Relaxation in BV of integral functionals defined on Sobolev functions with values in the unit sphere

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Abstract

In this paper we study the relaxation with respect to the L^1 norm of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx \quad u \in W^{1,1}(\Omega; S^{d-1}),$$

where Ω is a bounded open set of $\mathbb{R}^N$, S^{d-1} denotes the unite sphere in $\mathbb{R}^d$, N and d being any positive integers, and f satisfies linear growth conditions in the gradient variable. In analogy with the unconstrained case, we show that, if, in addition, f is quasiconvex in the gradient variable and satisfies some technical continuity hypotheses, then the relaxed functional $\overline{F}$ has an integral representation on $BV(\Omega; S^{d-1})$ of the type

$$\overline{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} + \int_\Omega f^\infty(x, u, dC(u)),$$

where the suface energy density K is defined by a suitable Dirichlet-type problem.

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1 Introduction

In this paper we study the relaxation with respect to the L^1 norm of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx \quad u \in W^{1,1}(\Omega; S^{d-1}), \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, S^{d-1} denotes the unite sphere in $\mathbb{R}^d$, N and d being any positive integers, and f satisfies linear growth conditions in the gradient variable. The relaxed functional $\overline{F}$ is defined by

$$\overline{F}(u) := \inf\{\liminf_{n\to+\infty} F(u_n) : \ u_n \in W^{1,1}(\Omega; S^{d-1}),\ u_n \to u \text{ in } L^1\}. \tag{1.2}$$

The main motivation for this analysis is the use of constrained energy functionals of this kind in variational models for the study of equilibria of magnetostrictive materials (see [7]).

A wide literature is available for analogous relaxation problems in the non constrained case. In particular, we refer to the work by Fonseca and Müller [12], where they study the relaxation with respect to the L^1 norm of integral functional of the type (1.1) but defined for $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, under the same growth conditions on f. They prove that, if, in addition, f is quasiconvex in the gradient variable and satisfies some technical continuity hypotheses (see Theorem 2.8), then the relaxed functional $\overline{F}$ has an integral representation on $BV(\Omega; \mathbb{R}^d)$ of the type

$$\begin{aligned}\overline{F}(u) \quad = \quad & \int_\Omega f(x, u, \nabla u)\, dx + \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} \\ & + \int_\Omega f^\infty(x, u, dC(u)),\end{aligned} \tag{1.3}$$

(see Section 2.2 for the definition of BV function and all the quantities above), where f^∞ is the recession function of f (see (2.9)) and the surface energy density $K(x, a, b, \nu)$ is defined by a Dirichlet-type problem on a unit cell in the direction ν (see (2.14)).

In the case the admissible functions are constrained to take values on a manifold, the problem of relaxation with respect to stronger topologies of functional of the type (1.1) has been faced by Dacorogna, Fonseca, Malý and Trivisa in [9]. More precisely, they study the relaxation with respect to the weak topology in $W^{1,p}(\Omega)$, $p \geq 1$, of functional of the form

$$\mathcal{F}(u) = \int_\Omega f(\nabla u)\, dx, \quad u \in W^{1,p}(\Omega; \mathcal{M}),$$

where f is a positive continuous function, $\mathcal{M}$ is any C^1 manifold in $\mathbb{R}^d$, including in particular the case $\mathcal{M} = S^{d-1}$. It turns out that the relaxed functional $\overline{\mathcal{F}}$ is of the form

$$\overline{\mathcal{F}}(u) = \int_\Omega Q_T f(u, \nabla u)\, dx, \quad u \in W^{1,p}(\Omega; \mathcal{M}), \tag{1.4}$$

where $Q_T f$ is the so called tangential quasiconvexification of f, defined by

$$Q_T f(y, \xi) := \inf \left\{ \int_{(0,1)^N} f(\nabla \varphi)\, dx,\ \varphi \in W_0^{1,\infty}((0,1)^N; T_y(\mathcal{M})) \right\},$$

for $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$, where $T_y(\mathcal{M})$ denotes the tangent space to $\mathcal{M}$ at y.

In [2] it was shown that, if F is of the form

$$F(u) = \int_\Omega f(u, \nabla u)\, dx, \quad u \in W^{1,1}(\Omega; S^{d-1}),$$

with f satisfying some technical continuity conditions and having linear growth in the gradient variable, then the relaxation of F with respect to the L^1 norm is still of the form (1.4) on $W^{1,1}(\Omega; S^{d-1})$. In particular, if f is tangentially quasiconvex, that is $f = Q_T f$, then F is lower semicontinuous with respect to the L^1 norm on $W^{1,1}(\Omega; S^{d-1})$. Anyway, this result is not satisfactory as far as we are concerned with minimum problems involving functionals of this kind, since sequences with bounded energies are not compact in $W^{1,1}(\Omega; S^{d-1})$.

In the case $d = 2$ and under convexity hypotheses in the gradient variable, the problem of relaxation in $BV(\Omega; S^1)$ of integral functionals with linear growth defined on $C^1(\Omega; S^1)$ has been faced by Giaquinta, Modica and Souček in [14] (see also Demengel and Hadiji [11]).

In this paper, we consider an integral functional of the type (1.1) and we give an integral representation in $BV(\Omega; S^{d-1})$ of the relaxed functional $\overline{F}$ defined by (1.2). More precisely, we prove that, under hypotheses on f and f^∞ analogous with those given in [12] (see (H1)-(H5) in Section 3) and if in addition f is tangentially quasiconvex, then $\overline{F}$ is still of the form (1.3) on $BV(\Omega; S^{d-1})$ (see Theorem 3.1). The main difference with respect to the result proved in [12] is that in the Dirichlet problem defining the surface energy density K the test functions take values on S^{d-1} instead of all the space $\mathbb{R}^d$ (see definition (3.18)). It turns out, for example, that if we consider the isotropic case $f(x, y, z) = |z|$, we obtain $K(x, a, b, \nu) = d_g(a, b)$, d_g denoting the geodetic distance on the unit sphere, while $K(x, a, b, \nu) = |a - b|$ in the non constrained case considered in [12] (see Remark 4.3).

It is worth noting that, thanks to a strong density result of smooth functions between manifolds in Sobolev Spaces proved by Bethuel and Zheng in [6] and in a more general version by Bethuel in [5], in (1.2) we can restrict to approximating sequences u_n belonging to a class of S^{d-1}-valued smooth functions. This class is $C^\infty(\Omega; S^{d-1})$ if $d \neq 2$, while, if $N > 1$ and $d = 2$ is given by all the S^1-valued functions which are C^∞ except at most on sets of codimension 2 (see Theorem 2.2

and Remark 3.3). On the other hand, in [14] it was shown that, in the case $d = 2$, if one restricts to approximating sequences in $C^1(\Omega; S^1)$, nonlocal terms appear in the relaxed functional.

The proof of our result closely follows the outline of the proof of the integral representation result in [12], based on blow-up techniques and a localization argument. The main difficulty to adapt the proof of [12] to our setting is that convolution and cut-off arguments cannot be applied in a standard way, due to the fact that the admissible functions are constrained to take values on the non convex set S^{d-1}. We overcome this difficulty by using a construction analogous with that used in the proof of the density results in [6] (see also [16] and [15]). It consists in suitably projecting a smooth function taking values on the unit ball onto S^{d-1}, without increasing too much its energy (see the proof of Lemmas 5.2 and 6.4 and Proposition 6.2).

We eventually derive a relaxation result for functionals of the type

$$\mathcal{G}(u) = \int_\Omega f(x, u, \nabla u) + \int_{\mathbb{R}^N} |h_u|^2 \, dx - \int_\Omega \langle h_{ext}, u\rangle \, dx, \quad u \in W^{1,1}(\Omega; S^{N-1}),$$

subject to the constraint

$$\begin{cases} \text{curl } h_u = 0 \\ \text{div } (h_u + u\chi_\Omega) = 0, \end{cases} \tag{1.5}$$

with f satisfying the hypotheses described above. Functionals of this kind generalize those involved in variational models for micromagnetics, where u represents the magnetization of a ferromagnetic material subject to an external magnetic field h_{ext} and h_u is the induced magnetic field related to u through the Maxwell's equations (1.5) (see [7], [17] for a detailed explanation of the model). We note that the additional terms are continuous and so they do not affect the form of the relaxed functional $\overline{\mathcal{G}}$, which is given by

$$\overline{\mathcal{G}}(u) = \overline{F}(u) + \int_{\mathbb{R}^N} |h_u|^2 \, dx - \int_\Omega \langle h_{ext}, u\rangle \, dx, \quad u \in BV(\Omega; S^{N-1}),$$

with h_u satisfying (1.5) and $\overline{F}$ given by (1.3) (see Theorem 7.2).

2 Preliminaries and notation

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^N$ and with $|\cdot|$ the usual euclidean norm, without specifying the dimension N when there is no risk of confusion. For every $t \in \mathbb{R}$, $[t]$ denotes its integer part. Given $\nu \in S^{N-1}$, Q_ν is an open unit cube centered at the origin with two of its faces normal to ν.

If Ω is a bounded open subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are the families of open and Borel subsets of Ω, respectively. We denote by $\mathcal{X}_B$ the characteristic function of the set $B \in \mathcal{B}(\Omega)$.

If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^N$ the Lebesgue measure in $\mathbb{R}^N$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, $k \geq 0$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified. We use standard notations for Lebesgue and Sobolev spaces.

2.1 Strong density result in $W^{1,1}(\Omega; S^{d-1})$

In this section we recall a result about the density of S^{d-1}-valued smooth functions in $W^{1,1}(\Omega; S^{d-1})$, which has been proved in [6] and, in a more general version, in [5]. We write it in a form which is suitable to our purposes.

Definition 2.1 *Given $N \in \mathbb{N}$, $N > 1$, we will denote by $\mathcal{G}$ the family of all closed subsets of C^∞ $(N-2)$-dimensional manifolds of $\mathbb{R}^N$.*

Theorem 2.2 *Let $N, d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be an open set and let $\mathcal{D}(\Omega; S^{d-1}) \subset W^{1,1}(\Omega; S^{d-1})$ be defined by*

$$\mathcal{D}(\Omega; S^{d-1}) := C^\infty(\Omega; S^{d-1}) \cap W^{1,1}(\Omega; S^{d-1}), \tag{2.6}$$

for $d \neq 2$, if $N > 1$, and for any $d \in \mathbb{N}$, if $N = 1$,

$$\mathcal{D}(\Omega; S^1) :=$$
$$\{u \in W^{1,1}(\Omega; S^1) : \exists k \in \mathbb{N}, \Gamma_i \in \mathcal{G}, i = 1, \dots, k : \ u \in C^\infty(\Omega \setminus \bigcup_{i=1}^{k} \Gamma_i; S^1)\}, \tag{2.7}$$

for $N > 1$. Then $\mathcal{D}(\Omega; S^{d-1})$ is dense in $W^{1,1}(\Omega; S^{d-1})$ for the $W^{1,1}$ norm.

2.2 Functions of bounded variation

We recall some definitions and basic results on functions with bounded variation. Our main reference is the book [3].

