} \mathbb{C}\mathcal{E}\tilde{v} : \mathcal{E}\tilde{v}\, dx. \tag{7.30}$$

We notice that the functions $\hat{v}_h$ are absolute minimisers for the functional $\mathcal{G}^h_{\varphi_h}$ defined in (7.15), by definition of local minimality. Therefore Lemma 7.4 ensures the L^2 convergence of $(\hat{v}_h)$ to the only minimiser of the functional $\mathcal{G}_\varphi$, that is exactly $\tilde{v}$, and the convergence of the energies. Now, if we let $h \to +\infty$ in (7.24) we obtain

$$1 = \int_{Q_{\bar{\delta}}} \mathbb{C}\mathcal{E}\tilde{v} : \mathcal{E}\tilde{v}\, dx \geq (1+\eta) \int_{Q_{\bar{\delta}}} \mathbb{C}\mathcal{E}\tilde{v} : \mathcal{E}\tilde{v}\, dx,$$

which gives a contradiction, therefore (7.20) is proved.

Let $\eta > 0$ be fixed and let $\beta > 0$ be such that the property (7.20) is satisfied. We are going to define a function $\hat{v}^\varepsilon$ by means of a minimisation problem similar to (LMin). More precisely, for every $\varepsilon > 0$ we consider the problem

$$(\mathrm{Min})^\varepsilon \quad \min\Big\{ \int_{Q_{\hat{\delta}}} \mathbb{C}\mathcal{E}w : \mathcal{E}w\, dx + \mathcal{H}^{n-1}(J_w) \quad : w \in SBD^2(Q_{\hat{\delta}}), J_w \subset I, \mathcal{H}^{n-1}(J_w) \leq \beta, \\ [w]\cdot\nu_w \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_w, w = v^\varepsilon \text{ on } \partial Q_{\hat{\delta}} \Big\}.$$

For a minimiser $\hat{v}^\varepsilon$ in $(\mathrm{Min})^\varepsilon$, let us consider the corresponding $\tilde{v}^\varepsilon$ defined by (Eul), with $\hat{v}$ replaced by $\hat{v}^\varepsilon$. As before

$$\int_{Q_{\hat{\delta}}} \mathbb{C}\mathcal{E}\tilde{v}^\varepsilon : \mathcal{E}\tilde{v}^\varepsilon dx \leq (1+\eta) \int_{Q_{\hat{\delta}}} \mathbb{C}\mathcal{E}\hat{v}^\varepsilon : \mathcal{E}\hat{v}^\varepsilon dx. \tag{7.31}$$

We extend the functions $\hat{v}^\varepsilon$ and $\tilde{v}^\varepsilon$ to the whole cube Q setting $\hat{v}^\varepsilon = \tilde{v}^\varepsilon = v^\varepsilon$ in $Q \setminus Q_{\hat{\delta}}$, where v^ε is the function in (7.19). This procedure can be repeated for every $k \in \{N_b(\varepsilon)+1,\ldots,N(\varepsilon)\}$ and leads to the definition of functions $(\hat{v}^\varepsilon_k)$ and $(\tilde{v}^\varepsilon_k)$ for $k \in \{N_b(\varepsilon)+1,\ldots,N(\varepsilon)\}$. Hence, for $k \in \{N_b(\varepsilon)+1,\ldots,N(\varepsilon)\}$

$$\begin{aligned} \int_{Q_k} \mathbb{C}\mathcal{E}v^\varepsilon : \mathcal{E}v^\varepsilon dx + \mathcal{H}^{n-1}(J_{v^\varepsilon} \cap Q_{\hat{\delta}}) &\geq \int_{Q_k} \mathbb{C}\mathcal{E}\hat{v}^\varepsilon_k : \mathcal{E}\hat{v}^\varepsilon_k dx + \mathcal{H}^{n-1}(J_{\hat{v}^\varepsilon} \cap Q_{\hat{\delta}}) \\ &\geq \Big(1 - \frac{\eta}{1+\eta}\Big) \int_{Q_k} \mathbb{C}\mathcal{E}\tilde{v}^\varepsilon_k : \mathcal{E}\tilde{v}^\varepsilon_k dx. \end{aligned} \tag{7.32}$$

Now, for $k \in \{N_b(\varepsilon)+1,\dots,N(\varepsilon)\}$, we define $\tilde u^\varepsilon_k$ as $\tilde u^\varepsilon_k(\varepsilon\, y) := \sqrt{\alpha_\varepsilon \varepsilon}\; \tilde v^\varepsilon_k(y)$. By (7.19) and (7.32) we obtain

$$\int_{Q^\varepsilon_k} \mathbb{C}\mathcal{E}u^\varepsilon : \mathcal{E}u^\varepsilon dx + \alpha_\varepsilon \mathcal{H}^{n-1}\big(J_{u^\varepsilon}\cap Q^\varepsilon_k\big) \geq \left(1-\frac{\eta}{1+\eta}\right)\int_{Q^\varepsilon_k} \mathbb{C}\mathcal{E}\tilde u^\varepsilon_k : \mathcal{E}\,\tilde u^\varepsilon_k dx. \tag{7.33}$$

