On one-dimensional continua uniformly approximating planar sets

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Abstract

Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^n$ satisfying length constraint $\mathcal{H}^1(\Sigma) \le l$ with given $l > 0$. The paper is concerned with the properties of minimizers of the uniform distance F_M of Σ to a given compact set $M \subset \mathbb{R}^n$,

$$F_M(\Sigma) := \max_{y \in M} \operatorname{dist}(y, \Sigma),$$

where $\operatorname{dist}(y, \Sigma)$ stands for the distance between y and Σ. The paper deals with the planar case $n = 2$. In this case it is proven that the minimizers (apart trivial cases) cannot contain closed loops. Further, some mild regularity properties as well as structure of minimizers is studied.

1 Introduction

Let $M \subset \mathbb{R}^n$ be a given compact set and consider the functional F_M defined over subsets of $\mathbb{R}^n$ by the formula

$$F_M(\Sigma) := \max_{y \in M} \operatorname{dist}(y, \Sigma),$$

where $\operatorname{dist}(y, \Sigma) := \inf_{x \in \Sigma} |x - y|$ and $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$. In this paper we focus our attention mainly on the following problem.

Problem 1. *Minimize F_M over all compact connected sets $\Sigma \subset \mathbb{R}^n$ with prescribed bound on the total length $\mathcal{H}^1(\Sigma) \le l$.*

One of the possible motivations for this problem is as follows. Suppose that M represent a populated area. One has to construct a highway Σ (or, generally speaking, a transportation network) of length not exceeding l (which is usually determined by the budget for construction), so that it be equally accessible to all the people living in M. This means that Σ has to be as near as possible to M in the uniform sense, i.e. it has to minimize F_M.

A similar problem on minimizing F_M over sets having prescribed cardinality, rather than having prescribed length, is somewhat better known. It can be interpreted as the problem of finding an optimal location of a prescribed number of production sites for the populated area M. In particular, when M consists of a finite number of points, $\#M = m$, then the problem of minimizing F_M over sets

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$\Sigma \subset M$ consisting of $k < m$ points is a well-known combinatorial problem called *k-center problem* (see e.g. [11, 12]).

Another related problem has also to be mentioned. Assume the density of the population is given by a finite Borel measure compactly supported in $\mathbb{R}^n$. The problem of constructing an optimal highway Σ of prescribed length can be then formulated with the help of another reasonable criterium, namely, that of minimizing the *average* distance (or some given function of the distance) to Σ. This problem would then read as follows: minimize over all compact connected Σ satisfying $\mathcal{H}^1(\Sigma) \le l$ the functional

$$F_{\varphi,A}(\Sigma) := \int_{\mathbb{R}^n} A(\mathrm{dist}\,(y,\Sigma))\, d\varphi(y),$$

where A: $\mathbb{R}^+ \to \mathbb{R}$ is some given nonnegative nondecreasing function and φ is some compactly supported finite Borel measure. Such minimization problems have been recently studied in [2, 3, 4] (see also [10] for the closely related so called *lazy traveling salesman problem*). Usually one takes $A(t) := t^p$ for $p \ge 1$ (with $p = 1$ or $p = 2$ most important cases in applications). In this case we will write $F_{\varphi,p}$ instead of $F_{\varphi,A}$. The analogue of this problem for the minimization of $F_{\varphi,A}$ in the class of sets consisting of a prescribed number of points (standing for production sites to be located) is called *optimal location* problem (for a survey see [6] as well as [9]). The "combinatorial analogue" of the latter ($\#\mathrm{supp}\,\varphi = m$, while $\Sigma \subset \mathrm{supp}\,\varphi$ consisting of $k < m$ points) is well-known under the name of *k-median* problem.

We find it useful to consider another problem which is in a certain sense dual to Problem 1, and reads as follows.

Problem 2. *Minimize $\mathcal{H}^1(\Sigma)$ over all compact connected sets $\Sigma \subset \mathbb{R}^n$ with prescribed bound on F_M, $F_M(\Sigma) \le r$.*

This problem also admits an easy interpretation. Namely, suppose that we have to provide a gas supply pipeline to every house located in some area M under the condition that the gas supply should reach each house at distance not greater than a given $r > 0$. The company constructing the pipeline will naturally try to minimize its length under the above restriction, which reduces to solving problem 2.

It is rather easy to show that both problems studied in this paper admit solutions, and, further, that Problem 1 can be considered in a certain sense a limiting problem for $F_{\varphi,p}$ as $p \to \infty$, with $M = \mathrm{supp}\,\varphi$. We will further study that Problems 1 and 2 in the planar case $n = 2$ and show that they are naturally equivalent in the sense they have the same set of minimizers. This will immediately follow once we prove that apart trivial cases, every minimizer Σ_{opt} of problem 1 must have the maximum possible length l. We further study the minimizers to the problems introduced and show that (again, trivial cases apart), they never not contain closed loops and possess some mild regularity properties.

2 Existence of minimizers and preliminaries

The first easy result regarding Problem 1 is the existence of minimizers.

Theorem 2.1. *Problem 1 admits a solution Σ_{opt} for any given $l \ge 0$.*

The proof of the above theorem is elementary, but we will omit it since this result can be also viewed as an immediate consequence of Proposition 2.3 below.

We introduce now the following notation: let $\mathrm{OPT}_\infty(M)$ stand for the set of compact connected $\Sigma \subset \mathbb{R}^n$ with $\mathcal{H}^1(\Sigma) < +\infty$ such that $\Sigma \not\supset M$ (note that this is always true, e.g., when $\mathcal{H}^1(M) = +\infty$) and for every compact connected $\Sigma' \subset \mathbb{R}^n$

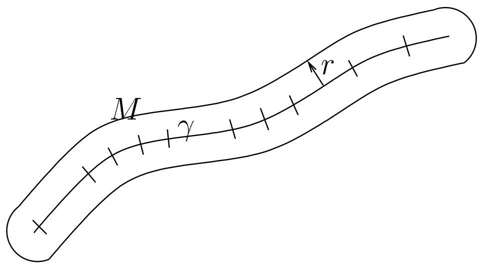

Figure 1: Possible nonlocal modification of γ to decrease the energy.

with $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma)$ one has $F_M(\Sigma') \geq F_M(\Sigma)$. In other words, the set $\mathrm{OPT}_\infty(M)$ consists of all the minimizers to Problem 1 for all the possible values of $l > 0$ except trivial ones (namely $\Sigma \not\supset M$ which are the only minimizers providing $F_M(\Sigma) = 0$). Theorem 2.1 shows therefore that $\mathrm{OPT}_\infty(M) \neq \emptyset$.

Analogously, we introduce the set $\mathrm{OPT}^*_\infty(M)$ consisting of all compact connected $\Sigma \subset \mathbb{R}^n$ with $\mathcal{H}^1(\Sigma) < +\infty$ such that $\Sigma \not\supset M$ and for every compact connected $\Sigma' \subset \mathbb{R}^n$ with $F_M(\Sigma') \leq F_M(\Sigma)$ one has $\mathcal{H}^1(\Sigma') \geq \mathcal{H}^1(\Sigma)$. This class is related to Problem 2 similarly to how $\mathrm{OPT}_\infty(M)$ is related to Problem 1. Namely, $\mathrm{OPT}^*_\infty(M)$ consists of all the minimizers to Problem 2 for all the possible values of $r > 0$ (the minimizers to Problem 2 with $r = 0$ are all closed connected $\Sigma \supset M$).