Definition 2.3 *Let $u \in L^1(\Omega; \mathbb{R}^d)$, we say that u is a function with* Bounded Variation *in Ω, we write $u \in BV(\Omega; \mathbb{R}^d)$, if the distributional derivative Du of u is representable by a $d \times N$ matrix valued measure on Ω with finite total variation $|Du|(\Omega)$ whose entries are denoted by $D_j u_i$, i.e., if $\varphi \in C_c^1(\Omega)$ then*

$$\int_\Omega u_i \partial_j \varphi \, dx = -\int_\Omega \varphi d D_j u_i.$$

Define the *approximate upper and lower limit* of each component u_i, $i = 1, \dots, d$, by

$$u_i^+(x) := \inf\left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N(\{u_i > t\} \cap B(x, \varepsilon)) = 0 \right\},$$

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N(\{u_i < t\} \cap B(x,\varepsilon)) = 0\right\}.$$

Then the *jump set* of u , denoted by $S(u)$, is defined by

$$S(u) := \cup_{i=1}^d \{x \in \Omega : u_i^- < u_i^+\}.$$

If $u \in BV\left(\Omega;\mathbb{R}^d\right)$, then $S(u)$ turns out to be countably $\left(\mathcal{H}^{N-1}, N-1\right)$ rectifiable, i.e.,

$$S(u) = N \cup \bigcup_{i\geq 1} K_i,$$

where $\mathcal{H}^{N-1}(N) = 0$ and each K_i is a compact subset of a C^1 manifold. If $x \in \Omega \setminus S(u)$, then $u(x)$ is understood as the common value of $(u_1^+,\ldots,u_d^+)$ and $(u_1^-,\ldots,u_d^-)$ with $u_i^\pm(x) \in [-\infty,+\infty]$ for $i = 1,\ldots,d$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus S(u)$.

Theorem 2.4 *If $u \in BV(\Omega;\mathbb{R}^d)$, then (i) for $\mathcal{L}^N$-a.e. $x \in \Omega$*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)} |u(y) - u(x) - \langle\nabla u(x), x - y\rangle|\,dy = 0,$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$; (ii) for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exists a unit vector $\nu(x) \in S^{N-1}$ and there exist $u^-(x), u^+(x) \in \mathbb{R}^d$ such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{\{y\in B(x,\varepsilon):\langle y-x,\nu(x)\rangle>0\}} |u(y) - u^+(x)|\,dx = 0,$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{\{y\in B(x,\varepsilon):\langle y-x,\nu(x)\rangle<0\}} |u(y) - u^-(x)|\,dx = 0;$$

(iii) for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega \setminus S(u)$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)} |u(y) - u(x)|\,dx = 0.$$

In what follows u^+ and u^- will denote the vectors introduced in (ii) above.

The next result will be used in Section 5.2.

Lemma 2.5 *For $\mathcal{H}^{N-1}$ a.e. $x_0 \in S(u)$,*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1}}\int_{S(u)\cap(x_0+\varepsilon Q_{\nu(x_0)})} |u^+(x) - u^-(x)|\,d\mathcal{H}^{N-1}(x) = |u^+(x_0) - u^-(x_0)|.$$

If $u \in BV(\Omega;\mathbb{R}^d)$, then Du can be decomposed into three orthogonal measure as

$$Du = \nabla u\, dx + (u^+ - u^-) \otimes \nu_u d\mathcal{H}^{N-1} \llcorner S(u) + C(u).$$

Here $C(u)$ is the so called Cantor part of Du and satisfies the property that $|C(u)|(B) = 0$ for any $B \in \mathcal{B}(\Omega)$ such that $\mathcal{H}^{N-1}(B) < +\infty$. We recall that, by a result of Alberti in [1], the density of the Cantor part $C(u)$ defined by

$$A(x) := \lim_{\varepsilon \to B(x,\varepsilon)} \frac{C(u)(B(x,\varepsilon))}{|C(u)|(B(x,\varepsilon))},$$

is a rank-one matrix for $|C(u)|$ a.e. $x \in \Omega$.

We denote by $BV(\Omega;S^{d-1})$ the space of functions $u \in BV(\Omega;\mathbb{R}^d)$ such that $u(x) \in S^{d-1}$ for a.e. $x \in \Omega$.

Remark 2.6 *It is easy to prove that, if $u \in BV(\Omega;S^{d-1})$, then, for a.e $x \in \Omega$, $\nabla u(x) \in [T_{u(x)}(S^{d-1})]^N$ and, for $|C(u)|$ a.e. $x \in \Omega$, $A(x) \in [T_{u(x)}(S^{d-1})]^N$, where, given $y \in S^{d-1}$, $T_y(S^{d-1})$ denotes the tangent space to S^{d-1} at y.*

The next lemma (see [4], Lemma 4.5) will be used in Section 6.

Lemma 2.7 *Let $u \in BV(\Omega : \mathbb{R}^d)$, let ρ be a convolution kernel, and let*

$$u_n(x) := (u * \rho_n)(x),$$

where $\rho_n(x) := n^N \rho(nx)$. Then

$$\int_{B(x_0,\varepsilon)} h(x)|\nabla u_n(x)|\, dx \leq \int_{B(x_0,\varepsilon+\frac{1}{n})} (h * \rho_n)(x)|Du(x)|,$$

whenever $\operatorname{dist}(x_0,\partial\Omega) > \varepsilon + \frac{1}{n}$ and h is a nonnegative Borel function;

$$\lim_{n\to+\infty} \int_{B(x_0,\varepsilon)} \theta(\nabla u_n(x))\, dx = \int_{B(x_0,\varepsilon)} \theta(Du(x)),$$

for every function θ positively homogeneous of degree one and for every $\varepsilon \in (0, \operatorname{dist}(x_0,\partial\Omega))$ such that $|Du|(\partial B(x_0,\varepsilon)) = 0$; if, in addition, $u \in L^\infty(\Omega;\mathbb{R}^d)$, then for every $x_0 \in \Omega \setminus S(u)$,

$$\lim_{n\to+\infty} u_n(x_0) = u(x_0), \qquad \lim_{n\to+\infty} (|u_n - u| * \rho_n)(x_0) = 0.$$

2.3 Quasiconvexity and relaxation results

A function $f : \mathbb{R}^{d\times N} \mapsto \mathbb{R}$ is said to be quasiconvex if

$$f(\xi) \leq \int_{(0,1)^N} f(\xi + \nabla\varphi)\, dx$$

for all $\xi \in \mathbb{R}^{d\times N}$ and for all $\varphi \in W_0^{1,\infty}((0,1)^N;\mathbb{R}^d)$. We recall that if f is quasi-convex and

$$|f(\xi)| \leq C(1+|\xi|), \tag{2.8}$$

then f is Lipschitz continuous (see [8]). We define the recession function of f by

$$f^\infty(\xi) := \limsup_{t\to+\infty} \frac{f(t\xi)}{t}. \tag{2.9}$$

Note that f^∞ is positively homogeneous of degree one and it can be proved that if f is quasiconvex and satisfies (2.8), then also f^∞ is quasiconvex (see [12]).

Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \mapsto [0,+\infty)$ and $F : L^1(\Omega;\mathbb{R}^d) \mapsto [0,+\infty]$ defined by

$$F(u) := \begin{cases} \int_\Omega f(x,u,\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

Let $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ the relaxation of F with respect to the L^1 topology, namely,

$$\overline{F}(u) = \inf\{\liminf_n F(u_n) : \; u_n \to u \text{ in } L^1(\Omega)\}. \tag{2.11}$$

If $(a,b,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$, let $\{\nu_1,\dots \nu_{N-1},\nu\}$ form an orthonormal basis of $\mathbb{R}^N$ and define

$$\mathcal{A}(a,b,\nu) := \big\{\varphi \in W^{1,1}(Q_\nu;\mathbb{R}^d) : \; \varphi(x) = a \text{ if } x\cdot\nu = -\tfrac{1}{2}, \; \varphi(x) = b \text{ if } x\cdot\nu = \tfrac{1}{2},$$

$$\varphi \text{ is periodic with period 1 in the } \nu_1,\dots,\nu_{N-1} \text{ directions}\big\}. \tag{2.12}$$

In [12] the following theorem was proved.

Theorem 2.8 *Let f satisfy the following hypotheses:*
(F1) f is continuous;
(F2) $f(x,u,\cdot)$ is quasiconvex;
(F3) There exist two positive constant c, C such that

$$c|\xi| \leq f(x,u,\xi) \leq C(1+|\xi|)$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$;
(F4) For every compact $J \subset\subset \Omega \times \mathbb{R}^d$ there exist a continuous function ω with $\omega(0) = 0$ such that

$$|f(x,u,\xi) - f(x',u',\xi)| \leq \omega(|x-x'| + |u-u'|)(1+|\xi|)$$

for all (x,u,ξ), $(x',u',\xi) \in J \times \mathbb{R}^{d\times N}$. In addition, for every $x_0 \in \Omega$ and for all $\delta > 0$ there exists $\varepsilon > 0$ such that if $|x - x_0| \leq \varepsilon$, then

$$f(x,u,\xi) - f(x_0,u,\xi) \geq -\delta(1+|\xi|)$$

for every $(u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$*;*
(F5) there exist $C' > 0$, $0 < m < 1$ *such that*

$$|f^\infty(x,u,\xi) - f(x,u,\xi)| \le C(1+|\xi|^{1-m})$$

for every $(x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$*. Then*

$$\overline{F}(u) = \begin{cases} \int_\Omega f(x,u,\nabla u)\,dx + \int_{S(u)} H(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega f^\infty(x,u,dC(u)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in BV(\Omega;\mathbb{R}^d) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{2.13}$$

where $H : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to \mathbb{R}$ *is defined by*

$$H(x,a,b,\nu) := \inf\left\{\int_{Q_\nu} f^\infty(x,\varphi,\nabla\varphi)\,dx : \ \varphi \in \mathcal{A}(a,b,\nu)\right\}. \tag{2.14}$$

Let, now, $\mathcal{M} \subseteq \mathbb{R}^d$ be a C^1 manifold and, given $y \in \mathcal{M}$, denote by $T_y(\mathcal{M})$ the tangent space to $\mathcal{M}$ at y.

The following definition has been introduced in [9].

Definition 2.9 *Given a function* $f : \mathbb{R}^{d\times N} \mapsto \mathbb{R}$*, the tangential quasiconvexification of* f *is defined by*

$$Q_T f(y,\xi) := \inf\{\int_{(0,1)^N} f(y,\xi+\nabla\varphi)\,dx : \ \varphi \in W_0^{1,\infty}((0,1)^N;T_y(\mathcal{M}))\}$$

for all $y \in \mathcal{M}$ *and* $\xi \in [T_y(\mathcal{M})]^N$*.*

In [9] the following relaxation result was proved.

