Small Strain Heterogeneous Elasto-Plasticity Revisited

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Note added in proof. In the context of Subsection 1.2 below, the definition of a *geometrically admissible multiphase domain* has to be completed with the following definition: the domain is a C^2-*geometrically admissible multiphase domain* if the interface $\Gamma \setminus S$ is a C^2-hypersurface, and not simply a C^1-hypersurface. Then, the assumption that the domain is a *geometrically admissible multiphase domain* should be replaced by the assumption that the domain is a C^2-*geometrically admissible multiphase domain* in all Theorems, Lemmata and Propositions of Sections 3, 4, except for Lemma 3.8 which stands as is. This is so that, in Theorem 3.5, the trace $(\sigma_D \nu)_\tau(x)$ at a point x on the interface $\Gamma \setminus S$, be defined independently of the approximation sequence $\{\sigma_n\}_n$ of σ (see the proof of that theorem). With this modification at hand, all other proofs remain unchanged.

1 Introduction

1.1 Introductory remarks

Small strain elasto-plasticity is an old and respectable theory in solid mechanics and it has been the topic of many scholarly books. Two landmark books are those of R. Hill [6] and of J. Lubliner [9]. It is a central theme in many fields of engineering and its offspring, limit analysis, is the bread and butter of the structural and soil engineers.

The modern mathematical treatment of plasticity finds its roots in the work P.-M. Suquet (see e.g. [15],[16]). This was completed by various works of R. Temam (see e.g. [18]) and of R.V. Kohn and R. Temam (see [8]). That last paper focusses on the duality, or apparent lack thereof, between stress fields (denoted by σ), or, more precisely, at least in perfect elasto-plasticity, between the deviatoric part σ_D of σ, and plastic strains (denoted by p). The (time derivative of the) plastic strain p is merely a measure, while the stress field σ_D is typically not continuous, so that their product is not a priori meaningful. But that product plays an important role in the analysis of the problem; it also represents the mechanical dissipation.

In any case, the interest of the mathematical community then subsided for about twenty years until G. Dal Maso, A. De Simone and M. G. Mora [3] revisited the existence of a quasi-static evolution, originally established by P.-M. Suquet, within the framework of the rapidly expanding variational theory of rate independent evolutions (see e.g. [11]).

That paper took the view that, in the absence of inertia, homogeneous elasto-plastic evolutions could be seen as a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation; see Section 2 for details. The minimizing triplet(s) in displacement $u(t)$, elastic strain $e(t)$, and plastic strain $p(t)$ should also be such that an energy conservation statement, amounting to a kind of first principle in thermodynamics, is satisfied throughout the evolution. The existence argument uses a mix of purely variational techniques and of weak convergence methods applied to the corresponding Euler-Lagrange equations, that is the equilibrium equations together with the stress admissibility constraint which forces the deviatoric part of the stress field $\sigma_D(t)$ to remain in the (compact) set K of admissible stresses.

In that paper and its sequels, one of the difficulties consists in showing that the obtained evolutions satisfy the more classical evolution laws envisioned in earlier works. In particular, the so-called flow rule states that, whenever the stress reaches the boundary of its admissible set, the plastic strain should flow in the direction normal to that set. For this, as well as for parts of the existence proof, the duality evoked earlier plays an essential role.

Following in the footstep of [3], F. Solombrino proposed in [13] an extension of the analysis to the case of heterogeneous materials. There, duality becomes even more cumbersome because of the spatial dependence of the dissipation functional. In particular, if the multi-function $x \multimap K(x)$, where $K(x)$ is the set of admissible stresses at x, is not continuous, then it becomes difficult to come up with a set of satisfactory conditions that will ensure the success of the method developed in [3]. In that paper, the case of a multi-phase material is discussed; the necessary assumptions on the map $x \multimap K(x)$ (see [13, Relations (3.2)-(3.9)]) are somewhat unnatural, except in very special cases like that where each phase exhibits a Von-Mises type behavior (see e.g. [9, Section 3.1]).

In this paper, we propose to revisit the heterogeneous case under more general, and perhaps natural assumptions. We pay close attention to duality and attempt to circumscribe its impact to those steps where it is truly needed. To do so, we revisit duality in Section 6 and show that all needed results can be derived for a Lipschitz domain, and not only for C^2-domains, the required regularity if one follows [8]. Actually, in all fairness, we do need further assumptions on the relative boundary of the "Dirichlet" part of the boundary, whenever a "Neumann" part is also present (see (6.20)). But this is to be expected, because it is the case even for the simplest elliptic problem in a similar setting. In Section 6 duality is defined globally, in contrast to what has been done up till now. Indeed, we do not need to define separately the duality in the interior of the domain and on its Dirichlet boundary. We include this as a last, rather than as a first section, because a reader familiar with the "classical" duality expounded in [8] and [3] could skip Section 6 without prejudice, provided that she is willing to replace the assumption of a Lipschitz boundary by that of a C^2-boundary.

Besides Section 6, the paper is organised as follows.

Section 2 is devoted to the derivation of the existence of a quasi-static evolution for a multi-phase composite (see Theorem 2.7). To that effect, we carefully dissect the correct stress admissibility assumptions for a multi-phase composite at the onset of the section. Essentially, besides smooth enough interfaces (piecewise C^1), we also require the following: at each point of the interface between two phases, say phases 1 and 2, and for each value of the plastic strain, the dissipation potential is the pointwise in space inf-convolution of that in either phase, but this for deviatoric matrices of the form $a \odot \nu$ only, where ν is the normal to the interface. This corresponds to choosing as admissibility set for the tangential part $(\sigma\nu)_\tau$ of the normal stresses $\sigma\nu$ to the interface the intersection of the admissibility sets $(K_1\nu)_\tau$ and $(K_2\nu)_\tau$, where K_i is the admissibility set for phase i. Such a condition seems to be known in limit analysis, although the only reference that we could locate is [17, Paragraph 5.4, Page 334]. To our knowledge, it has never been reconciled with elasto-plastic evolution. In particular, that condition is *not* akin to taking the intersection of the sets K_i on the boundary, which would produce a dissipation which is not maximal in the sense of Hill (see below). Also, there is no need to impose some kind of ordering of the admissible sets of the various phases. In the latter case the interface can be much rougher (e.g. Lipschitz) and we recover the result of [13] established for C^2-interfaces; see Remark 5.2.

The existence result is then contingent upon a lower semi-continuity result (Proposition 2.3) for the dissipation which is proved in Section 5. The required lower semi-continuity is carefully tailored to the kinematic structure of elasto-plasticity and not a blanket lower semi-continuity à la Reshetnyak. No reference is made to duality in Section 2, and this is to be contrasted with all prior existence results known to us.

In Section 3, we strive to recover a more classical evolution from that derived in the previous section. It is easily shown that the quasi-static evolution obtained in Section 2 satisfies equilibrium, the natural boundary conditions, as well as the admissibility constraint of a classical evolution (see Theorem 3.6). Note that the situation will be different in the case of homogenization as discussed in [5]. In the course of that investigation, we uncover an admissibility constraint on the interfaces (as well as on the Dirichlet boundary of the domain) which seems to be lacking in the existing literature; see Theorem 3.5.

Subsection 3.2 demonstrates that we can recover the classical flow rule (see Theorem 3.13). Once again, we also derive a flow rule at the interfaces and on the Dirichlet part of the boundary which seems to be a missing ingredient in the mechanics literature on elasto-plasticity (see Theorem 3.13 again). We also recover in Theorem 3.12 the maximality of the plastic work during the evolution. This, which is often called Hill's principle [6], is a usual statement in plasticity theory and it is a direct consequence of the flow rule once one agrees on the correct definition of plastic work; but that is in turn what the duality between stresses and plastic strains is about.

Note that Hill's principle actually contains more information than the derived flow rules. This is so because the latter only activate the Lebesgue-absolutely continuous part of the plastic strain rate, or the interfacial/boundary plastic strain rate. In any case, Theorem 3.13 is the best one can hope for in the absence of spatial regularity of the stress field; see Remark 3.15.

Section 4 demonstrates that our assumptions on the dissipation potential at the interfaces are the natural ones from the standpoint of small hardening. Indeed, introduce a vanishingly small amount of linear isotropic hardening in the model. The resulting evolution is smoothed out by the hardening, resulting in a plastic strain that cannot concentrate, so that the value of the dissipation potential along the interfaces is irrelevant. Yet, the zero-hardening limit is proved in Theorem 4.5 to coincide with the elasto-plastic evolution established in the previous sections. But the proof strongly uses Proposition 3.9 which in turn does not hold true unless the dissipation potential is exactly of the announced form as emphasized in Remark 3.10.

As alluded to before, Section 5 is devoted to the proof of the lower semi-continuity of the dissipation potential; see Proposition 2.3.

Finally, the present work provides what we believe to be a sound foundation for a study of the periodic homogenization of elasto-plastic evolutions. This is the topic of [5].

1.2 Preliminaries

Here, we detail the mathematical notation, as well as a few mathematical remarks that will be of relevance. We also detail the geometry of the multi-phase material and the loads that are applied.

Throughout the paper, we refer to [2] for background material, especially concerning finer measure theoretical points. We do so because of our familiarity with that reference, which is not to say that those results cannot be found elsewhere. In particular, [4, Chapter 6] contains many of the needed results with simpler proofs at times.

General notation. For $A \subseteq \mathbb{R}^N$, χ_A denotes the characteristic function of A, *i.e.*, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. The indicator function of A, denoted by $\mathbb{I}_A$, is defined as $\mathbb{I}_A(x) = 0$ for $x \in A$, and $\mathbb{I}_A(x) = +\infty$ for $x \notin A$. For $B \subseteq \mathbb{R}^N$, the symbol $A \subset\subset B$ means that the closure of A is compact and contained in B. Finally the symbol $\lfloor_A$ stands for "restricted to A".

Matrices. We denote by $\mathrm{M}^N_{\mathrm{sym}}$ the set of $N \times N$-symmetric matrices and by M^N_D the set of trace-free elements of $\mathrm{M}^N_{\mathrm{sym}}$. The identity matrix in $\mathrm{M}^N_{\mathrm{sym}}$ is denoted by i. If M is an element of $\mathrm{M}^N_{\mathrm{sym}}$, then M_D denotes its deviatoric part, *i.e.*, its projection onto the subspace M^N_D of $\mathrm{M}^N_{\mathrm{sym}}$ orthogonal to i for the Frobenius inner product. The symbol $\cdot$ denotes that inner product. We denote by $\mathscr{L}_s(\mathrm{M}^N_D)$ the set of symmetric endomorphisms on M^N_D. For $a, b \in \mathbb{R}^N$, $a \odot b$ stands for the symmetric matrix such that $(a \odot b)_{ij} := (a_i b_j + a_j b_i)/2$.

Depending on the context, we will denote by $B(x,r)$ the open ball of center x and radius r in $\mathbb{R}^N$, or that in M^N_D.

Measures. If $E \subseteq \mathbb{R}^N$ is locally compact and Y a finite dimensional normed space, $\mathscr{M}_b(E;Y)$ will denote the space of finite Radon measures with values in Y. For $\mu \in \mathscr{M}_b(E;Y)$, we denote by $|\mu|$ its total variation. The space $\mathscr{M}_b(E;Y)$ is the topological dual of $C^0_c(E;Y^*)$, the set of continuous functions u from E to the vector dual Y^* of Y which "vanish at the boundary", *i.e.*, such that for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|u(x)| < \varepsilon$ for $x \notin E$. Besides the associated weak* convergence, we also use strict convergence. We say that

$$\mu_n \xrightarrow{s} \mu \quad \text{strictly in } \mathscr{M}_b(E;Y)$$

iff

$$\mu_n \xrightharpoonup{*} \mu \text{ weakly}^* \text{ in } \mathscr{M}_b(E;Y) \text{ and } |\mu_n|(E) \to |\mu|(E).$$

Functional spaces. Given $E \subseteq \mathbb{R}^N$ measurable, $1 \le p < +\infty$, and Y a finite dimensional normed space, $L^p(E;Y)$ stands for the space of p-summable functions on E with values in Y, with associated norm denoted by $\|\cdot\|_p$. Given $A \subseteq \mathbb{R}^N$ open, $H^1(A;Y)$ is the Sobolev space of functions in $L^2(A;Y)$ with distributional derivatives in L^2.

Finally, let X be a normed space. We denote by $BV(a,b;X)$ and $AC(a,b;X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a,b]$ to X, respectively. We recall that the total variation of $f \in BV(a,b;X)$ is defined as

$$\mathscr{V}_X(f;a,b) := \sup\left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \, : \, a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

Without detailing at this point the geometry of our problem, we note that, throughout the paper, $\Omega \subset \mathbb{R}^N$ will refer to an open bounded domain with (at least) Lipschitz boundary. The following two spaces will play an essential role.

The (kinematic) space BD. In this paper as in previous works on elasto-plasticity the displacement field u lies in $BD(\Omega)$, the space of functions of bounded deformations. We refer the reader to e.g. [18, Chapter 2], and [1] for background material.

Besides elementary properties of $BD(\Omega)$, we will only appeal to two "finer" results. Firstly, Poincaré-Korn's inequality states in particular that, given $\Gamma_d \subseteq \partial\Omega$ with $\mathscr{H}^{N-1}(\Gamma_d) > 0$, there exists $C > 0$, such that

$$\|u\|_{BD(\Omega)} \le C \left(\int_{\Gamma_d} |u| \, d\mathscr{H}^{N-1} + \|Eu\|_{\mathscr{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}})} \right),$$

where Eu denotes the symmetrized gradient of u, and the integral on Γ_d involves the trace of u on $\partial\Omega$ which is well defined as an element of $L^1(\partial\Omega;\mathbb{R}^N)$; see [18, Chapter 2, Remark 2.5(ii)]. Secondly, the measure Eu does not charge $\mathscr{H}^{N-1}$-negligible sets; see [1, Remark 3.3].

We say that

$$u_n \stackrel{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BD(\Omega)$$

iff

$$u_n \to u, \text{ strongly in } L^1(\Omega;\mathbb{R}^N) \text{ and } Eu_n \stackrel{*}{\rightharpoonup} Eu \text{ weakly}^* \text{ in } \mathscr{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}}).$$

Bounded sequences in $BD(\Omega)$ always admit a weakly* converging subsequence. Finally, we will use the fact that if $u_n \stackrel{*}{\rightharpoonup} u$ weakly* in $BD(\Omega)$, and $Eu_n \stackrel{s}{\to} Eu$ strictly in $\mathscr{M}_b(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, then strong convergence of the traces holds, *i.e.*, $u_n \to u$ strongly in $L^1(\partial\Omega;\mathbb{R}^N)$; see [18, Chapter 2, Theorem 3.1].

The (static) space Σ. It is defined as

$$\Sigma := \left\{ \sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) : \mathrm{div}\,\sigma \in L^2(\Omega;\mathbb{R}^N) \text{ and } \sigma_D \in L^\infty(\Omega;\mathbb{R}^N) \right\}.$$

It is classical that, if $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ with div $\sigma \in L^2(\Omega;\mathbb{R}^N)$, $\sigma\nu$ is well defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$, ν being the outer normal to $\partial\Omega$.

More generally, consider an arbitrary Lipschitz subdomain $A \subset \Omega$ with outer normal ν, and $\Delta \subset \partial A$ open in the relative topology. We can define the restriction of $\sigma\nu$ "on Δ" by testing against functions in $H^{1/2}(\partial A;\mathbb{R}^N)$ with compact support in Δ. This amounts to viewing $\sigma\nu$ as an element of the dual to $H^{1/2}_{00}(\Delta;\mathbb{R}^N)$.

If $\sigma \in \Sigma$, then, in the spirit of [8, Lemma 2.4], we can define a tangential component $(\sigma\nu)_\tau$ of $\sigma\nu$ on Δ such that

$$(\sigma\nu)_\tau \in L^\infty(\Delta;\mathbb{R}^N) \qquad \text{with} \qquad \|(\sigma\nu)_\tau\|_\infty \le \|\sigma_D\|_\infty.$$

Indeed, consider any regularization $\sigma_n \in C^\infty(\bar{A};\mathrm{M}^N_{\mathrm{sym}})$ of σ on $\bar{A}$ such that

$$\begin{cases} \sigma_n \to \sigma & \text{strongly in } L^2(A;\mathrm{M}^N_{\mathrm{sym}}) \\ \mathrm{div}\sigma_n \to \mathrm{div}\sigma & \text{strongly in } L^2(A;\mathbb{R}^N) \\ \|(\sigma_n)_D\|_\infty \le \|\sigma_D\|_\infty. & \end{cases} \tag{1.1}$$

Define the tangential component $(\sigma_n\nu)_\tau := (\sigma_n)\nu - ((\sigma_n)\nu \cdot \nu)\nu$. It is readily seen that $(\sigma_n\nu)_\tau = ((\sigma_n)_D\nu)_\tau$ (the tangential component of $(\sigma_n)_D$ is defined analogously). Since $x \mapsto \nu(x)$ is an $L^\infty(\Delta;\mathbb{R}^N)$-mapping, there exists an $L^\infty(\Delta;\mathbb{R}^N)$-function $(\sigma\nu)_\tau$ such that, up to a subsequence,

$$(\sigma_n\nu)_\tau \stackrel{*}{\rightharpoonup} (\sigma\nu)_\tau \text{ weakly}^* \text{ in } L^\infty(\Delta;\mathbb{R}^N). \tag{1.2}$$

If $\sigma_D \equiv 0$ then, clearly, $(\sigma\nu)_\tau \equiv 0$, so that $(\sigma\nu)_\tau$ is only a function of $(\sigma_n)_D$ which we will denote henceforth by $(\sigma_D\nu)_\tau$. Notice that $(\sigma_D\nu)_\tau$ may depend upon the approximation sequence σ_n (or at least upon $(\sigma_n)_D$).

If Δ is a C^2-hypersurface, then $(\sigma_D\nu)_\tau$ is uniquely determined as an element of $L^\infty(\Delta;\mathbb{R}^N)$. Indeed, for every $\varphi \in H^{1/2}(\partial A;\mathbb{R}^N)$ with support compactly contained in Δ (that is by density $\varphi \in H^{1/2}_{00}(\Delta;\mathbb{R}^N)$),

$$\int_\Delta (\sigma\nu)_\tau \cdot \varphi \, d\mathscr{H}^{N-1} = \langle \sigma\nu, \varphi\rangle - \langle (\sigma\nu)_\nu, \varphi\rangle,$$

where

$$\langle (\sigma\nu)_\nu, \varphi\rangle := \langle \sigma\nu, (\varphi\cdot\nu)\nu\rangle.$$

Since the normal component $(\varphi\cdot\nu)\nu$ of φ with respect to Δ belongs to $H^{1/2}(\partial A;\mathbb{R}^N)$ in view of the regularity of ν on Δ, the definition of $(\sigma\nu)_\nu$ is meaningful.

Geometrically admissible multiphase domains. We now detail the minimal geometric assumptions that will hold throughout the paper. As already stated, $\Omega \subseteq \mathbb{R}^N$ is an open, bounded set with (at least) Lipschitz boundary.

The Dirichlet part of the boundary $\Gamma_d \subseteq \partial\Omega$ is assumed to be a non empty *open* set in the relative topology of $\partial\Omega$ with boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ in $\partial\Omega$. We mostly refer to the complement of $\bar{\Gamma}_d$ in $\partial\Omega$ as Γ_t.

We further assume that Ω is made up of finitely many phases Ω_i, together with their interfaces. Those phases are pairwise disjoint open sets with Lipschitz boundary. We have $\overline{\Omega} = \cup\overline{\Omega}_i$ and denote by Γ the inner interfaces, *i.e.*,

$$\Gamma := \bigcup_{i,j} \partial\Omega_i \cap \partial\Omega_j \cap \Omega.$$

We finally assume the existence of a compact set $S \subset \Gamma$ with $\mathscr{H}^{N-1}(S) = 0$ and such that

$$\Gamma \setminus S \text{ is a } C^1\text{-hypersurface},$$

i.e., $\Gamma \setminus S$ is a C^1-submanifold of $\mathbb{R}^N$ of dimension $N-1$.

Finally, setting

$$S' := \{x \in \partial\Omega : x \in \partial\Omega_i \cap \partial\Omega_j \text{ for some } i, j\},$$

S' is taken to be compact and such that

$$\mathscr{H}^{N-1}(S') = 0.$$

We will write

$$\Gamma = \bigcup_{i\neq j} \Gamma_{ij},$$

where Γ_{ij} stands for the inner interface between Ω_i and Ω_j.

A domain Ω that satisfies the collection of those (minimal) assumptions will be referred to henceforth as a *geometrically admissible multiphase domain*.

Also, for notational convenience, we set $\mathscr{V}(0,t;f) := \mathscr{V}_{\mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)}(0,t;f)$ (the total variation of f on $[0,t]$ as a function with values in $\mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)$; see the paragraph "Functional spaces" above).

Loads. Throughout this paper, we assume that the body is only submitted to a hard device on Γ_d, that is that the only solicitation is a displacement field w applied to Γ_d. Specifically, we are given a field $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$. Its trace on Γ_d, still denoted by w, is in particular in $L^1(\Gamma_d;\mathbb{R}^N)$.

The remainder of the boundary Γ_t is traction free, and no body forces are present. This is of course a great simplification because the issue of safe loads, a somewhat delicate problem in elasto-plasticity is completely eschewed. We chose to do so not out of laziness, but because it is our belief that the essential features (and difficulties) of elasto-plasticity are preserved in the absence of loads. Doing so has additionally the arguable merit of greatly streamlining the mathematical arguments without major prejudice to the mathematical generality of the obtained results. The concerned reader is invited to heed Remark 2.9 below which briefly addresses the issue of safe loads.