It is rather easy to prove that $\mathrm{OPT}^*_\infty(M) \subset \mathrm{OPT}_\infty(M)$ (see Proposition 3.1). We will show that the reverse inclusion is still true, though its proof is much more tricky and is based on showing that every solution of Problem 1 must have maximum possible length l (see Theorem 3.7). Though this fact might seem natural, its proof is not quite obvious. To understand the difficulty, consider the following situation. Let γ stand for the trace of an injective smooth curve in $\mathbb{R}^2$ connecting two given points a and b, let $r < \mathcal{H}^1(\gamma)$, and let M stand for the r-neighborhood of γ. For each $l > 0$ let $\Sigma_l \subset \mathbb{R}^2$ stand for a solution to Problem 1. One is tempted to conjecture that (at least for reasonable γ) for $l := \mathcal{H}^1(\gamma)$ one has $\Sigma_l = \gamma$ (so that $F_M(\Sigma_l) = r$). But if it is so, then how should Σ_l look like for l just slightly greater than $\mathcal{H}^1(\gamma)$? It is clear that changing locally γ (e.g. attaching to γ somewhere a piece of small length $\delta := l - \mathcal{H}^1(\gamma)$) would not decrease the energy F_M. The reasonable way of decreasing the energy is that of attaching pieces of small length (say, small segments) to γ in many points, so that the those pieces be distributed more or less everywhere along γ (see figure 1). Reasoning in this way, one observes however, that the attached segments should be denser where the curvature of γ is high, and that the length of the segments clearly decreases once their density increases. It is thus not clear whether a similar procedure can be fulfilled even in rather simple situations.

Another way of looking at similar difficulties is observing that in the absence of the mentioned result, i.e. when some solutions to Problem 1 can have length strictly less than that allowed by the problem statement, then there is no hope to obtain any regularity result on solutions to this problem. In fact, if Σ_{opt} solves Problem 1 but $\mathcal{H}^1(\Sigma_{opt}) < l$, then any closed connected Σ containing Σ_{opt} and satisfying $\mathcal{H}^1(\Sigma) \leq l$ solves the same problem.

It is worth mentioning that $\mathrm{OPT}_\infty(M)$ contains in fact minimizers for the much larger class of functionals of the type

$$G(\Sigma) := \Phi(F_M(\Sigma)) + H(\mathcal{H}^1(\Sigma)),$$

where Φ and H are nondecreasing functions. Recalling our interpretation of Σ as a highway or a general public transportation network, the cost $G(\Sigma)$ is naturally interpreted as a sum of the cost $\Phi(F_M(\Sigma))$ on getting to the network (which therefore reveals the social benefit of the network) and the cost of construction of Σ represented by $H(\mathcal{H}^1(\Sigma))$. We may claim the following easy result.

Proposition 2.2. *The minimizers Σ_{opt} of G (if exist) among all compact connected sets belong to $\mathrm{OPT}_\infty(M)$, if $\Sigma \not\supset M$ and either of the functions Φ or H is strictly increasing.*

Proof. If Φ is strictly increasing and H is non decreasing then the minimizers of G among all compact connected sets belong to $\mathrm{OPT}_\infty(M)$. On the other hand, if H is strictly increasing then the minimizers of G all belong to $\mathrm{OPT}^*_\infty(M)$. It remains to mention that $\mathrm{OPT}^*_\infty(M) = \mathrm{OPT}_\infty(M)$ as it will be shown in the sequel. $\square$

Finally, we mention the following remarkable result.

Proposition 2.3. *Consider a sequence $\{\Sigma_p\}_{p=1}^\infty$, where each Σ_p is a minimizer to $F_{\varphi,p}$ among compact connected sets $\Sigma \subset \mathbb{R}^n$ satisfying the length constraint $\mathcal{H}^1(\Sigma) \leq l$. Then, up to a subsequence (not relabeled), $\Sigma_p \to \Sigma_\infty$ in Hausdorff distance as $p \to \infty$, where Σ_∞ minimizes F_M with $M = \operatorname{supp}\varphi$ over the same set of admissible Σ.*

Proof. Let Ω stand for the convex hull of M and observe that all sets Σ_p, being minimizers of $F_{\varphi,p}$, are contained in the convex hull of M as proven in [4]. Therefore in view of the Blaschke theorem [1] there exists a subsequence of Σ_p (not relabeled) which converges to some compact set Σ_∞. Since all Σ_p are connected, then so is also Σ_∞ and besides we have

$$\mathcal{H}^1(\Sigma_\infty) \leq \liminf_p \mathcal{H}^1(\Sigma_p) \leq l$$

due to the Golab theorem. Thus Σ_∞ is an admissible set and we have only to prove that $F_M(\Sigma_\infty) \leq F_M(\Sigma)$ for all compact connected Σ with $\mathcal{H}^1(\Sigma) \leq l$.

Define

$$F_p(\Sigma) := F_{\varphi,p}(\Sigma)^{1/p} = \left[\int_M d(y,\Sigma)^p \, d\varphi(y)\right]^{1/p}.$$

We denote with $d_H(\Sigma, \Sigma')$ the Hausdorff distance between compact sets Σ and Σ', so that $d_H(\Sigma_p, \Sigma_\infty) \to 0$ as $p \to \infty$. Also we notice that given any two compact sets Σ and Σ' one has

$$|F_p(\Sigma) - F_p(\Sigma')| \leq \left[\int_M |d(y,\Sigma) - d(y,\Sigma')|^p \, d\varphi(y)\right]^{1/p} \leq d_H(\Sigma,\Sigma')\varphi(M)^{1/p}.$$

Recall that for a fixed compact Σ we have $F_p(\Sigma) \to F_M(\Sigma)$ as $p \to \infty$. Hence,

$$\begin{aligned}
&\liminf_{p\to\infty} |F_p(\Sigma_p) - F_M(\Sigma_\infty)| \\
&\leq \liminf_{p\to\infty} |F_p(\Sigma_p) - F_p(\Sigma_\infty)| + \liminf_{p\to\infty} |F_p(\Sigma_\infty) - F_M(\Sigma_\infty)| \\
&\leq \liminf_{p\to\infty} d_H(\Sigma_p, \Sigma_\infty)\varphi(M) = 0,
\end{aligned}$$

i.e. $\liminf_p F_p(\Sigma_p) = F_M(\Sigma_\infty)$.

We now argue by contradiction supposing the existence of an admissible Σ_0 with $F_M(\Sigma_0) \leq F_M(\Sigma_\infty) - \varepsilon$ for some $\varepsilon > 0$. Then we would have

$$\liminf_p F_p(\Sigma_p) = F_M(\Sigma_\infty) > F_M(\Sigma_0) = \lim_p F_p(\Sigma_0).$$

Thus there would exist some large p such that $F_p(\Sigma_p) > F_p(\Sigma_0)$ or, equivalently, $F_{\varphi,p}(\Sigma_p) > F_{\varphi,p}(\Sigma_0)$. The latter contradiction with the minimality of Σ_p concludes the proof. $\square$

3 Fundamental properties of minimizers

We start with the following easy result stating that $\mathrm{OPT}^*_\infty(M) \subset \mathrm{OPT}_\infty(M)$. The idea of the proof is to show that every minimizer Σ of Problem 2 must have maximum possible energy $F_M(\Sigma) = r$.

Proposition 3.1 (maximal energy). *Let* $\Sigma \in \mathrm{OPT}^*_\infty(M)$. *Then* $\Sigma \in \mathrm{OPT}_\infty(M)$.

Proof. Let Σ' be a compact connected set such that $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma)$ and suppose by contradiction that $F_M(\Sigma') < F_M(\Sigma)$. Let $R > 0$ be such that $\Sigma' \subset B_R(0)$. If $\lambda\Sigma'$ is the λ-rescaling of Σ' we notice that

$$F_M(\lambda\Sigma') \leq F_M(\Sigma') + \mathrm{dist}\,(\Sigma', \lambda\Sigma') \leq F_M(\Sigma') + R|1-\lambda|$$

and

$$\mathcal{H}^1(\lambda\Sigma') = \lambda\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma') - (1-\lambda)\mathcal{H}^1(\Sigma').$$

Hence, if we choose $\lambda < 1$ such that $R(1-\lambda) \leq F_M(\Sigma) - F_M(\Sigma')$ we have found that $F_M(\lambda\Sigma') \leq F_M(\Sigma)$ and $\mathcal{H}^1(\lambda\Sigma') < \mathcal{H}^1(\Sigma)$. So we have a contradiction with the assumption $\Sigma \in \mathrm{OPT}^*_\infty(M)$. □

Given an $x \in \Sigma$, a straight line $\Pi \subset \mathbb{R}^n$ such that $x \in \Pi$, and a number $\rho > 0$, we define

$$\beta_{\Sigma,\Pi}(x,\rho) := \sup_{y \in \Sigma \cap B_\rho(x)} \frac{\mathrm{dist}\,(y,\Pi)}{\rho}.$$

Define then the flatness β_Σ of a set Σ by the formula

$$\beta_\Sigma(x,\rho) = \inf_\Pi \beta_{\Sigma,\Pi}(x,\rho)$$

where Π varies among all straight lines of $\mathbb{R}^n$ passing through x. We are able to announce now the following auxiliary technical result.