Theorem 2.10 *If* $f : \mathbb{R}^{d\times N} \to [0,+\infty)$ *is a continuous function satisfying*

$$0 \le f(\xi) \le C(1+|\xi|^p)$$

for some $p \ge 1$, $C > 0$*, and all* $\xi \in \mathbb{R}^{d\times N}$*, then*

$$\mathcal{F}(u) = \int_\Omega Q_T f(u,\nabla u)\,dx, \quad u \in W^{1,p}(\Omega;\mathcal{M}),$$

where

$$\mathcal{F}(u) := \inf\{\liminf_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx : \ u_n \in W^{1,p}(\Omega;\mathcal{M}), \ u_n \rightharpoonup u \text{ in } W^{1,p}\}.$$

Definition 2.11 *A function* $f : \mathcal{M} \times \mathbb{R}^{d\times N} \mapsto \mathbb{R}$*, is said to be tangentially quasiconvex if*

$$f(y,\xi) := \int_{(0,1)^N} f(y,\xi+\nabla\varphi)\,dx$$

for all $y \in \mathcal{M}$ *and* $\xi \in [T_y(\mathcal{M})]^N$ *and* $\varphi \in W_0^{1,\infty}((0,1)^N;T_y(\mathcal{M})$*.*

Remark 2.12 *In [2], it was observed that the result given by Theorem 2.10 still holds true if f depends continuously also on u. In particular, we infer that if f is tangentially quasiconvex, then the functional*

$$\mathcal{F}(u) := \int_\Omega f(u, \nabla u)\, dx, \quad u \in W^{1,p}(\Omega; \mathcal{M})$$

is $W^{1,p}$-sequentially weakly lower semicontinuous.

Let P_y the orthogonal projection of $\mathbb{R}^d$ onto the tangent space $T_y(\mathcal{M})$ and consider the function $\overline{f} : \mathcal{M} \times \mathbb{R}^{d\times N} \mapsto \mathbb{R}$ defined by

$$\overline{f}(y, \xi) := f(y, P_y \xi),$$

with $P_y\xi := (P_y\xi^1, \ldots, P_y\xi^N)$ and ξ^i the i^{th} columns of $\xi \in \mathbb{R}^{d\times N}$. In [9] it was proved that for any $y \in \mathcal{M}$ and $\xi \in [T_y(\mathcal{M})]^N$

$$Q_T f(y, \xi) = Q\overline{f}(y, \xi),$$

where $Q\overline{f}$ is the quasiconvex envelope of $\overline{f}$ defined by

$$Q\overline{f}(y, \xi) := \inf\{\int_{(0,1)^N} f(y, \xi + \nabla\varphi)\, dx : \varphi \in W_0^{1,\infty}((0,1)^N; \mathbb{R}^d)\}.$$

In particular, if f is tangentially quasiconvex, then $\overline{f}$ is quasiconvex.

In the rest of the paper a tangentially quasiconvex function f will be identified by the function $\overline{f}$ defined above. So we may think a tangentially quasiconvex function as the restriction of a quasiconvex function on the set $T(\mathcal{M}) \subseteq \mathcal{M} \times \mathbb{R}^{d\times N}$ defined by

$$T(\mathcal{M}) := \{(y, \xi) : y \in \mathcal{M},\ \xi \in [T_y(\mathcal{M})]^N\}. \tag{2.15}$$

3 Statement of the main result

Let $N, d \in \mathbb{N}$, with $d \geq 2$. Given a bounded open subset Ω of $\mathbb{R}^N$ and a function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$, define the functional $F : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ as

$$F(u) := \begin{cases} \int_\Omega f(x, u, \nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega; S^{d-1}) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

Note that, in order that the definition of F is well posed, it suffices that the integrand f is defined only on $\Omega \times T(S^{d-1})$, where $T(S^{d-1})$ is given by (2.15) with $\mathcal{M} = S^{d-1}$.

On f we will consider the following set of hypotheses:

(H1) f is continuous;

(H2) $f(x,\cdot,\cdot)$ is a tangentially quasiconvex function according to Definition 2.11 with $\mathcal{M} = S^{d-1}$;
(H3) there exist two positive constants $c_1,\ c_2$ such that

$$c_1|\xi| \leq f(x,y,\xi) \leq c_2(|\xi|+1)$$

for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$;
(H4) For every compact $J \subset \Omega$, there exist a continuous function ω with $\omega(0) = 0$ such that

$$|f(x,y,\xi) - f(x',y',\xi)| \leq \omega(|x-x'|+|y-y'|)(1+|\xi|)$$

for every $(x,y,\xi),\ (x',y',\xi) \in J \times S^{d-1} \times \mathbb{R}^{d\times N}$;
(H5) there exist $C > 0$, $0 \leq m < 1$ such that

$$|f^\infty(x,y,\xi) - f(x,y,\xi)| \leq C(1+|\xi|^{1-m})$$

for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$.

The main result of the paper is the integral representation for the relaxation $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to \bar{[}0,+\infty)$ of F with respect to the L^1 topology (see (2.11)).

Define

$$\mathcal{P}(a,b,\nu) := \big\{\varphi \in W^{1,1}(Q_\nu;S^{d-1}) :\ \varphi(x) = a \text{ if } x\cdot\nu = -\tfrac{1}{2},\ \varphi(x) = b \text{ if } x\cdot\nu = \tfrac{1}{2},$$
$$\varphi \text{ is periodic with period 1 in the } \nu_1,\nu_2,\ldots,\nu_{N-1} \text{ directions}\big\}.$$

Theorem 3.1 *If (H1)–(H5) hold, then*

$$\overline{F}(u) = \begin{cases} \int_\Omega f(x,u,\nabla u)\,dx + \int_{S(u)} K(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega f^\infty(x,u,dC(u)) \\ \qquad\qquad\qquad\qquad \textit{if } u \in BV(\Omega;S^{d-1}) \\ +\infty \qquad\qquad\qquad\quad \textit{otherwise,} \end{cases} \tag{3.17}$$

where $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to \mathbb{R}$ *is defined by*

$$K(x,a,b,\nu) := \inf\left\{\int_{Q_\nu} f^\infty(x,\varphi,\nabla\varphi)\,dx :\ \varphi \in \mathcal{P}(a,b,\nu)\right\} \tag{3.18}$$

Remark 3.2 *If f satisfies (H2)–(H4), then, using the definition of recession function, one can easily prove that:*
(H2') $f^\infty(x,\cdot,\cdot)$ is tangentially quasiconvex;
(H3') $c_1|\xi| \leq f^\infty(x,y,\xi) \leq c_2|\xi|$ for every $x \in \Omega,\ y \in S^{d-1},\ \xi \in [T_y(S^{d-1})]^N$;
(H4') For every compact $J \subset \Omega$, there exist a continuous function ω with $\omega(0) = 0$ such that

$$|f^\infty(x,y,\xi) - f^\infty(x',y',\xi)| \leq \omega(|x-x'|+|y-y'|)|\xi|$$

for every $(x,y,\xi),\ (x',y',\xi) \in J \times S^{d-1} \times \mathbb{R}^{d\times N}$.

Remark 3.3 *By Theorem 2.2 and by (H1) and (H3), we can restrict to sequences of smooth approximating functions in the definition of* $\overline{F}$, *that is*

$$\overline{F}(u) = \inf\{\liminf_n \int_\Omega f(x, u_n, \nabla u_n)\, dx : \ u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d),\ u_n \in \mathcal{D}(\Omega; S^{d-1})\},$$

where $\mathcal{D}(\Omega; S^{d-1})$ *is defined in (2.6) and (2.7).*

4 Properties of the surface density function

Before proving Theorem 3.1 we state some properties of the surface energy density K, we will need in the sequel, and we show a more explicit characterization of it under isotropy assumption on f^∞.

The following lemma is the analogue of [12] Lemma 2.15 .

Lemma 4.1 *Let (H1)–(H4) hold and let* $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0, +\infty)$ *be defined by (3.18). Then*

(a) $|K(x, a, b, \nu) - K(x, a', b', \nu)| \leq c(|a - a'| + |b - b'|)$ *for every* (x, a, b, ν), $(x, a', b', \nu) \in \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$;

(b) $(x, \nu) \mapsto K(x, a, b, \nu)$ *is upper semicontinuous for every* $(a, b) \in S^{d-1} \times S^{d-1}$;

(c) K *is upper semicontinuous in* $\Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$;

(d) $K(x, a, b, \nu) \leq C|b - a|$ *for every* $(x, a, b, \nu) \in \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1}$;

(e) for all $x \in \Omega$, $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $|x' - x| < \delta$, *then* $|K(x', a, b, \nu) - K(x, a, b, \nu)| \leq \varepsilon C|b - a|$.

Proof. (a) Let $\varphi \in \mathcal{P}(a, b, \nu)$ and let $\gamma_1, \gamma_2 : [1/4, 1/2] \to S^{d-1}$ be smooth and such that

$$\gamma_1(1/4) = b,\ \gamma_1(1/2) = b', \quad \gamma_2(1/4) = a,\ \gamma_2(1/2) = a',$$

$$\int_{1/4}^{1/2} |\gamma_1'(t)|\, dt \leq C|b - b'|, \quad \int_{1/4}^{1/2} |\gamma_2'(t)|\, dt \leq C|a - a'|. \tag{4.19}$$

Then define $\varphi^* \in \mathcal{P}(a', b', \nu)$ by

$$\varphi^*(y) = \begin{cases} \varphi(2y) & \text{if } |\langle y, \nu\rangle| < 1/4 \\ \gamma_1(\langle y, \nu\rangle) & \text{if } 1/4 < \langle y, \nu\rangle < 1/2 \\ \gamma_2(-\langle y, \nu\rangle) & \text{if } -1/2 < \langle y, \nu\rangle < -1/4. \end{cases}$$

So far, arguing as in the proof of [12] Lemma 2.15 (a), by using the periodicity of φ, the growth condition (H3') on f^∞ and (4.19), we get

$$K(x, a', b', \nu) \quad \leq \quad \int_{Q_\nu} f^\infty(x, \varphi^*(y), \nabla\varphi^*(y)\, dy$$

$$\leq \int_{Q_\nu} f^\infty(x, \varphi(y), \nabla\varphi(y)\, dy + C(|a-a'| + |b-b'|).$$

Then, by the arbitrariness of $\varphi \in \mathcal{P}(a,b,\nu)$, we conclude that

$$K(x, a', b', \nu) \leq K(x, a, b, \nu) + C(|a-a'| + |b-b'|).$$

The proof of (b) is exactly analogous to that of [12] Lemma 2.15 (b) and hypotheses (H3') and (H4') on f^∞ are needed. Note that (c) is an immediate consequence of (a) and (b).
(d) Use the growth condition (H3') and consider the characterization of K given by Lemma 4.2 and Remark 4.3 below when $f^\infty(x,u,\xi) = |\xi|$.

Finally the proof of (e) can be carried out exactly as in Proposition 2.4 (ii) of [13]. □

The following lemma is the analogue of [13] Proposition 2.6 (iii) and show that, if f^∞ satisfies an isotropy assumption, then the surface density K can be calculated by restricting the infimum to functions with one-dimensional profile. We omit the proof since it is exactly the same of that of [13] Proposition 2.6. (iii).