However, note that both body forces and surface tractions are present, with appropriate regularity, in Section 6 which deals with the duality between stress and plastic strain. This is because there is no need there to refer to any kind of safe load condition and also because we wish to treat the duality in its full generality for future use.

2 Quasi-static evolution for multiphase composites

In this section, we propose to revisit and expand upon [13] where the existence of a quasi-static evolution for an elasto-plastic material, originally considered in [3], is extended to the case of a heterogeneous material, provided that certain restrictions are met by the admissible yield surfaces; see [13, Theorem 3.14]. The arguments presented in [13] are considerably simplified and lead to an existence result for a much larger class of yield surfaces and domains.

As already briefly mentioned in the introduction, by quasi-static evolution, we mean a globally minimizing conservative energetic path in the sense of [11]. The extent to which that notion of evolution amounts to a more classical elasto-plastic evolution will be the topic of Section 3. Also note that, as further elaborated upon in Remark 2.8, our proof of the existence of a quasi-static evolution is somewhat different from the usual proof found in e.g. [3] because it is purely variational.

Throughout this section, we consider a *geometrically admissible multiphase domain* and adopt the following

Definition 2.1 (Admissible configurations). $\mathscr{A}(w)$, the family of admissible configurations relative to $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$, is defined as

$$\mathscr{A}(w) := \big\{(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \times \mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D) : \\ Eu = e + p \text{ in } \Omega,\ p = (w-u) \odot \nu\, \mathscr{H}^{N-1} \text{ on } \Gamma_d\big\},$$

where ν denotes the outer normal to $\partial\Omega$ and u denotes the trace of u on $\partial\Omega$. We also define

$$\mathscr{P} := \{p \in \mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D) : \exists (u,e,w) \text{ with } (u,e,p) \in \mathscr{A}(w)\}$$

and

$$\mathscr{P}(w) := \{p \in \mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D) : \exists (u,e) \text{ with } (u,e,p) \in \mathscr{A}(w)\}. \tag{2.1}$$

Given $(u,e,p) \in \mathscr{A}(w)$, we have, for every $i \neq j$,

$$p\lfloor\Gamma_{ij} = (u^i - u^j) \odot \nu\, \mathscr{H}^{N-1}\lfloor\Gamma_{ij}, \tag{2.2}$$

where u^i, u^j are the traces of u on Γ_{ij} from Ω_i and Ω_j respectively, ν pointing from Ω_j to Ω_i. Since p takes its values in the space of deviatoric matrices, this implies that only plastic strains of the form $a \odot \nu$ with $a \perp \nu$ are activated along Γ by admissible configurations.

The constitutive properties of the material occupying Ω are as follows.

The elasticity tensor: At a.e. $x \in \Omega$, the elasticity tensor (the Hooke's law) is of the form

$$\mathbb{C}(x)M := \mathbb{C}_D(x)M_D + k(x)tr(M)\mathrm{i},$$

with $\mathbb{C}_D \in L^\infty(\Omega;\mathscr{L}_s(\mathrm{M}^N_D))$ and $k \in L^\infty(\Omega)$ such that

$$\begin{cases} c_1|M|^2 \le \mathbb{C}_D(x)M \cdot M \le c_2|M|^2 \text{ for every } M \in \mathrm{M}^N_D, \\ c_1 \le k(x) \le c_2, \end{cases} \tag{2.3}$$

for some $c_1, c_2 > 0$.

For every $e \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ we set

$$\mathscr{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}(x)e \cdot e\,dx.$$

The set of admissible stresses: In elasto-plasticity, the deviatoric part of the stress σ is assumed to be restricted by the yield condition. For heterogeneous materials, this means that, at a.e. $x \in \Omega$, there exists a convex compact set $K(x) \subset \mathrm{M}^N_D$ such that $\sigma_D(x) \in K(x)$. We further assume that those sets cannot be too small or too large, *i.e.*, there exist $c_3, c_4 > 0$ such that

$$B(0,c_3) \subset K(x) \subset B(0,c_4) \qquad \text{for a.e. } x \in \Omega. \tag{2.4}$$

Our formulation of the problem uses the Legendre transform of $\mathbb{I}_{K(x)}$, which is often referred to as the dissipation potential.

The dissipation potential: For a.e. $x \in \Omega$, we define the dissipation to be

$$H(x,\xi) := \sup\{\tau \cdot \xi : \tau \in K(x)\}. \tag{2.5}$$

Definition (2.5) produces, for a.e. $x \in \Omega$, a convex, one-homogeneous function in ξ which further satisfies

$$c_3|\xi| \leq H(x,\xi) \leq c_4|\xi| \qquad \text{for a.e. } x \in \Omega.$$

This is not however sufficient for our purpose because we need the dissipation potential to act upon the plastic strain (or plastic strain rate) which, being a measure, may concentrate on Lebesgue-negligible sets. Moreover, plastic strains can concentrate on the inner interfaces, activating only particular strain-directions. We thus have to extend H to $(\Omega \cup \Gamma_d) \times \mathrm{M}^N_D$. In doing so, we will avoid the imposition of any ordering between the admissible yield surfaces of the various phases, carefully defining H on the inner interfaces.

The dissipation potential $H : (\Omega \cup \Gamma_d) \times \mathrm{M}^N_D \mapsto [0,+\infty]$ of a *geometrically admissible multiphase domain* is constructed as follows.

(a) The dissipation potential in the phases: We take

$$H(x,\xi) = H_i(x,\xi) \ \text{ for } x \in \Omega_i$$

with H_i continuous on $\overline{\Omega}_i \times \mathrm{M}^N_D$ and also such that

$$\xi \mapsto H_i(x,\xi) \text{ is convex and positively one-homogeneous in } \xi \tag{2.6}$$

with

$$c_3|\xi| \leq H_i(x,\xi) \leq c_4|\xi|, \tag{2.7}$$

where $c_3, c_4 > 0$ are independent of the phase i.

(b) The dissipation potential on the inner interfaces: At a point $x \in \Gamma \setminus S$ on the interface between Ω_i and Ω_j such that the associated normal $\nu(x)$ points from Ω_j to Ω_i, we set, for every $\xi = a \odot \nu(x) \in \mathrm{M}^N_D$, $a \perp \nu(x)$,

$$\begin{aligned} H(x, a \odot \nu(x)) := \inf\{ & H_i(x, a_i \odot \nu(x)) + H_j(x, -a_j \odot \nu(x)) : \\ & a = a_i - a_j,\ a_i, a_j \in \mathbb{R}^N,\ a_i \perp \nu(x), a_j \perp \nu(x)\} \end{aligned} \tag{2.8}$$

and

$$H(x,\xi) = +\infty \qquad \text{otherwise on } \mathrm{M}^N_D.$$

Remark that $\xi \mapsto H(x,\xi)$ is convex and positively one-homogeneous and that, for every $a \odot \nu(x) \in \mathrm{M}^N_D$,

$$c_3|a \odot \nu(x)| \leq H(x, a \odot \nu(x)) \leq c_4|a \odot \nu(x)|. \tag{2.9}$$

Also, since H_i, H_j and ν are continuous functions of x and ξ, while, by coercivity, the infimum in the inf-convolution is actually a minimum, $H(x,\xi)$ is actually lower semicontinuous on $(\Gamma \setminus S) \times \mathrm{M}^N_D$.

(c) The dissipation potential on the Dirichlet part of the boundary: All points on $\Gamma_d \setminus S'$ belong to the boundary of a single Ω_i and we take $H = H_i$.

(d) Finally, we define H arbitrarily on $S \cup S'$ for example as $c_3|\xi|$, since those points will not be activated by admissible plastic strains because $\mathscr{H}^{N-1}(S \cup S') = 0$.

Remark that the dissipation potential $H : (\Omega \cup \Gamma_d) \times \mathrm{M}^N_D \to [0, +\infty]$ is a Borel function. Indeed, H is continuous in the phases and on $\Gamma_d \setminus S'$, lower semicontinuous on $\Gamma \setminus S$; moreover S, S' are closed sets, and H is continuous on $S \cup S'$. We will call *admissible dissipation potential* for a *geometrically admissible multiphase domain* any dissipation potential that satisfies conditions (a)-(d) above.

Remark 2.2. The assumptions above can be rephrased in terms of the multifunction $x \multimap K(x)$. Denote by $K_i(x)$ the admissible set at $x \in \Omega_i$. The multimap $x \multimap K_i(x)$ is continuous on $\overline{\Omega}_i$: it satisfies the lower semi-continuity condition

$$\forall \varepsilon > 0, \exists U_x \text{ open s.t. } x \in U_x \text{ and } K_i(x) \subset K_i(y) + \varepsilon B(0,1) \text{ for every } y \in U_x, \tag{2.10}$$

together with the upper semi-continuity condition

$$\forall \varepsilon > 0, \exists U_x \text{ open s.t. } x \in U_x \text{ and } K_i(y) \subset K_i(x) + \varepsilon B(0,1) \text{ for every } y \in U_x. \tag{2.11}$$

At a point x at the interface between Ω_i and Ω_j, admissible plastic strains only activate directions of the form $a \odot \nu(x) \in \mathrm{M}^N_D$. The dissipation potential is defined in (2.8) only for those matrices (it is $+\infty$ elsewhere), and it has the form of an inf-convolution between $H_i(x,\cdot)$ and $H_j(x,\cdot)$. By convex conjugation, we can associate to the dissipation at x a set $K(x) \subseteq \mathrm{M}^N_D$. It is readily seen that $K(x)$ is

$$K(x) = \{\sigma_D \in \mathrm{M}^N_D : (\sigma_D \nu(x))_\tau \in (K_i(x)\nu(x))_\tau \cap (K_j(x)\nu(x))_\tau\},$$

where $(\cdot)_\tau$ denotes the orthogonal projection to the hyperplane tangent to Γ at x. Notice that $K(x)$ is a cylinder in M^N_D. We take the view that this is a constraint on the vector $(\sigma_D \nu(x))_\tau$, rather than on the matrix σ_D. Set

$$K_\Gamma(x) := (K_i(x)\nu(x))_\tau \cap (K_j(x)\nu(x))_\tau \subseteq \mathbb{R}^N. \tag{2.12}$$

That way, $\mathbb{I}_{K_\Gamma(x)}$ is the Legendre transform of the map $a \mapsto H(x, a \odot \nu(x))$ with $a \perp \nu(x)$, and conversely. ¶

The dissipation functional: For every admissible plastic strain $p \in \mathscr{P}$, we define the dissipation functional as

$$\mathscr{H}(p) := \int_{\Omega \cup \Gamma_d} H\left(x, \frac{p}{|p|}\right) d|p|,$$

where $p/|p|$ denotes the Radon-Nikodym derivative of p with respect to its total variation $|p|$.

For every $p \in \mathscr{P}$,

$$c_3 |p|(\Omega \cup \Gamma_d) \leq \mathscr{H}(p) \leq c_4 |p|(\Omega \cup \Gamma_d). \tag{2.13}$$

This is a consequence of the bounds (2.7) and (2.9), taking into account the form (2.2) of p along Γ and also the fact that $\mathscr{H}^{N-1}(S \cup S') = 0$.

The existence of a quasi-static evolution for a multiphase material will be based on the following lower semi-continuity result for $\mathscr{H}$, the proof of which will be given in Section 5.

Proposition 2.3 (Lower semi-continuity of $\mathscr{H}$). *For every $(u_n, e_n, p_n) \in \mathscr{A}(w_n)$ and $(u, e, p) \in \mathscr{A}(w)$,*

$$\left.\begin{aligned} &u_n \overset{*}{\rightharpoonup} u \text{ weakly}^* \text{ in } BD(\Omega) \\ &e_n \rightharpoonup e \text{ weakly in } L^1(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \\ &p_n \overset{*}{\rightharpoonup} p \text{ weakly}^* \text{ in } \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D) \end{aligned}\right\} \Rightarrow \quad \mathscr{H}(p) \le \liminf_n \mathscr{H}(p_n). \tag{2.14}$$

Remark 2.4. If $H(x,\xi)$ is lower semicontinuous and $\xi \mapsto H(x,\xi)$ is convex and positively one-homogeneous, the lower semi-continuity of $\mathscr{H}$ is a consequence of Reshetnyak's theorem (see e.g. [2, Theorem 2.38], or [14, Theorem 1.7]).

In terms of admissible sets for the stresses, this entails the lower semi-continuity property (2.10) for the multi-function $x \in \Omega\cup\Gamma_d \multimap K(x)$. This condition is rather restrictive, at least as far as the behavior near the interface of the two phases is concerned. Take the case of a two-phase material. It then amounts to requiring that the yield surface of one phase be included in that of the other phase.

Our conditions on H encompass much more general situations, and consequently H is not lower semicontinuous in (x,ξ). The ensuing lower semi-continuity of the dissipation functional $\mathscr{H}$ does not hold for any sequence of measures, but Proposition 2.3 ensures that it does hold for those that are plastic strains corresponding to admissible configurations in the sense of Definition 2.1. ¶

The total dissipation: If $t \mapsto p(t)$ is a map from $[0,T]$ to $\mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)$, we define, for every $[a,b] \subseteq [0,T]$,

$$\mathscr{D}(a,b;p) := \sup\left\{ \sum_{j=1}^{k} \mathscr{H}\left(p(t_j) - p(t_{j-1})\right) \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\} \tag{2.15}$$

to be the *total dissipation* over the time interval $[a,b]$. Thanks to (2.13), the total dissipation satisfies for every $t \in [0,T]$

$$c_3 \mathscr{V}(0,t;p) \le \mathscr{D}(0,t;p) \le c_4 \mathscr{V}(0,t;p). \tag{2.16}$$

In what follows, the energetic formulation of the quasi-static evolution is detailed. To that aim, we prescribe a boundary displacement w on Γ_d for the time interval $[0,T]$ as the trace on Γ_d of some

$$w \in AC(0,T;H^1(\mathbb{R}^N;\mathbb{R}^N)). \tag{2.17}$$

There is by now a large body of literature on quasi-static evolution and the reader is referred to e.g. [11] for a general overview of the topic, or still to [3] for the case of interest to us here, *i.e.*, that of elasto-plasticity: the two ingredients of such evolutions are a stability statement at each time, together with an energy conservation statement that relates the total energy of the system to the work of the loads applied to that system.

Definition 2.5 (Quasi-static evolution). The mapping

$$t \mapsto (u(t), e(t), p(t)) \in \mathscr{A}(w(t))$$

is a quasi-static evolution relative to w iff the following conditions hold for every $t \in [0,T]$:

(a) Global stability: for every $(v,\eta,q) \in \mathscr{A}(w(t))$

$$\mathscr{Q}(e(t)) \leq \mathscr{Q}(\eta) + \mathscr{H}(q - p(t)). \tag{2.18}$$

(b) Energy equality: $p \in BV\left(0,T;\mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D)\right)$ and

$$\mathscr{Q}(e(t)) + \mathscr{D}(0,t;p) = \mathscr{Q}(e(0)) + \int_0^t \int_\Omega \sigma(\tau)\cdot E\dot{w}(\tau)\,dx\,d\tau,$$

where $\sigma(t) := \mathbb{C}e(t)$.

Remark 2.6. The time integral appearing on the right hand-side of energy equality is well defined under the BV-regularity assumption on p.

Indeed, the global stability of $(u,e,p) \in \mathscr{A}(w)$ is equivalent, thanks to the one-homogeneous character of $\mathscr{H}$, to the following set of inequalities

$$-\mathscr{H}(q) \leq \int_\Omega \mathbb{C}(x)e\cdot\eta\;dx \leq \mathscr{H}(-q) \quad \text{for every } (v,\eta,q) \in \mathscr{A}(0).$$

One implication is proved in [3, Theorem 3.4] while the other is immediate by convexity of the quadratic form $\mathscr{Q}(e)$. Then, if $(u',e',p') \in \mathscr{A}(w')$ is an other globally stable configuration, we obtain

$$-\mathscr{H}(p'-p) \leq \int_\Omega \mathbb{C}(x)e\cdot\big((e'-e)-(Ew'-Ew)\big)\;dx$$

and

$$\int_\Omega \mathbb{C}(x)e'\cdot\big((e'-e)-(Ew'-Ew)\big)\;dx \leq \mathscr{H}(p-p').$$

We deduce

$$\int_\Omega \mathbb{C}(x)(e'-e)\cdot(e'-e)\,dx \leq \int_\Omega \mathbb{C}(x)(e'-e)\cdot(Ew'-Ew)\,dx + \mathscr{H}(p-p') + \mathscr{H}(p'-p),$$

and, appealing to (2.3), (2.13), we conclude that

$$\|e'-e\|_2 \leq C\left(\|Ew'-Ew\|_2 + |p'-p|^{1/2}(\Omega\cup\Gamma_d)\right), \tag{2.19}$$

for some $C > 0$. The above inequality implies in particular the uniqueness of $e \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ and, by application of Poincaré-Korn's inequality, of $u \in BD(\Omega)$ for a globally stable configuration $(u,e,p) \in \mathscr{A}(w)$ once p is fixed.

The summability in time of

$$t \mapsto \int_\Omega \sigma(t)\cdot E\dot{w}(t)\,dx \tag{2.20}$$

is then a consequence of the following considerations. If $t \mapsto p(t)$ has bounded variation, then p is bounded, and global stability implies that $\sup_{t\in[0,T]} \|e(t)\|_2 < +\infty$, hence $\sup_{t\in[0,T]} \|\sigma(t)\|_2 < +\infty$ (choose $(w(t), Ew(t), 0)$ as a comparison configuration). Moreover, since p has at most a countable number of discontinuity points, relation (2.19), together with the absolute continuity of $t \mapsto w(t)$, entails that $t \mapsto e(t)$, hence also $t \mapsto \sigma(t)$, is continuous, up to a countable set, as a map with values in $L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$. We conclude that the map (2.20) is measurable and bounded, hence summable on $[0,T]$. ¶

The following theorem is a generalization of [13, Theorem 3.14]. It provides an existence result for a general multiphase material. Actually, it is more general to the extent that, for the theorem to hold true, it suffices that the heterogeneous material be such that its dissipation potential H defines in turn a functional $\mathscr{H}$ for which the lower semi-continuity property (2.14) holds true.

Theorem 2.7 (**Existence of quasi-static evolutions).** *Consider a* geometrically admissible multiphase domain Ω. *Assume that* (2.3) *and* (2.17) *are satisfied, and that H is an* admissible dissipation potential. *Let* $(u_0, e_0, p_0) \in \mathscr{A}(w(0))$ *satisfy the global stability condition* (2.18). *Then there exists a quasi-static evolution* $\{t \mapsto (u(t), e(t), p(t)), t \in [0,T]\}$ *relative to the boundary displacement w such that* $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$.

Proof. We quickly sketch the proof of the above theorem.

As a first step, we note that, if, for some $w_k \to w$ strongly in $H^1(\mathbb{R}^N; \mathbb{R}^N)$, $(u_k, e_k, p_k) \in \mathscr{A}(w_k)$ minimizes $\mathscr{Q}(\eta) + \mathscr{H}(q - p_k)$ among all $(v, \eta, q) \in \mathscr{A}(w_k)$, and if, further,

$$\begin{aligned} u_k &\stackrel{*}{\rightharpoonup} u && \text{weakly}^* \text{ in } BD(\Omega) \\ e_k &\rightharpoonup e && \text{weakly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) \\ p_k &\stackrel{*}{\rightharpoonup} p && \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^N_D), \end{aligned}$$

then $(u, e, p) \in \mathscr{A}(w)$ minimizes $\mathscr{Q}(\eta) + \mathscr{H}(q - p)$ among all $(v, \eta, q) \in \mathscr{A}(w)$. Indeed, compare (u_k, e_k, p_k) with $(u_k + v - u, e_k + \eta - e, p_k + q - p) \in \mathscr{A}(w_k)$. We obtain

$$\mathscr{Q}(e_k) \le \mathscr{Q}(e_k + \eta - e) + \mathscr{H}(q - p).$$

Then, factoring out the elastic energy on the right hand-side,

$$0 \le \mathscr{Q}(\eta - e) + \int_\Omega \mathbb{C}(x) e_k \cdot (\eta - e)\, dx + \mathscr{H}(q - p),$$

so that, in the limit,

$$0 \le \mathscr{Q}(\eta - e) + \int_\Omega \mathbb{C}(x) e \cdot (\eta - e)\, dx + \mathscr{H}(q - p).$$

Adding $\mathscr{Q}(e)$ to both sides, we conclude to the global stability

$$\mathscr{Q}(e) \le \mathscr{Q}(\eta) + \mathscr{H}(q - p).$$

Now, as in [3], we use an incremental process in time, defining, for $i = 0, \ldots., n$, the times $t_i^n := iT/n$ and a triplet $(u_i^n, e_i^n, p_i^n) \in \mathscr{A}(w(t_i^n))$ such that, for $i > 0$,

$$(u_i^n, e_i^n, p_i^n) \in \text{ Argmin } \{\mathscr{Q}(\eta) + \mathscr{H}(q - p_{i-1}^n) : (v, \eta, q) \in \mathscr{A}(w(t_i^n))\}. \tag{2.21}$$

The existence of such a minimizer is immediately obtained thanks to the coercivity and lower semi-continuity in (η, q) of the functional. This in turn is a consequence of the quadratic character of $\mathscr{Q}$ and of the the lower semi-continuity property (2.14) for $\mathscr{H}$. Coercivity in the displacement v comes from the Poincaré-Korn's inequality in $BD(\Omega)$.