Lemma 3.2. *Let* $I_0 \subset \mathbb{R}$ *be a compact neighborhood of* t_0 *and let* $\gamma : I \to \mathbb{R}^n$, $I_0 \subset I$, *be a continuous curve such that there is a* $\gamma'(t_0) \neq 0$ *and* $\#\gamma^{-1}(x_0) = 1$, *where* $x_0 := \gamma(t_0)$. *Let* $v = \gamma'(t_0)$, $\Pi := \{x_0 + vs : s \in \mathbb{R}\}$ *and* $\Sigma_0 := \gamma(I_0)$. *Then*

$$\lim_{\rho\to 0+} \beta_{\Sigma_0,\Pi}(x_0,\rho) = 0.$$

Proof. Step 1. We first claim that

$$d_\rho := \mathrm{diam}\,\gamma^{-1}(B_\rho(x_0)) \to 0 \text{ as } \rho \to 0^+.$$

In fact, otherwise there is an $\varepsilon > 0$ and a sequence $\{t_\nu\} \subset I_0$ such that $\gamma(t_\nu) \to \gamma(t_0)$ as $\nu \to \infty$ and $|t_\nu - t_0| > \varepsilon$. Then, up to a subsequence (not relabeled), we have $t_\nu \to t \in I$ and in view of continuity of γ one has $\gamma(t_\nu) \to \gamma(t)$ as $\nu \to \infty$. Then $t \neq t_0$ but $\gamma(t) = \gamma(t_0)$ which contradicts the assumption $\#\gamma^{-1}(x_0) = 1$.

Step 2. One has

$$\mathrm{dist}\,(\gamma(t),\Pi) \leq |\gamma(t) - (x_0 + v(t-t_0))|$$

for all $t \in I$. Therefore,

$$\frac{\mathrm{dist}\,(\gamma(t),\Pi)}{t-t_0} \leq \frac{|\gamma(t) - (x_0 + v(t-t_0))|}{t-t_0},$$

and hence, minding the definition of a derivative of γ in t_0, one gets

$$\lim_{t\to t_0} \frac{\mathrm{dist}\,(\gamma(t),\Pi)}{t-t_0} = 0. \tag{1}$$

Observe now that

$$\begin{aligned}\beta_{\Sigma_0,\Pi}(x_0,\rho) &= \sup_{\gamma(t)\in B_\rho(x_0)} \frac{\operatorname{dist}(\gamma(t),\Pi)}{\rho}\\ &= \sup_{\gamma(t)\in B_\rho(x_0)} \frac{\operatorname{dist}(\gamma(t),\Pi)}{|t-t_0|}\frac{|t-t_0|}{\rho}.\end{aligned}$$

Now, if $\rho \to 0^+$, then for $t \in \gamma^{-1}(B_\rho(x_0))$ one has $t \to t_0$. But, for t sufficiently close to t_0 one has

$$\gamma(t) - x_0 = v(t-t_0) + o(t-t_0),$$

and hence

$$|\gamma(t) - x_0| \geq \frac{1}{2}|v| \cdot |t-t_0|.$$

Minding that $v \neq 0$ according to our assumption, we get

$$|t-t_0| \leq 2\frac{|\gamma(t)-x_0|}{|v|} \leq 2\frac{\rho}{|v|}.$$

Therefore, for all sufficiently small $\rho > 0$ one has

$$\begin{aligned}\beta_{\Sigma,\Pi}(x_0,\rho) &= \sup_{\gamma(t)\in B_\rho(x_0)} \frac{\operatorname{dist}(\gamma(t),\Pi)}{\rho}\\ &\leq \frac{2}{|v|}\sup_{|t-t_0|<d_\rho} \frac{\operatorname{dist}(\gamma(t),\Pi)}{|t-t_0|} \to 0\end{aligned}$$

when $\rho \to 0^+$ in view of (1). □

We need also the following lemma from [5].

Lemma 3.3. *Let $\Sigma \subset \mathbb{R}^n$ be a closed connected set satisfying $\mathcal{H}^1(\Sigma) < +\infty$. Then there is a surjective (but not necessarily injective) Lipschitz arc-length parameterization γ: $[0,L] \to \Sigma$ with $|\gamma'| = 1$ a.e. over $[0,L]$, where $L \leq 2\mathcal{H}^1(\Sigma)$.*

In the sequel we will extensively use the result below which in a certain sense provides the existence of "classical" (rather than approximate) tangent lines to a one-dimensional continuum Σ.

Proposition 3.4 (existence of tangent lines). *Let $\Sigma \subset \mathbb{R}^n$ be a closed connected set such that $\mathcal{H}^1(\Sigma) < +\infty$. Then in $\mathcal{H}^1$-a.e $x \in \Sigma$ there exists a "tangent" line Π to Σ at x in the sense that $x \in \Pi$ and*

$$\lim_{\rho\to 0^+} \beta_{\Sigma,\Pi}(x,\rho) = 0.$$

Proof. In view of Lemma 3.3 there is a surjective Lipschitz parameterization γ: $[0,L] \to \Sigma$ with $|\gamma'| = 1$ a.e. over $[0,L]$, where $L < +\infty$. Let

$$\begin{aligned}\Sigma_0 &= \{x \in \Sigma \,:\, t \in (0,L),\ \gamma'(t) \text{ exists and } |\gamma'(t)| = 1 \text{ whenever } \gamma(t) = x\},\\ \Sigma_1 &= \{x \in \Sigma_0 \,:\, \gamma^{-1}(x) \text{ is finite}\},\\ \Sigma_2 &= \{x \in \Sigma_1 \,:\, \text{if } \gamma(t) = \gamma(s) = x \text{ then } \gamma'(t) = \pm\gamma'(s)\}.\end{aligned}$$

Clearly $\mathcal{H}^1(\Sigma \setminus \Sigma_0) = 0$ by the definition of γ. Also $\mathcal{H}^1(\Sigma_0 \setminus \Sigma_1) = 0$ since otherwise we would have

$$\int_0^L |\gamma'(t)|\,dt = \int_\Sigma \#\gamma^{-1}(x)\,d\mathcal{H}^1(x) \geq \int_{\Sigma_0\setminus\Sigma_1} \#\gamma^{-1}(x)\,d\mathcal{H}^1(x) = \infty.$$

Finally, we claim that $\mathcal{H}^1(\Sigma_1 \setminus \Sigma_2) = 0$. In fact, given $x \in \Sigma_1 \setminus \Sigma_2$ we note that in a sufficiently small neighborhood of x there are two different arcs Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \{x\}$ and x is an internal point both of Γ_1 and of Γ_2. Thus one has for the upper density

$$\Theta^*(\Sigma, x) := \limsup_{\rho \to 0^+} \frac{\mathcal{H}^1(\Sigma \cap B_\rho(x))}{2\rho} \geq 2.$$

On the other hand, $\Theta^*(\Sigma, x) = 1$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$ in view of Besicovitch-Marstrand-Mattila Theorem [1, Theorem 2.63].

Let now $x \in \Sigma_2$ be given and let $\{t_1, \ldots, t_N\} = \gamma^{-1}(x)$. We define

$$\Pi := \{x + \lambda \gamma'(t_i) \,:\, \lambda \in \mathbb{R}\}$$

which, by the definition of Σ_2, does not depend on $i \in \{1, \ldots, N\}$. Let $I_1, \ldots, I_N$ be compact neighborhoods of the points $t_1, \ldots, t_N$, such that $I_1 \cup \ldots \cup I_N = [0, L]$ and such that $t_i \in I_j$, if and only if $i = j$. Set $\Sigma^i := \gamma(I_i)$ and define

$$\beta_{\Sigma,\Pi}(x, \rho) := \max_{i \in \{1,\ldots,N\}} \beta_{\Sigma^i,\Pi}(x, \rho)$$

and hence, applying Lemma 3.2, we find that $\beta_{\Sigma,\Pi}(x,\rho) \to 0$ as $\rho \to 0^+$. This is true for all $x \in \Sigma_2$ and hence for $\mathcal{H}^1$-a.e. $x \in \Sigma$. □

The following technical lemma will be crucial for our constructions in the sequel.