Lemma 4.2 *Let $K : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0,+\infty)$ be defined by (3.18) and let f^∞ isotropic, i.e., for every $(x,u,z) \in \Omega \times S^{d-1} \times \mathbb{R}^{d\times N}$ and $\nu \in S^{n-1}$ there holds*

$$f^\infty(x, u, z\nu \otimes \nu) \leq f^\infty(x, u, z). \tag{4.20}$$

Then

$$K(x,a,b,\nu) = \inf\left\{\int_0^1 f^\infty(x, \gamma(t), \gamma'(t) \otimes \nu)\, dt : \gamma \in W^{1,1}((0,1); S^{d-1}),\ \gamma(0) = a,\ \gamma(1) = b\right\}.$$

Remark 4.3 *In the particular case $f(x,u,\xi) = h(|\xi|)$ with $\lim_{t\to+\infty} \frac{h(t)}{t} = 1$, then $f^\infty(x,u,\xi) := |\xi|$ and so condition (4.20) is satisfied. Thus, by Lemma 4.2, we get*

$$\begin{aligned} K(x,a,b,\nu) &= \inf\left\{\int_0^1 |\gamma'(t)|\, dt : \gamma \in W^{1,1}((0,1); S^{d-1}),\ \gamma(0) = a,\ \gamma(1) = b\right\} \\ &=: d_g(a,b), \end{aligned}$$

where d_g denotes the geodesic distance on S^{d-1}.

5 Estimate from below

Set, for $u \in BV(\Omega; S^{d-1})$, $B \in \mathcal{B}(\Omega)$

$$G(u,B) := \int_B f(x,u,\nabla u)\, dx + \int_{S(u)\cap B} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}$$

$$+\int_B f^\infty(x,u,dC(u)).$$

By simplicity, we denote $G(u)=G(u,\Omega)$.

Proposition 5.1 *Let (H1)–(H5) hold and let $u_n\in W^{1,1}(\Omega;S^{d-1})$ such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\liminf_n F(u_n)<+\infty$. Then $u\in BV(\Omega;S^{d-1})$ and*

$$\liminf_n F(u_n)\geq G(u) \tag{5.21}$$

Proof. We may assume without loss of generality that

$$\liminf_n F(u_n)=\lim_n F(u_n)<+\infty.$$

Then by the growth hypothesis (H3), we get

$$\sup_n \|u_n\|_{W^{1,1}(\Omega;S^{d-1})}<+\infty,$$

from which we immediately derive that $u\in BV(\Omega;S^{d-1})$.

Since $f\geq 0$, up to passing to a subsequence, we may assume that there exists a non-negative finite Radon measure μ on Ω such that

$$f(\cdot,(u_n(\cdot),\nabla u_n(\cdot))\mathcal{L}^N\llcorner\Omega\to\mu$$

weakly* in the sense of measures. Using the Radon-Nykodim's Theorem we decompose μ in the sum of four mutually orthogonal measures

$$\mu=\mu_a\mathcal{L}^N+\mu_c|D^cu|+\mu_S|u^+-u^-|\mathcal{H}^{N-1}\llcorner S(u)+\mu_o,$$

we claim that

$$\mu_a(x_0)\geq f\left(x_0,u(x_0),\nabla u(x_0)\right) \tag{5.22}$$

for a.e. $x_0\in\Omega$;

$$\mu_c(x_0)\geq f^\infty\left(x_0,\tilde{u}(x_0),\frac{dC(u)}{d\|C(u)\|}(x_0)\right) \tag{5.23}$$

for $\|C(u)\|$ a.e. $x_0\in\Omega$;

$$\mu_S(x_0)\geq\frac{1}{|u^+(x_0)-u^-(x_0)|}K\left(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)\right) \tag{5.24}$$

for $|u^+-u^-|\mathcal{H}^{N-1}\llcorner S(u)$ a.e. $x_0\in\Omega$.

Assuming the previous inequalities shown, to conclude consider an increasing sequence of smooth cut-off functions $(\varphi_i)\subset C_0^\infty(\Omega)$ such that $0\leq\varphi_i\leq 1$ and $\sup_i\varphi_i(x)=1$ on Ω, then for every $i\in\mathbb{N}$ we have

$$\begin{aligned}\lim_n F(u_n)&\geq\liminf_n\int_\Omega f\left(x,u_n,\nabla u_n\right)\varphi_i\,dx\\&=\int_\Omega\varphi_i d\mu\geq\int_\Omega f\left(x,u,\nabla u\right)\varphi_i\,dx+\int_\Omega f^\infty\left(x,\tilde{u},\frac{dC(u)}{d\|C(u)\|}\right)\varphi_i d\|C(u)\|\\&+\int_{S(u)}K\left(x,u^+,u^-,\nu_u\right)\varphi_i\,d\mathcal{H}^{N-1}.\end{aligned}$$

Eventually, by letting $i \to +\infty$ and applying the Monotone Convergence Theorem, we get (5.21). □

In the following subsections we prove (5.22), (5.23) and (5.24).

5.1 The density of the diffuse part

Let $\varphi : [0, +\infty) \to [0, 1]$ a Lipschitz function such that $\varphi \equiv 0$ on $[0, 1/2]$ and $\varphi \equiv 1$ on $[1, +\infty)$, and consider the function $\tilde{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ defined by

$$\tilde{f}(x, u, \xi) := \varphi(|u|)f(x, \frac{u}{|u|}, P_u\xi).$$

Then $\tilde{f}$ is an extension of f and, for any $\varepsilon > 0$, it can be easily verified that hypotheses (H1)–(H5) on f imply that the function

$$f_\varepsilon(x, u, \xi) := \tilde{f}(x, u, \xi) + \varepsilon|\xi|$$

satisfies hypotheses (F1)–(F5) of Theorem 2.8. Hence, given u_n as in Proposition 5.1, for every $A \in \mathcal{A}(\Omega)$ there holds, by Theorem 2.8,

$$\begin{aligned}\liminf_n \int_A f(x, u_n, \nabla u_n)\, dx &\geq \int_A f_\varepsilon(x, u_n, \nabla u_n)\, dx - \varepsilon \sup_n \|\nabla u_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})} \\ &\geq \int_A f_\varepsilon(x, u, \nabla u)\, dx + \int_A f_\varepsilon^\infty(x, u, dC(u)) - \varepsilon \sup_n \|\nabla u_n\|_{L^1(\Omega;\mathbb{R}^{d\times N})}\end{aligned}$$

Then, letting ε tend to 0, we get

$$\liminf_n \int_A f(x, u_n, \nabla u_n)\, dx \geq \int_A f(x, u, \nabla u)\, dx + \int_A f^\infty(x, u, dC(u))$$

for every $A \in \mathcal{A}(\Omega)$. From this, it is easy to infer (5.22) and (5.23).

5.2 The density of the jump part

To prove (5.24) we apply the same blow-up argument of [12] Section 3. Recall that Lemma 2.5, Theorem 3.77 [3] and Radon-Nykodym's Theorem yield for $\mathcal{H}^{N-1}$ a.e. $x_0 \in J_u$

$$\lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{S(u)\cap\left(x_0+tQ_{\nu(x_0)}\right)} \left|u^+(x) - u^-(x)\right|\, d\mathcal{H}^{N-1} = \left|u^+(x_0) - u^-(x_0)\right|, \tag{5.25}$$

$$\lim_{t\to 0^+} \frac{1}{t^n} \int_{x_0+tQ^\pm_{\nu(x_0)}} \left|u(x) - u^\pm(x_0)\right|\, dx = 0, \tag{5.26}$$

$$\mu_S(x_0) = \lim_{t\to 0^+} \frac{\mu\left(x_0 + tQ_{\nu(x_0)}\right)}{|u^+ - u^-|\, \mathcal{H}^{N-1}\left(S(u) \cap \left(x_0 + tQ_{\nu(x_0)}\right)\right)}, \tag{5.27}$$

exists and is finite.

By (5.25) and (5.27), and since the function $\mathcal{X}_{x_0+tQ_{\nu(x_0)}}$ is upper semicontinuous and with compact support in Ω if t is sufficiently small, we get

$$\begin{aligned}
&|u^+(x_0) - u^-(x_0)|\mu_S(x_0) = \lim_{t\to 0^+} \frac{1}{t^{n-1}} \int_{x_0+tQ_{\nu(x_0)}} d\mu(x) \\
\geq\ & \limsup_{t\to 0^+} \limsup_{n} \frac{1}{t^{n-1}} \int_{x_0+tQ_{\nu(x_0)}} f(x, u_n, \nabla u_n)\, dx \\
=\ & \limsup_{t\to 0^+} \limsup_{n} \int_{Q_{\nu(x_0)}} t f(x_0+ty, u_n(x_0+ty), \nabla u_n(x_0+ty)\, dy \\
=\ & \limsup_{t\to 0^+} \limsup_{n} \int_{Q_{\nu(x_0)}} t f(x_0+ty, u_{n,t}(y), \frac{1}{t}\nabla u_{n,t}(y)\, dy, \qquad (5.28)
\end{aligned}$$

where

$$u_{n,t}(y) := u_n(x_0+ty).$$

Note that, by (5.26), we get that, set

$$u_0(y) := \begin{cases} u^+(x_0) & \text{if } \langle y, \nu_u(x_0)\rangle \geq 0 \\ u^-(x_0) & \text{if } \langle y, \nu_u(x_0)\rangle < 0 \end{cases},$$

then

$$\lim_{t\to 0^+} \lim_{n} \int_{Q_{\nu(x_0)}} |u_{n,t}(y) - u_0(y)|\, dx = 0. \qquad (5.29)$$

So far, from (5.28), by using hypotheses (H3)–(H5) and following the same steps of the proof in [12] Section 3, we get

$$|u^+(x_0) - u^-(x_0)|\mu_S(x_0) \geq \limsup_{t\to 0^+} \limsup_{n} \int_{Q_{\nu(x_0)}} f^\infty(x_0, u_{n,t}(y), \nabla u_{n,t}(y))\, dy. \qquad (5.30)$$

Then, by (5.29) and (5.30), using a standard diagonalization procedure we construct a sequence (v_k) such that $v_k \to u_0$ in $L^1(Q_{\nu(x_0)}; \mathbb{R}^d)$ and

$$|u^+(x_0) - u^-(x_0)|\mu_S(x_0) \geq \lim_{k\to +\infty} \int_{Q_{\nu(x_0)}} f^\infty(x_0, v_k(y), \nabla v_k(y))\, dy.$$

In order to establish (5.24), by the definition of K, it suffices to replace (v_k) by a sequence in $\mathcal{P}(u^-(x_0), u^+(x_0), \nu_u(x_0))$ whithout increasing the energy in the limit. Assuming Lemma 5.2 below proved, we are done.