For $i = 0$, we choose $(u_0^n, e_0^n, p_0^n) \equiv (u_0, e_0, p_0)$. Upon testing the minimality of (u_i^n, e_i^n, p_i^n) in (2.21) with $(u_{i-1}^n + w(t_i^n) - w(t_{i-1}^n), e_{i-1}^n + Ew(t_i^n) - Ew(t_{i-1}^n), p_{i-1}^n) \in \mathscr{A}(w(t_i^n))$ and upon iterating, we easily get

$$\begin{aligned} \mathscr{Q}(e_i^n) + \sum_{1 \le k \le i} \mathscr{H}(p_k^n - p_{k-1}^n) & \\ \le \mathscr{Q}(e_0) + \sum_{1 \le k \le i} \int_{t_{k-1}^n}^{t_k^n} \int_\Omega \mathbb{C}(x) e_{k-1}^n \cdot E\dot{w}(s) dx\, ds + \delta_n, & \end{aligned} \tag{2.22}$$

for some $\delta_n \to 0$.

Define $(u^n(t), e^n(t), p^n(t))$ to be the right-continuous and piecewise in time constant interpolation of the (u_i^n, e_i^n, p_i^n)'s. Then (2.22) can be rewritten as

$$\mathscr{Q}(e^n(t)) + \mathscr{D}(0, t; p^n) \le \mathscr{Q}(e_0) + \int_0^{t_{i_n(t)}^n} \int_\Omega \mathbb{C}(x) e^n(s) \cdot E\dot{w}(s) dx\, ds + \delta_n, \tag{2.23}$$

where $i_n(t)$ is the largest index i such that $t \in [t_i^n, t_{i+1}^n)$. We deduce that there exists $C > 0$ independent of n such that

$$\sup_{t \in [0,T]} \|e^n(t)\|_2 + \mathscr{D}(0, T; p^n) \le C. \tag{2.24}$$

In view of (2.16), a generalized version of Helly's theorem (see [10, Theorem 3.2]) implies the existence of a subsequence of $\{p^n\}_{n \in \mathbb{N}}$, still indexed by n, such that, for all $t \in [0, T]$,

$$p^n(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}_D^N)$$

for a suitable $p \in BV(0, T; \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}_D^N))$. By (2.24) and by Poincaré-Korn's inequality in $BD(\Omega)$, there exists a t-dependent subsequence $\{(u^{n_t}(t), e^{n_t}(t))\}_{n_t \in \mathbb{N}}$ such that

$$u^{n_t}(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega), \tag{2.25}$$

$$e^{n_t}(t) \rightharpoonup e(t) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}_{\mathrm{sym}}^N), \tag{2.26}$$

with $(u(t), e(t), p(t)) \in \mathscr{A}(w(t))$. Further,

$$\mathbb{C} e^{n_t}(t) \rightharpoonup \mathbb{C} e(t) =: \sigma(t) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}_{\mathrm{sym}}^N). \tag{2.27}$$

It remains to show that the obtained evolution satisfies items (a) and (b) of Definition 2.5.

The global stability (item (a)) is a direct application of the first step of this proof, upon remarking that, by sub-additivity of $\mathscr{H}$,

$$(u^{n_t}(t), e^{n_t}(t), p^{n_t}(t)) \in \operatorname{Argmin}\{\mathscr{Q}(\eta) + \mathscr{H}(q - p^{n_t}(t)) : (v, \eta, q) \in \mathscr{A}(w(t^{n_t}_{i_{n_t}(t)}))\}.$$

As already noticed in Remark 2.6, the pair $(u(t), e(t))$ is uniquely defined, once $p(t)$ is known; thus, there is no need to extract t-dependent subsequences and n_t can be replaced by n in (2.25), (2.26) and (2.27) above.

Concerning the energy equality (item (b)), the lower semi-continuity of $\mathscr{H}$, together with the very definition of the dissipation on $[0,t]$, yields

$$\mathscr{D}(0,t;p) \leq \liminf_n \mathscr{D}(0,t;p^n) \tag{2.28}$$

Now, for a.e. $s \in [0,T]$,

$$\int_\Omega \mathbb{C}(x) e^n(s) \cdot E\dot{w}(s) dx \to \int_\Omega \sigma(s) \cdot E\dot{w}(s) dx$$

and is dominated by a constant, independently of s in view of the L^∞-bound (2.24) on $\|e^n(t)\|_2$. Since $w \in AC(0,T; H^1(\mathbb{R}^N;\mathbb{R}^N))$ and $t - t^n_{i_n(t)} \overset{n}{\searrow} 0$, Lebesgue's dominated convergence theorem implies that

$$\int_\Omega \sigma(s) \cdot E\dot{w}(s) dx \in L^1(0,T),$$

and, for $t \in [0,T]$,

$$\int_0^{t^n_{i_n(t)}} \int_\Omega \mathbb{C}(x) e^n(s) \cdot E\dot{w}(s) dx\, ds \longrightarrow \int_0^t \int_\Omega \sigma(s) \cdot E\dot{w}(s) dx\, ds.$$

The weak-L^2-lower semicontinuous character of $\mathscr{Q}$, together with (2.23) and (2.28), implies the energy inequality

$$\mathscr{Q}(e(t)) + \mathscr{D}(0,t;p) \leq \mathscr{Q}(e(0)) + \int_0^t \int_\Omega \sigma(\tau) \cdot E\dot{w}(\tau)\, dx d\tau. \tag{2.29}$$

The opposite inequality is then deduced from the global minimality (item (a)). Let us set $s^i_n := it/n$ for $i = 0, \ldots, n$. Testing the minimality of $(u(s_i), e(s_i), p(s_i)) \in \mathscr{A}(w(s_i))$ by $(u(s_{i+1}) + w(s_i) - w(s_{i+1}), e(s_{i+1}) + Ew(s_i) - Ew(s_{i+1}), p(s_{i+1})) \in \mathscr{A}(w(s_i))$ yields

$$\begin{aligned}
&\mathscr{Q}(e(s_{i+1})) + \mathscr{H}(p(s_{i+1}) - p(s_i)) \\
&\quad \geq \mathscr{Q}(e(s_i)) + \int_\Omega \sigma(s_{i+1}) \cdot (Ew(s_{i+1}) - Ew(s_i))\, dx - \mathscr{Q}(Ew(s_{i+1}) - Ew(s_i)) \\
&\qquad \geq \mathscr{Q}(e(s_i)) + \int_{s_i}^{s_{i+1}} \int_\Omega \sigma(s_{i+1}) \cdot E\dot{w}(s)\, dx ds - \mathscr{Q}(Ew(s_{i+1}) - Ew(s_i)).
\end{aligned}$$

Summing over i and using the definition of the dissipation, we arrive at

$$\mathscr{Q}(e(t)) + \mathscr{D}(0,t;p) \geq \mathscr{Q}(e(0)) + \int_0^t \int_\Omega \bar{\sigma}^n(s) \cdot E\dot{w}(s)\, dx ds - \delta_n \tag{2.30}$$

where $\bar{\sigma}^n$ denotes the left continuous piecewise constant interpolation of the $\sigma(s_n^i)$'s, and $\delta_n \to 0$. From Remark 2.6 we know that the BV regularity in time for p entails that $t \mapsto \sigma(t)$ has at most a countable number of discontinuity points. We deduce that, for every $s \in [0,t]$ up to a countable set,

$$\bar{\sigma}^n(s) \to \sigma(s) \qquad \text{strongly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}).$$

Taking into account the uniform bound on $\|\sigma(s)\|_2$ coming from (2.24), we can pass to the limit in (2.30) obtaining

$$\mathscr{Q}(e(t)) + \mathscr{D}(0,t;p) \geq \mathscr{Q}(e(0)) + \int_0^t \int_\Omega \sigma(s) \cdot E\dot{w}(s)\, dx ds,$$

which, together with (2.29), yields the energy equality. □

Remark 2.8. Our proof is quite different from that in [13] because we do not seek to interpret the global minimality condition in terms of its associated Euler-Lagrange equations and, consequently, do not attempt to prove global minimality by passing to the limit in those. In refraining from doing so, we bypass the delicate notions of duality between σ_D and p that must be accounted for if using the latter. In turn, this allows us to prove a more general existence theorem. Duality will make a comeback in a later part of this paper when attempting to recover the flow rule described in item (3) of Definition 3.1. ¶

Remark 2.9 (**Safe load conditions**). The reader familiar with [3] may rightfully object that our derivation of the existence of a quasi-static evolution, which does not use any kind of duality between stress and plastic strain, is somewhat deceiving because, if we were in the presence of body loads of the form $f \in AC(0,T;L^N(\Omega;\mathbb{R}^N))$ and/or surface tractions of the form $g \in AC(0,T;L^\infty(\Gamma_t;\mathbb{R}^N))$, then an additional term

$$\mathscr{L}(u) := \int_\Omega f \cdot u\, dx - \int_{\Gamma_t} g \cdot u\, d\mathscr{H}^{N-1}$$

that represents the work of the loads would have to be added to both sides of (2.18) (and an additional term would also appear in the energy equality). In order to obtain the existence of a minimizer as in (2.21), one would then have to impose uniform safe load conditions (see [3, Equations (2.17), (2.18)]), that is the existence of a stress field $\pi \in AC(0,T;L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}))$ with $\pi_D \in AC(0,T;L^\infty(\Omega;\mathrm{M}^N_D))$ and

$$\pi_D(t,x) + B(0,\alpha) \subset K(x) \tag{2.31}$$

for some $\alpha > 0$, which is such that, for every $t \in [0,T]$,

$$\operatorname{div} \pi(t) = f(t) \text{ in } \Omega \qquad \text{and} \qquad \pi(t)\nu = g(t) \text{ on } \Gamma_t.$$

In [3], the coercivity of the functional to minimize is then obtained through a rewriting of $\mathscr{L}(u)$ as $-\int_\Omega \pi\cdot(e-Ew)\,dx-\langle\pi_D,p\rangle+\mathscr{L}(w)$ (see (6.3) with $\varphi\equiv 1$). The meaning of the term $\langle\pi_D,p\rangle$ uses the duality; see [3, Lemmata 3.1, 3.2].

However, note that the absence of spatial regularity of π is the only reason for appealing to the duality in that argument because the product of π_D with p is not meaningful under the assumed regularity of π_D. But, in all fairness, the regularity assumptions on π are a somewhat collateral issue. Assuming e.g. that $\pi_D\in AC(0,T;C^0(\overline{\Omega};\mathrm{M}^N_D))$ completely alleviates the need for a duality argument and would easily be seen to allow one to recover the existence result in the presence of loads.

The true difficulty in imposing loads lies elsewhere, namely in the feasibility, for a given pair (f,g) of loads, of finding a stress tensor π that satisfies (2.31). This, which is the purpose of limit analysis, cannot be argued for or against on duality grounds. ¶

In order to address the issue of the flow rule in Section 3, we need some regularity in time of the triplet $(u(t),e(t),p(t))$ which can be established with a proof identical – modulo the absence of force loads – to that of [3, Theorem 5.2]. The existence of the limits of the difference quotients follows from [3, Remark 5.1], while the relation $(\dot u(t),\dot e(t),\dot p(t))\in\mathscr{A}(\dot w(t))$ is proved in [3, Lemma 5.5]. We thus have the following

Proposition 2.10 (Regularity in time). *Let $\{t\mapsto(u(t),e(t),p(t)):t\in[0,T]\}$ be a quasi-static evolution relative to the boundary displacement w according to Definition 2.5. Then*

$$(u,e,p)\in AC\left(0,T;BD(\Omega)\times L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})\times\mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D)\right)$$

and for a.e. $t\in[0,T]$ the following limits exist

$$\dot u(t):=\lim_{s\to t}\frac{u(s)-u(t)}{s-t}\qquad \textit{weakly}^*\ \textit{in } BD(\Omega),$$

$$\dot e(t):=\lim_{s\to t}\frac{e(s)-e(t)}{s-t}\qquad \textit{strongly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}),$$

$$\dot p(t):=\lim_{s\to t}\frac{p(s)-p(t)}{s-t}\qquad \textit{strictly in } \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D),$$

with $(\dot u(t),\dot e(t),\dot p(t))\in\mathscr{A}(\dot w(t))$. Finally $\mathscr{D}(0,t;p)\in AC(0,T)$ and, for a.e. $t\in[0,T]$,

$$\dot{\mathscr{D}}(0,t;p)=-\int_\Omega\sigma(t)\cdot(\dot e(t)-E\dot w(t))\,dx. \tag{2.32}$$

3 Energetic solutions vs. classical evolutions

We consider throughout this section a *geometrically admissible multiphase domain*, unless otherwise specified.

In the absence of force loads, quasi-static evolutions in elasto-plasticity are written more classically in terms of $(u(t),e(t),p(t),\sigma(t))$ as the following set of conditions:

Definition 3.1 (Classical evolutions). The triplet $(u(t),e(t),p(t))$ is a classical evolution relative to w iff, for every $t\in[0,T]$,

(1) *Compatibility:* $(u(t),e(t),p(t))\in\mathscr{A}(w(t))$;
(2) *Balance equations and stress admissibility:* $\sigma(t)\in\mathscr{K}$, where $\sigma(t):=\mathbb{C}e(t)$ and

$$\mathscr{K}:=\big\{\sigma\in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}):\operatorname{div}\sigma=0\text{ in }\Omega;\ \sigma\nu=0\text{ on }\partial\Omega\setminus\overline{\Gamma}_d;\\ \sigma_D(x)\in K(x)\text{ for a.e. }x\in\Omega\big\}. \tag{3.1}$$

(3) *Flow rule:* $\dot p(t,x)\in N_{K(x)}(\sigma_D(t,x))$ a.e. in Ω, where $N_{K(x)}(\sigma_D(t,x))$ denotes the normal cone to $K(x)$ at $\sigma_D(t,x)$.

Remark 3.2. As far as item (2) is concerned, the last condition in the definition of $\mathscr{K}$ implies that $\sigma(t)$ satisfies the admissibility condition in each of the phases Ω_i, that is

$$\sigma_D(t,x)\in K_i(x)\qquad\text{for a.e. }x\in\Omega_i.$$

We show in Theorem 3.5 that this actually entails an admissibility condition for the stress along the inner interfaces and on the Dirichlet boundary Γ_d.

The first two conditions in the definition of $\mathscr{K}$ are precisely the balance equations for $\sigma(t)$.

In view of (2.4), if $\sigma\in\mathscr{K}$, then $\sigma_D\in L^\infty(\Omega;\mathrm{M}^N_D)$. As a consequence,

$$\mathscr{K}\subset L^r(\Omega;\mathrm{M}^N_{\mathrm{sym}}),\text{ for all }1\le r<\infty.$$

In the case of a C^2-boundary, this fact is a well-known result to be found in e.g. [8, Proposition 2.5 and Corollary 2.6]. Here, we contend that such is also the case for a Lipschitz domain and quickly derive the result in Section 6 (see Proposition 6.1). We do not claim paternity of that result, but have merely been unable to locate a similar result in our perusal of the existing literature on elasto-plasticity. ¶

Remark 3.3. If body loads of the form $f\in AC(0,T;L^N(\Omega;\mathbb{R}^N))$ and/or surface tractions of the form $g\in AC(0,T;L^\infty(\partial\Omega\setminus\overline{\Gamma}_d;\mathbb{R}^N))$ were present, the definition of the set $\mathscr{K}$ would be altered. We would replace $\mathscr{K}$ by

$$\mathscr{K}_{f,g}(t):=\big\{\sigma\in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}):\operatorname{div}\sigma=f(t)\text{ in }\Omega;\ \sigma\nu=g(t)\text{ on }\partial\Omega\setminus\overline{\Gamma}_d;\\ \sigma_D(x)\in K(x)\text{ for a.e. }x\in\Omega\big\},$$

the remainder of Definition 3.1 being unchanged. ¶

Remark 3.4. As far as item (3) is concerned, the meaning of the flow rule is to be further elaborated upon since $\dot p(t)$ is only a measure. In any case, the sense in which the quasi-static evolution described in Definition 2.5 satisfies the flow rule in Definition 3.1 will be the focus of Subsection 3.2 below. ¶

As announced in Remark 3.2, we show that the admissibility for the stress in the phases entails an admissibility condition for the associated trace on the inner interfaces Γ that involves the set K_Γ defined in (2.12); there is also an admissibility condition on the Dirichlet boundary Γ_d. More precisely, the admissibility condition involves the tangential trace $(\sigma_D\nu)_\tau$ of $\sigma\nu$ introduced in (1.2). Recall that this trace may depend upon the approximation, but that it is uniquely determined if Γ and Γ_d are of class C^2.

Theorem 3.5 (Stress admissibility; interfaces and Dirichlet boundary). *Consider a* geometrically admissible multiphase domain. *For every $\sigma \in \mathcal{K}$,*

$$(3.2)\qquad \begin{aligned} &(\sigma_D\nu)_\tau(x) \in K_\Gamma(x) &&\textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x \in \Gamma\\ &(\sigma_D\nu)_\tau \in [K_i(x)\nu(x)]_\tau &&\textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x \in \Gamma_d \cap \overline{\Omega}_i, \end{aligned}$$

with $K_\Gamma(x)$ defined in (2.12).

Proof. Let $x \in \Gamma_{ij} \setminus S$ be a Lebesgue point for $(\sigma_D(t)\nu)_\tau$, and let $\varepsilon > 0$. Let us choose $r > 0$ such that $B(x,r) \subset\subset \Omega$, $\Gamma_{ij} \cap B(x,r)$ is a C^1-graph and

$$(3.3)\qquad \|\nu(y) - \nu(x)\|_\infty \le \varepsilon \qquad \text{for every } y \in B(x,r).$$

In view of the upper semi-continuity condition (2.11), we can assume, at the possible expense of shrinking r, that

$$(3.4)\qquad K_i(y) \subset K_i(x) + \varepsilon B(0,1) \text{ for every } y \in B(x,r) \cap \overline{\Omega}_i.$$

Similar inclusions hold for K_j and $\overline{\Omega}_j$ in place of K_i and $\overline{\Omega}_i$.

Translating infinitesimally $\sigma(t)$ on $B(x,r)$ in the direction $-\nu(x)$ ($\nu(x)$ pointing from Ω_j to Ω_i), and regularizing by convolution, we obtain a smooth approximation σ_n of σ satisfying (1.1) with the choice $A := B(x,r/2) \cap \overline{\Omega}_i$. Since

$$(\sigma_n)_D(y) \in K_i(x) + \varepsilon B(0,1) \qquad \text{for every } y \in B(x,r/2) \cap \Gamma_{ij},$$

passing to the tangent components of $(\sigma_n)_D\nu$ on Γ_{ij} we obtain, in view of (1.2), (3.4), (3.3) and (2.4),

$$(\sigma_D\nu)_\tau(y) \in [K_i(x)\nu(x)]_\tau + c_\varepsilon B(0,1) \qquad \text{for a.e. } y \in B(x,r/2) \cap \Gamma_{ij},$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0$.

Since x is a Lebesgue point for $(\sigma\nu)_\tau$, the arbitrariness of ε yields

$$(\sigma_D\nu)_\tau(x) \in [K_i(x)\nu(x)]_\tau.$$

By the same construction, but translating this time in the direction $\nu(x)$ instead of $-\nu(x)$, we conclude that

$$(\sigma_D\nu)_\tau(x) \in [K_j(x)\nu(x)]_\tau.$$

In view of the very definition of K_Γ, and since Lebesgue points have full measure in Γ while $\mathcal{H}^{N-1}(S) = 0$, we conclude that the first inclusion in (3.2) holds true.

The proof of the admissibility on Γ_d is identical, upon replacing (3.3) by the condition that x should also be a Lebesgue point for ν. $\square$

We should point out that an admittedly cursory review of the mechanics literature has failed to evidence any awareness of the fact that interface admissibility is a necessary byproduct of the classical formulation of elasto-plasticity.

Fix a quasi-static evolution

$$\{t \mapsto (u(t), e(t), p(t)) : t \in [0,T]\}$$

relative to the boundary displacement w (see Definition 2.5). We set, as before, $\sigma(t) := \mathbb{C}e(t)$. We wish to investigate the extent to which such an evolution can be considered a classical evolution in the sense of Definition 3.1. This will be the focus of the following two subsections.

3.1 Equilibrium and stress admissibility

A first link between quasi-static evolutions in the sense of Definition 2.5 and classical evolutions is an easy consequence of global stability.

Theorem 3.6 (Equilibrium and stress admissibility). *Consider a* geometrically admissible multiphase domain *and an evolution $t \mapsto (u(t), e(t), p(t))$ in the sense of Definition 2.5. Then, for every $t \in [0,T]$, $\sigma(t) = \mathbb{C}e(t) \in \mathscr{K}$, i.e., $\sigma(t)$ satisfies the balance equations and the admissibility constraint in the phases*

$$\sigma_D(t,x) \in K(x) \text{ for a.e. } x \in \Omega.$$

Along the inner interfaces Γ and on the Dirichlet boundary Γ_d, the following holds:

$$\begin{aligned} &(\sigma_D(t)\nu)_\tau(x) \in K_\Gamma(x) && \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in \Gamma \\ &(\sigma_D(t)\nu)_\tau \in [K_i(x)\nu(x)]_\tau && \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in \Gamma_d \cap \overline{\Omega}_i, \end{aligned} \tag{3.5}$$

where $K_\Gamma(x)$ is defined in (2.12).