Lemma 3.5. *Let $R = [-a, a] \times [-b, b]$, $\bar{x} = (0, 0)$ and suppose $r \geq \max\{8a, 32b\}$. Then there exist two compact connected sets X^+, X^- such that $X^\pm \supset \{\pm a\} \times [-b, b]$ (see Figure 2), and denoting $X := X^+ \cup X^-$ one has that*

$$\mathcal{H}^1(X) \leq C_1(b + a^2/r)$$

(one can take $C_1 = 48$), while given an arbitrary $y \in \mathbb{R}^2$ such that $|y - \bar{x}| \geq r/2$ one has

$$\operatorname{dist}(y, X) \leq \operatorname{dist}(y, R) - b.$$

Proof. Let

$$L := 4\left(b + a^2/r\right).$$

We remark that

$$L \leq r/4. \tag{2}$$

In fact, minding that $r \geq 8a$ and $r \geq 32b$, we have $L = 4b + 4a^2/r \leq r/8 + r/16 < r/4$. Define now

$$z^\pm := (\pm a, 0), \qquad X^\pm := \{\pm a\} \times [-L, L] \cup \partial B_{2b}(z^\pm).$$

Clearly, $\mathcal{H}^1(X) = 4L + 8\pi b \leq 48(b + a^2/r) = C_1(b + a^2/r)$. Let $y = (\alpha, \beta)$ be a point such that $|y - \bar{x}| \geq r/2$. We consider two cases.

Case A: $|\alpha| \geq a$. Suppose first that $\alpha \geq a$. Since $|y - \bar{x}| \geq r/2$, then

$$|y - z^+| \geq |y - x| - |z^+ - x| \geq r/2 - a = r/4 + r/4 - a \geq a + 2b - a = 2b.$$

Hence we have $y \notin B_{2b}(z^+)$ and therefore

$$\operatorname{dist}(y, \partial B_{2b}(z^+)) \leq \operatorname{dist}(y, R) - b.$$

The analogous claim holds for $\alpha \leq -a$, namely, in this case

$$\operatorname{dist}(y, \partial B_{2b}(z^-)) \leq \operatorname{dist}(y, R) - b.$$

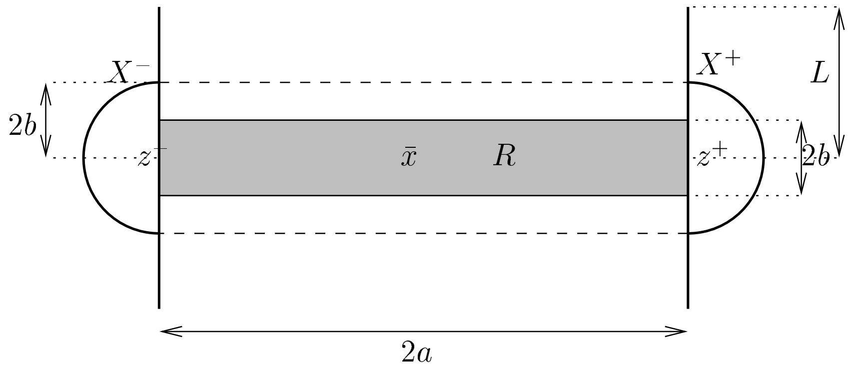

Figure 2: The rectangle R and the corresponding set $X = X^- \cup X^+$ in strong lines.

Therefore, we have

$$\operatorname{dist}(y, X) \leq \operatorname{dist}(y, R) - b.$$

Case B: $|\alpha| \leq a$. Minding that $\alpha^2 + \beta^2 \geq r^2/4$ and $\alpha^2 \leq a^2 \leq r^2/64$, we clearly have $\beta \geq r/4$ and hence $\beta \geq 2b$. Also we have $L = 4b + 4a^2/r \leq r/8 + r/8 \leq r/4$. We claim that

$$(\beta - L)^2 + a^2 \leq (\beta - 2b)^2. \tag{3}$$

In fact,

$$\begin{aligned}
(\beta - L)^2 + a^2 - (\beta - 2b)^2 &= \beta^2 - 2\beta L + L^2 + a^2 - \beta^2 - 4b^2 + 4b\beta \\
&\leq -2\beta(L - 2b) + L^2 + a^2 \\
&\leq -\frac{r}{2}(L - 2b) + L^2 + a^2 \qquad \text{(because } \beta \geq r/4) \\
&= -\left(\frac{r}{2} - L\right) L + br + a^2 \\
&\leq -\frac{r}{4} L + br + a^2 = 0 \qquad \text{(due to (2))}.
\end{aligned}$$

By (3) we conclude that

$$\begin{aligned}
\operatorname{dist}(y, X) &\leq \sqrt{(\beta - L)^2 + (\alpha - a)^2} \leq \sqrt{(\beta - L)^2 + a^2} \\
&\leq \beta - 2b = (\beta - b) - b \leq \operatorname{dist}(y, R) - b.
\end{aligned}$$

□

We will also use the following easy covering result.

Lemma 3.6 (covering). *Let $\Sigma \subset \mathbb{R}^n$ be a bounded set. Then, given $\rho > 0$, there is a finite set of points (called further ρ-lattice of Σ) $\{x_1, \ldots, x_N\} \subset \Sigma$ such that*

$$\bigcup_{j=1}^{N} B_\rho(x_j) \supset \Sigma,$$

while $B_{\rho/2}(x_j)$, $j = 1, \ldots, N$, are pairwise disjoint.

Proof. Take an $R > 0$ such that $\Sigma \subset B_R(0)$. Consider the family $\mathfrak{F}$ of all sets $X \subset \Sigma$ such that for all different $x_1, x_2 \in X$ one has $B_{\rho/2}(x_1) \cap B_{\rho/2}(x_2) = \emptyset$. Clearly for each set $X \in \mathfrak{F}$ one has

$$\sum_{x \in X} |B_{\rho/2}(x)| \leq |B_{R+\rho/2}(0)|,$$

which implies that $\#X \le (2R+\rho)^n/\rho^n$, i.e. the number of elements in each X is estimated from above by a unique constant independent of X. Therefore there is an $X_0 \in \mathfrak{F}$ which has the maximum cardinality among all elements of $\mathfrak{F}$. Then for some $N \in \mathbb{N}$ one has $X_0 = \{x_1, \dots, x_N\}$, and

$$\bigcup_{j=1}^N B_\rho(x_j) \supset \Sigma,$$

since otherwise there is a $x' \in \Sigma$ such that $|x_j - x'| \ge \rho$ for all $j = 1, \dots, N$, and hence $X_0 \cup \{x'\} \in \mathfrak{F}$ while having cardinality strictly greater than $\#X_0$. □

Now we are able to prove that every minimizer Σ_{opt} to Problem 1 must have maximum available length $\mathcal{H}^1(\Sigma_{opt}) = l$.

Theorem 3.7 (maximal length). *Let $\Sigma \subset \mathbb{R}^2$ be a compact connected set with $\mathcal{H}^1(\Sigma) < \infty$ and with $F_M(\Sigma) > 0$. Then for each $\lambda > 0$ there exists a compact connected Σ' such that $\mathcal{H}^1(\Sigma') \le \mathcal{H}^1(\Sigma) + \lambda$ and $F_M(\Sigma') < F_M(\Sigma)$. In particular, if Σ_{opt} solves Problem 1, then $\mathcal{H}^1(\Sigma_{opt}) = l$.*

Proof. In view of Proposition 3.4 one has $\lim_{k\to\infty} \beta_\Sigma(x, 1/k) = 0$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$. Choose $\varepsilon = \lambda/4\pi$ and let $r = F_M(\Sigma)$. By Egorov Theorem there exists a set $\Sigma_\varepsilon \subset \Sigma$ such that $\mathcal{H}^1(\Sigma_\varepsilon) \le \varepsilon$ and

$$\lim_{k\to\infty} \sup_{x\in\Sigma\setminus\Sigma_\varepsilon} \beta_\Sigma(x, 1/k) = 0.$$

Choose

$$\alpha := \min\left\{\lambda/(8C_1\mathcal{H}^1(\Sigma)), 1/8\right\} \tag{4}$$

where C_1 is the constant introduced in Lemma 3.5. Choose also $\rho = 1/k > 0$ such that

$$\rho \le \alpha r, \quad \rho \le \operatorname{diam}\Sigma/2 \quad \text{and} \quad \sup_{x\in\Sigma\setminus\Sigma_\varepsilon} \beta_\Sigma(x,\rho) \le \alpha/2. \tag{5}$$

Consider now a ρ-lattice $\{x_1, \dots, x_N\}$ of $\Sigma \setminus \Sigma_\varepsilon$ as provided by Lemma 3.6 so that the balls of radius $\rho/2$ centered in these points are all disjoint while the balls of radius ρ cover the whole set $\Sigma \setminus \Sigma_\varepsilon$.