Lemma 5.2 *Let $f : \Omega \times S^{d-1}\mathbb{R}^{d\times N}$ be a Carathéodory function such that*

$$0 \le f(x, u, \xi) \le c(1 + |\xi|)$$

for some $C > 0$ and for all $(x, u) \in \Omega \times S^{d-1}$ and $\xi \in [T_u(S^{d-1})]^N$. Let $a, b \in S^{d-1}$ and let $v_n \in W^{1,1}(Q_\nu; S^{d-1})$ converge in $L^1(Q_\nu; \mathbb{R}^d)$ to the function u_0 defined by

$$u_0(x) := \begin{cases} b & \text{if } \langle x, \nu\rangle \ge 0 \\ \\ a & \text{if } \langle x, \nu\rangle < 0 \end{cases}.$$

Then there exists a sequence $w_n \in \mathcal{P}(a, b, \nu)$ such that $w_n \to u_0$ in $L^1(Q_\nu; \mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} \int_{Q_\nu} f(x, v_n, \nabla v_n)\, dx \ge \limsup_{n\to+\infty} \int_{Q_\nu} f(x, w_n, \nabla w_n)\, dx.$$

Proof. For simplicity of notations, assume $\nu = e_N$ and set $Q := Q_{e_N}$. Without loss of generality, we may suppose that

$$\liminf_{n\to+\infty} \int_Q f(x, v_n, \nabla v_n)\, dx = \lim_{n\to+\infty} \int_Q f(x, v_n, \nabla v_n)\, dx < +\infty.$$

Moreover, by Theorem 2.2, we may assume $v_n \in \mathcal{D}(\Omega; S^{d-1})$, defined by (2.6) and (2.7). Let ρ be a mollifier and set $\rho_n := n^N \rho(nx)$. Then define

$$\tilde{\psi}_n(x) := (\rho_n * u_0)(x) = \int_{B(x,1/n)} \rho_n(x - y) u_0(y)\, dy.$$

Note that for all $x \in \mathbb{R}^N$, $\tilde{\psi}_n(x) \in \overline{ab} := \{ta + (1 - t)b : \ t \in [0, 1]\}$. Let $\pi : \overline{ab} \to S^{d-1}$ a C^1 function such that $\pi(a) = a$ and $\pi(b) = b$ and set

$$\psi_n := \pi \circ \tilde{\psi}_n.$$

It can be easily seen that $\psi_n \in \mathcal{P}(a, b, e_N)$. Moreover

$$\psi_n(x) = \begin{cases} b & \text{if } x_N > 1/n \\ a & \text{if } x_N < 1/n, \end{cases} \quad \|\nabla \psi_n\|_\infty = O(n).$$

So far, we argue as in the proof of [12] Lemma 3.1. Let

$$\alpha_n := \sqrt{\|v_n - \psi_n\|_{L^1(Q)}}, \quad k_n := n[1 + \|v_n\|_{1,1} + \|\psi_n\|_{1,1}], \quad s_n := \frac{\alpha_n}{k_n}$$

where $[k]$ denotes the largest integer less than or equal to k. Since $\alpha_n \to 0^+$ we may assume $0 \le \alpha_n < 1$ and we set

$$Q_0 := (1 - \alpha_n)Q, \quad Q_i := (1 - \alpha_n + i s_n), \quad i = 1, \dots, k_n.$$

Then, let φ_i be a cut-off function between Q_{i-1} and Q_i, with $\|\nabla\varphi_i\|_\infty = O(\frac{1}{s_n})$ for $i = 1, \dots, k_n$, and define

$$w_n^i := \varphi_i v_n + (1 - \varphi_i)\psi_n.$$

Note that $w_n^i \in W^{1,1}(Q; B^d(0,1))$, and $w_n^i \equiv v_n$ on Q_{i-1}, $w_n^i \equiv \psi_n$ on $Q \setminus Q_i$. Moreover, since

$$\nabla w_n^i = \varphi_i \nabla v_n + (1 - \varphi_i)\nabla\psi_n + (v_n - \psi_n) \otimes \nabla\varphi_i,$$

we get

$$\int_{Q_i \setminus Q_{i-1}} |\nabla w_n^i|\, dx \le C \int_{Q_i \setminus Q_{i-1}} (|\nabla v_n| + |\nabla\psi_n| + \frac{1}{s_n}|v_n - \psi_n|)\, dx. \tag{5.31}$$

We now need to suitably project the functions w_n^i on S^{d-1}. First of all observe that if $N = 1$, the sets $w_n^i(Q)$ are embedded curves so that you can find a sequence of points in the ball $B^d(0,1)$ from which the projection of w_n^i into the sphere S^{d-1} is in $W^{1,1}(Q, S^{d-1})$ and its $W^{1,1}$-norm is uniformly controlled by $\|w_n^i\|_{W^{1,1}}$.

Let us deal now with $N > 1$. To this purpose, given $y \in B^d(0, 1/2)$, let $\pi_y : B^d(0,1) \setminus \{y\} \to S^{d-1}$ the function projecting $x \in B^d(0,1)$ on S^{d-1} along the direction $x - y$. An easy computation shows that π_y is given by

$$\pi_y(x) = y + \frac{-\langle y, x - y\rangle + \sqrt{(\langle y, x - y\rangle)^2 + |x - y|^2(1 - |y|^2)}}{|x - y|^2}(x - y). \tag{5.32}$$

Note that

$$\pi_{y_{|S^{d-1}}} = Id_{S^{d-1}}. \tag{5.33}$$

Moreover, it is easy to show that

$$|\nabla\pi_y(x)| \le \frac{C}{|x - y|},\ \forall x \in B^d(0,1), \tag{5.34}$$

with C independent on $y \in B^d(0, 1/2)$. Let G_n^i the set of critical values in $B^d(0, 1/2)$ of w_n^i, that is

$$G_n^i := \{y \in B^d(0, 1/2) : \exists x \in Q \text{ with } w_n^i(x) = y \text{ and rank}\,(\nabla w_n^i(x)) < N \wedge d\},$$

and set

$$G := \cup_{n,i} G_n^i.$$

By Sard's Lemma, $\mathcal{H}^d(G) = 0$. Then, for $y \in B^d(0, 1/2) \setminus G$, the function $\pi_y \circ w_{n,i}$ is smooth except on a submanifold of $\mathbb{R}^N$ of codimension greater than 2. Moreover, by Fubini's Theorem and by (5.34), we get

$$\int_{B^d(0,1/2)} \int_{Q_i \setminus Q_{i-1}} |\nabla\pi_y \circ w_n^i|\, dx\, dy$$

$$\leq C\int_{Q_i\setminus Q_{i-1}} |\nabla w_n^i(x)|\left(\int_{B^d(0,1/2)} |w_n^i(x)-y|^{-1}\,dy\right)dx$$
$$\leq C\left(\int_{B^d(0,3/2)} |z|^{-1}\,dz\right)\int_{Q_i\setminus Q_{i-1}} |\nabla w_n^i(x)|\,dx \qquad (5.35)$$

Then, we may find $y_n^i \in B^d(0,1/2)\setminus G$ such that

$$\int_{Q_i\setminus Q_{i-1}} |\nabla \pi_{y_n^i}\circ w_{n,i}|\,dx \leq C\int_{Q_i\setminus Q_{i-1}} |\nabla w_{n,i}|\,dx. \qquad (5.36)$$

Set, then,

$$\tilde{w}_n^i := \pi_{y_n^i}\circ w_n^i.$$

Observe that, by (5.33), $\tilde{w}_n^i \to u_0$ in $L^1(Q,\mathbb{R}^d)$ and

$$\tilde{w}_n^i \equiv v_n \text{ on } Q_{i-1},$$

$$\tilde{w}_n^i \equiv \psi_n \text{ on } Q\setminus Q_i.$$

Moreover, by (5.36) $\tilde{w}_n^i \in W^{1,1}(Q;S^{d-1})$ and so $\tilde{w}_n^i \in \mathcal{P}(a,b,e_N)$. Hence, by the growth condition on f, (5.31) and (5.36), we get

$$\begin{aligned}\int_Q f(x,\tilde{w}_n^i,\nabla\tilde{w}_n^i)\,dx \quad \leq \quad & \int_Q f(x,v_n,\nabla v_n)\,dx \\ & +C\int_{Q_i\setminus Q_{i-1}} \left(1+|\nabla v_n|+|\nabla\psi_n|+\frac{1}{s_n}|v_n-\psi_n|\right)dx \\ & +C\int_{Q\setminus Q_i}(1+|\nabla\psi_n|)\,dx.\end{aligned}$$

So far, proceeding exactly as in the proof of Lemma 3.1 [12], one proves that for any $n\in\mathbb{N}$ there exists an index $i(n)\in\{1,\ldots,k_n\}$ such that

$$\int_Q f(x,\tilde{w}_n^{i(n)},\nabla\tilde{w}_n^{i(n)})\,dx \leq \int_Q f(x,v_n,\nabla v_n)\,dx + O(1).$$

To conclude, it suffices to set

$$w_n := \tilde{w}_n^{i(n)}.$$

□

6 Estimate from above

In this section we conclude the proof of Theorem 3.1, by showing that

$$\overline{F}(u) \leq G(u). \qquad (6.37)$$

To do this, we follow the same argument of [12] Section 5.

As a first step, we localize the functional $\overline{F}$ by setting, for any $u \in BV(\Omega; S^{d-1})$ and $A \in \mathcal{A}(\Omega)$,

$$\begin{aligned}\overline{F}(u, A) \quad &:= \\ &\inf\{\liminf_n \int_A f(x, u_n, \nabla u_n)\, dx : u_n \in W^{1,1}(A; S^{d-1}),\ u_n \to u \text{ in } L^1(A; \mathbb{R}^d)\}.\end{aligned}$$

We claim that

$$\overline{F}(u, A) \le C(|A| + |Du|(A)) \quad \forall (u, A) \in BV(\Omega; S^{d-1}) \times \mathcal{A}(\Omega), \tag{6.38}$$

and that $\overline{F}(u, A)$ is a *variational functional with respect to the* L^1 *topology*, that is (i) $\overline{F}(\cdot, A)$ is local, *i.e.*, ,

$$\overline{F}(u, A) = \overline{F}(v, A) \quad \text{if } u = v \text{ a.e. in } A;$$

(ii) for every $A \in \mathcal{A}(\Omega)$, $\overline{F}(\cdot, A)$ is lower semicontinuous with respect to the $L^1(A; \mathbb{R}^d)$ topology; (iii) for every $u \in BV(\Omega; S^{d-1})$, the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$.

Inequality (6.38) follows by the growth hypothesis (H3) and by the following Lemma.

Lemma 6.1 *For every $u \in BV(\Omega; S^{d-1})$ and $A \in \mathcal{A}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,1}(A; S^{d-1})$ such that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$ and*

$$\limsup_n |Du_n|(A) \le C|Du|(A),$$

where $C > 0$ is a constant independent on u and A.