Proof. As noted in Remark 2.6, global stability entails that for every $(v, \eta, q) \in \mathscr{A}(0)$

$$-\mathscr{H}(q) \le \int_\Omega \sigma(t) \cdot \eta \, dx \le \mathscr{H}(-q).$$

Choose (v, η, q) to be $(\varphi, E\varphi, 0)$ with $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$, then with $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ s.t. $\varphi \equiv 0$ on $\overline{\Gamma}_d$. We obtain

$$\operatorname{div}\sigma(t) = 0 \text{ in } \Omega, \qquad \sigma(t)\nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}_d.$$

Finally, choose (v, η, q) to be $(0, \chi_B\xi, -\chi_B\xi)$ with $\xi \in \mathrm{M}^N_D$ and χ_B the characteristic function of an arbitrary Borel subset B of Ω. Letting ξ vary first in a countable and dense set in M^N_D, and then using the continuity of $\xi \mapsto H(x, \xi)$ for a.e. $x \in \Omega$, we obtain that

$$-H(x, -\xi) \le \sigma_D(t,x) \cdot \xi \le H(x, \xi), \text{ a.e. in } \Omega,$$

so that, in view of (2.5),

$$\sigma_D(t,x) \in K(x), \text{ a.e. in } \Omega.$$

We conclude that $\sigma(t) \in \mathscr{K}$. Relations (3.5) follow in view of Theorem 3.5. □

Remark 3.7. The admissibility constraints (3.5) hold for *any* tangential trace $(\sigma_D\nu)_\tau$ of $\sigma_D\nu$ along Γ and Γ_d: as shown in Subsection 1.2, such a trace is uniquely determined if Γ and Γ_d are of class C^2. ¶

3.2 Hill's principle of maximum plastic work and the flow rule

In this subsection we propose to investigate the validity of the flow rule, *i.e.*, of item (3) in Definition 3.1. In view of Theorem 3.6, this is the missing item if we are to establish that the quasi-static evolutions whose existence has been secured through Theorem 2.7 are also classical evolutions in the sense of Definition 3.1. We also show that an energetic quasi-static evolution satisfies Hill's principle of maximum plastic work.

We will make use of the *duality pairing between admissible stresses σ and admissible plastic strains p* which allows one to view their product $\langle\sigma_D, p\rangle$ as a measure on $\Omega\cup\Gamma_d$. This was one of the main issues addressed in [8] and the results in [3, Subsection 2.3] are direct consequences of those in that earlier paper.

We revisit that duality in Section 6 in the case where the boundary is merely Lipschitz and re-derive the needed results in that case. As seen there, $\langle\sigma_D, p\rangle$ is a finite Radon measure on $\Omega\cup\Gamma_d$. However, in order to compute the mass of $\langle\sigma_D, p\rangle$ explicitly in terms of u, e and w, we need to further assume a condition on the relative boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ of Γ_d in $\partial\Omega$, namely (6.20).

The following result deals with the behavior of $\langle\sigma_D, p\rangle$ on the inner interfaces Γ, as well as on Γ_d (see also a related result [13, Proposition 2.6]).

Lemma 3.8. *Consider a* geometrically admissible multiphase domain. *For every $\sigma\in\mathscr{K}$, every $p\in\mathscr{P}$ with associated u,e,w, and for every $i\neq j$,*

$$\langle\sigma_D, p\rangle\lfloor\Gamma_{ij} = (\sigma_D\nu)_\tau\cdot(u^i-u^j)\,\mathscr{H}^{N-1}\lfloor\Gamma_{ij},$$

where u^i and u^j are the traces on Γ_{ij} of the restrictions of u on Ω_i and Ω_j respectively, assuming that ν points from Ω_j to Ω_i.

Similarly,

$$\langle\sigma_D, p\rangle\lfloor\Gamma_d = (\sigma_D\nu)_\tau\cdot(w-u)\,\mathscr{H}^{N-1}\lfloor\Gamma_d,$$

where u is the trace on Γ_d of u, assuming that ν is the outer normal to Ω.

Proof. Let $\varphi\in C^1_c(\Omega)$ be such that its support is contained in $\Omega_i\cup\Omega_j\cup\Gamma_{ij}$. Consider a smooth regularization σ_n of $\sigma(t)$ satisfying (1.1) with $A\subset\subset\Omega$ containing $\mathrm{supp}(\varphi)$, but with the convergences taking place in L^N rather than in L^2. This is possible, thanks to the summability properties of $\sigma(t)$ (see Remark 3.2) and to its divergence free character. In view of (6.3), and since $\varphi u\in BD(\Omega)$,

$$\begin{aligned}
\langle\sigma_D, p\rangle(\varphi) &= -\ \lim_n \textstyle\int_\Omega\{\sigma_n\cdot(e-Ew)\varphi+\sigma_n\cdot[(u-w)\odot\nabla\varphi]+\varphi\ \mathrm{div}\sigma_n\cdot(u-w)\}dx\\
&= -\ \lim_n \textstyle\int_{\Omega_i\cup\Omega_j}\{\sigma_n\cdot e\ \varphi\ dx+\sigma_n\cdot[u\odot\nabla\varphi]+\varphi\ \mathrm{div}\sigma_n\cdot u\}\ dx\\
&= \quad \lim_n\left[\textstyle\int_{\Omega_i\cup\Omega_j}-\sigma_n\,dE(\varphi u)+\int_{\Omega_i\cup\Omega_j}\varphi\sigma_n\,dp-\int_{\Omega_i\cup\Omega_j}\mathrm{div}\sigma_n\cdot\varphi u\,dx\right]\\
&= \quad \lim_n\left[\textstyle\int_{\Gamma_{ij}}\sigma_n\nu\cdot(u^i-u^j)\varphi\,d\mathscr{H}^{N-1}+\int_{\Omega_i\cup\Omega_j}\varphi(\sigma_n)_D\,dp\right].
\end{aligned}$$

Since $p\lfloor\Gamma_{ij} = (u^i - u^j) \odot \nu\, \mathscr{H}^{N-1}\lfloor\Gamma_{ij}$ has values in $\mathbb{M}^N_D$, $(u^i - u^j) \perp \nu$ a.e. on Γ_{ij}, so that, recalling (1.2)

$$\langle \sigma_D, p\rangle(\varphi) = \int_{\Gamma_{ij}} (\sigma_D \nu)_\tau \cdot (u^i - u^j)\varphi \, d\mathscr{H}^{N-1} + \lim_n \int_{\Omega_i \cup \Omega_j} \varphi(\sigma_n)_D \, dp. \tag{3.6}$$

Notice that $\lambda_n \in \mathscr{M}(\Omega_i \cup \Omega_j \cup \Gamma_{ij})$ defined as

$$\lambda_n(\varphi) := \int_{\Omega_i \cup \Omega_j} \varphi(\sigma_n)_D \, dp$$

satisfies $|\lambda_n| \le \|\sigma_D\|_\infty |p|\lfloor(\Omega_i \cup \Omega_j)$. In view of (3.6) we infer that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ weakly* in $\mathscr{M}(\Omega_i \cup \Omega_j \cup \Gamma_{ij})$ with

$$|\lambda| \le \|\sigma_D\|_\infty |p|\lfloor(\Omega_i \cup \Omega_j)$$

and

$$\langle \sigma_D, p\rangle(\varphi) = \int_{\Gamma_{ij}} (\sigma_D \nu)_\tau \cdot (u^i - u^j)\varphi \, d\mathscr{H}^{N-1} + \lambda(\varphi).$$

The result follows by restricting the previous equality to Γ_{ij} since $\lambda\lfloor\Gamma_{ij} = 0$.

The representation formula for $\langle \sigma_D, p\rangle\lfloor\Gamma_d$ follows with similar arguments. □

The next result involves the dissipation functional and the duality between admissible stresses and plastic strains.

Proposition 3.9. *Consider a* geometrically admissible multiphase domain. *For every $\sigma \in \mathscr{K}$ and $p \in \mathscr{P}$,*

$$H\left(x, \frac{p}{|p|}\right)|p| \ge \langle \sigma_D, p\rangle \qquad \textit{as measures on } \Omega \cup \Gamma_d. \tag{3.7}$$

Proof. We can establish the inequality considering the behavior of the measures on Ω_i, Γ and Γ_d successively.

Consider an open set A such that $\bar{A} \subset \Omega_i$ for some i. Let us regularize σ by convolution: we obtain $\sigma_n \in C^\infty(\bar{A}; \mathbb{M}^N_{\mathrm{sym}})$ such that

$$\sigma_n \to \sigma \qquad \text{strongly in } L^N(A; \mathbb{M}^N_{\mathrm{sym}})$$

with

$$\operatorname{div} \sigma_n \to 0 \qquad \text{strongly in } L^N(A; \mathbb{M}^N_{\mathrm{sym}}).$$

In view of the continuity of the multimap $x \multimap K_i(x)$ on Ω_i, one easily gets that, for every $\varepsilon > 0$ and $x \in \bar{A}$,

$$(\sigma_n)_D(x) \in K_i(x) + \varepsilon B(0,1)$$

for n large enough. As a consequence, for $|p|$-a.e. $x \in A$ we have

$$H\left(x, \frac{p}{|p|}(x)\right) = H_i\left(x, \frac{p}{|p|}(x)\right) \ge (\sigma_n)_D(x) \cdot \frac{p}{|p|}(x) - \varepsilon.$$

Recalling that, since σ_n is smooth, (6.3) implies that

$$\langle (\sigma_n)_D, p\rangle = (\sigma_n)_D \cdot p$$

and that, using (6.3) again,

$$\langle (\sigma_n)_D, p\rangle \xrightarrow{*} \langle \sigma_D, p\rangle \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(A),$$

for every $\varphi \in C^1_c(A)$ with $\varphi \geq 0$ we obtain

$$\int_A \varphi H\left(x, \frac{p}{|p|}\right)|p| \geq \lim_n \int_A \varphi(\sigma_n)_D \cdot p - \varepsilon \int_A \varphi\, dx = \lim_n \langle (\sigma_n)_D, p\rangle(\varphi) - \varepsilon \int_A \varphi\, dx$$
$$= \langle \sigma_D, p\rangle(\varphi) - \varepsilon \int_A \varphi\, dx.$$

Inequality (3.7) on the phase Ω_i follows upon letting $\varepsilon \to 0$.

For every $i \neq j$,

$$H\left(x, \frac{p}{|p|}(x)\right)|p| \lfloor \Gamma_{ij} = H(x, (u^i - u^j) \odot \nu)\mathscr{H}^{N-1} \lfloor \Gamma_{ij},$$

and, in view of Lemma 3.8,

$$\langle \sigma_D, p\rangle \lfloor \Gamma_{ij} = (\sigma_D \nu)_\tau \cdot (u^i - u^j)\,\mathscr{H}^{N-1} \lfloor \Gamma_{ij}.$$

Above, u^i and u^j are the traces on Γ_{ij} of the restrictions of u on Ω_i and Ω_j respectively, assuming that ν points from Ω_j to Ω_i. Since p has values in M^N_D, $(u^i(x) - u^j(x)) \perp \nu(x)$ for $\mathscr{H}^{N-1}$-a.e. $x \in \Gamma_{ij}$. Inequality (3.7) on Γ_{ij} then follows since $(\sigma_D \nu)_\tau(x) \in K_\Gamma(x)$ for $\mathscr{H}^{N-1}$-a.e. $x \in \Gamma$ in view of Theorem 3.5, and since $\mathbb{I}_{K_\Gamma(x)}$ is the convex conjugate of $a \mapsto H(x, a \odot \nu(x))$, $a \perp \nu(x)$, so that

$$H(x, a \odot \nu(x)) = \sup\{b \in K_\Gamma(x) : b \cdot a\}. \tag{3.8}$$

The inequality (3.7) on Γ_d can be established in an identical manner. □

Remark 3.10. Note that it is precisely here that we are forced to choose the correct H, lest Proposition 3.9 not hold; think for example of replacing H by $1/2H$ along the interfaces, which would then prevent (3.8) from holding true. ¶

The reinterpretation of the energy equality in terms of more classical dissipation statements is based on the following result.

Proposition 3.11. *Consider a* geometrically admissible multiphase domain. *Also assume that $\partial \lfloor_{\partial\Omega} \Gamma_d$ is admissible (see* (6.20)*). For a.e. $t \in [0, T]$,*

$$H\left(x, \frac{\dot p(t)}{|\dot p(t)|}\right)|\dot p(t)| = \langle \sigma_D(t), \dot p(t)\rangle \qquad \textit{as measures on } \Omega \cup \Gamma_d. \tag{3.9}$$

Proof. Differentiating the energy equality in time and recalling (2.32) we get, for a.e. $t \in [0, T]$,

$$\dot{\mathscr{D}}(0, t; p) = -\int_\Omega \sigma(t) \cdot (\dot e(t) - E\dot w(t))\, dx = \langle \sigma_D(t), \dot p(t)\rangle(\Omega \cup \Gamma_d),$$

the last equality coming from (6.3) with the choice $\varphi \in C_c(\mathbb{R}^N)$, $\varphi \equiv 1$ on $\overline{\Omega}$.

Take t to be a point where u, e, p, w and $\mathscr{D}(0, \cdot; p)$ are differentiable (see Proposition 2.10), and also where the derivative of $\mathscr{D}(0, t; p)$ is given by $\langle \sigma_D(t), \dot p(t)\rangle(\Omega \cup$

Γ_d). Since $\mathscr{D}(0,t_n;p)$ is a total variation, and thanks to the positive one-homogeneous character of $\mathscr{H}$ we get that, for $t_n > t$,

$$\mathscr{H}\left(\frac{p(t_n)-p(t)}{t_n-t}\right) \leq \frac{\mathscr{D}(0,t_n;p)-\mathscr{D}(0,t;p)}{t_n-t}.$$

Now, $\dfrac{p_n(t)-p(t)}{t_n-t} \in \mathscr{P}\left(\dfrac{w(t_n)-w(t)}{t_n-t}\right)$, while $(w(t_n)-w(t))/(t_n-t)$ converges strongly in $H^1(\mathbb{R}^N;\mathbb{R}^N)$ to $\dot{w}(t)$, so that the lower semi-continuity property (2.14) implies that

$$\mathscr{H}(\dot{p}(t)) \leq \dot{\mathscr{D}}(0,t;p) = \langle \sigma_D(t), \dot{p}(t)\rangle(\Omega \cup \Gamma_d).$$

Since $(\dot{u}(t),\dot{e}(t),\dot{p}(t)) \in \mathscr{A}(\dot{w}(t))$ and $\sigma(t) \in \mathscr{K}$, Proposition 3.9 implies that

$$H\left(x, \frac{\dot{p}(t)}{|\dot{p}(t)|}\right)|\dot{p}(t)| \geq \langle \sigma_D(t), \dot{p}(t)\rangle \qquad \text{as measures on } \Omega \cup \Gamma_d,$$

so that the result easily follows. □

Hill's principle is an immediate consequence of (3.9), (3.7).

Theorem 3.12 (Hill's maximum plastic work principle). *Consider a* geometrically admissible multiphase domain. *Also assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible (see* (6.20)*). Plastic work is maximal for a.e. $t \in [0,T]$, i.e.,*

$$\langle \sigma_D(t), \dot{p}(t)\rangle(\Omega \cup \Gamma_d) = \max\{\langle \tau_D, \dot{p}(t)\rangle(\Omega \cup \Gamma_d) : \tau \in \mathscr{K}\},$$

where the set of admissible stresses $\mathscr{K}$ has been defined in (3.1).

We are now in a position to investigate the validity of the flow rule. The following result demonstrates that the flow rule is indeed satisfied $\mathscr{L}^N$-a.e. on the support $\{|\dot{p}(t)| > 0\}$ of the measure $\dot{p}(t)$; as such, it is a form of the flow rule in the phases. Moreover, a flow rule involving the plastic slip on the inner interfaces and the set K_Γ introduced in (2.12) can be established at $\mathscr{H}^{N-1}$-a.e. point.

Theorem 3.13 (Flow rule). *Consider a* geometrically admissible multiphase domain. *Also assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible (see* (6.20)*). For a.e. $t \in [0,T]$,*

$$\frac{\dot{p}(t,x)}{|\dot{p}(t,x)|} \in N_{K(x)}(\sigma_D(t,x)) \quad \textit{for } \mathscr{L}^N \textit{ a.e. } x \in \{|\dot{p}(t| > 0\}. \tag{3.10}$$

Moroever, for every $i \neq j$,

$$\frac{\dot{u}^i(t,x)-\dot{u}^j(t,x)}{|\dot{u}^i(t,x)-\dot{u}^j(t,x)|} \in \vec{N}_{K_\Gamma(x)}((\sigma_D(t)\nu)_\tau(x)) \qquad \textit{for } \mathscr{H}^{N-1} \textit{ a.e. } x \in \{\dot{u}^i(t) \neq \dot{u}^j(t)\}, \tag{3.11}$$

where $\dot{u}^i(t)$ and $\dot{u}^j(t)$ are the traces on Γ_{ij} of the restrictions of $\dot{u}(t)$ to Ω_i and Ω_j respectively, assuming that ν points from Ω_j to Ω_i, and where $\vec{N}_{K_\Gamma(x)}(\tau)$ denotes the normal cone – a cone of vectors – to $K_\Gamma(x)$ at a vector $\tau \perp \nu(x)$.

Finally, for $\mathscr{H}^{N-1}$-a.e. $x \in \Gamma_d \cap \overline{\Omega}_i$ with $\dot w(t,x) \neq \dot u(t,x)$,

$$\frac{\dot w(t,x) - \dot u(t,x)}{|\dot w(t,x) - \dot u(t,x)|} \in \vec N_{(K_i(x)\nu(x))_\tau}((\sigma_D(t)\nu)_\tau(x)). \tag{3.12}$$

Proof. Concerning the flow rule in the phases, it suffices to prove that

$$\sigma_D(t,x) \in \partial H(x, \dot p_a(t,x)) \qquad \text{for } \mathscr{L}^N \text{ a.e. } x \in \Omega,$$

where $\dot p_a(t)$ is the density of the $\mathscr{L}^N$-absolutely continuous part of p. Proposition 3.11 implies that, for a.e. $t \in [0,T]$,

$$H\left(x, \frac{\dot p(t)}{|\dot p(t)|}\right) |\dot p(t)| = \langle \sigma_D(t), \dot p(t)\rangle \qquad \text{as measures on } \Omega \cup \Gamma_d. \tag{3.13}$$

Taking the absolutely continuous parts and invoking Theorem 6.2, we obtain

$$H(x, \dot p_a(t,x)) = \sigma_D(t,x) \cdot \dot p_a(t,x) \qquad \text{for } \mathscr{L}^N\text{-a.e. } x \in \Omega,$$

which, since $\sigma_D(t,x) \in K(x)$ for a.e. $x \in \Omega$ thanks to Theorem 3.6, yields (3.10) by convex duality.

Concerning the flow rule on the interfaces, taking the restriction on Γ_{ij} in (3.13) yields, in view of Lemma 3.8,

$$H(x, (\dot u^i(t) - \dot u^j(t)) \odot \nu))\, \mathscr{H}^{N-1} \lfloor \Gamma_{ij} = (\sigma_D(t)\nu)_\tau \cdot (\dot u^i(t) - \dot u^j(t))\, \mathscr{H}^{N-1} \lfloor \Gamma_{ij}.$$

Since, by Theorem 3.5,

$$(\sigma_D(t)\nu)_\tau(x) \in K_\Gamma(x) \qquad \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in \Gamma,$$

and since $\mathbb{I}_{K_\Gamma(x)}$ is the convex conjugate of $a \mapsto H(x, a \odot \nu(x))$, $a \perp \nu(x)$, the flow rule (3.11) follows again by convex analysis arguments.

An identical proof would yield (3.12). □

Remark 3.14. The flow rules (3.11) and (3.12) hold for *any* tangential trace $(\sigma_D\nu)_\tau$ of $\sigma_D\nu$ along Γ and Γ_d: as shown in Subsection 1.2, such a trace is uniquely determined if Γ and Γ_d are of class C^2. ¶

Once again, we should point out that an equally cursory review of the mechanics literature has failed to evidence any awareness of the fact that a boundary (or interface) flow rule should also be imposed in the classical formulation of elasto-plasticity as demonstrated above.

Remark 3.15. Under strict convexity assumption for the set of admissible stresses in the phases, a proof similar to that of [3, Theorem 6.6] would establish that a precise representative of $\sigma(t)$ that satisfies the flow rule $\mathscr{L}^N + |\dot p(t)|$-a.e. can be defined in an intrinsic manner. In other words, a flow rule on the support of the singular part of $\dot p(t)$ in the phases can be established. ¶

At the close of this section, we can unambiguously assert that, modulo adequate smoothness assumptions on the Dirichlet boundary and/or interfaces and on the relative boundary of the Dirichlet boundary, the quasi-static evolution evidenced in Theorem 2.7 is also a classical evolution in the sense of Definition 3.1. In doing so, we have also uncovered byproducts of a classical evolution which are, to our knowledge, missing items in the mechanics literature on elasto-plasticity:

- A stress admissibility condition on any smooth enough interface and also on any "Dirichlet boundary", *i.e.*, that where a hard device is applied (see Theorem 3.5);
- A flow rule on any smooth enough interface and also on any "Dirichlet boundary", *i.e.*, that where a hard device is applied (see Theorem 3.13).

It would be premature to gauge the impact of such a lapse on e.g. numerical models for elasto-plastic evolutions.

4 Quasi-static evolution via vanishingly small linear isotropic hardening

In this section we establish that the model of quasi-static evolution of multi-phase materials studied in the previous sections arises naturally as limit of models with vanishing hardening.