Third step: estimate on boundary cubes. Let $\Omega_0 \supset \Omega$ be such that $\operatorname{dist}(\Omega,\partial\Omega_0) > 1$. To simplify the notation we define the set $B_Q(\varepsilon)$ and $B_I(\varepsilon)$ of *boundary cubes* and *boundary inclusions*, respectively, as follows:

$$B_Q(\varepsilon) := \bigcup_{k=N(\varepsilon)+1}^{N(\varepsilon)+N_r(\varepsilon)} Q^\varepsilon_k, \quad B_I(\varepsilon) := \bigcup_{k=N(\varepsilon)+1}^{N(\varepsilon)+N_r(\varepsilon)} I^\varepsilon_k.$$

We extend the sequence (u^ε) to Ω_0 simply setting $u^\varepsilon = 0$ in $\Omega_0\setminus\Omega$. Notice that, since by assumption $\operatorname{tr}(u^\varepsilon) = 0$ on $\partial\Omega$, this trivial extension does not introduce any additional jump set. Let $(\hat u^\varepsilon)$ the sequence defined in Theorem 4.1. Then we have

$$\mathcal{F}^\varepsilon(u^\varepsilon, B_Q(\varepsilon)) = \mathcal{F}^\varepsilon(u^\varepsilon,(\Omega_0\setminus\Omega)\cup B_Q(\varepsilon)) \geq \int_{(\Omega_0\setminus\Omega)\cup(B_Q(\varepsilon)\setminus B_I(\varepsilon))} \mathbb{C}\mathcal{E}\hat u^\varepsilon : \mathcal{E}\hat u^\varepsilon dx. \tag{7.34}$$

Fourth step: final estimate. Let us define the new sequence $w^\varepsilon \in SBD^2_0(\Omega_0)$ as

$$w^\varepsilon := \begin{cases} \hat u^\varepsilon_k & \text{in } Q^\varepsilon_k, \text{ for } k\in\{1,\dots,N_b(\varepsilon)\},\\ \tilde u^\varepsilon_k & \text{in } Q^\varepsilon_k, \text{ for } k\in\{N_b(\varepsilon)+1,\dots,N(\varepsilon)\},\\ \check u^\varepsilon & \text{in } (\Omega_0\setminus\Omega)\cup B_Q(\varepsilon).\end{cases}$$

Notice that $w^\varepsilon \in H^1_0(\Omega_0;\mathbb{R}^n)$ and that $w^\varepsilon \in H^1_0(\Omega';\mathbb{R}^n)$ for every $\overline{\Omega}\subset\Omega'\subset\Omega_0$.

Define also the function $a^\varepsilon : \Omega_0 \to \mathbb{R}$ as

$$a^\varepsilon(x) := \begin{cases} 0 & \text{in } \left(\bigcup_{k=1}^{N_b(\varepsilon)} I^\varepsilon_k\right)\cup B_I(\varepsilon),\\ 1 & \text{otherwise in } \Omega_0.\end{cases}$$

From what we proved in the previous steps we can write

$$\mathcal{F}^\varepsilon(u^\varepsilon,\Omega) = \mathcal{F}^\varepsilon(u^\varepsilon,\Omega_0) \geq \left(1-\frac{\eta}{1+\eta}\right)\int_{\Omega_0} a^\varepsilon(x)\,\mathbb{C}\mathcal{E}w^\varepsilon : \mathcal{E}w^\varepsilon dx. \tag{7.35}$$

It remains to apply Lemma 7.1 to (7.35). First of all we show the convergence of a^ε. We have

$$\int_\Omega |a^\varepsilon - 1|\,dx = \mathcal{L}^n\left(\left(\bigcup_{k=1}^{N_b(\varepsilon)} I^\varepsilon_k\right)\cup B_I(\varepsilon)\right) = (N_b(\varepsilon)+N_R(\varepsilon))\varepsilon^n\mathcal{L}^n(I) \leq c\,\frac{\varepsilon}{\alpha_\varepsilon},$$

hence $a^\varepsilon \to 1$ strongly in $L^1(\Omega_0)$. Once we prove that $w^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_0;\mathbb{R}^n)$ and that $u_{|\Omega} \in H^1_0(\Omega;\mathbb{R}^n)$, as $u=0$ in $\Omega_0\setminus\Omega$, it turns out that

$$\liminf_{\varepsilon\to0}\mathcal{F}^\varepsilon(u^\varepsilon) \geq \left(1-\frac{\eta}{1+\eta}\right)\int_{\Omega_0}\mathbb{C}\mathcal{E}u:\mathcal{E}u\,dx = \left(1-\frac{\eta}{1+\eta}\right)\int_\Omega \mathbb{C}\mathcal{E}u:\mathcal{E}u\,dx,$$

and the thesis follows letting η converge to zero.