Proof. By a standard density result in BV theory, there exists $v_n \in C^\infty(A; B^d(0,1))$ such that $v_n \to u$ in $L^1(A; \mathbb{R}^d)$ and

$$\lim_n |Dv_n|(A) = |Du|(A).$$

For $y \in B^d(0, 1/2)$, let $\pi_y : B^d(0,1) \setminus \{y\} \to S^{d-1}$ defined by (5.32). Then, as in the proof of Lemma 5.2, we may find $y_n \in B^d(0, 1/2)$ such that $\pi_{y_n} \circ v_n \in W^{1,1}(A; S^{d-1})$ and

$$\int_A |\nabla \pi_{y_n} \circ v_n|\, dx \le C \int_A |\nabla v_n|\, dx.$$

Then, it suffices to set

$$u_n := \pi_{y_n} \circ v_n.$$

□

Properties (i) and (ii) are direct consequence of the definition of $\overline{F}(u,A)$. To prove (iii), thanks to De Giorgi-Letta criterion (see [10]), we have to show that the set function $F(u,\cdot)$ is superadditive, subadditive and inner regular. The proof of the superadditivity property is straightforward. By (6.38) and using a standard argument (see for example the proof of Theorem 4.3 in [4]), the proof of the last two properties follows by the following Proposition, in which we establish the so called weak subbaditivity for $\overline{F}(u,\cdot)$.

Proposition 6.2 *Let $u \in BV(\Omega;S^{d-1})$. Then, for any $A', A, B \in \mathcal{A}(\Omega)$ such that $A' \subset\subset A$, there holds*

$$\overline{F}(u, A' \cup B) \leq \overline{F}(u,A) + \overline{F}(u,B). \tag{6.39}$$

Proof. Let $u_n \in W^{1,1}(A;S^{d-1})$, $v_n \in W^{1,1}(B;S^{d-1})$ be such that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$, $v_n \to u$ in $L^1(B;\mathbb{R}^d)$ and

$$\lim_n \int_A f(x,u_n,\nabla u_n)\,dx = \overline{F}(u,A),$$

$$\lim_n \int_B f(x,v_n,\nabla u_n)\,dx = \overline{F}(u,B).$$

By Theorem 2.2, we may suppose $u_n \in C^\infty(A;S^{d-1})$, $v_n \in C^\infty(B;S^{d-1})$. Set

$$d := \mathrm{dist}\,(A', A^c)$$

and, given $M \in \mathbb{N}$, for any $i \in \{1,\ldots,M\}$ define

$$A_i := \{x \in A : \mathrm{dist}\,(x,A') < i\frac{d}{M}\},$$

$$C_i = (A_{i+1} \setminus \overline{A}_i) \cap B.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \leq 2\frac{M}{d}$, and set

$$w_n^i := \varphi_i u_n + (1-\varphi_i)v_n.$$

Then, for any $i \in \{1,\ldots,M\}$, $w_n^i \in W^{1,1}(A'\cup B; B^d(0,1)) \cap C^\infty(A' \cup B; B^d(0,1))$ and $w_n^i \to u$ in $L^1(A'\cup B;\mathbb{R}^d)$. Moreover, since

$$\nabla w_n^i(x) = \varphi_i(x)\nabla u_n(x) + (1-\varphi_i(x))\nabla v_n(x) + \nabla\varphi_i(x) \otimes (u_n(x) - v_n(x)),$$

we get

$$\int_{C_i} |\nabla w_n^i|\,dx \leq \int_{C_i} (|\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|)\,dx. \tag{6.40}$$

In order to find a good recovery sequence for $\overline{F}(u,A'\cup B)$, we now argue as in the proof of Lemma 5.2. For $y \in B^d(0,1/2)$, let $\pi_y : B^d(0,1)\setminus\{y\} \to S^{d-1}$ defined by (5.32). Then, as in the proof of Lemma 5.2, we may find $y_n^i \in B^d(0,1/2)$ such that $\pi_{y_n^i} \circ w_n^i \in W^{1,1}(A' \cup B; S^{d-1})$ and

$$\int_{C_i} |\nabla \pi_{y_n^i} \circ w_{n,i}| \, dx \leq C \int_{C_i} |\nabla w_{n,i}| \, dx. \tag{6.41}$$

Set, then,

$$\tilde{w}_n^i := \pi_{a_n^i} \circ w_n^i.$$

Observe that $w_n^i \to u$ in $L^1(A' \cup B; \mathbb{R}^d)$ and

$$\tilde{w}_n^i \equiv u_n \text{ on } A^i,$$

$$\tilde{w}_n^i \equiv v_n \text{ on } B \setminus A^{i+1}.$$

Hence, by (H3), (6.40) and (6.41), we get

$$\begin{aligned} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i) \, dx \quad &\leq \quad \int_A f(x, u_n, \nabla u_n) \, dx + \int_B f(x, v_n, \nabla v_n) \, dx \\ &+ C \int_{C_i} (1 + |\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|) \, dx. \end{aligned}$$

Thus, summing over $i \in \{1, \dots M\}$ and averaging, we get

$$\begin{aligned} \frac{1}{M} \sum_{i=1}^{M} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i) \, dx &\leq \int_A f(x, u_n, \nabla u_n) \, dx + \int_B f(x, v_n, \nabla v_n) \, dx \\ &+ \frac{C}{M} \int_{A \cap B} (1 + |\nabla u_n| + |\nabla v_n| + 2\frac{M}{d}|u_n - v_n|) \, dx. \end{aligned} \tag{6.42}$$

For any $n \in \mathbb{N}$ there exists $i(n) \in \{1, \dots M\}$ such that

$$\int_{A' \cup B} f(x, \tilde{w}_n^{i(n)}, \nabla \tilde{w}_n^{i(n)}) \, dx \leq \frac{1}{M} \sum_{i=1}^{M} \int_{A' \cup B} f(x, \tilde{w}_n^i, \nabla \tilde{w}_n^i) \, dx \tag{6.43}$$

Then, since $\tilde{w}_n^{i(n)}$ still converges to u in $L^1(A' \cup B; \mathbb{R}^d)$ and, by (H3),

$$\sup_n \int_{A \cup B} |\nabla u_n| + |\nabla v_n| \, dx < +\infty,$$

from (6.42) and (6.43) we deduce that

$$\overline{F}(u, A' \cup B) \leq \liminf_n \int_{A' \cup B} f(x, \tilde{w}_n^{i(n)}, \nabla \tilde{w}_n^{i(n)}) \, dx \leq \overline{F}(u, A) + \overline{F}(u, B) + \frac{C}{M}.$$

Eventually, letting $M \to +\infty$, we obtain the thesis. □

Remark 6.3 $\overline{F}$ *enjoys the following locality property on* $\mathcal{B}(\Omega)$*:*
let $u, v \in BV(\Omega; S^{d-1})$ *and let* $B \in \mathcal{B}(\Omega)$ *be such that*

$$B \subseteq S(u) \cap S(v), \qquad (u^-(x), u^+(x), \nu_u(x)) = (v^-(x), v^+(x), \nu_v(x)) \ \forall x \in B,$$

then

$$\overline{F}(u, B) = \overline{F}(v, B).$$

The proof of this property can be carried out as in Step 1 of the proof of Proposition 4.4 in [4], where the same property is stated in the non constrained case.

So far, we can obtain inequality (6.37) by showing that

$$\overline{F}(u, \Omega \setminus S(u)) \leq \int_\Omega f(x, u, \nabla u)\, dx + \int_\Omega f^\infty(x, u, dC(u)), \tag{6.44}$$

and

$$\overline{F}(u, S(u)) \leq \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{6.45}$$

Inequality (6.44) will follow by Lemma 6.4 below, while (6.45) will be proved in Lemma 6.5.

Lemma 6.4 *If* $u \in BV(\Omega, S^{d-1})$*, then for* $\mathcal{L}^N$ *a.e.* $x_0 \in \Omega$*,*

$$\frac{d\overline{F}(u, \cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)), \tag{6.46}$$

and for $|C(u)|$ *a.e* $x_0 \in \Omega$*,*

$$\frac{d\overline{F}(u, \cdot)}{d|C(u)|}(x_0) \leq f^\infty(x_0, u(x_0), A(x_0)). \tag{6.47}$$

Proof. The proof follows the lines of *Step 2* of Theorem 2.16 in [12]. We will enter into details only when the changes are significant, otherwise reminding to [12].

Let us prove first (6.46). By Theorem 2.4, and by Theorems 2.7-2.8 in [12], for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|(1 + |\nabla u(x)|)\, dx = 0, \tag{6.48}$$

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0,\varepsilon)} |\nabla u(x) - \nabla u(x_0)|\, dx = 0, \tag{6.49}$$

$$\lim_{\varepsilon \to 0} \frac{|D_s u|(B(x_0, \varepsilon))}{|B(x_0, \varepsilon)|} = 0, \quad \lim_{\varepsilon \to 0} \frac{|Du|(B(x_0, \varepsilon))}{|B(x_0, \varepsilon)|} \ \text{ exists and is finite}, \tag{6.50}$$

$$\frac{d\overline{F}(u, \cdot)}{d\mathcal{L}^N}(x_0) \ \text{ exists and is finite.} \tag{6.51}$$

Let $\{u_n\}$ be the sequence defined in Lemma 2.7. Fix a sequence of numbers $\varepsilon \in (0, \frac{\mathrm{dist}(x_0,\partial\Omega)}{2})$ such that $|Du|(\partial B(x_0,\varepsilon)) = 0$, and a subsequence of u_n, not relabeled, such that $w_n := \pi_{a_n^\varepsilon} \circ u_n$, defined as in the proof of Lemma 5.2 and Proposition 6.2, is such that $w_n \in W^{1,1}(\Omega, S^{d-1})$ and, for some $\delta \in (\frac{3}{4}, 1)$,

$$\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|\le\delta\}} |\nabla w_n|\,dx \le C \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|\le\delta\}} |\nabla u_n|\,dx, \tag{6.52}$$

where $a_n^\varepsilon \in B^d(0, \frac{1}{2})$ satisfies

$$\lim_{\varepsilon\to 0} \lim_{n\to+\infty} a_n^\varepsilon = \lim_{\varepsilon\to 0} a^\varepsilon = a_0.$$

Thanks to this choice of δ we have also that

$$|\nabla w_n|\chi_{\{|u_n|>\delta\}} \le C|\nabla u_n|\chi_{\{|u_n|>\delta\}}, \tag{6.53}$$

so

$$\int_{B(x_0,\varepsilon)} |\nabla w_n|\,dx \le C \int_{B(x_0,\varepsilon)} |\nabla u_n|\,dx. \tag{6.54}$$

Then

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0} \frac{\overline{F}(u, B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \\ &\le \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} f(x, w_n(x), \nabla w_n(x))\,dx. \end{aligned} \tag{6.55}$$

Introducing, as in [12], the Yosida transforms of f, given by

$$f_\lambda(x,u,\xi) := \sup\{f(x',u',\xi) - \lambda[|x-x'| + |u-u'|](1+|\xi|) : (x',u') \in \Omega\times\mathbb{R}^d\},$$

we have

$$\begin{aligned} f(x, w_n(x), \nabla w_n(x)) &\le f(x_0, u(x_0), \nabla w_n(x)) + \eta(1+|\nabla w_n(x)|) \\ &+ \lambda[|x-x_0| + C|w_n(x) - u(x_0)|](1+|\nabla w_n(x)|), \end{aligned}$$

for $x \in \overline{B}\left(x_0, \frac{\mathrm{dist}(x_0,\partial\Omega)}{2}\right)$ and $\eta > 0$. Thus, by (6.55) and (6.54),