We thus consider, once again, a *geometrically admissible multiphase domain* Ω, assume that the Hooke's law satisfies (2.3) and that the set of admissible stresses $K(x)$ satisfies (2.4), while the multimap

$$x \multimap K_i(x) \text{ is continuous on each phase } \overline{\Omega}_i. \tag{4.1}$$

When hardening is present, the plastic strains cannot concentrate and create plastic slips. Thus, interfacial conditions – like those that prescribe the value of the dissipation functional or the set of admissible stresses – are no longer necessary. Yet, by obtaining the perfect elasto-plastic evolution for a multi-phase material as limit of a vanishingly small hardening evolution, we demonstrate that the interfacial conditions that were imposed on H at the onset of Section 2 are precisely those that are compatible with vanishingly small hardening models.

Hardening is usually modeled through an internal variable $\zeta \leq 1$ which quantifies the increase of the yield surfaces. The dependence of the set of admissible stresses upon ζ is as follows in the case of isotropic hardening:

$$\sigma_D(x) \in (1-\zeta(x))K(x) \qquad \text{for a.e. } x \in \Omega.$$

This condition can be viewed as a condition on the pair $(\sigma_D(x), \zeta(x))$ similar to that of perfect plasticity, *i.e.*,

$$(\sigma_D(x), \zeta(x)) \in \hat{K}(x) \qquad \text{for a.e. } x \in \Omega,$$

where $\hat{K}(x) := \{(\sigma,\zeta) \in \mathrm{M}^N_D \times]-\infty,1] : \sigma \in (1-\zeta)K(x)\} \subseteq \mathrm{M}^N_D \times \mathbb{R}$ is a convex set. The associated Legendre transform of its indicator function is readily shown to be given by the following positively one-homogeneous function

$$\hat{H}(x,p,z) := \begin{cases} z & \text{if } H(x,p) \le z \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2}$$

where z is the dual variable to ζ and $H(x,p) := \sup_{\sigma \in K(x)} \sigma \cdot p$ is the support function of $K(x)$. As one would expect, the function $\hat{H}$ plays the role of the dissipation.

The functional framework required to study evolutions in presence of isotropic hardening is much simpler than that in the case of perfect elasto-plasticity, since no concentration of the plastic strains will occur. As a consequence p is an L^2-function, and the displacement u acquires Sobolev regularity. More precisely,

Definition 4.1 (Admissible configurations with hardening). Given an element $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$,

$$(u,e,p,z) \in H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \times L^2(\Omega;\mathrm{M}^N_D) \times L^2(\Omega)$$

is an admissible configuration for w iff

$$Eu = e + p \qquad \text{in } \Omega, \qquad u = w \qquad \text{on } \Gamma_d$$

and

$$H(x,p(x)) \le z(x) \qquad \text{for a.e. } x \in \Omega. \tag{4.3}$$

We denote the set of admissible configurations for w by $\hat{\mathscr{A}}(w)$.

In order to formulate the notion of quasi-static evolutions in the presence of linear isotropic hardening, we need to introduce the dissipation functional $\hat{\mathscr{H}}$. For every $(p,z) \in L^2(\Omega;\mathrm{M}^N_D) \times L^2(\Omega)$ we set

$$\hat{\mathscr{H}}(p,z) := \int_\Omega \hat{H}(x,p(x),z(x))\,dx.$$

In view of (4.2) and (4.3), it is immediate that, for $(u,e,p,z) \in \mathscr{A}(w)$,

$$\hat{\mathscr{H}}(p,z) = \int_\Omega z\,dx = \|z\|_1. \tag{4.4}$$

Finally, consider $t \mapsto (p(t),z(t)) \in L^2(\Omega;\mathrm{M}^N_D) \times L^2(\Omega)$, $t \in [0,T]$. For $[a,b] \subseteq [0,T]$, we set, as usual,

$$\hat{\mathscr{D}}(a,b;p,z) := \sup\left\{ \sum_{j=1}^k \hat{\mathscr{H}}\left(p(t_j)-p(t_{j-1}), z(t_j)-z(t_{j-1})\right) : \right. \\ \left. a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

Notice that $t \mapsto p(t)$ can be seen in particular as a map from $[0,T]$ to $\mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D)$, and that, for every $t \in [0,T]$,

$$\mathscr{D}(0,t;p) \le \hat{\mathscr{D}}(0,t;p,z), \tag{4.5}$$

where $\mathscr{D}(0,t;p)$ is given in (2.15).

The simplest kind of isotropic hardening is that where the free energy, in general a function of both e and z, is uncoupled as $\mathscr{Q}(e)+\mathscr{Q}_h(z)$, with $\mathscr{Q}_h$ convex, and linear hardening corresponds to taking $\mathscr{Q}_h$ to be a quadratic functional, *i.e.,*

$$\mathscr{Q}_h(z) := \frac{h^2}{2}\int_\Omega z^2\,dx,$$

where $h>0$ is a hardening parameter.

Let us assume that the boundary displacement w lies in $AC(0,T;H^1(\mathbb{R}^N;\mathbb{R}^N))$. We are now in a position to define a quasi-static evolution for linear isotropic hardening.

Definition 4.2 (Quasi-static evolution for linear isotropic hardening). A mapping

$$t \mapsto (u(t),e(t),p(t),z(t)) \in \hat{\mathscr{A}}(w(t))$$

is a quasi-static evolution relative to w iff the following conditions hold for every $t\in[0,T]$:

(a) Global stability: for every $(v,\eta,q,\beta)\in\hat{\mathscr{A}}(w(t))$

$$\mathscr{Q}(e(t))+\mathscr{Q}_h(z(t)) \le \mathscr{Q}(\eta)+\mathscr{Q}_h(\beta)+\hat{\mathscr{H}}(q-p(t),\beta-z(t)). \tag{4.6}$$

(b) Energy equality: $(p,z)\in BV\left(0,T;L^1(\Omega;\mathbb{M}^N_D)\times L^1(\Omega)\right)$ and

$$\mathscr{Q}(e(t))+\mathscr{Q}_h(z(t))+\hat{\mathscr{D}}(0,t;p,z)=\mathscr{Q}(e(0))+\mathscr{Q}_h(z(0))+\int_0^t\int_\Omega \sigma(\tau)\cdot E\dot{w}(\tau)\,dxd\tau,$$

where $\sigma(t):=\mathbb{C}e(t)$.

With arguments similar to those of Remark 2.6, it is easily shown that the integral appearing in the right-hand side of the energy equality is indeed well defined.

We state the following existence result for a quasistatic evolution without proof; see e.g [7] for an equivalent derivation in a more classical setting. The proof in our specific setting could easily be performed along the lines of that of Theorem 2.7.

Proposition 4.3. *Assume that Ω is a* geometrically admissible multiphase domain *and that assumptions* (2.3), (2.4), (4.1) *are satisfied. Let $(u_0,e_0,p_0,z_0)\in\hat{\mathscr{A}}(w(0))$ be a globally stable configuration. Then there exists a unique quasistatic evolution $\{t\mapsto(u(t),e(t),p(t),z(t)) : t\in[0,T]\}$ relative to w such that $(u(0),e(0),p(0),z(0))=(u_0,e_0,p_0,z_0)$. Moreover,*

$$(u,e,p,z)\in AC\left(0,T;H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega;\mathbb{M}^N_{\rm sym})\times L^2(\Omega;\mathbb{M}^N_D)\times L^2(\Omega)\right),$$

and

$$\hat{\mathscr{D}}(0,t;p,z)=\int_0^t \hat{\mathscr{H}}(\dot{p}(s),\dot{z}(s))\,ds=\int_0^t\int_\Omega \dot{z}(s)\,dx\,ds.$$

Remark 4.4 (**Classical conditions**). The quasi-static evolution of Definition 4.2 can be reinterpreted in more classical terms. We omit the details of the derivation.

(a) *Balance equations and stress admissibility.* By taking the variation with respect to $(v,\eta,q,\beta)\in\hat{\mathscr{A}}(0)$ global stability implies that

$$\int_\Omega (\sigma(t)\cdot\eta+h^2 z(t)\beta)\,dx+\hat{\mathscr{H}}(q,\beta)\geq 0 \quad \text{for every } (v,\eta,q,\beta)\in\hat{\mathscr{A}}(0),$$

so that

$$\int_\Omega (-\sigma(t)\cdot\eta+\zeta(t)\beta)\,dx\leq\hat{\mathscr{H}}(q,\beta),$$

where we have set $\zeta(t):=-h^2 z(t)$.

Considering $q=0$ and $\beta=0$, we obtain the balance equations

$$\begin{cases}\operatorname{div}\sigma(t)=0 & \text{in } \Omega\\ \sigma(t)\nu=0 & \text{on } \partial\Omega\setminus\bar{\Gamma}_d.\end{cases} \tag{4.7}$$

Taking $v=0$ and $\eta=-q$ with $q\in C_c^\infty(\Omega;\mathrm{M}_D^N)$ we get

$$\int_\Omega \sigma_D(t)\cdot q+\zeta(t)\beta\,dx\leq\hat{\mathscr{H}}(q,\beta),$$

from which we infer the constraint

$$(\sigma_D(t,x),\zeta(t,x))\in\hat{K}(x) \qquad \text{for a.e. } x\in\Omega,$$

i.e.,

$$\sigma_D(t,x)\in K(t,x) \qquad \text{for a.e. } x\in\Omega, \tag{4.8}$$

with $K(t,x):=(1-\zeta(t,x))K(x)$.

(b) *Flow rule.* Note that the flow rules derived below will not be used in the sequel. Differentiating the energy equality and recalling (4.3) and (4.4) eventually leads to

$$\dot{p}(t,x)\in N_{K(t,x)}(\sigma_D(t,x)) \qquad \text{for a.e. } x\in\{\dot{p}(t)\neq 0\}, \tag{4.9}$$

and

$$z(t,x)=\int_0^t H(x,\,\dot{p}(\tau,x))\,d\tau \qquad \text{for a.e. } x\in\Omega.$$

That last equality implies that

$$K(t,x)=\left(1+h^2\int_0^t H(x,\,\dot{p}(\tau,x))\,d\tau\right)K(x),$$

so that the yield set increases isotropically depending linearly on the accumulated plastic strain measure $\int_0^t H(x,\,\dot{p}(\tau,x))d\tau$. ¶

So as to prove that evolutions with vanishing hardening approach evolutions for our elasto-plastic model, we take a hardening constant h that tends to 0.

Moreover, for simplicity sake, we also assume that

$$w(0)=0, \qquad (u_0,e_0,p_0,z_0)=(0,0,0,0), \tag{4.10}$$

i.e., that the initial state of the evolution is undeformed. This simplification is so that to there be no conditions on the initial minimizing state $(u_0,e_0,p_0)\in\mathscr{A}(w(0))$

for that state to also be the first three entries of a minimizer for the hardening problems in the sense of (4.6).

We denote by $t \mapsto (u_h(t), e_h(t), p_h(t), z_h(t))$ the associated quasistatic evolution delivered by Proposition 4.3. The following result holds.

Theorem 4.5. *Assume that Ω is a* geometrically admissible multiphase domain *and that assumptions* (2.3), (2.4), (4.1) *are satisfied. Also assume* (4.10) *and the admissibility of $\partial\lfloor_{\partial\Omega}\Gamma_d$ (see* (6.20)*).*

There exists $h_n \to 0$ and a quasistatic evolution $t \mapsto (u(t), e(t), p(t))$ relative to w in the sense of Definition 2.5 with

$$(u(0), e(0), p(0)) = (0,0,0)$$

such that, setting $(u_n, e_n, p_n) := (u_{h_n}, e_{h_n}, p_{h_n})$, then

$$\begin{aligned} u_n(t) &\overset{*}{\rightharpoonup} u(t) && \textit{weakly}^* \textit{ in } BD(\Omega) \\ e_n(t) &\to e(t) && \textit{strongly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) \\ p_n(t) &\overset{*}{\rightharpoonup} p(t) && \textit{weakly}^* \textit{ in } \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^N_D), \end{aligned}$$

for every $t \in [0,T]$. Finally, for every $t \in [0,T]$,

$$\lim_n \hat{\mathscr{D}}(0,t;p_n,z_n) = \mathscr{D}(0,t;p).$$

Proof. Fix $h_n \to 0$. We divide the proof into several steps.

Step 1. As a first step, we deduce some compactness properties of the solution-sequences. Under our assumptions, the energy equality reads

$$\mathscr{Q}(e_n(t)) + \mathscr{Q}_{h_n}(z_n(t)) + \hat{\mathscr{D}}(0,t;p_n,z_n) = \int_0^t \int_\Omega \sigma_n(\tau) \cdot E\dot{w}(\tau)\, dx\, d\tau,$$

where $\sigma_n(\tau) := \mathbb{C}e_n(\tau)$. In view of the admissibility condition $H(x, p_n(t)) \leq z_n(t)$ and of (4.5) one readily infers the existence of $C > 0$ such that, for every $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\|e_n(t)\|_2 + \|h_n z_n(t)\|_2 + \mathscr{V}(0,t;p_n) \leq C. \tag{4.11}$$

Let $\Omega' \subseteq \mathbb{R}^N$ be open, bounded and such that $\Omega \cup \Gamma_d = \overline{\Omega} \cap \Omega'$. We extend $(u_n(t), e_n(t), p_n(t))$ to Ω' by setting

$$u_n(t) = w(t), \qquad e_n(t) = Ew(t), \qquad p_n(t) = 0 \qquad \text{on } \Omega' \setminus \Omega.$$

Clearly

$$Eu_n(t) = e_n(t) + p_n(t) \qquad \text{on } \Omega'.$$

By a generalized version of Helly's theorem (see [10, Theorem 3.2]), there exists a subsequence, not relabeled, such that, for every $t \in [0,T]$,

$$p_n(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega'; \mathrm{M}^N_D),$$

for some $p \in BV(0,T;\mathscr{M}_b(\Omega';\mathrm{M}^N_D))$. For every $t \in [0,T]$, there exists a further subsequence $\{n_t\}$ such that

$$e_{n_t}(t) \rightharpoonup e(t) \qquad \text{weakly in } L^2(\Omega';\mathrm{M}^N_{\mathrm{sym}}),$$

and, appealing to Korn's inequality in BD,

$$u_{n_t}(t) \rightharpoonup u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega'),$$

for some $u(t) \in BD(\Omega')$ and $e(t) \in L^2(\Omega';\mathrm{M}^N_{\mathrm{sym}})$ with

$$Eu(t) = e(t) + p(t) \qquad \text{on } \Omega'.$$

Clearly $u(t) = w(t)$, $e(t) = Ew(t)$ and $p(t) = 0$ on $\Omega' \setminus \overline{\Omega}$, so that we deduce

$$p(t)\lfloor\Gamma_d = (w(t) - u(t)) \odot \nu\, \mathscr{H}^{N-1}\lfloor\Gamma_d.$$

As a consequence, by restricting $(u(t), e(t))$ to Ω and $p(t)$ to $\Omega \cup \Gamma_d$, we get

$$(u(t), e(t), p(t)) \in \mathscr{A}(w(t))$$

with

$$\begin{cases} u_{n_t}(t) \overset{*}{\rightharpoonup} u(t) & \text{weakly}^* \text{ in } BD(\Omega), \\ e_{n_t}(t) \rightharpoonup e(t) & \text{weakly in } L^2(\Omega';\mathrm{M}^N_{\mathrm{sym}}), \\ p_n(t) \overset{*}{\rightharpoonup} p(t) & \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega \cup \Gamma_d;\mathrm{M}^N_D). \end{cases} \tag{4.12}$$

Setting

$$\zeta_{n_t}(t) := -h_n^2 z_{n_t}(t),$$

we get, in view of the bound (4.11),

$$\zeta_{n_t}(t) \to 0 \qquad \text{strongly in } L^2(\Omega). \tag{4.13}$$

Step 2. A second step is devoted to global stability. Since

$$\sigma_{n_t}(t) = \mathbb{C}e_{n_t}(t) \rightharpoonup \sigma(t) := \mathbb{C}e(t) \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}),$$

we deduce, using the balance equations (4.7), that

$$\begin{cases} \operatorname{div}\sigma(t) = 0 & \text{in } \Omega \\ \sigma(t)\nu = 0 & \text{on } \partial\Omega \setminus \bar{\Gamma}_d. \end{cases}$$

Concerning the stress constraint, (4.8) implies that

$$(\sigma_{n_t})_D(t,x) \in (1 - \zeta_{n_t}(t,x))K(x) \qquad \text{for a.e. } x \in \Omega.$$

Since convex combinations of elements of $\{(\sigma_{n_t})_D(t)\}$ converge to $\sigma(t)$, strongly in $L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, (4.13) implies in turn that

$$\sigma_D(t,x) \in K(x) \qquad \text{for a.e. } x \in \Omega.$$

We conclude that $\sigma(t) \in \mathscr{K}$, so that Proposition 3.9 applies and, for every $(v,\eta,q) \in \mathscr{A}(0)$,

$$H\left(x, \frac{q}{|q|}\right) d|q| \geq \langle \sigma_D(t), q \rangle \qquad \text{as measures on } \Omega \cup \Gamma_d.$$

Thanks to the admissibility of $\partial_{\partial\Omega}\Gamma_d$, we can compute the masses and we obtain, in view of (6.3) (with $f \equiv g \equiv 0$),

$$\mathscr{H}(q) \geq -\int_\Omega \sigma(t)\cdot\eta\,dx$$

so that

$$-\mathscr{H}(q) \leq \int_\Omega \sigma(t)\cdot\eta\,dx \leq \mathscr{H}(-q).$$

In view of the beginning of Remark 2.6, we conclude that the triplet $(u(t),e(t),p(t))$ in $\mathscr{A}(w(t))$ is a globally stable configuration. In particular, $(u(t),e(t))$ is uniquely determined by $p(t)$, so that the convergences in (4.12) hold without passing to a subsequence.

Step 3. We now derive the energy equality. For every $t\in[0,T]$, using (4.5) together with Proposition 2.3, we get

$$\begin{aligned}
\mathscr{Q}(e(t))+\mathscr{D}(0,t;p) &\leq \liminf\nolimits_n \mathscr{Q}(e_n(t)) + \liminf\nolimits_n \hat{\mathscr{D}}(0,t;p_n,z_n)\\
&\leq \liminf\nolimits_n \left[\mathscr{Q}(e_n(t))+\mathscr{Q}_{h_n}(z_n(t))+\hat{\mathscr{D}}(0,t;p_n,z_n)\right]\\
&\leq \limsup\nolimits_n \left[\mathscr{Q}(e_n(t))+\mathscr{Q}_{h_n}(z_n(t))+\hat{\mathscr{D}}(0,t;p_n,z_n)\right]\\
&= \textstyle\int_0^t\int_\Omega \sigma(\tau)\cdot E\dot{w}(\tau)\,dx\,d\tau \leq \mathscr{Q}(e(t))+\mathscr{D}(0,t;p).
\end{aligned}$$

Above, the last equality is obtained by dominated convergence and the last inequality is a consequence of the global stability of $(u(t),e(t),p(t))\in\mathscr{A}(w(t))$ proved in step 2; see the end of the proof of Theorem 2.7 after (2.29).

We conclude that the energy equality holds, so that $t\mapsto(u(t),e(t),p(t))$ is a quasistatic evolution for the multi-phase material according to Definition 2.5. Moreover, the previous inequalities entail that

$$\lim_n \left[\mathscr{Q}(e_n(t))+\mathscr{Q}_{h_n}(z_n(t))+\hat{\mathscr{D}}(0,t;p_n,z_n)\right] = \mathscr{Q}(e(t))+\mathscr{D}(0,t;p)$$

from which we infer

$$\lim_n \mathscr{Q}(e_n(t)) = \mathscr{Q}(e(t)) \qquad \text{and} \qquad \lim_n \hat{\mathscr{D}}(0,t;p_n,z_n) = \mathscr{D}(0,t;p).$$

Thus in particular

$$e_n(t)\to e(t) \qquad \text{strongly in } L^2(\Omega;\mathbb{M}^N_{\mathrm{sym}}),$$

which concludes the proof. □

5 Semicontinuity of the dissipation functional of multiphase materials

This section is devoted to the proof of Proposition 2.3 on which we based the existence of a quasi-static evolution for a multiphase material.

We will make use of the following general observation:

Lemma 5.1. *Let $A \subseteq \mathbb{R}^N$ be open, bounded and with C^1-boundary. Assume that $u_n \in BD(A)$ is such that*

$$u_n \stackrel{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BD(A).$$

Considering Eu_n as a measure on $\mathbb{R}^N$, let us assume that

$$Eu_n \stackrel{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\mathbb{R}^N;\mathrm{M}^N_{\rm sym}).$$

Then μ is supported in $\bar{A}$ and

$$\mu\lfloor\partial A = a \odot \nu\, \lambda, \tag{5.1}$$

where λ is a finite positive measure supported on ∂A, $a : \partial A \to \mathbb{R}^N$ is a Borel function, and ν is the exterior normal to ∂A.