Fifth step: convergence of w^ε. First of all it is clear that the sequence $(w^\varepsilon)\subset H^1(\Omega_0;\mathbb{R}^n)$ converges weakly in H^1, as $||\mathcal{E}w^\varepsilon||_{(L^2(\Omega_0))^{n\times n}} \leq c$. Let us denote by w its weak limit; the fact that $u=v$ a.e. on Ω follows from a similar argument to the one used in the proof of Lemma 4.1. Moreover, since $w^\varepsilon \in H^1_0(\Omega';\mathbb{R}^n)$ for every $\overline{\Omega}\subset\Omega'\subset\Omega_0$, then $u\in H^1_0(\Omega;\mathbb{R}^n)$. □

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References

[1] Acerbi E., Chiadò Piat V., Dal Maso G., Percivale D.: An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal.*, **18**/5 (1992), 481–496.

[2] Ambrosio L., Coscia A., Dal Maso G.: Fine properties of functions with bounded deformation. *Arch. Ration. Mech. Anal.*, **139** (1997), 201–238.

[3] Ambrosio L., Fusco N., Pallara D.: Functions of bounded variations and Free Discontinuity Problems. Clarendon Press, Oxford, 2000.

[4] Attouch H.: Variational Convergence for Functions and Operators. Pitman Advanced Publishing Program, Boston, 1984.

[5] Barchiesi M., Dal Maso G.: Homogenization of fiber reinforced brittle materials: the extremal cases. *SIAM J. Math. Anal.*, accepted for publication.

[6] Bellettini G., Coscia A., Dal Maso G.: Compactness and lower semicontinuity properties in $SBD(\Omega)$. *Math. Z.*, **228** (1998), 337–351.

[7] Braides A., Defranceschi A.: Homogenization of Multiple Integrals. Oxford University Press, New York, 1998.

[8] Braides A., Defranceschi A., Vitali E.: Homogenization of Free Discontinuity Problems. *Arch. Ration. Mech. Anal.*, **135** (1996), 297–356.

[9] Braides A., Truskinovsky L.: Asymptotic expansion via Γ-convergence. *Cont. Mech. Therm.*, **20** (2008), 21–62.

[10] Chambolle A.: An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.*, **83**/7 (2004), 929–954.

[11] Chambolle A., Giacomini A., Ponsiglione M.: Crack initiation in elastic bodies. *Arch. Ration. Mech. Anal.*, **188**/2 (2008), 309–349.

[12] Charlotte M., Laverne J., Marigo J.-J.: Initiation of cracks with cohesive force models: a variational approach. *Eur. J. Mech. A/Solids*, **25** (2006), 649–669.

[13] Cioranescu D., Donato P.: An Introduction to Homogenization. Oxford University Press, New York, 1999.

[14] Dal Maso G.: An introduction to Γ-convergence. Birkhäuser, Boston, 1993.

[15] Dal Maso G., Modica L.: A general theory of variational functionals. Topics in functional analysis, 1980–81, Quaderni, Scuola Norm. Sup. Pisa, Pisa, 1981, 149–221.

[16] Dal Maso G., Morel J.M., Solimini S.: A variational method in image segmentation: Existence and approximation results. *Acta Mathematica*, **168** (1992), 89–151.

[17] Dal Maso G., Zeppieri C.I.: Homogenization of fiber-reinforced brittle materials: The intermediate case. *Adv. Calc. Var.*, accepted for publication.

[18] Griffith A.: The phenomena of rupture and flow in solids. *Philos. Trans. Roy. Soc. London Ser. A*, **221** (1920), 163–198.

[19] El Hajji, M.: Homogenization of linearized elasticity systems with traction condition in perforated domains. *Electronic Journal of Differential Equations*, **1999**/41 (1999), 1–11.

[20] Oleinik O.A., Shamaev A.S., Yosifian G.A.: Mathematical problems in elasticity and homogenization. North-Holland, Amsterdam, 1992.

[21] Scardia, L.: Damage as Γ-limit of microfractures in anti-plane linearized elasticity. *Math. Models Methods Appl. Sci.*, **18**/10 (2008), 1703–1740.

[22] Temam, R.: Problemes mathematiques en plasticité. Paris, Gauthier-Villars, 1983.

[23] Truskinovsky L.: Fracture as a phase transition. *Contemporary research in the mechanics and mathematics of materials* (R.C. Batra and M.F. Beatty eds.) CIMNE, Barcelona (1996), 322–332.

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