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &\le \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla w_n(x))\,dx \\ &+ C(\eta+\lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n|\,dx + (\lambda\varepsilon+\eta)|B(x_0,\varepsilon)| \\ &+ C\lambda \int_{B(x_0,\varepsilon)} |w_n(x) - u(x_0)|(1+|\nabla w_n|)\,dx \Big]. \end{aligned}$$

Since, by (6.53) and (6.54),

$$\begin{aligned}
& \int_{B(x_0,\varepsilon)} |w_n(x) - u(x_0)||\nabla w_n|\, dx \\
\leq\ & C \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\
=\ & C \int_{\{x \in B(x_0,\varepsilon): |u_n(x)| \leq \delta\}} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\
+\ & C \int_{\{x \in B(x_0,\varepsilon): |u_n(x)| > \delta\}} |u_n(x) - u(x_0)||\nabla w_n|\, dx \\
\leq\ & (1+\delta) C \int_{\{x \in B(x_0,\varepsilon): |u_n(x)| \leq \delta\}} |\nabla u_n|\, dx \\
+\ & C \int_{\{x \in B(x_0,\varepsilon): |u_n(x)| > \delta\}} |u_n(x) - u(x_0)||\nabla u_n|\, dx \\
\leq\ & C \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla u_n|\, dx,
\end{aligned}$$

we deduce

$$\begin{aligned}
\frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \quad \leq \quad & \liminf_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla w_n(x))\, dx \\
+ \quad & C(\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n|\, dx + (\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| \\
+ \quad & C\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|)\, dx \Big].
\end{aligned}$$

Taking into account that $f(x_0, u(x_0), \cdot)$ is a Lipschitz function we get

$$\begin{aligned}
& \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq \liminf_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{|B(x_0,\varepsilon)|} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla u_n(x))\, dx \\
& + C \int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n|\, dx + (C\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n|\, dx \\
& + (\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| + C\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|)\, dx \Big].
\end{aligned} \tag{6.56}$$

Now, the first and the third term of (6.56) can be treated as in *Step 2* of Theorem

2.16 in [12], getting

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)) + C\eta$$
$$+C \liminf_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{|B(x_0,\varepsilon)|}\Big[\int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n|\, dx \tag{6.57}$$
$$+\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|)\, dx\Big].$$

Splitting $B(x_0,\varepsilon)$ into the sets where $|u_n| \leq \delta$ and where $|u_n| > \delta$, the term $I^n_\varepsilon := \int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n|\, dx$ can be estimated in the following way

$$I^n_\varepsilon \leq C \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|\leq\delta\}} |u_n(x) - u(x_0)||\nabla u_n|\, dx$$
$$+ \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |\nabla u_n - \nabla w_n|\, dx, \tag{6.58}$$

the firts term being of the same type of the last term of (6.57). Let us deal with the second term. It yields

$$\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |\nabla u_n - \nabla w_n|\, dx$$
$$\leq \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |\nabla \pi_{a^\varepsilon_n}(u_n) - \nabla \pi_{a^\varepsilon_n}(u(x_0))||\nabla u_n|\, dx$$
$$+ \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |(I - \nabla \pi_{a^\varepsilon_n}(u(x_0)))\nabla u_n|\, dx$$
$$\leq C \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla u_n|\, dx + \int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L^\varepsilon_n \nabla u_n|\, dx, \tag{6.59}$$

where $L^\varepsilon_n : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ defined by $L^\varepsilon_n = I - \nabla \pi_{a^\varepsilon_n}(u(x_0))$ is such that

$$\lim_{\varepsilon\to 0} \lim_{n\to+\infty} L^\varepsilon_n = \lim_{\varepsilon\to 0} L^\varepsilon = L_0,$$

with $L^\varepsilon = I - \nabla \pi_{a^\varepsilon}(u(x_0))$ and $L_0 = I - \nabla \pi_{a_0}(u(x_0))$. Let us note that

$$\begin{aligned} L^\varepsilon_n \nabla u_n &= L^\varepsilon_n(\rho_n * Du) = L^\varepsilon_n \left(\int_{B(x,\frac{1}{n})} \rho_n(x-y) Du(y) \right) \\ &= \int_{B(x,\frac{1}{n})} \rho_n(x-y) L^\varepsilon_n Du(y) = \rho_n * (L^\varepsilon_n Du), \end{aligned}$$

so that, by Lemma 2.7,

$$\begin{aligned}&\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon \nabla u_n|\,dx\\ \leq\ &\int_{B(x_0,\varepsilon)} |\rho_n * (L_n^\varepsilon Du)|\,dx \leq |L_n^\varepsilon Du|(B(x_0,\varepsilon+\frac{1}{n})).\end{aligned}$$

Taking into account that $|Du|(\partial B(x_0,\varepsilon))=0$, passing to the limit as n goes to infinity in the previous inequality, we get

$$\limsup_{n\to+\infty}\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon \nabla u_n|\,dx \leq |L^\varepsilon Du|(B(x_0,\varepsilon)). \tag{6.60}$$

Let us divide by $|B(x_0,\varepsilon)|$ and denote by μ^ε the measure $\mu^\varepsilon = L^\varepsilon((u^+-u^-)\otimes \nu_u)\mathcal{H}^{N-1}\lfloor S(u) + L^\varepsilon(A(x))|C(u)|$, obtaining

$$\begin{aligned}&\frac{|L^\varepsilon Du|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} + \fint_{B(x_0,\varepsilon)} |L^\varepsilon \nabla u|\,dx\\ &\leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} + \fint_{B(x_0,\varepsilon)} |L^\varepsilon\nabla u - L_0\nabla u|\,dx\\ &+\fint_{B(x_0,\varepsilon)} |L_0\nabla u|\,dx \leq \frac{|\mu^\varepsilon|(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|}\\ &+|L^\varepsilon - L_0|\fint_{B(x_0,\varepsilon)} |\nabla u|\,dx + |L_0|\fint_{B(x_0,\varepsilon)} |\nabla u(x)-\nabla u(x_0)|\,dx,\end{aligned} \tag{6.61}$$

where we have used Remark 2.6 for the last term. Putting together (6.60) and (6.61) and using (6.49) and $(6.50)_1$, we get

$$\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |L_n^\varepsilon\nabla u_n|\,dx = 0,$$

so that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\Big[\int_{\{x\in B(x_0,\varepsilon):|u_n(x)|>\delta\}} |\nabla u_n - \nabla w_n|\,dx\\ &-\int_{B(x_0,\varepsilon)} |u_n(x)-u(x_0)||\nabla u_n|\,dx\Big] = 0\\ &.\end{aligned}$$

At this point to prove (6.46) it remains to show that

$$\liminf_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)} |u_n(x)-u(x_0)|(1+|\nabla u_n|)\,dx = 0,$$

and this can be done exactly as in *Step 2* of Theorem 2.16 in [12], so the proof of (6.46) is complete.

Next we prove (6.47). Denoting by ν the measure $\nu = |Du| - |C(u)|$, by Theorems 2.7, 2.8 and 2.11 in [12], for $|C(u)|$ a.e. $x_0 \in \Omega$ we have

$$\lim_{\varepsilon\to 0} \frac{\nu(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))} = 0, \qquad \lim_{\varepsilon\to 0} \frac{|Du|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))} \text{ exists and is finite,} \tag{6.62}$$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^N}{|C(u)|(B(x_0,\varepsilon))} = 0, \tag{6.63}$$

$$\lim_{\varepsilon\to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |u(x) - u(x_0)|\, d|C(u)| = 0, \tag{6.64}$$

$$\lim_{\varepsilon\to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |A(x) - A(x_0)|\, d|C(u)| = 0, \tag{6.65}$$

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f^\infty(x_0, u(x_0), A(x))\, d|C(u)| \\ &= f^\infty(x_0, u(x_0), A(x_0)), \end{aligned} \tag{6.66}$$

$$\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) \text{ exists and is finite.} \tag{6.67}$$

As for (6.56), we get

$$\begin{aligned} \frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) &\le \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \Big[\int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla u_n(x))\, dx \\ &+C \int_{B(x_0,\varepsilon)} |\nabla u_n - \nabla w_n|\, dx + (C\eta + \lambda\varepsilon) \int_{B(x_0,\varepsilon)} |\nabla u_n|\, dx \\ &+(\lambda\varepsilon + \eta)|B(x_0,\varepsilon)| + C\lambda \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)|(1 + |\nabla u_n|)\, dx \Big]. \end{aligned} \tag{6.68}$$

Using (6.58), (6.59), (6.60), and the fact that

$$\begin{aligned} \limsup_{n\to+\infty} \int_{B(x_0,\varepsilon)} |u_n(x) - u(x_0)||\nabla u_n|\, dx &\le \int_{\overline{B}(x_0,\varepsilon)\setminus S(u)} |u(x) - u(x_0)||Du|(x) \\ &+ 4|Du|(\overline{B}(x_0,\varepsilon) \cap S(u)), \end{aligned}$$

which is proved in [12], since $C(B(x_0,\varepsilon) \cap S(u)) = 0$, we conclude that

$$\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0) \le \liminf_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{|C(u)|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x_0, u(x_0), \nabla u_n(x))\, dx$$

$$
\begin{aligned}
&+C\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))\\
&+\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}(C\eta+\lambda\varepsilon)(|Du|(B(x_0,\varepsilon))+|B(x_0,\varepsilon)|)\\
&+C\lambda\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)}|u(x)-u(x_0)|\,d|C(u)|\\
&+2|B(x_0,\varepsilon)|+6|\nu|(B(x_0,\varepsilon))\Big]\\
&\leq \liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}f(x_0,u(x_0),\nabla u_n(x))\,dx\\
&+C\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))\\
&+\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}(C\eta+\lambda\varepsilon)(|Du|(B(x_0,\varepsilon))+|B(x_0,\varepsilon)|)\\
&+C\lambda\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\Big[\int_{B(x_0,\varepsilon)}|u(x)-u(x_0)|\,d|C(u)|\\
&+6\nu(B(x_0,\varepsilon))+2|B(x_0,\varepsilon)|\Big].
\end{aligned}
\tag{6.69}
$$

Now, denoting by $\tilde{\mu}^\varepsilon$ the measure $\tilde{\mu}^\varepsilon = L^\varepsilon(\nabla u)\mathcal{L}^N + L^\varepsilon((u^+-u^-)\otimes\nu_u)\mathcal{H}^{N-1}\lfloor S(u)$, as for (6.61), one sees that

$$
\begin{aligned}
&\frac{|L^\varepsilon Du|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}\leq\frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L^\varepsilon A(x)||C(u)|\\
&\leq\frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L^\varepsilon A(x)-L_0A(x)|C(u)|\\
&+\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|L_0A(x)||C(u)|\\
&\leq\frac{|\tilde{\mu}^\varepsilon|(B(x_0,\varepsilon))}{|C(u)|(B(x_0,\varepsilon))}+\frac{|L^\varepsilon-L_0|}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|A(x)||C(u)|\\
&+\frac{|L_0|}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}|A(x)-A(x_0)||C(u)|,
\end{aligned}
$$

where we have used Remark 2.6 for the last term. Therefore, by $(6.62)_1$ and (6.65), we obtain

$$
\limsup_{\varepsilon\to 0}\frac{1}{|C(u)|(B(x_0,\varepsilon))}|L^\varepsilon Du|(B(x_0,\varepsilon))=0. \tag{6.70}
$$

Now, applying (6.62)-(6.64) to (6.69), and using (6.70) we deduce

$$
\frac{d\overline{F}(u,\cdot)}{d|C(u)|}(x_0)\leq\liminf_{\varepsilon\to 0}\liminf_{n\to+\infty}\frac{1}{|C(u)|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}f(x_0,u(x_0),\nabla u_n(x))\,dx+C\eta, \tag{6.71}
$$

and the same arguments of *Step 2* of Theorem 2.16 in [12] lead to (6.47). □

Eventually the following Lemma provides us the inequality of the surface term that concludes the proof of Theorem 3.1.