Proof. Since Eu_n is supported in $\bar{A}$, the support of μ is contained in $\bar{A}$. Given $\varphi \in C^1_c(\mathbb{R}^N;\mathrm{M}^N_{\rm sym})$ and using the integration by parts formula for *BD* functions (see [18, Chapter 2, Theorem 2.1]), we can write

$$\int_{\bar{A}} \varphi\, d\mu = \lim_n \int_A \varphi\, dEu_n = \lim_n \int_{\partial A} \varphi \cdot (u_n \odot \nu)\, d\mathscr{H}^{N-1} - \int_A \mathrm{div}\varphi \cdot u_n\, dx.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $BD(A)$, the traces of $\{u_n\}$ are bounded in $L^1(\partial A;\mathbb{R}^N)$. Up to a subsequence,

$$u_n \mathscr{H}^{N-1}\lfloor\partial A \stackrel{*}{\rightharpoonup} \eta \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\partial A;\mathbb{R}^N),$$

for a suitable $\eta \in \mathscr{M}_b(\partial A;\mathbb{R}^N)$. Since $\nu \in C^0(\partial A;\mathbb{R}^N)$, we obtain

$$\begin{aligned}\int_{\bar{A}} \varphi\, d\mu &= \int_{\partial A} \varphi \cdot \left(\frac{\eta}{|\eta|} \odot \nu\right)\, d|\eta| - \int_A \mathrm{div}\varphi \cdot u\, dx\\ &= \int_{\partial A} \varphi \cdot \left(\frac{\eta}{|\eta|} \odot \nu\right)\, d|\eta| - \int_{\partial A} \varphi \cdot (u \odot \nu)\, d\mathscr{H}^{N-1} + \int_A \varphi\, dEu.\end{aligned}$$

We conclude that

$$\mu\lfloor\partial A = \frac{\eta}{|\eta|} \odot \nu\ |\eta| - u \odot \nu\, \mathscr{H}^{N-1}\lfloor\partial A,$$

so that (5.1) follows by choosing $\lambda := |\eta| + \mathscr{H}^{N-1}\lfloor\partial A$, and $a := \eta/\lambda - bu$, where η/λ and b are the Radon-Nikodym derivatives of η and $\mathscr{H}^{N-1}\lfloor\partial A$ with respect to λ. □

We are now in a position to prove that $\mathscr{H}$ satisfies the required lower semicontinuity property.

Let $(u_n, e_n, p_n) \in \mathscr{A}(w_n)$ and $(u, e, p) \in \mathscr{A}(w)$ satisfy the convergences in (2.14). Setting $\Gamma^i_d := (\Gamma_d \setminus S') \cap \overline{\Omega}_i$, we can write

$$p_n = \sum_i p^i_n + \sum_{i\neq j} p^{ij}_n$$

where $p_n^i := p_n \lfloor (\Omega_i \cup \Gamma_d^i)$, and $p_n^{ij} := p_n \lfloor \Gamma_{ij}$. We can assume that, up to a subsequence,

$$p_n^i \overset{*}{\rightharpoonup} p^i \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}_D^N),$$

and

$$p_n^{ij} \overset{*}{\rightharpoonup} p^{ij} \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega \cup \Gamma_d; \mathrm{M}_D^N).$$

Clearly,

$$p = \sum_i p^i + \sum_{i \neq j} p^{ij},$$

with $\mathrm{supp}(p^i) \subseteq \overline{\Omega}_i$ and $\mathrm{supp}(p^{ij}) \subseteq \Gamma_{ij} \cup S'$.

By Reshetnyak lower semi-continuity theorem (see e.g. [2, Theorem 2.38], or [14, Theorem 1.7]), we get

$$\begin{aligned}
&\liminf_n \int_{\Omega \cup \Gamma_d} H\left(x, \frac{p_n^i}{|p_n^i|}\right) d|p_n^i| = \liminf_n \int_{\overline{\Omega}_i} H_i\left(x, \frac{p_n^i}{|p_n^i|}\right) d|p_n^i| \geq \\
&\quad \int_{\overline{\Omega}_i} H_i\left(x, \frac{p^i}{|p^i|}\right) d|p^i| \geq \int_{\Omega_i \cup \Gamma_d^i} H_i\left(x, \frac{p^i}{|p^i|}\right) d|p^i| + \int_{\Gamma} H_i\left(x, \frac{p^i}{|p^i|}\right) d|p^i| \\
&\qquad = \int_{\Omega_i \cup \Gamma_d^i} H\left(x, \frac{p^i}{|p^i|}\right) d|p^i| + \sum_{j \neq i} \int_{(\Gamma_{ij} \setminus S)} H_i\left(x, \frac{p^i}{|p^i|}\right) d|p^i|.
\end{aligned}$$

By assumption $e_n \rightharpoonup e$ weakly in $L^1(\Omega; \mathrm{M}_{\mathrm{sym}}^N)$, so that

$$Eu_n \lfloor \Omega_i \overset{*}{\rightharpoonup} e \mathscr{L}^N \lfloor \Omega_i + p^i \qquad \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega; \mathrm{M}_{\mathrm{sym}}^N).$$

For every $j \neq i$, Lemma 5.1 implies that, with a normal ν on Γ_{ij} pointing from Ω_j to Ω_i,

$$p^i \lfloor (\Gamma_{ij} \setminus S) = -a^{ij} \odot \nu \, \lambda^{ij}, \tag{5.2}$$

for suitable λ^{ij}'s finite positive measures supported on $\Gamma_{ij} \setminus S$, and suitable a^{ij}'s Borel functions on $\Gamma_{ij} \setminus S$ with values in $\mathbb{R}^N$ such that $a^{ij}(x) \perp \nu(x)$ for λ^{ij}-a.e. $x \in (\Gamma_{ij} \setminus S)$ (recall that p^i takes its values in M_D^N). Thus,

$$\begin{aligned}
&\liminf_n \int_{\Omega \cup \Gamma_d} H\left(x, \frac{p_n^i}{|p_n^i|}\right) d|p_n^i| \\
&\qquad \geq \int_{\Omega_i \cup \Gamma_d^i} H\left(x, \frac{p^i}{|p^i|}\right) d|p^i| + \sum_{j \neq i} \int_{\Gamma_{ij} \setminus S} H_i\left(x, -a^{ij} \odot \nu\right) d\lambda^{ij}.
\end{aligned} \tag{5.3}$$

Concerning p_n^{ij}, we have

$$p_n^{ij} = (u_n^i - u_n^j) \odot \nu \, \mathscr{H}^{N-1} \lfloor \Gamma_{ij},$$

where u_n^i and u_n^j are the traces of u_n on Γ_{ij} coming from Ω_i and Ω_j respectively. In view of the definition of H on Γ_{ij} (see (2.8)), and since the inf-convolution is indeed attained as a minimum, we get

$$\int_{\Gamma_{ij}} H\left(x, \frac{p_n^{ij}}{|p_n^{ij}|}\right) d|p_n^{ij}| = \int_{\Gamma_{ij}\setminus S} H(x, (u_n^i - u_n^j) \odot \nu)\, d\mathscr{H}^{N-1}$$
$$= \int_{\Gamma_{ij}\setminus S} H_i(x, b_{i,n}^{ij} \odot \nu) + H_j(x, -b_{j,n}^{ij} \odot \nu)\, d\mathscr{H}^{N-1}, \tag{5.4}$$

for suitable Borel functions $b_{i,n}^{ij}, b_{j,n}^{ij} : \Gamma_{ij} \to \mathbb{R}^N$ such that

$$b_{i,n}^{ij}(x) - b_{j,n}^{ij}(x) = u_n^i(x) - u_n^j(x) \text{ for } \mathscr{H}^{N-1}\text{-a.e. } x \in \Gamma_{ij}$$

with

$$b_{i,n}^{ij}(x) \perp \nu(x), b_{j,n}^{ij}(x) \perp \nu(x) \text{ for } \mathscr{H}^{N-1}\text{-a.e. } x \in \Gamma_{ij}. \tag{5.5}$$

Note that the Borel character of the functions $b_{i,n}^{ij}, b_{j,n}^{ij}$ can be argued by approximating $u_n^i - u_n^j$ along Γ_{ij} by simple functions, and upon recalling that ν is continuous.

In view of the coercivity estimate (2.7) we get

$$\int_{\Gamma_{ij}\setminus S} \left\{ |b_{i,n}^{ij} \odot \nu| + |b_{j,n}^{ij} \odot \nu| \right\} d\mathscr{H}^{N-1} \le C,$$

for a suitable constant $C > 0$. Thanks to (5.5), the bound above actually implies that the measures

$$\eta_{i,n}^{ij} := b_{i,n}^{ij} \mathscr{H}^{N-1} \lfloor (\Gamma_{ij} \setminus S) \qquad \text{and} \qquad \eta_{j,n}^{ij} := b_{j,n}^{ij} \mathscr{H}^{N-1} \lfloor (\Gamma_{ij} \setminus S)$$

are bounded in n. Thus, we can assume that, up to a subsequence that will not be relabeled,

$$\begin{cases} \eta_{i,n}^{ij} \overset{*}{\rightharpoonup} \eta_i^{ij} = b_i^{ij} |\eta_i^{ij}| & \text{weakly}^* \text{ in } \mathscr{M}_b(\Gamma_{ij} \setminus S; \mathbb{R}^N) \\ \eta_{j,n}^{ij} \overset{*}{\rightharpoonup} \eta_j^{ij} = b_j^{ij} |\eta_j^{ij}| & \text{weakly}^* \text{ in } \mathscr{M}_b(\Gamma_{ij} \setminus S; \mathbb{R}^N). \end{cases}$$

Since the normal vector field ν is continuous,

$$\begin{cases} b_{i,n}^{ij} \odot \nu\, \mathscr{H}^{N-1} \lfloor ((\Gamma_{ij} \setminus S) \overset{*}{\rightharpoonup} b_i^{ij} \odot \nu\, |\eta_i^{ij}| & \text{weakly}^* \text{ in } \mathscr{M}_b(\Gamma_{ij} \setminus S; \mathrm{M}_D^N) \\ b_{j,n}^{ij} \odot \nu\, d\mathscr{H}^{N-1} \lfloor (\Gamma_{ij} \setminus S) \overset{*}{\rightharpoonup} b_j^{ij} \odot \nu\, |\eta_j^{ij}| & \text{weakly}^* \text{ in } \mathscr{M}_b(\Gamma_{ij} \setminus S; \mathrm{M}_D^N). \end{cases}$$

Moreover, in view of (5.4), Reshetnyak's lower semi-continuity theorem yields

$$\liminf_n \int_{\Gamma_{ij}} H\left(x, \frac{p_n^{ij}}{|p_n^{ij}|}\right) d|p_n^{ij}|$$
$$\ge \liminf_n \int_{\Gamma_{ij}\setminus S} H_i(x, b_{i,n}^{ij} \odot \nu) + H_j(x, -b_{j,n}^{ij} \odot \nu)\, d\mathscr{H}^{N-1}$$
$$\ge \int_{\Gamma_{ij}\setminus S} H_i(x, b_i^{ij} \odot \nu)\, d|\eta_i^{ij}| + \int_{\Gamma_{ij}\setminus S} H_j(x, -b_j^{ij} \odot \nu)\, d|\eta_j^{ij}|. \tag{5.6}$$

Recalling (5.2) we clearly have

$$\text{(5.7)}\quad p\lfloor((\Gamma_{ij}\setminus S)) = -a^{ij}\odot\nu\,\lambda^{ij} + a^{ji}\odot\nu\,\lambda^{ji} + b_i^{ij}\odot\nu\,|\eta_i^{ij}| - b_j^{ij}\odot\nu\,|\eta_j^{ij}| \\ = (c^i - c^j)\odot\nu\,\zeta^{ij},$$

where $\zeta^{ij} := \lambda^{ij} + \lambda^{ji} + |\eta_i^{ij}| + |\eta_j^{ij}|$, and c^i, c^j are suitable Borel functions on $\Gamma_{ij}\setminus S$ with values in $\mathbb{R}^N$ such that

$$c^i\odot\nu\,\zeta^{ij} = -a^{ij}\odot\nu\,\lambda^{ij} + b_i^{ij}\odot\nu\,|\eta_i^{ij}|,$$

idem for c^j. Further,

$$c^i(x)\perp\nu(x),\ c^j(x)\perp\nu(x) \qquad \text{for } \zeta^{ij}\text{-a.e. } x\in\Gamma_{ij}\setminus S.$$

Since p_n does not charge $S\cup S'$ we get

$$\mathscr{H}(p_n) = \sum_i \int_{\Omega_i\cup\Gamma_d^i} H\left(x, \frac{p_n^i}{|p_n^i|}\right) d|p_n^i| + \sum_{i\neq j}\int_{\Gamma_{ij}\setminus S} H\left(x, \frac{p_n^{ij}}{|p_n^{ij}|}\right) d|p_n^{ij}|,$$

so that, thanks to (5.3) and (5.6),

$$\begin{aligned}
&\liminf_n \mathscr{H}(p_n)\\
&\quad\geq \sum_i \liminf_n \int_{\Omega_i\cup\Gamma_d^i} H\left(x, \frac{p_n^i}{|p_n^i|}\right) d|p_n^i| + \sum_{i\neq j}\liminf_n \int_{\Gamma_{ij}\setminus S} H\left(x, \frac{p_n^{ij}}{|p_n^{ij}|}\right) d|p_n^{ij}|\\
&\quad\geq \sum_i\left(\int_{\Omega_i\cup\Gamma_d^i} H\left(x, \frac{p^i}{|p^i|}\right) d|p^i| + \sum_{j\neq i}\int_{\Gamma_{ij}\setminus S} H_i\left(x, -a^{ij}(x)\odot\nu(x)\right) d\lambda^{ij}(x)\right)\\
&\qquad + \sum_{i\neq j}\left(\int_{\Gamma_{ij}\setminus S} H_i(x, b_i^{ij}\odot\nu)\, d|\eta_i^{ij}| + \int_{\Gamma_{ij}\setminus S} H_j(x, -b_j^{ij}\odot\nu)\, d|\eta_j^{ij}|\right)\\
&\quad = \int_{\cup_i(\Omega_i\cup\Gamma_d^i)} H\left(x, \frac{p}{|p|}\right) d|p| + \sum_{i\neq j}\left(\int_{\Gamma_{ij}\setminus S} H_i\left(x, -a^{ij}(x)\odot\nu(x)\right) d\lambda^{ij}(x)\right.\\
&\qquad\qquad + \int_{\Gamma_{ij}\setminus S} H_j\left(x, +a^{ji}(x)\odot\nu(x)\right) d\lambda^{ji}(x)\\
&\qquad\qquad \left. + \int_{\Gamma_{ij}\setminus S} H_i(x, b_i^{ij}\odot\nu)\, d|\eta_i^{ij}| + \int_{\Gamma_{ij}\setminus S} H_j(x, -b_j^{ij}\odot\nu)\, d|\eta_j^{ij}|\right).
\end{aligned}$$

In view of (5.7), by the definition of H on $\Gamma_{ij} \setminus S$ and the sub-additive character of H_i and H_j, and since p does not charge $S \cup S'$, we deduce that

$$\begin{aligned}
\liminf_n \mathscr{H}(p_n) \geq & \int_{\cup_i \Omega_i \cup (\Gamma_d \setminus S')} H\left(x, \frac{p}{|p|}\right) d|p| \\
& + \sum_{i \neq j} \int_{\Gamma_{ij} \setminus S} H_i\left(x, c^i(x) \odot \nu(x)\right) + H_j(x, -c^j(x) \odot \nu(x))\, d\zeta^{ij}(x) \\
\geq & \int_{\cup_i \Omega_i \cup (\Gamma_d \setminus S')} H\left(x, \frac{p}{|p|}\right) d|p| + \sum_{i \neq j} \int_{\Gamma_{ij} \setminus S} H(x, (c^i(x) - c^j(x)) \odot \nu(x))\, d\zeta^{ij}(x) \\
= & \int_{\cup_i \Omega_i \cup (\Gamma_d \setminus S')} H\left(x, \frac{p}{|p|}\right) d|p| + \sum_{i \neq j} \int_{\Gamma_{ij} \setminus S} H\left(x, \frac{p}{|p|}\right) d|p| = \mathscr{H}(p),
\end{aligned}$$

so that the proof is concluded.

Remark 5.2. In the case where the yield surfaces satisfy an ordering assumption as in [13] (for example when the phases exhibit a Von Mises type behaviour), the dissipation potential H on Γ_{ij} coincides either with H_i or H_j on matrices of the form $a \odot \nu$ with $a \perp \nu$. As far as the values of $\mathscr{H}$ on admissible plastic strains are concerned, we may thus replace H with H_i or H_j on Γ_{ij}, avoiding the value $+\infty$ and gaining lower semi-continuity in (x, ξ). As a consequence, even if Γ is only Lipschitz regular, the lower semi-continuity of $\mathscr{H}$ turns out to be a direct consequence of Reshetnyak lower-semi-continuity theorem. The results of Section 3 can then be established within this framework by treating the interfaces Γ like we did the Dirichlet boundary Γ_d. ¶

6 Duality revisited: the case of Lipschitz boundaries

In this section, we reexamine the duality first investigated in [8] and revisited more recently in [3] between admissible stresses and plastic strains. In both papers, it is assumed that $\partial\Omega$ is of class C^2, and that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is a $(N-2)$-dimensional C^2-manifold. The purpose of what follows is to show that a Lipschitz regularity of $\partial\Omega$, together with appropriate regularity assumptions on $\partial\lfloor_{\partial\Omega}\Gamma_d$, are sufficient as far as duality is concerned.

As in those previous works, we will consider body forces in $L^N(\Omega;\mathbb{R}^N)$ and surface tractions in $L^\infty(\Gamma_t;\mathbb{R}^N)$, with

$$\Gamma_t := \partial\Omega \setminus \overline{\Gamma}_d.$$

Our first result in this direction implies that announced in Remark 3.2.

Proposition 6.1. *The set*

$$\left\{\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) : \mathrm{div}\sigma \in L^N(\Omega;\mathbb{R}^N);\ \sigma_D \in L^\infty(\Omega;\mathrm{M}^N_D)\right\}$$

is a subset of $L^r(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, *for every* $1 \leq r < \infty$, *and*

$$\|\sigma\|_r \leq C\left\{\|\sigma_D\|_\infty + \|\mathrm{div}\sigma\|_N + \|\sigma\|_2\right\}. \tag{6.1}$$

Proof. The proof relies on the following result – sometimes referred to as Korn's theorem – which holds true for all $L^p, 1 < p < \infty$, on a Lipschitz domain (see [12]): if $u \in W^{-1,p}(\Omega)$ with $\nabla u \in W^{-1,p}(\Omega)$, then $u \in L^p(\Omega)$ with

$$\|u\|_p \le C(\|u\|_{W^{-1,p}} + \|\nabla u\|_{W^{-1,p}}),$$

for some $C > 0$ depending on Ω.

Decomposing σ as

$$\sigma = \hat{\sigma}\mathrm{i} + \sigma_D$$

and remarking that, by Sobolev embedding, $L^N(\Omega;\mathbb{R}^N) \subset W^{-1,r}(\Omega;\mathbb{R}^N)$, for $1 \le r < \infty$, the assumption $\operatorname{div}\sigma \in L^N(\Omega;\mathbb{R}^N)$ implies that

$$\nabla\hat{\sigma} \in W^{-1,r}(\Omega) + W^{-1,\infty}(\Omega) \qquad \text{for every } 1 \le r < \infty. \tag{6.2}$$

If $N = 2$, the regularity $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ implies, by Sobolev embedding, that $\hat{\sigma} \in W^{-1,\infty}(\Omega)$. By Korn's theorem and (6.2), $\hat{\sigma} \in L^r(\Omega)$ for every $r \ge 1$.

If $N > 2$, the regularity $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ implies that $\hat{\sigma} \in W^{-1,2N/(N-2)}(\Omega)$. By Korn's theorem and (6.2), $\hat{\sigma} \in L^{2N/(N-2)}(\Omega)$.

Since $2N/(N-2) > 2$, reiteration of the argument proves that $\hat{\sigma} \in L^r(\Omega)$ for every $1 \le r < \infty$. The estimate (6.1) is a direct consequence of the continuity of all mappings involved in this argument. □

Let $f \in L^N(\Omega;\mathbb{R}^N)$ and $g \in L^\infty(\Gamma_t;\mathbb{R}^N)$ be given, and introduce the set

$$\mathscr{K}(f,g) := \left\{\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) : \operatorname{div}\sigma = f \text{ in } \Omega;\ \sigma\nu = g \text{ on } \Gamma_t;\ \sigma_D \in L^\infty(\Omega;\mathrm{M}^N_D)\right\}.$$

Given $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$, define $\mathscr{A}(w)$ and $\mathscr{P}(w)$ as in Definition 2.1, *requiring only that the associated elastic strain e belong to $L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ in lieu of* $L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$. Every $p \in \mathscr{P}(w)$ determines a measure $\tilde{p}$ on all of $\mathbb{R}^N$ through the relation

$$\tilde{p}(B) := p(B \cap (\Omega \cup \Gamma_d))$$

for every Borel set $B \subseteq \mathbb{R}^N$. In the following, we still write p for $\tilde{p}$.

For $\sigma \in \mathscr{K}(f,g)$ and $p \in \mathscr{P}(w)$ with an associated pair $(u,e) \in BD(\Omega) \times L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, we consider, for all $\varphi \in C^\infty_c(\mathbb{R}^N)$,

$$\begin{aligned}\langle\sigma_D, p\rangle(\varphi) := &-\int_\Omega \varphi\sigma\cdot(e - Ew)\,dx - \int_\Omega \varphi f\cdot(u-w)\,dx \\ &- \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,dx + \int_{\Gamma_t}\varphi g\cdot(u-w)\,d\mathscr{H}^{N-1}.\end{aligned} \tag{6.3}$$

The above expression defines a meaningful distribution on $\mathbb{R}^N$ since, according to Proposition 6.1, $\sigma \in L^N(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, while $u \in L^{N/N-1}(\Omega;\mathbb{R}^N)$ in view of the embedding of $BD(\Omega)$ into $L^{N/N-1}(\Omega;\mathbb{R}^N)$. Further, u has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega;\mathbb{R}^N)$. Finally note that, if σ is the restriction to Ω of a C^1-function and if $\mathscr{H}^{N-1}(\partial\lfloor_{\partial\Omega}\Gamma_d) = 0$, then, performing an integration by parts in

BD (see [18, Chapter 2, Theorem 2.1]), the right hand side of (6.3) coincides with the integral of φ with respect to the (well defined) measure $\sigma_D p$.