Note that since Σ is connected, we have $\mathcal{H}^1(\Sigma \cap B_{\rho/2}(x_i)) \ge \rho/2$ and hence

$$\frac{N\rho}{2} \le \sum_{i=1}^N \mathcal{H}^1\left(\Sigma \cap B_{\rho/2}(x_i)\right) \le \mathcal{H}^1(\Sigma) \tag{6}$$

i.e.

$$\rho \le 2\mathcal{H}^1(\Sigma)/N. \tag{7}$$

Let now $i \in \{1, \dots, N\}$ be fixed and consider the line Π through x_i such that $\operatorname{dist}(x, \Pi) \le \rho\beta_\Sigma(x_i, \rho) \le \alpha\rho/2$ for all $x \in \Sigma \cap B_\rho(x_i)$. Consider now an orthonormal system of coordinates such that $x_i = (0,0)$ and such that the line Π is horizontal. We have $\Sigma \cap B_\rho(x_i) \subset [-\rho, \rho] \times [-\alpha\rho, \alpha\rho]$ (see Figure 3).

Then define $R_i := [-s_i, t_i] \times [-\alpha\rho, \alpha\rho]$ where $0 \le s_i, t_i \le \rho$ are such that $\Sigma \cap B_\rho(x_i) \subset R_i$ but also such that both the sides $\{-s_i\} \times [-\alpha\rho, \alpha\rho]$ and $\{t_i\} \times [-\alpha\rho, \alpha\rho]$ intersect Σ. Then let X_i be the set constructed in Lemma 3.5 with respect to R_i (by (5) both $a := (t_i + s_i)/2 \le \rho \le r/8$ and $b := \alpha\rho \le r/32$ verify the conditions of the lemma). Since the two components of X_i contain the left and right sides of R_i

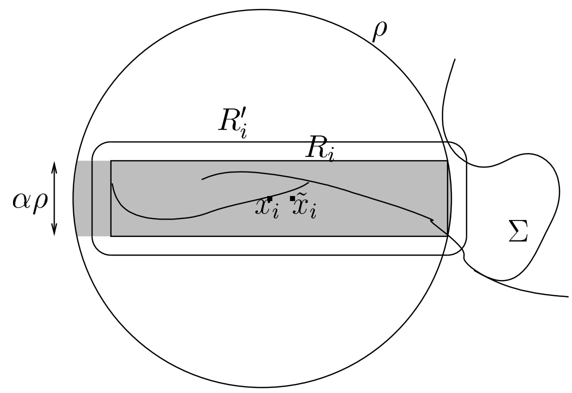

Figure 3: The construction of Theorem 3.7. We know that $\Sigma \cap B_\rho(x_i)$ is contained in the shaded region.

we know that $\Sigma \cup X_i$ is connected. Moreover, X_i has been constructed so that (by means of (4), (5) and (7))

$$\begin{aligned} \mathcal{H}^1(X_i) &\le C_1\left(\alpha\rho + \frac{((t_i+s_i)/2)^2}{r}\right) \le C_1(\alpha\rho + \frac{\rho^2}{r}) \\ &\le 2C_1\alpha\rho \qquad \text{by (5)} \\ &\le \frac{4C_1\alpha\mathcal{H}^1(\Sigma)}{N} \qquad \text{by (7)} \\ &\le \frac{\lambda}{2N} \qquad \text{by (4)} \end{aligned} \tag{8}$$

We denote by $\tilde{x}_i$ the center of the rectangle R_i. We know from Lemma 3.5 that if $|y - \tilde{x}_i| \ge r/2$ then $\mathrm{dist}\,(y, X_i) \le \mathrm{dist}\,(y, R_i) - \alpha\rho$.

Let now

$$R_i' := \{x \in \mathbb{R}^2 \ : \ \mathrm{dist}\,(x, R_i) < \alpha\rho/2\}$$

stand for the open $\alpha\rho/2$-neighborhood of R_i. Since

$$\bigcup_{i=1}^N B_\rho(x_i) \supset \Sigma \setminus \Sigma_\varepsilon \text{ and } \Sigma \cap B_\rho(x_i) \subset R_i \subset R_i',$$

then one has

$$\bigcup_{i=1}^N R_i' \supset \bigcup_{i=1}^N (\Sigma \cap B_\rho(x_i)) \supset \Sigma \setminus \Sigma_\varepsilon.$$

Further, if $|y - \tilde{x}_i| \ge r/2$ we conclude that $\mathrm{dist}\,(y, X_i) \le \mathrm{dist}\,(y, R_i') - \alpha\rho/2$.

Consider the set

$$Z := \Sigma \setminus \bigcup_{i=1}^N R_i' \subset \Sigma_\varepsilon.$$

Since all R_i' are open sets and Σ is compact, then Z is a compact set.

Choose

$$\delta := \min\{(\mathrm{diam}\,\Sigma)/2, r/4\}. \tag{9}$$

Since the spherical Hausdorff measure of the rectifiable set is equal to the usual Hausdorff measure, then there exists an at most countable number of balls $B_{\delta_i}(z_i)$

with $z_i \in Z$ and $\delta_i < \delta$ such that

$$\bigcup_i B_{\delta_i}(z_i) \supset Z \text{ and } \sum_i 2\delta_i \leq \mathcal{H}^1(Z) \leq \mathcal{H}^1(\Sigma_\varepsilon) \leq \varepsilon \leq \frac{\lambda}{4\pi} \tag{10}$$

The compactness of Z permits us to assume that there is only a finite number M of such balls.

Consider now the circles $Y_i := \partial B_{2\delta_i}(z_i)$. It is clear that each $\Sigma \cup Y_i$ is connected: in fact, $z_i \in \Sigma$ and $\operatorname{diam} \Sigma > 2\delta_i$, hence $\Sigma \cap Y_i \neq \emptyset$.

We finally define

$$\Sigma' := \Sigma \cup \bigcup_{i=1}^N X_i \cup \bigcup_{i=1}^M Y_i.$$

By the properties of X_i and Y_i we know that Σ' is compact and connected.

Let us prove that $F_M(\Sigma') < r = F_M(\Sigma)$. Let $y \in M$ be given. If $\operatorname{dist}(y, \Sigma) < 3r/4$, we obviously have $\operatorname{dist}(y, \Sigma') \leq \operatorname{dist}(y, \Sigma) < r - r/4$. So suppose instead that $\operatorname{dist}(y, \Sigma) \geq 3r/4$. Clearly we also know $\operatorname{dist}(y, \Sigma) \leq r$ (since $r = F_M(\Sigma)$). Consider a point $x \in \Sigma$ such that $|x - y| = \operatorname{dist}(y, \Sigma)$. Only two cases may happen: either $x \in R'_i$ for some $i \in \{1, \ldots, N\}$ or $x \in B_{\delta_i}(z_i)$ for some $i \in \{1, \ldots, M\}$.

In the first case ($x \in R'_i$) we have (recall (4) and (5))

$$|y - \tilde{x}_i| \geq |y - x| - |x - \tilde{x}_i| \geq 3r/4 - \sqrt{(\alpha\rho)^2 + \rho^2} - \alpha\rho/2 \geq r/2.$$

Therefore

$$\operatorname{dist}(y, X_i) \leq \operatorname{dist}(y, R'_i) - \alpha\rho/2 \leq |y - x| - \alpha\rho/2 \leq r - \alpha\rho/2.$$

In the second case ($x \in B_{\delta_i}(z_i)$) we know that $y \notin B_{2\delta_i}(z_i)$ since, by (9)

$$|y - x_i| \geq |y - x| - |x - z_i| \geq 3r/4 - \delta \geq 2\delta.$$

Thus

$$\operatorname{dist}(y, Y_i) \leq |y - x| - \delta_i \leq r - \gamma,$$

where γ is the minimum of δ_i for $i = 1, \ldots, M$.