Lemma 6.5 *For any $u \in BV(\Omega; S^{d-1})$ there holds*

$$\overline{F}(u, S(u)) \leq \int_{S(u)} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{6.72}$$

Proof. The proof closely follows that of Step 3 in Section 5 of [12]. Here we outline the main steps and enter into details of the proof only when significant changes occur.

Step 1. If $u(x) = a\chi_E(x) + b(1 - \chi_E(x))$ with $Per_\Omega(E) < +\infty$, then (6.72) holds. As in [12], it is enough to prove that for every $A \in \mathcal{A}(\Omega)$,

$$\overline{F}(u, A) \leq \int_A f(x, u, 0)\, dx + \int_{S(u)\cap A} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}. \tag{6.73}$$

(i) Suppose first $E = \{x \in \mathbb{R}^N : \langle x, \nu\rangle = 0\}$ for some $\nu \in S^{N-1}$. Without loss of generality we set $\nu := e_N$. In case f does not depend on x, by the definition of K let $\varphi \in \mathcal{P}(a, b, \nu)$ be such that

$$K(a, b, e_N) + \eta \geq \int_Q f^\infty(\varphi, \nabla\varphi)\, dx, \tag{6.74}$$

for some $\eta > 0$. Then, for $n \in \mathbb{N}$, define $u_n \in W^{1,1}(A; S^{d-1})$ by

$$u_n(x) := \begin{cases} b & \text{if } x_N > 1/2n \\ \varphi(nx) & \text{if } |x_N| \leq 1/2n \\ a & \text{if } x_N < -1/2n. \end{cases}$$

Then it is easy to see that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$. Moreover, set $A_n := \{x \in A : |x_N| \leq 1/2n\}$, $A'_n := \pi(A_n)$, π denoting the orthogonal projection onto E, by Fubini Theorem and by a change of variables, we get

$$\begin{aligned}
\int_A f(u_n, \nabla u_n)\, dx \;=\; & |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
& + \int_{A_n} f(\varphi(nx), n\nabla\varphi(nx))\, dx \\
& \leq |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
& + \int_{A'_n} dx' \int_{-1/2n}^{1/2n} f(\varphi(nx', nt), n\nabla\varphi(nx', nt))\, dt \\
& = |A \cap \{x_N \geq 1/2n\}| f(b, 0) + |A \cap \{x_N \leq -1/2n\}| f(a, 0) \\
& + \int_{A'_n} dx' \int_{-1/2}^{1/2} \frac{1}{n} f(\varphi(nx', s), n\nabla\varphi(nx', s))\, ds
\end{aligned}$$

$$=: I_n^1 + I_n^2 + I_n^3.$$

Thus, one easily gets that

$$\lim_n (I_n^1 + I_n^2) = \int_A f(u,0)\,dx,$$

while, by (H5) and Riemann-Lebesgue Theorem, there holds

$$\lim_n I_n^3 = \mathcal{H}^{N-1}(S(u)\cap A)\int_Q f^\infty(\varphi,\nabla\varphi)\,dx.$$

The conclusion follows by (6.74) and the arbitrariness of η.

In the general case, when f depends also on x, we can argue as in the proof of Proposition 4.1 in [13], by using assumption (H4), property (a) of Lemma 4.1 and Lemma 5.2.

(ii) Suppose E is a polyhedral set, that is E is a bounded strongly Lipschitz domain and $\partial E = H_1 \cup \ldots \cup H_m$, H_i being closed subsets of hyperplanes. Then the proof of (6.73) can be obtained as in Step 3 (c) of Section 5 in [12], by using the same argument of the proof of Lemma 5.2.

(iii) Finally, if E is an arbitrary set of finite perimeter in Ω, the proof of (6.73) is exactly the same of Step 3 (f) of Section 5 in [12], where property (a) and (b) of Lemma 4.1 are needed.

Step 2. Inequality (6.72) holds if

$$u(x) = \sum_{i=1}^h a_i \chi_{E_i}(x), \tag{6.75}$$

with $h\in\mathbb{N}$, $a_1,\ldots,a_h \in S^{d-1}$, $E_1,\ldots,E_h$ mutually disjoint sets of finite perimeter in Ω covering Ω. The proof can be done exactly as in Step 1 of the proof of Proposition 4.8 in [4], where the property of $\overline{F}$ stated in Remark 6.3 is needed.

Step 3. Inequality (6.72) holds if $u\in BV(\Omega;S^{d-1})$. First of all, note that the function K can be extended to a function $\tilde{K}:\Omega\times\mathbb{R}^d\setminus\{0\}\times\mathbb{R}^d\setminus\{0\}\times S^{N-1}\to [0,+\infty)$ as

$$\tilde{K}(x,a,b,\nu) := K(x,\frac{a}{|a|},\frac{b}{|b|},\nu),$$

so that $\tilde{K}$ inherites the properties of K stated in Lemma 4.1 on $\Omega\times J\times J\times S^{N-1}$, for every compact set J in $\mathbb{R}^d$. Given $A\in\mathcal{A}(\Omega)$, as in Step 2 of the proof of Proposition 4.8 in [4], by using the upper semicontinuity of $\tilde{K}$, one constructs a sequence $(u_n)\subset BV(\Omega;\mathbb{R}^d)$ of the type (6.75), such that

$$\lim_n \|u_n - u\|_\infty = 0 \tag{6.76}$$

and

$$\liminf_n \int_{S(u_n)\cap A} \tilde{K}(x,u_n^-,u_n^+,\nu_{u_n})\,d\mathcal{H}^{N-1} \quad\leq\quad C|Du|(A\setminus S(u)$$

$$+ \quad \int_{S(u)\cap A} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}$$

Thus, by (6.76), for n large enough, the sequence

$$v_n := \frac{u_n}{|u_n|}$$

is in $BV(\Omega; S^{d-1})$ of the type (6.75) and, thanks to Lemma 4.1(a),

$$\begin{aligned}
&\liminf_n \int_{S(v_n)\cap A} K(x, v_n^-, v_n^+, \nu_{v_n})\, d\mathcal{H}^{N-1} \\
&\leq \liminf_n \int_{S(u_n)\cap A} \tilde{K}(x, u_n^-, u_n^+, \nu_{u_n})\, d\mathcal{H}^{N-1} + o(1).
\end{aligned}$$

Then, by the previous Step and the lower semicontinuity of $\overline{F}$, there holds

$$\begin{aligned}
\overline{F}(u, A) \quad &\leq \quad \liminf_n \overline{F}(v_n, A) \leq \liminf_n \int_{S(v_n)\cap A} K(x, v_n^-, v_n^+, \nu_{v_n})\, d\mathcal{H}^{N-1} \\
&\leq \quad C|Du|(A \setminus S(u) + \int_{S(u)\cap A} K(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}.
\end{aligned}$$

From this and since $\overline{F}(u, \cdot)$ is a bounded measure, we easily infer (6.72). □

7 Relaxation of energies in micromagnetics

In this section we study the relaxation of constrained functionals $\mathcal{G} : L^1(\Omega; \mathbb{R}^N) \to [0, +\infty]$ of the type

$$\mathcal{G}(u) = \begin{cases} \int_\Omega f(x, u, \nabla u) + \int_{\mathbb{R}^N} |h_u|^2\, dx - \int_\Omega \langle h_{ext}, u\rangle\, dx & \text{if } u \in W^{1,1}(\Omega; S^{N-1}) \\ \infty & \text{otherwise,} \end{cases} \tag{7.1}$$

where $h_{ext} \in L^1(\Omega; \mathbb{R}^N)$ and $h_u \in L^2(\mathbb{R}^N; \mathbb{R}^N)$ is defined by

$$\begin{cases} \text{curl } h_u = 0 \\ \text{div } (h_u + u\chi_\Omega) = 0. \end{cases} \tag{7.2}$$

Functionals of this kind generalize those involved in variational models for micromagnetics, where u represents the magnetization of a ferromagnetic material subject to an external magnetic field h_{ext} and h_u is the induced magnetic field related to u through the Maxwell's equations (7.2) (see [7], [17] for a detailed explanation of the model).

System (7.2) is understood in the following way. Using the first equation of (7.2) it can be introduced the potential v with $h_u = -\nabla v$. Thus the second equation can be written as

$$\mathrm{div}(-\nabla v + u) = 0 \text{ in } \mathbb{R}^N, \tag{7.3}$$

where we have extended $u = 0$ outside Ω. The equation (7.3) means that

$$\int_{\mathbb{R}^N} (-\nabla v + u)\nabla w \, dx = 0 \quad \forall w \in V, \tag{7.4}$$

where $V = \{w \in H^1(B) : \nabla w \in L^2(\mathbb{R}^N) \text{ and } \int_B w \, dx = 0\}$ is a Hilbert space with inner product $(v, w) = \int_{\mathbb{R}^N} \nabla v \nabla w \, dx + \int_B vw \, dx$, $B \subset \mathbb{R}^N$ a fixed ball with $\overline{\Omega} \subset B$.

In [17] (see Lemma 3.1) the following lemma is proved.

Lemma 7.1 *Let $u \in L^2(\Omega, \mathbb{R}^N)$. The equation (7.4) admits a unique solution $v \in V$. The mapping $T : L^2(\Omega, \mathbb{R}^N) \to V$, defined by $T(u) = v$ is linear and continuous.*

We are now in position to prove the following integral representation result for the relaxation $\overline{\mathcal{G}} : L^1(\Omega; \mathbb{R}^N) \to [0, +\infty)$ of the functional $\mathcal{G}$, given by (7.1), with respect to the L^1 topology.

Theorem 7.2 *Let f satisfy (H1)–(H5). Then*

$$\overline{\mathcal{G}}(u) = \overline{F}(u) + \int_{\mathbb{R}^N} |h_u|^2 \, dx - \int_\Omega \langle h_{ext}, u \rangle \, dx,$$

where $\overline{F}(u)$ is given by (3.17).

Proof. Observe that a sequence $u_n \in W^{1,1}(\Omega, S^{N-1})$ converging with respect to the L^1 norm is also compact in the strong topology of L^2. So, thanks to Lemma 7.1, the result follows by the continuity of the last two terms of the functional $\mathcal{G}$ and by Theorem 3.1. □

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