The following result holds:

Theorem 6.2. *The restriction of the distribution defined by* (6.3) *to the open set* $\mathbb{R}^N \setminus \partial \lfloor_{\partial\Omega} \Gamma_d$ *is a bounded Radon measure such that*

$$|\langle \sigma_D, p\rangle| \leq \|\sigma_D\|_\infty |p| \qquad \text{on } \mathbb{R}^N \setminus \partial \lfloor_{\partial\Omega} \Gamma_d. \tag{6.4}$$

Moreover, the density of its $\mathscr{L}^N$*-absolutely continuous part is* $\sigma_D \cdot p_a$*, where* p_a *is the density of the* $\mathscr{L}^N$*-absolutely continuous part of* p*.*

Proof. Clearly, translating u by w, it is enough to prove the theorem for $w \equiv 0$. We thus consider the distribution on $\mathbb{R}^N \setminus \partial \lfloor_{\partial\Omega} \Gamma_d$ given for every $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \partial \lfloor_{\partial\Omega} \Gamma_d)$ by

$$\begin{aligned} \langle \sigma_D, p\rangle(\varphi) := -\int_\Omega \varphi \sigma \cdot e\, dx - \int_\Omega \varphi f \cdot u\, dx \\ - \int_\Omega \sigma \cdot [u \odot \nabla \varphi]\, dx + \int_{\Gamma_t} \varphi g \cdot u\, d\mathscr{H}^{N-1}. \end{aligned} \tag{6.5}$$

The first part of the theorem follows if we prove that for every $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \partial \lfloor_{\partial\Omega} \Gamma_d)$

$$\langle \sigma_D, p\rangle(\varphi) \leq \|\sigma_D\|_\infty \int_{\Omega \cup \Gamma_d} |\varphi|\, d|p|. \tag{6.6}$$

In order to prove this inequality, let us consider $V \subseteq \mathbb{R}^N$ open and bounded with $\partial \lfloor_{\partial\Omega} \Gamma_d \subseteq V$ and $\bar{V} \cap supp(\varphi) = \emptyset$. Let $W \subseteq \mathbb{R}^N$ be an open neighborhood of $\Gamma_t \setminus V$ with $W \cap \partial\Omega \subset\subset \Gamma_t$, and such that

$$|p|(\partial W) = 0.$$

Let $\psi \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \psi \leq 1$ be such that

$$\begin{cases} \psi = 1 & \text{on an open neighborhood of } \Gamma_t \setminus V, \\ \psi = 0 & \text{outside } W, \end{cases}$$

and let us decompose σ as

$$\sigma = \psi\sigma + (1-\psi)\sigma = \sigma^1 + \sigma^2.$$

By Proposition 6.1, $\operatorname{div} \sigma^i \in L^N(\Omega; \mathbb{R}^N)$. Further,

$$\sigma^1 \nu = \psi g \text{ on } \partial\Omega \qquad \text{and} \qquad \sigma^2 = 0 \text{ on a neighborhood of } \Gamma_t \setminus V. \tag{6.7}$$

We can write

$$\langle \sigma_D, p\rangle(\varphi) = \lambda^1(\varphi) + \lambda^2(\varphi),$$

where

$$\begin{aligned}
\lambda^1(\varphi) \quad &:= \quad -\textstyle\int_\Omega \varphi\sigma^1\cdot e\,dx - \int_\Omega \varphi\,\mathrm{div}\sigma^1\cdot u\,dx - \int_\Omega \sigma^1\cdot[u\odot\nabla\varphi]\,dx \\
&\qquad + \textstyle\int_{\Gamma_t} \varphi g\cdot u\,d\mathscr{H}^{N-1}, \\
\lambda^2(\varphi) \quad &:= \quad -\textstyle\int_\Omega \varphi\sigma^2\cdot e\,dx - \int_\Omega \varphi\,\mathrm{div}\sigma^2\cdot u\,dx - \int_\Omega \sigma^2\cdot[u\odot\nabla\varphi]\,dx.
\end{aligned}$$

We claim that

$$\lambda^1(\varphi) \le \|\sigma_D\|_\infty\|\varphi\|_\infty|p|(W) \tag{6.8}$$

and

$$\lambda^2(\varphi) \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\,d|p|. \tag{6.9}$$

In view of these inequalities,

$$\langle\sigma_D,p\rangle(\varphi) \le \|\sigma_D\|_\infty\|\varphi\|_\infty|p|(W) + \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\,d|p|.$$

Since W is an arbitrary neighborhood of $\Gamma_t\setminus V$ and p does not charges Γ_t, we deduce that

$$\langle\sigma_D,p\rangle(\varphi) \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\,d|p|,$$

so that (6.6) follows.

In order to conclude the proof, we need to prove claims (6.8) and (6.9), together with the form of the absolutely continuous part of $\langle\sigma_D,p\rangle$. We divide the proof into three steps.

Step 1. In a first step we prove claim (6.8). Let us consider $(u_n,e_n,p_n)\in C^\infty(\overline{\Omega};\mathbb{R}^N\times \mathrm{M}^N_{\mathrm{sym}}\times\mathrm{M}^N_D)$ with

$$Eu_n = e_n + p_n \qquad \text{in } \Omega \tag{6.10}$$

and

$$\begin{cases} u_n \to u & \text{strongly in } L^{N/N-1}(\Omega;\mathbb{R}^N) \\ e_n \to e & \text{strongly in } L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \\ p_n \xrightarrow{s} p & \text{strictly in } \mathscr{M}_b(\Omega;\mathrm{M}^N_D). \end{cases} \tag{6.11}$$

Then, because the trace operator is continuous under strict convergence (see e.g. [18, Chapter 2, Theorem 3.1]), $u_n\to u$ strongly in $L^1(\partial\Omega;\mathbb{R}^N)$. Set

$$\begin{aligned}
\lambda^1_n(\varphi) := -\int_\Omega \varphi\sigma^1\cdot e_n\,dx - \int_\Omega \varphi\mathrm{div}\sigma^1\cdot u_n\,dx - \int_\Omega \sigma^1\cdot[u_n\odot\nabla\varphi]\,dx \\
+\int_{\Gamma_t} \varphi g\cdot u_n\,d\mathscr{H}^{N-1}.
\end{aligned}$$

In view of (6.11),

$$\lim_n \lambda^1_n(\varphi) = \lambda^1(\varphi).$$

Integrating by parts, we obtain immediately, in view of (6.7) and since $\psi \equiv 1$ on $supp(\varphi) \cap \Gamma_t$ with $\psi = 0$ on Γ_d,

$$\begin{aligned}\lambda_n^1(\varphi) &= \int_\Omega \varphi\sigma_D^1 \cdot p_n\,dx - \langle \sigma^1\nu, \varphi u_n\rangle + \int_{\Gamma_t} \varphi g\cdot u_n\,d\mathscr{H}^{N-1} \\ &= \int_\Omega \varphi\sigma_D^1 \cdot p_n\,dx - \int_{\partial\Omega} \varphi\psi g\cdot u_n\,d\mathscr{H}^{N-1} + \int_{\Gamma_t} \varphi g\cdot u_n\,d\mathscr{H}^{N-1} \\ &= \int_\Omega \varphi\sigma_D^1 \cdot p_n\,dx = \int_\Omega \varphi\psi\sigma_D \cdot p_n\,dx.\end{aligned}$$

We conclude that

$$\lambda_n^1(\varphi) \le \|\sigma_D\|_\infty \|\varphi\|_\infty \int_W |p_n|\,dx.$$

Passing to the limit, we get

$$\lambda^1(\varphi) \le \|\sigma_D\|_\infty \|\varphi\|_\infty |p|(\bar{W}) = \|\sigma_D\|_\infty \|\varphi\|_\infty |p|(W),$$

so that claim (6.8) follows.

The existence of $(u_n, e_n, p_n) \in C^\infty(\overline{\Omega}; \mathbb{R}^N \times \mathrm{M}^N_{\mathrm{sym}} \times \mathrm{M}^N_D)$ satisfying (6.10) and (6.11) can be obtained in the following manner.

Let us consider a finite covering $\{Q_{\nu_k}(x_k, r_k)\}_{k\in I}$ of $\partial\Omega$ made of open cubes with centers $x_k \in \partial\Omega$, side $2r_k$ with $r_k > 0$ and with a face orthogonal to $\nu_k \in \mathbb{R}^N$, such that $\Omega \cap Q_{\nu_k}(x_k, r_k)$ is a Lipschitz subgraph in the direction ν_k. Let $\{\phi_k\}_{k\in I}$ be an associated partition of unity of $\partial\Omega$. We can write

$$u = \sum_{k\in I} \phi_k u + \left(1 - \sum_{k\in I} \phi_k\right) u,$$

the last term having a support compactly contained in Ω. Set

$$e_k := \phi_k e + \nabla\phi_k \odot u \qquad \text{and} \qquad p_k = \phi_k p, \tag{6.12}$$

so that

$$E(\phi_k u) = e_k + p_k \qquad \text{in } \Omega.$$

Given $a_h \searrow 0$ with $h \nearrow \infty$, we consider, for every $k \in I$ and every $x \in [\Omega \cap Q_{\nu_k}(x_k, r_k)] + a_h\nu_k$,

$$u_{k,h}(x) := \phi_k(x - a_h\nu_k) u(x - a_h\nu_k),$$

and define $e_{k,h}, p_{h,k}$ following (6.12). In particular $p_{k,h}$ is the push-forward along the translation given by $a_h\nu_k$ of the measure p_k. We can choose $a_h \to 0$ in such a way that

$$p_{k,h}(\partial\Omega \cap Q_{\nu_k}(x_k, r_k)) = 0.$$

Let us set

$$u_h := \sum_{k\in I} u_{k,h} + \left(1 - \sum_{k\in I} \phi_k\right) u$$

$$e_h := \sum_{k\in I} e_{k,h} + \left(1 - \sum_{k\in I}\phi_k\right) e - \sum_{k\in I} \nabla\phi_k \odot u$$

and

$$p_h := \sum_{k\in I} p_{k,h} + \left(1 - \sum_{k\in I}\phi_k\right) p.$$

Notice that (u_h, e_h, p_h) is well defined on an open neighborhood Ω_h of $\overline{\Omega}$, *i.e.*, $(u_h, e_h, p_h) \in BD(\Omega_h) \times L^{N/N-1}(\Omega_h; \mathbb{M}^N_{\mathrm{sym}}) \times \mathscr{M}_b(\Omega; \mathbb{M}^N_D)$ with

$$Eu_h = e_h + p_h \qquad \text{on } \Omega_h.$$

By construction we also have

$$p_h(\partial\Omega) = 0. \tag{6.13}$$

Restricting to Ω, and remarking that we only used local translations,

$$\begin{cases} u_h \to u & \text{strongly in } L^{N/N-1}(\Omega; \mathbb{R}^N) \\ e_h \to e & \text{strongly in } L^{N/N-1}(\Omega; \mathbb{M}^N_{\mathrm{sym}}) \\ p_h \xrightarrow{*} p & \text{weakly}^* \text{ in } \mathscr{M}_b(\Omega; \mathbb{M}^N_D), \end{cases} \tag{6.14}$$

as $h \to \infty$. Moreover, since translations point outside Ω, we get

$$\begin{aligned} \limsup_h |p_h|(\Omega) &\le \sum_{k\in I} \limsup_h |p_{k,h}|(\Omega) + \int_\Omega \left(1 - \sum_{k\in I}\phi_k\right) d|p| \\ &= \sum_{k\in I} |p_k|(\Omega) + \int_\Omega \left(1 - \sum_{k\in I}\phi_k\right) d|p| \\ &= \sum_{k\in I} \int_\Omega \phi_k \, d|p| + \int_\Omega \left(1 - \sum_{k\in I}\phi_k\right) d|p| = |p|(\Omega), \end{aligned}$$

so that indeed

$$p_h \xrightarrow{s} p \qquad \text{strictly in } \mathscr{M}_b(\Omega; \mathbb{M}^N_D). \tag{6.15}$$

If we regularize by convolution, we obtain, since $\overline{\Omega} \subset \Omega_h$, that

$$(u_h^m, e_h^m, p_h^m) \in C^\infty(\overline{\Omega}; \mathbb{R}^N \times \mathbb{M}^N_{\mathrm{sym}} \times \mathbb{M}^N_D),$$

with

$$Eu_h^m = e_h^m + p_h^m \qquad \text{in } \Omega \tag{6.16}$$

and

$$\begin{cases} u_h^m \to u_h & \text{strongly in } L^{N/N-1}(\Omega; \mathbb{R}^N) \\ e_h^m \to e_h & \text{strongly in } L^{N/N-1}(\Omega; \mathbb{M}^N_{\mathrm{sym}}). \end{cases} \tag{6.17}$$

Concerning p_h^m, recall that regularization by convolution of a measure entails local weak* convergence and, in addition, strict convergence on open subsets whose boundaries are not charged by the measure itself (see [2, Theorem 2.2]). In view of (6.13),

$$p_h^m \xrightarrow{s} p_h \qquad \text{strictly in } \mathscr{M}_b(\Omega;\mathrm{M}^N_D). \tag{6.18}$$

The desired configurations $(u_n, e_n, p_n) \in C^\infty(\overline{\Omega};\mathbb{R}^N \times \mathrm{M}^N_{\mathrm{sym}} \times \mathrm{M}^N_D)$ follow by choosing $h = h_n$ and $m = m_n$ and using a diagonal argument. The compatibility relation (6.10) follows by (6.16). The convergences (6.11) are a consequence of (6.14), (6.15), (6.17) and (6.18).

Step 2. In a second step we prove claim (6.9). Let $U \subseteq \mathbb{R}^N$ be open with $\Gamma_t \setminus V \subseteq U$ and

$$\sigma^2 = 0 \qquad \text{on } U \cap \Omega.$$

We now approximate σ^2 through a procedure involving local translations and convolutions (see e.g. [3, Lemma 2.3]). For every $x \in \partial\Omega \setminus U$, let us consider $r_x > 0$ such that $B(x,r_x) \cap \partial\Omega$ is a Lipschitz graph and $B(x,r_x) \cap (\Gamma_t \setminus V) = \emptyset$. Let $\{B(x_j,r_j)\}_{j=1,\dots,m}$ be a finite subordinated covering of $\partial\Omega \setminus U$, and let $\{\varphi_j\}_{j=1,\dots,m}$ be an associated partition of unity.

We write

$$\sigma^2 = \sum_{j=1}^m \varphi_j \sigma^2 + \left(1 - \sum_{j=1}^m \varphi_j\right)\sigma^2.$$

We translate each $\varphi_j\sigma^2$ by $\tau_n^j := a_n \nu(x_i)$ with $a_n \searrow 0$, where $\nu(x_j)$ is the outer normal to Ω at x_j, and then regularize by convolution, getting $\sigma^2_{j,n}$ with $\sigma^2_{j,n} = 0$ near $\Gamma_t \setminus V$. The last contribution in the above identity has its support compactly contained in Ω; it is simply regularized by convolution. Proceeding so produces a sequence $\sigma_n^2 \in C^\infty(\overline{\Omega};\mathrm{M}^N_{\mathrm{sym}})$ such that, as $n \nearrow \infty$,

$$\begin{cases} \sigma_n^2 \to \sigma^2 & \text{strongly in } L^r(\Omega;\mathrm{M}^N_{\mathrm{sym}}), \text{ for every } 1 \le r < \infty \\ \operatorname{div}\sigma_n^2 \to \operatorname{div}\sigma^2 & \text{strongly in } L^N(\Omega;\mathbb{R}^N) \\ \sigma_n^2 \nu = 0 & \text{on } \Gamma_t \setminus V \\ \limsup_n \|(\sigma_n^2)_D\|_\infty \le \|\sigma_D^2\|_\infty. \end{cases}$$

Above, the last estimate is obtained as follows. For every $x \in \Omega$

$$(\sigma_n)_D(x) = \sum_{i=1}^m \varphi_i(\cdot - \tau_n^i)\sigma(\cdot - \tau_n^i) \star \rho_n(x) + \left(1 - \sum_{i=1}^m \varphi_i\right)\sigma \star \rho_n(x).$$

where ρ_n denotes the convolution kernel. Then,

$$|(\sigma_n)_D(x)| \le \|\sigma_D\|_\infty \left[\sum_{i=1}^m \varphi_i(\cdot - \tau_n^i) + 1 - \sum_{i=1}^m \varphi_i\right] \star \rho_n(x).$$

The result is the obtained upon remarking that

$$\sum_{i=1}^m \varphi_i(\cdot - \tau_n^i) + 1 - \sum_{i=1}^m \varphi_i \to 1 \qquad \text{uniformly on } \mathbb{R}^N.$$

Consider

$$\lambda_n^2(\varphi) := -\int_\Omega \varphi\sigma_n^2 \cdot e\, dx - \int_\Omega \varphi \,\mathrm{div}\sigma_n^2 \cdot u\, dx - \int_\Omega \sigma_n^2 \cdot [u \odot \nabla\varphi]\, dx.$$

Clearly

$$\lim_n \lambda_n^2(\varphi) = \lambda^2(\varphi).$$

Since σ_n^2 is smooth, it is immediately checked, using integration by parts in BD (see [18, Chapter 2, Theorem 2.1]), the facts that $\varphi\sigma_n^2\nu = 0$ on $\overline{\Gamma}_t$ and $p\lfloor\Gamma_d = -u\odot\nu\mathscr{H}^{N-1}\lfloor\Gamma_d$, that

$$\lambda_n^2(\varphi) = \int_{\Omega\cup\Gamma_d} \varphi\, \sigma_n^2 \cdot dp = \int_{\Omega\cup\Gamma_d} \varphi\, (\sigma_n^2)_D \cdot dp.$$

In view of the L^∞-bound on $(\sigma_n^2)_D$, taking the limit for $n \to \infty$ yields

$$\lambda^2(\varphi) \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\, d|p|,$$

so that claim (6.9) follows.

Note that we cannot argue as for the proof of (6.8), that is through a regularization of (u,e,p), because the integrations by part would produce a term in $\int_{\partial\Omega} \sigma^2\nu \cdot u_n\, d\mathscr{H}^{N-1}$ which would not converge because we only know that $\sigma^2\nu$ is in $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$.

Step 3. In a third step, we compute the absolutely continuous part of $\langle\sigma_D, p\rangle$. For every A open set with $A \subset\subset \Omega$, regularization by convolution of σ yields a sequence $\sigma_n \in C^\infty(\bar{A};\mathrm{M}^N_{\mathrm{sym}})$ such that

$$\sigma_n \to \sigma \qquad \text{strongly in } L^r(A;\mathrm{M}^N_{\mathrm{sym}}) \text{ for every } 1 \le r < \infty$$

$$\mathrm{div}\sigma_n \to \mathrm{div}\sigma \qquad \text{strongly in } L^N(A;\mathbb{R}^N)$$

and

$$(\sigma_n)_D \overset{*}{\rightharpoonup} \sigma_D \qquad \text{weakly}^* \text{ in } L^\infty(A;\mathrm{M}^N_D) \qquad \text{with} \qquad \|(\sigma_n)_D\|_\infty \le \|\sigma_D\|_\infty.$$

By (6.5), taking $\varphi \in C^1_c(A)$, we infer that

$$\langle(\sigma_n)_D, p\rangle = (\sigma_n)_D p \qquad \text{as measures on } A.$$

The decomposition of $(\sigma_n)_D p$ with respect to the Lebesgue measure $\mathscr{L}^N$ is given by

$$(\sigma_n)_D p = (\sigma_n)_D \cdot p_a \mathscr{L}^N + (\sigma_n)_D \cdot p^s, \tag{6.19}$$

where $p = p_a\mathscr{L}^N + p^s$ is the associated decomposition of p, with $p_a \in L^1(\Omega;\mathrm{M}^N_D)$. But, thanks to (6.3), $\langle(\sigma_n)_D, p\rangle \overset{*}{\rightharpoonup} \langle(\sigma)_D, p\rangle$ weakly* in $\mathscr{M}_b(A)$, so that, since

$$|(\sigma_n)_D p^s| \le \|\sigma_D\|_\infty |p^s|,$$

passing to the limit in (6.19) yields

$$\langle\sigma_D, p\rangle = \sigma_D \cdot p_a \mathscr{L}^N + \lambda^s \qquad \text{on } A,$$

with λ^s singular with respect to $\mathscr{L}^N$, since $|\lambda^s| \leq \|\sigma_D\|_\infty |p^s|$. By the arbitrariness of A, we conclude that the density of the $\mathscr{L}^N$-absolutely continuous part of $\langle \sigma_D, p\rangle$ is given by $\sigma_D \cdot p_a$, which concludes the proof. □

Remark 6.3. Remark that, if $\Gamma_d = \partial\Omega$, that is in the case of a hard device applied to the whole boundary, then Theorem 6.2 fully characterizes the duality between σ_D and p because $\langle \sigma_D, p\rangle$ is then a bounded Radon measure on $\mathbb{R}^N$ satisfying $|\langle \sigma_D, p\rangle| \leq \|\sigma_D\|_\infty |p|$ on $\mathbb{R}^N$. Thus, in that case, we have obtained a generalization of the duality established in [8] to the case of a Lipschitz domain. ¶

In the following, we focus on conditions on $\partial\lfloor_{\partial\Omega}\Gamma_d$ that allow one to extend the result of Theorem 6.2 to all of $\mathbb{R}^N$ in lieu of $\mathbb{R}^N \setminus \partial\lfloor_{\partial\Omega}\Gamma_d$. If such is the case, we will say that

$$\partial\lfloor_{\partial\Omega}\Gamma_d \text{ is admissible.} \tag{6.20}$$

Remark 6.4. Assume that (6.20) holds true, *i.e.*, that, for every $f \in L^N(\Omega;\mathbb{R}^N)$, $g \in L^\infty(\Gamma_t;\mathbb{R}^N)$ and for every $\sigma \in \mathscr{K}(f,g)$ and $p \in \mathscr{P}(w)$ with associated $(u,e) \in BD(\Omega) \times L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, the distribution $\langle \sigma_D, p\rangle$ is a bounded Radon measure on $\mathbb{R}^N$ with $|\langle \sigma_D, p\rangle| \leq \|\sigma_D\|_\infty |p|$ and mass given by taking $\varphi = 1$ in (6.3), then we conjecture that $\mathscr{H}^{N-1}(\partial\lfloor_{\partial\Omega}\Gamma_d) = 0$, although we have been unable to find a satisfactory proof of that statement. ¶

In what follows we demonstrate that (6.20) holds true, provided that the assumptions of either Theorem 6.5, or Theorem 6.6 below are met.