So in either case $\operatorname{dist}(y, \Sigma') \leq r - \min\{r/4, \alpha\rho/2, \gamma\}$ and hence $F_M(\Sigma') < F_M(\Sigma)$.

Finally, by (8) and (10) we have

$$\mathcal{H}^1(\Sigma') - \mathcal{H}^1(\Sigma) \leq \sum_{i=1}^N \mathcal{H}^1(X_i) + \sum_{i=1}^M \mathcal{H}^1(Y_i) \leq \frac{\lambda}{2} + \sum_{i=1}^M 4\pi\delta_i \leq \lambda,$$

concluding the proof. □

An immediate consequence of the above proven Theorem 3.7 is the equivalence of problems 1 and 2.

Corollary 3.8. *One has* $\mathrm{OPT}_\infty(M) = \mathrm{OPT}^*_\infty(M)$.

4 Topological properties

In this section we show that the optimal sets contain no loop (homeomorphic image of S^1).

Theorem 4.1. *Let* $\Sigma \in \mathrm{OPT}^*_\infty(M)$. *Then* Σ *contains no simple closed curve (homeomorphic image of* S^1*). Therefore,* $\mathbb{R}^2 \setminus \Sigma$ *is connected.*

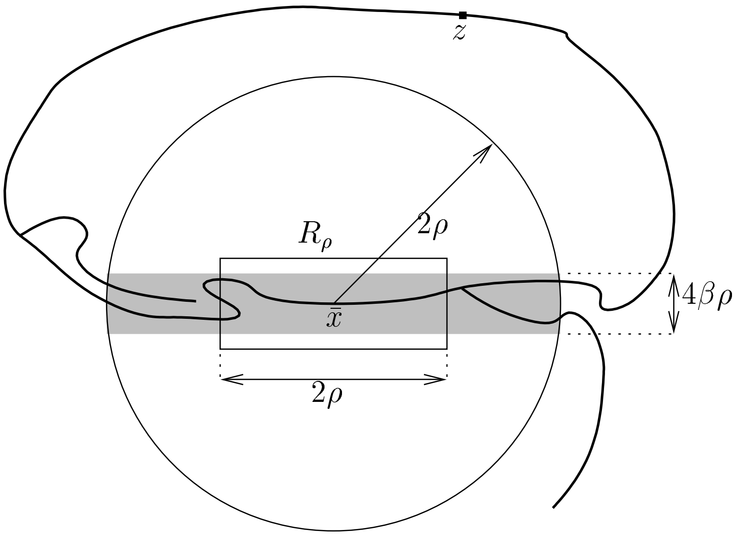

Figure 4: The construction of Theorem 4.1: the set Σ in strong lines.

Proof. Suppose by contradiction that there is a continuous curve $\gamma : [0,1] \to \Sigma$ such that $\gamma(0) = \gamma(1)$ and $\gamma : [0,1) \to \Sigma$ is injective. We set $z := \gamma(0)$. Take a point $\bar{t} \in (0,1)$ such that there exists a "tangent" line Π to Σ at $\bar{x} = \gamma(\bar{t})$ (in the sense of Proposition 3.4), $\Pi := \{x + \lambda\gamma'(\bar{t}) : \lambda \in \mathbb{R}\}$, so that

$$\lim_{\rho \to 0^+} \beta_{\Sigma,\Pi}(\bar{x}, \rho) = 0.$$

The existence of such a point is guaranteed by Proposition 3.4. Consider a system of orthonormal coordinates such that $\bar{x} = (0,0)$, $\gamma'(\bar{t}) = (|\gamma'(\bar{t})|, 0)$ (i.e. $\gamma'(\bar{t})$ is directed along the first coordinate axis and consequently $\Pi = \mathbb{R} \times \{0\}$). Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be the two components of γ with respect to our system of coordinates. Since $\gamma_1'(\bar{t}) > 0$, then there exists an $h > 0$ such that for all $t \in (\bar{t}, \bar{t} + h]$ we have $\gamma_1(t) > 0$ and for all $t \in [\bar{t} - h, \bar{t})$ we have $\gamma_1(t) < 0$. Let

$$\Sigma_0 := \gamma([0, \bar{t} - h]) \cup \gamma([\bar{t} + h, 1])$$

and define $\rho_0 := \operatorname{dist}(\bar{x}, \Sigma_0)$. Observe that $\rho_0 > 0$ since $\bar{x} \notin \Sigma_0$.

Choose a $\rho > 0$ such that

$$\rho < \rho_0/2, \quad \rho < r/C_1, \quad \rho < r/96 \quad \text{and} \quad \beta := \beta_{\Sigma,\Pi}(\bar{x}, 2\rho) < \frac{1}{3C_1}, \tag{11}$$

where C_1 is the constant defined in Lemma 3.5 and $r := F_M(\Sigma)$. Consider the rectangle $R_\rho := [-\rho, \rho] \times [-3\beta\rho, 3\beta\rho]$ and let $Y := Y^+ \cup Y^-$, $Y^\pm := \{\pm\rho\} \times [-3\beta\rho, 3\beta\rho]$ be the two short edges of R_ρ.

By definition of β we know that $\operatorname{dist}(y, \Pi) \le 2\beta\rho < 3\beta\rho$ for all $y \in \Sigma \cap B_{2\rho}(\bar{x})$ and hence $\Sigma \cap \partial R_\rho \subset Y$. Define

$$t_0 = \min\{t \in [\bar{t} - h, \bar{t}] : \gamma(t) \in R_\rho\}, \quad t_1 = \max\{t \in [\bar{t}, \bar{t} + h] : \gamma(t) \in R_\rho\}.$$

Clearly $t_0 > \bar{t} - h$ (because $\gamma(\bar{t} - h) \in \Sigma_0$, while $\Sigma_0 \cap R_\rho = \emptyset$ by construction) and analogously $t_1 < \bar{t} + h$. We thus conclude that both $\gamma(t_0) \in \partial R_\rho$ and $\gamma(t_1) \in \partial R_\rho$ and hence, minding that $\gamma_1(t_0) < 0$ and $\gamma_1(t_1) > 0$, we get

$$\gamma(t_0) \in Y^- \quad \text{and} \quad \gamma(t_1) \in Y^+.$$

Let $X := X^+ \cup X^-$ be the set constructed in Lemma 3.5 with respect to the rectangle R_ρ and define

$$\Sigma' := (\Sigma \setminus R_\rho) \cup X.$$

Clearly Σ' is compact (recall that X is compact and that $\Sigma \cap \partial R_\rho \subset Y \subset X$).

We claim that Σ' is also connected. Observe to this end that the curves $\gamma([0,t_0])$ and $\gamma([t_1,1])$ connect respectively Y^- (hence X^-) and Y^+ (hence X^+) to the point z and that both curves stay in Σ'. In fact, $\gamma([0,\bar{t}-h])$ and $\gamma([\bar{t}+h,1])$ do not intersect $B_{\rho_0}(\bar{x})$ by the definition of ρ_0, while $\gamma([\bar{t}-h,t_0])$ and $\gamma([t_1,\bar{t}+h])$ do not intersect the interior of R_ρ by the definition of t_0 and t_1. Therefore, every $x \in X \subset \Sigma'$ is connected to z by a curve contained in Σ'. To conclude the proof of the claim, it remains to consider the case of an $x \in \Sigma \setminus R_\rho \subset \Sigma'$. We know in this case that, in view of arcwise connectedness of Σ, there exists a continuous curve $\varphi : [0,1] \to \Sigma$ such that $\varphi(0) = x$ and $\varphi(1) = z$. If this curve is not completely contained in Σ', consider the $s \in [0,1]$ such that

$$s := \min\{t \in [0,1] \,:\, \varphi(t) \in \partial R_\rho\}.$$

We have then $\varphi(s) \in Y \subset X \subset \Sigma'$, and hence the curve $\varphi([0,s])$ connects x to X staying in Σ'. But since as shown above both X^+ and X^- are connected to z in Σ', then x is connected to z in Σ' and thus we finally conclude that Σ' is connected.