The following condition was proposed in [8].

Theorem 6.5 (The Kohn-Temam condition). *Assume the following setting: the relative boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ is a $(N-2)$-dimensional manifold, and $\partial\Omega$ is of class C^2 in a neighborhood of $\partial\lfloor_{\partial\Omega}\Gamma_d$.*

Then the distribution $\langle \sigma_D, p\rangle$ is a bounded Radon measure on $\mathbb{R}^N$ such that

$$|\langle \sigma_D, p\rangle| \leq \|\sigma_D\|_\infty |p|.$$

Its mass is given by taking $\varphi = 1$ in (6.3). *Finally, the density of its $\mathscr{L}^N$-absolutely continuous part is $\sigma_D \cdot p_a$, where p_a is the density of the $\mathscr{L}^N$-absolutely continuous part of p.*

Proof. Translating u by w, it is enough to prove the theorem for $w \equiv 0$. In [8, Lemma 3.5], it is proved that, in the setting of the theorem,

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{\Omega\cap U_\delta} |\sigma||u|\,dx = 0,$$

with $U_\delta := \{x \in \mathbb{R}^N : d(x, \partial\lfloor_{\partial\Omega}\Gamma_d) < \delta\}$.

Introduce $\psi_\delta \in C_c^\infty(U_\delta)$ with $0 \le \psi_\delta \le 1$, $\|\nabla\psi_\delta\|_\infty \le 2/\delta$ and $\psi_\delta \equiv 1$ in a neighborhood of $\partial\lfloor_{\partial\Omega}\Gamma_d$. For every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\text{(6.21)}\quad \langle\sigma_D, p\rangle((1-\psi_\delta)\varphi) = -\int_\Omega (1-\psi_\delta)\varphi\sigma\cdot e\,dx - \int_\Omega (1-\psi_\delta)\varphi f\cdot u\,dx$$
$$-\int_\Omega (1-\psi_\delta)\sigma\cdot[u\odot\nabla\varphi]\,dx + \int_{\Gamma_t}(1-\psi_\delta)\varphi g\cdot u\,d\mathscr{H}^{N-1} + \int_\Omega \nabla\psi_\delta\sigma\cdot[u\odot\nabla\varphi]\,dx.$$

In view of the properties of ψ_δ, the last term can be estimated as follows:

$$\limsup_{\delta\to 0}\left|\int_\Omega \nabla\psi_\delta\sigma\cdot[u\odot\nabla\varphi]\,dx\right| \le \limsup_{\delta\to 0}\frac{2\|\nabla\varphi\|_\infty}{\delta}\int_{\Omega\cap U_\delta}|\sigma||u|\,dx = 0.$$

Passing to the limit in (6.21) we thus obtain

$$\lim_{\delta\to 0}\langle\sigma_D, p\rangle((1-\psi_\delta)\varphi) = \langle\sigma_D, p\rangle(\varphi).$$

But, in view of Theorem 6.2,

$$\langle\sigma_D, p\rangle(\varphi) = \lim_{\delta\to 0}\langle\sigma_D, p\rangle((1-\psi_\delta)\varphi) \le \|\sigma_D\|_\infty \limsup_{\delta\to 0}\int_{\Omega\cup\Gamma_d}(1-\psi_\delta)|\varphi|\,d|p|$$
$$\le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d}|\varphi|\,d|p|,$$

so that the first part of the theorem follows.

The mass of $\langle\sigma_D, p\rangle$ is clearly given by taking $\varphi = 1$ in (6.3). As far as its absolutely continuous part is concerned, the result follows by Theorem 6.2. □

The following result can be used to handle a certain class of non-smooth geometries, as explained below.

Theorem 6.6. *Assume that, for every $(u,e,p) \in \mathscr{A}(0)$, there exists $(u_n,e_n,p_n) \in \mathscr{A}(0)$ with*

$$\text{(6.22)}\qquad (u_n,e_n,p_n) \equiv 0 \qquad \text{on a neighborhood of } \partial\lfloor_{\partial\Omega}\Gamma_d,$$

and such that

$$\text{(6.23)}\qquad \begin{cases} u_n \rightharpoonup u & \text{weakly in } L^{N/N-1}(\Omega;\mathbb{R}^N) \\ e_n \rightharpoonup e & \text{weakly in } L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \\ p_n \xrightarrow{s} p & \text{strictly in } \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^N_D). \end{cases}$$

Then the distribution $\langle\sigma_D, p\rangle$ is a bounded Radon measure on $\mathbb{R}^N$ such that

$$|\langle\sigma_D, p\rangle| \le \|\sigma_D\|_\infty |p|.$$

Its mass is given by taking $\varphi = 1$ in (6.3). Finally, the density of its $\mathscr{L}^N$-absolutely continuous part is $\sigma_D\cdot p_a$, where p_a is the density of the $\mathscr{L}^N$-absolutely continuous part of p.

Proof. Let us approximate $(u-w, e-Ew, p) \in \mathscr{A}(0)$ through $(u_n, e_n, p_n) \in \mathscr{A}(0)$ satisfying (6.22) and (6.23). Denote by U_n the neighborhood of $\partial\lfloor_{\partial\Omega}\Gamma_d$ on which the configuration vanishes. In view of the convergences for u_n and e_n, it is readily shown that, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\langle \sigma_D, p\rangle(\varphi) = \lim_n \langle \sigma_D, p_n\rangle(\varphi).$$

Consider $\psi_n \in C_c^\infty(U_n)$ such that $0 \le \psi_n \le 1$, with $\psi_n \equiv 1$ in a neighborhhood of $\partial_{\partial\Omega}\Gamma_d$, and apply Theorem 6.2. Then,

$$\begin{aligned}\langle \sigma_D, p_n\rangle(\varphi) = \langle \sigma_D, p_n\rangle((1-\psi_n)\varphi) \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} (1-\psi_n)|\varphi|\, d|p_n| \\ \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\, d|p_n|.\end{aligned}$$

By taking the limit for $n \to \infty$, we conclude, thanks to the strict convergence of p_n to p, that

$$\langle \sigma_D, p\rangle(\varphi) \le \|\sigma_D\|_\infty \int_{\Omega\cup\Gamma_d} |\varphi|\, d|p|$$

so that the first part of the theorem follows.

The mass of $\langle \sigma_D, p\rangle$ is clearly given by taking $\varphi = 1$ in (6.3). As far as its absolutely continuous part is concerned, the result follows by Theorem 6.2. □

The density condition of Theorem 6.6 can be established in some non-smooth situations by approximating $(u, e, p) \in \mathscr{A}(0)$ by means of a local translation technique near $\partial\lfloor_{\partial\Omega}\Gamma_d$.

We finally discuss two examples that demonstrate the range of applicability of Theorem 6.6.

1. A two dimensional example: Let us assume that $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain, and that $\partial_{\partial\Omega}\Gamma_d$ is composed of a finite number of points $\{q_i\}_{i\in I}$. For every $i \in I$, there exists a suitable orthogonal coordinate system (x_1, x_2) and $\delta > 0$ such that

$$\Omega \cap Q(q_i, \delta) = \{(x_1, x_2) \in]-\delta, \delta[^2 : x_2 > f_i(x_1)\},$$

where $Q(q_i, \delta)$ is an open cube of center q_i and side 2δ, while $f_i :]-\delta, \delta[\to \mathbb{R}$ is a Lipschitz function with $f_i(0) = 0$. Set

$$\begin{aligned}\Gamma_d^{i,\delta} &:= \Gamma_d \cap Q(q_i, \delta) = \{(x_1, f_i(x_1)) : x_1 \in]-\delta, 0[\} \\ \Gamma_t^{i,\delta} &:= \Gamma_t \cap Q(q_i, \delta) = \{(x_1, f_i(x_1)) : x_1 \in]0, \delta[\}.\end{aligned}$$

We assume that

$$\begin{cases} f_i \ge 0, \text{ i.e., } \Omega \text{ is locally on one side of the "tangent" line in } q_i \\ f_i \text{ is strictly decreasing on }]-\delta, 0[\text{ and strictly increasing on }]0, \delta[. \end{cases}$$

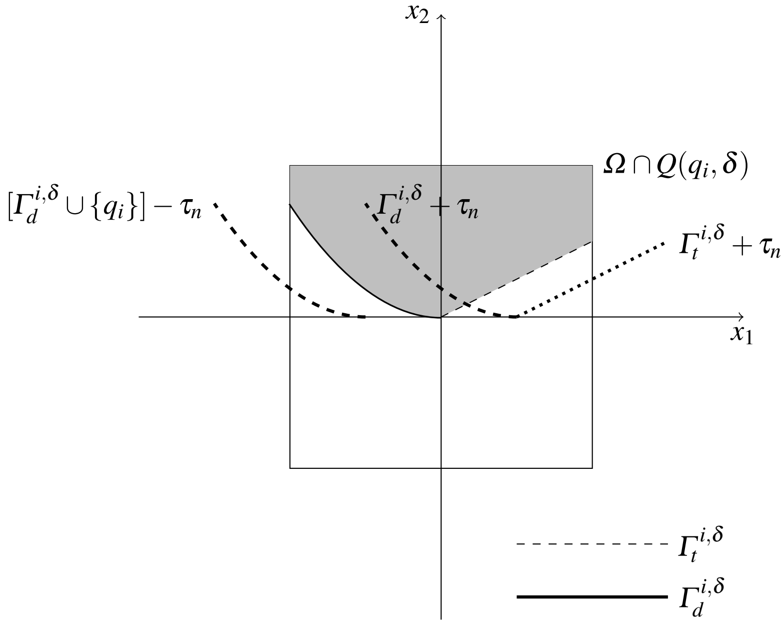

FIGURE 6.1. The boundary and its translates around a point in $\partial_{\partial\Omega}\Gamma_d$.

If we translate in the direction $\tau_n \in \mathbb{R}^2$ given in the coordinate system by $(a_n, 0)$ with $a_n > 0$ and $a_n \to 0$, we clearly have (see Figure 6.1)

$$[\Gamma_t^{i,\delta} + \tau_n] \cap Q(q_i,\delta) \cap (\Omega \cup \Gamma_d) = \emptyset \tag{6.24}$$

and

$$[(\Gamma_d^{i,\delta} \cup \{q_i\}) - \tau_n] \cap Q(q_i,\delta) \cap (\Omega \cup \Gamma_d) = \emptyset. \tag{6.25}$$

Moreover,

$$[\Gamma_d^{i,\delta} + \tau_n] \cap (\Omega \cup \Gamma_d) = \Gamma_d^{i,\delta,n} + \tau_n \qquad \text{with } \Gamma_d^{i,\delta,n} \nearrow \Gamma_d^{i,\delta}. \tag{6.26}$$

Let us fix $(u,e,p) \in \mathscr{A}(0)$. By extending (u,e) and p to $\mathbb{R}^N$ by zero and $-u \odot \nu \mathscr{H}^{N-1} \lfloor \Gamma_t$ outside Ω and $\Omega \cup \Gamma_d$ respectively, we obtain $u \in BD(\mathbb{R}^2)$, $e \in L^2(\Omega; M^2_{\mathrm{sym}})$ and $p \in \mathscr{M}_b(\mathbb{R}^2; M^2_{\mathrm{sym}})$ with

$$Eu = e + p \qquad \text{in } \mathbb{R}^2.$$

Remark that in general p is no longer M^2_{dev}-valued since the admissibility condition for the zero boundary value displacement is only imposed on Γ_d, so that u is not necessarily orthogonal to ν along Γ_t.

Let δ be so small that the cubes $Q(q_i,\delta)$ are disjoint, and let $\varphi_i \in C_c^\infty(Q(q_i,\delta))$ be such that $\varphi_i \equiv 1$ in a neighborhood of q_i. We can write

$$u = \sum_{i\in I} \varphi_i u + \left(1 - \sum_{i\in I} \varphi_i\right) u.$$

Set

$$e_i := \varphi_i e + \nabla\varphi_i \odot u \qquad \text{and} \qquad p_i := \varphi_i p. \tag{6.27}$$

Let us then set

$$u_{i,n} := \varphi_i(x-\tau_n)u(x-\tau_n),$$

with associated elastic and plastic strains $(e_{i,n}, p_{i,n})$ defined following (6.27). In particular, $p_{i,n}$ is the push forward along the translation τ_n of the measure p_i. We restrict $(u_{i,n}, e_{i,n})$ to Ω and $p_{i,n}$ to $\Omega \cup \Gamma_d$. Clearly,

$$Eu_{i,n} = e_{i,n} + p_{i,n} \qquad \text{in } \Omega,$$

and (6.25) implies that

$$(u_{i,n}, e_{i,n}, p_{i,n}) \equiv 0 \qquad \text{in a neighborhood of } \Gamma_d \cup \{q_i\}. \tag{6.28}$$

Moreover, in view of (6.24), $p_{i,n}$ has values in M^2_D so that

$$(u_{i,n}, e_{i,n}, p_{i,n}) \in \mathscr{A}(0). \tag{6.29}$$

Finally notice that

$$\begin{cases} u_{i,n} \to u_i & \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ e_{i,n} \to e_i & \text{strongly in } L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}}) \\ p_{i,n} \xrightarrow{s} p_i & \text{strictly in } \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^2_D). \end{cases} \tag{6.30}$$

The first two convergences follow since only translations are involved. Concerning the convergence for the plastic strains, since $p_{i,n}$ is the push-forward of p_i along the translation τ_n, the strict convergence in $\Omega\cup\Gamma_d$ results from the fact that $p_{i,n}$ has a concentrated part on the intersection of $\Gamma_d^{i,\delta} + \tau_n$ with $\Omega\cup\Gamma_d$, which satisfies (6.26).

In order to approximate $(u,e,p) \in \mathscr{A}(0)$, we can consider the configurations

$$(u_n, e_n, p_n) \in BD(\Omega) \times L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}}) \times \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^2_D)$$

given by

$$u_n := \sum_{i\in I} u_{i,n} + \left(1 - \sum_{i\in I}\varphi_i\right) u$$

$$e_n := \sum_{i\in I} e_{i,n} + \left(1 - \sum_{i\in I}\varphi_i\right) e - \sum_{i\in I} \nabla\varphi_i \odot u$$

and

$$p_n := \sum_{i\in I} p_{i,n} + \left(1 - \sum_{i\in I}\varphi_i\right) u.$$

In view of (6.28), (6.29) and (6.30), $(u_n, e_n, p_n) \in \mathscr{A}(0)$ with $(u_n, e_n, p_n) \equiv 0$ in a neighborhood of $\partial_{\partial\Omega}\Gamma_d$ and

$$\begin{cases} u_n \to u & \text{strongly in } L^2(\Omega;\mathbb{R}^2) \\ e_n \to e & \text{strongly in } L^2(\Omega;\mathrm{M}^2_{\mathrm{sym}}) \\ p_n \xrightarrow{s} p & \text{strictly in } \mathscr{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^2_D), \end{cases}$$

so that the density result follows.

We emphasize that the previous construction is based on the geometric properties (6.24), (6.25) and (6.26) of $\partial\Omega$ near a point of $\partial_{\partial\Omega}\Gamma_d$.

2. The case of a cube: $\Omega \subseteq \mathbb{R}^3$ is now a cube, with Γ_d given by one of its faces. We consider two opposite sides l_1 and l_2 of Γ_d, and let V_1, V_2 two bounded open neighborhoods of l_1, l_2 such that $\bar V_1 \cap \bar V_2 = \emptyset$. Also consider $\psi_i \in C^\infty_c(V_i)$ with $\psi_i = 1$ in a neighborhood of l_i. Finally, for $i = 1, 2$, let π_i be a plane containing l_i such that $\overline{\Omega} \cap \pi_i = l_i$.

Given $(u, e, p) \in \mathscr{A}(0)$, we write

$$u = \psi_1 u + \psi_2 u + (1-\psi_1-\psi_2)u = u_1 + u_2 + u_3,$$

with associated elastic and plastic strains given by

$$\begin{aligned} &e_1 := \psi_1 e + \nabla\psi_1 \odot u, \qquad e_2 := \psi_2 e + \nabla\psi_2 \odot u, \\ &e_3 := (1-\psi_1-\psi_2)e - (\nabla\psi_1 + \nabla\psi_2)\odot u \end{aligned}$$

and

$$p_1 := \psi_1 p, \qquad p_2 := \psi_2 p, \qquad p_3 := (1-\psi_1-\psi_2)p.$$

We can approximate $(u_i, e_i, p_i) \in \mathscr{A}(0)$, for $i = 1, 2$, by translating with respect to $\tau^i_n \to 0$ with $\tau^i_n \in \pi_i$ and $\tau^i_n \perp l_i$. Indeed, setting $\Gamma^i_d := \Gamma_d \cap V_i$ and $\Gamma^i_t := \Gamma_t \cap V_i$, a suitable choice of the direction of τ^i_n produces the analogue of (6.24), (6.25) and (6.26), *i.e.*,

$$[\Gamma^i_t + \tau^i_n] \cap (\Omega\cup\Gamma_d) = \emptyset$$

$$[(\Gamma^i_d \cup \{l_i\}) - \tau^i_n] \cap (\Omega\cup\Gamma_d) = \emptyset$$

and

$$[\Gamma^i_d + \tau^i_n] \cap (\Omega\cup\Gamma_d) = \Gamma^{i,n}_d + \tau^i_n \qquad \text{with } \Gamma^{i,n}_d \nearrow \Gamma^i_d.$$

The configuration $(u_3, e_3, p_3) \in \mathscr{A}(0)$ is approximated by translating with respect to $\tau^3_n = -a_n\nu$, where ν is the exterior normal to Ω on Γ_d, and $a_n \searrow 0$.

The density result follows.

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Bibliography

[1] Ambrosio, L.; Coscia, A.; Dal Maso, G. Fine Properties of Functions with Bounded Deformations. *Arch. Rat. Mech. Anal.* **139** (1997), 201–238.

[2] Ambrosio, L.; Fusco, N.; Pallara, D. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, Oxford, 2000.

[3] Dal Maso, G.; DeSimone, A.; Mora, M. G. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.* **180** (2006), no. 2, 237–291.

[4] Demengel, F.; Demengel, G. *Espaces fonctionnels*. Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)], EDP Sciences, Les Ulis, 2007. Utilisation dans la résolution des équations aux dérivées partielles. [Application to the solution of partial differential equations].

[5] Francfort, G. A.; Giacomini, A. On periodic homogenization in perfect plasticity (to appear).

[6] Hill, R. *The Mathematical Theory of Plasticity*. Oxford, at the Clarendon Press, 1950.

[7] Johnson, C. On plasticity with hardening. *J. Math. Anal. Appl.* **62** (1978), no. 2, 325–336.

[8] Kohn, R.; Temam, R. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.* **10** (1983), no. 1, 1–35.

[9] Lubliner, J. *Plasticity Theory*. Macmillan Publishing, New York, 1990.

[10] Mainik, A.; Mielke, A. Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations* **22** (2005), no. 1, 73–99.

[11] Mielke, A. Evolution of rate-independent systems. In *Evolutionary equations. Vol. II*, edited by A. Dafermos; E. Feireisl, Handb. Differ. Equ., pp. 461–559, Elsevier/North-Holland, Amsterdam, 2005.

[12] Nečas, J. Sur les normes équivalentes dans $W^k(\Omega)$ et sur la coercivité des formes formellement positives. *Séminaire de mathématiques supérieures, Équations aux derivées partielles* **19** (1966), 101–108.

[13] Solombrino, F. Quasistatic evolution problems for nonhomogeneous elastic plastic materials. *J. Convex Anal.* **16** (2009), no. 1, 89–119.

[14] Spector, D. Simple proofs of some results of Reshetnyak. *Proc. Amer. Math. Soc.* **139** (2011), no. 5, 1681–1690.

[15] Suquet, P.-M. Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sci. Toulouse Math. (5)* **1** (1979), no. 1, 77–87.

[16] Suquet, P.-M. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique* **20** (1981), no. 1, 3–39.

[17] Suquet, P. M. Discontinuities and Plasticity. In *Nonsmooth mechanics and applications*, *International Center for Mechanical Sciences, Courses and Lectures*, vol. 302, edited by P. P. J.-J. Moreau, pp. 280–340, Springer-Verlag Wien-New York, 1988.

[18] Temam, R. *Problèmes mathématiques en plasticité*, *Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science]*, vol. 12. Gauthier-Villars, Montrouge, 1983.