By Lemma 3.5 we know that

$$\begin{aligned}\mathcal{H}^1(\Sigma') &\le \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap R_\rho) + \mathcal{H}^1(X)\\ &\le \mathcal{H}^1(\Sigma) - 2\rho + C_1(3\beta\rho + \rho^2/r) < \mathcal{H}^1(\Sigma),\end{aligned}$$

the latter estimate being valid in view of (11).

We claim that $F_M(\Sigma') \le r = F_M(\Sigma)$. In fact, consider an arbitrary $y \in M$. Let $x \in \Sigma$ be such that $\operatorname{dist}(y,\Sigma) = |y-x|$. Then, if $x \in \Sigma'$, we have automatically

$$\operatorname{dist}(y,\Sigma') \le |y-x| = \operatorname{dist}(y,\Sigma).$$

Otherwise, $x \in R_\rho$. Consider first the case $|y-x| > r/2$. Then $|x-\bar{x}| > r/2$ since $\bar{x} \in \Sigma$. By Lemma 3.5 we get therefore that $\operatorname{dist}(y,X) < \operatorname{dist}(y,R_\rho)$. We observe now that $\operatorname{dist}(y,R_\rho) \le |y-x| = \operatorname{dist}(y,\Sigma)$, which still implies $\operatorname{dist}(y,\Sigma') \le \operatorname{dist}(y,\Sigma)$. At last, it remains to consider the case $|y-x| \le r/2$. Observe that $\operatorname{dist}(y,\Sigma') \le |y-x| + 2\rho \le r/2 + 2\rho \le r$ since $\Sigma \setminus \Sigma' \subset R_\rho \subset B_{2\rho}(x_0)$.

Finally, we conclude that $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma)$, while $F_M(\Sigma') \le F_M(\Sigma)$, which contradicts the assumption $\Sigma \in \mathrm{OPT}^*_\infty(M)$. This contradiction proves the absence of simple closed curves in Σ. This also implies that $\mathbb{R}^2 \setminus \Sigma$ is connected (see [4]). $\square$

5 Ahlfors regularity

We show now that minimizers of Problem 2 (hence also of Problem 1 in view of Corollary 3.8) possess some mild regularity properties. In particular, we show that every $\Sigma \in \mathrm{OPT}^*_\infty(M)$ is Ahlfors regular in the sense that there exist two constants $c > 0$ and $C > 0$ such that for every positive $\rho < \operatorname{diam}\Sigma$ and for every $x \in \Sigma$ one has

$$c\rho \le \mathcal{H}^1(\Sigma \cap B_\rho(x)) \le C\rho$$

(while a singleton is considered to be Ahlfors regular by definition). It is worth mentioning that Ahlfors regularity of a closed connected set Σ implies the so-called *uniform rectifiability* on Σ, which, as it has been shown in [5], provides several nice analytical properties of Σ. This condition can be considered a kind of "quantitative rectifiability" which is somewhat stronger than the classical rectifiability used in geometric measure theory.

Theorem 5.1. *Given $\Sigma \in \mathrm{OPT}^*_\infty(M)$, there exists such a $\rho_0 > 0$ that for all $x \in \Sigma$ and all $\rho < \rho_0$ one has*

$$\rho \le \mathcal{H}^1(\Sigma \cap B_\rho(x)) \le 2\pi\rho.$$

In particular, Σ is Ahlfors regular.

Proof. Let $\rho_0 := \min\{\operatorname{diam}\Sigma/2, F_M(\Sigma)\}$. Given $\rho < \operatorname{diam}\Sigma/2$ and $x \in \Sigma$ we have $\Sigma \cap \partial B_\rho(x) \neq \emptyset$. Thus there exists a curve $\Gamma \subset \Sigma \cap \bar{B}_\rho(x)$ which joins x to $\partial B_\rho(x)$ and hence

$$\mathcal{H}^1(\Sigma \cap B_\rho(x)) \ge \mathcal{H}^1(\Gamma \cap B_\rho(x)) \ge \rho.$$

On the other hand, setting

$$\Sigma' := \Sigma \setminus B_\rho(x) \cup \partial B_\rho(x)$$

for $\rho < \operatorname{diam}\Sigma$, we observe that the compact set Σ' is connected. If also $\rho < F_M(\Sigma)$, we have $F_M(\Sigma') \le F_M(\Sigma)$, while

$$\mathcal{H}^1(\Sigma') \le \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B_\rho(x)) + 2\pi\rho$$

But since $\Sigma \in \mathrm{OPT}^*_\infty(M)$, we have $\mathcal{H}^1(\Sigma) \le \mathcal{H}^1(\Sigma')$, and hence $\mathcal{H}^1(\Sigma \cap B_\rho(x)) \le 2\pi\rho$. □

6 Structure of minimizers

Let us consider a minimizer $\Sigma \in \mathrm{OPT}^*_\infty(M)$ with energy $r = F_M(\Sigma)$. In this section we show that the set Σ can be split in three parts which turn out to have very different properties. We need for this purpose the following notions.

Definition 6.1. *A point $x \in \Sigma$ is called* energetic, *if for all $\rho > 0$ one has*

$$F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma).$$

Let G_Σ stand for the set of energetic points of Σ. Given a point $x \in G_\Sigma$ we say that x is an isolated energetic point, *if there exists such a $\rho > 0$ that $B_\rho(x) \cap G_\Sigma = \{x\}$. Further, we define $X_\Sigma \subset G_\Sigma$ to be the set of isolated energetic points of Σ and let $E_\Sigma := G_\Sigma \setminus X_\Sigma$ to be the set of* non isolated energetic points. *The remaining set $S_\Sigma := \Sigma \setminus G_\Sigma$ is the set of* non energetic points *of Σ.*

In this way a set Σ can be split into three disjoint sets:

$$\Sigma = E_\Sigma \cup X_\Sigma \cup S_\Sigma, \qquad G_\Sigma = E_\Sigma \cup X_\Sigma.$$

In the theorem below we collect the results which will be proved later in Propositions 6.3, 6.6 and 6.7.

Theorem 6.2 (structure of minimizers). *Let $\Sigma \in \mathrm{OPT}^*_\infty(M)$, $r := F_M(\Sigma)$ and $E := E_\Sigma$, $X = X_\Sigma$ and $S := S_\Sigma$ be defined as above. Then the sets E, X and S have the following properties.*

1. *X is a discrete set (i.e. all the points of X are isolated, or, in other words, the topological dimension $\dim X = 0$). For any point $x \in X$ there exists $y \in M$ such that $|x - y| = r$ and $B_r(y) \cap \Sigma = \emptyset$. If X is not finite, the limit points of X are always points of E.*

2. *E is a compact set with distance r from M in the following sense: for each $x \in E$ there exists an $y \in M$ with $|x - y| = r$, $B_r(y) \cap \Sigma = \emptyset$ and there exists a sequence $y_k \to y$, $y_k \neq y$, $y_k \in M$ such that*

$$\lim_{k\to\infty} \frac{\langle y - x, y_k - y\rangle}{|y_k - y|} = 0.$$

3. *For all $x \in S$ there exists $\varepsilon > 0$ such that $S \cap B_\varepsilon(x)$ is either a segment or a triple point i.e. the union of three segments with an endpoint in x and relative angles of 120 degrees.*

In the next section we will give some comments on the above structure theorem. The rest of the section is devoted to its proof. We start from the following easy statement.

Proposition 6.3. *Let G_Σ, E_Σ, X_Σ and S_Σ be defined as before. Then G_Σ is compact, E_Σ is compact, X_Σ is discrete and relatively open in G_Σ with $\bar{X}_\Sigma \setminus X_\Sigma \subset E_\Sigma$, and S_Σ is relatively open in Σ.*

Proof. Let $\{x_k\} \subset G_\Sigma$ be a sequence of points $x_k \neq x$ which converges to a point $x \in \Sigma$. Given $\varepsilon > 0$ we choose a k such that $|x_k - x| < \varepsilon/2$. Minding $B_{\varepsilon/2}(x_k) \subset B_\varepsilon(x)$, we get

$$F_M(\Sigma \setminus B_\varepsilon(x)) \geq F_M(\Sigma \setminus B_{\varepsilon/2}(x_k)) > F_M(\Sigma)$$

which means that $x \in G_\Sigma$. Thus G_Σ is a closed set and, since Σ is compact, then so is G_Σ.

The set X_Σ is relatively open in G_Σ and is discrete by definition. Also, possible accumulation points of X_Σ belong to G_Σ and hence to E_Σ, since X_Σ is discrete. As a consequence, E_Σ is closed and hence compact. Since G_Σ closed, we also deduce that S_Σ is relatively open in Σ. □

The two technical lemmata below will be used in the proof of Proposition 6.6.

Lemma 6.4. *Let M and Σ be given compact subsets of $\mathbb{R}^2$, and Σ is connected. Let G_Σ be defined as above. Then there exists a map $\tau : G_\Sigma \to M$ such that for each $x \in G_\Sigma$ one has*

$$|x - \tau(x)| = \operatorname{dist}(\tau(x), \Sigma) = F_M(\Sigma), \tag{12}$$

and $\#\tau^{-1}(\tau(x)) \leq 4$. In particular, $B_r(\tau(x)) \cap \Sigma = \emptyset$ with $r := F_M(\Sigma)$.

Proof. Step 1. Let $x \in G_\Sigma$ and $r := F_M(\Sigma)$. Consider a sequence of positive numbers $\varepsilon_k \to 0$ and $\varepsilon_k < \operatorname{diam}\Sigma/2$. Since Σ is connected, $x \in \Sigma$ and $\operatorname{diam}\Sigma > 2\varepsilon_k$, then $\Sigma \cap \partial B_{\varepsilon_k} \neq \emptyset$. Therefore we can choose a sequence $x_k \in \Sigma \cap \partial B_{\varepsilon_k}(x)$.

Since $x \in G_\Sigma$, we know that $F_M(\Sigma \setminus B_{\varepsilon_k}(x)) > r$ for all k. In particular, there exists an $y_k \in M$ such that

$$\operatorname{dist}(y_k, \Sigma \setminus B_{\varepsilon_k}(x)) = F_M(\Sigma \setminus B_{\varepsilon_k}(x)) > r. \tag{13}$$

But $\operatorname{dist}(y_k, \Sigma \setminus B_{\varepsilon_k}(x)) \leq |y_k - x_k|$ since $x_k \in \Sigma \setminus B_{\varepsilon_k}(x)$. Thus

$$|y_k - x| \geq |y_k - x_k| - |x_k - x| > r - \varepsilon_k. \tag{14}$$

On the other hand, we know that $\operatorname{dist}(y_k, \Sigma) \leq F_M(\Sigma) = r$. Hence there exists an $\tilde{x}_k \in \Sigma$ such that $|y_k - \tilde{x}_k| = \operatorname{dist}(y_k, \Sigma) \leq r$. Moreover we have $\tilde{x}_k \in B_{\varepsilon_k}(x)$, since otherwise we would have $\operatorname{dist}(y_k, \Sigma \setminus B_{\varepsilon_k}(x)) \leq |y_k - \tilde{x}_k| \leq r$ which would contradict the choice of y_k. We conclude therefore that

$$|y_k - x| \leq |y_k - \tilde{x}_k| + |\tilde{x}_k - x| \leq r + \varepsilon_k. \tag{15}$$

Up to a subsequence, not relabeled, $y_k \to y \in M$ as $k \to \infty$ and hence passing to the limit as $k \to \infty$ in equations (14) and (15), we get $|y - x| = r$. We then set $\tau(x) := y$. Notice that

$$\operatorname{dist}(y_k, \Sigma) = |y_k - \tilde{x}_k| \geq |y_k - x| - |x - \tilde{x}_k| \geq |y_k - x| - \varepsilon_k$$

which, after passing to the limit $k \to \infty$, gives $\operatorname{dist}(y, \Sigma) \geq |y - x| = r$. The property (12) is therefore proven.

Step 2. We now prove that $\#\tau^{-1}(y) \leq 4$. By (13), we have

$$\bar{B}_r(y_k) \cap \Sigma \subset B_{\varepsilon_k}(x). \tag{16}$$

If $y_k = y$ for infinitely many indices k we deduce that $\bar{B}_r(y) \cap \Sigma = \{x\}$ and hence necessarily $\tau^{-1}(y) = \{x\}$. Therefore we will suppose without loss of generality that $y_k \neq y$ for all k. Thus, up to a subsequence (not relabeled), there exists at least one unit vector v_x such that

$$\frac{y_k - y}{|y_k - y|} \to v_x.$$

In the next step we will prove that for all $x' \in \tau^{-1}(y)$, $x' \neq x$ one has

$$\begin{aligned} \langle v_x, x - y\rangle &\geq 0, \\ \langle v_x, x' - y\rangle &\leq 0. \end{aligned} \tag{17}$$

Once (17) is proven we are able to prove the remaining claim. In fact, suppose by contradiction that $\#\tau^{-1}(y) \geq 5$. Set in this case $v_i := v_{x_i}$, $w_i := x_i - y$, $i = 1, \dots, 5$, where $x_i \in \tau^{-1}(y)$. Then (17) provides

$$\langle v_i, w_i\rangle \geq 0, \quad \langle v_i, w_j\rangle \leq 0, \quad i, j = 1, \dots, 6, \quad i \neq j.$$

We claim now that there exists a $\xi \in \mathbb{R}^2$ and at least three indices $\{i_1, i_2, i_3\} \subset \{1, \dots, 5\}$ such that $\langle \xi, v_{i_j}\rangle > 0$. In fact, let ξ' be any vector satisfying $\langle \xi', v_i\rangle \neq 0$ for all $i = 1, \dots, 5$. If among the products $\langle \xi', v_i\rangle$ there are three positive ones, then choose $\xi := \xi'$, otherwise choose $\xi := -\xi'$.

Without loss of generality we may now suppose (up to renumbering) that $i_1 = 1$, $i_2 = 2$, $i_3 = 3$ and the vector v_2 is between v_1 and v_3 (this assumption makes sense in view of the claim just proven). Then $\langle w_2, v_1\rangle \leq 0$ and $\langle w_2, v_3\rangle \leq 0$, which means that both v_1 and v_3 belong to a half-plane $\{v : \langle w_2, v\rangle \leq 0\}$. Then v_1 must belong to the same half-space, which contradicts the condition $\langle w_2, v_2\rangle > 0$.

Step 3. It remains to prove (17). Since $|\tilde{x}_k - y| \geq \operatorname{dist}(y, \Sigma) = r$ and $|\tilde{x}_k - y_k| \leq r$, we have

$$\begin{aligned} 2\langle y_k - y, \tilde{x}_k - y_k\rangle &= |\tilde{x}_k - y|^2 - |y_k - y|^2 - |\tilde{x}_k - y_k|^2 \\ &\geq r^2 - |y_k - y|^2 - r^2 = -|y_k - y|^2 \end{aligned}$$

and hence

$$\begin{aligned} \langle y_k - y, x - y\rangle &= \langle y_k - y, \tilde{x}_k - y_k\rangle + |y_k - y|^2 + \langle y_k - y, x - \tilde{x}_k\rangle \\ &\geq -\frac{|y_k - y|^2}{2} + |y_k - y|^2 - |y_k - y| \cdot |x - \tilde{x}_k| \\ &\geq -|y_k - y| \cdot |x - \tilde{x}_k|. \end{aligned}$$

Dividing by $|y_k - y|$ and passing to the limit we obtain the first part of (17).

Similarly, given $x' \neq x$, $x' \in \tau^{-1}(y)$ we have $|y - x'| = r$ in view of (12). On the other hand, for all sufficiently large $k \in \mathbb{N}$ one has $x' \notin B_{\varepsilon_k}(x)$ and hence by (13) we get $|y_k - x'| > r$. Therefore,

$$\begin{aligned} 2\langle y_k - y, x' - y\rangle &= |y - x'|^2 + |y_k - y|^2 - |y_k - x'|^2 \\ &< r^2 + |y_k - y|^2 - r^2 = |y_k - y|^2. \end{aligned}$$

Again we divide by $|y_k - y|$ and pass to the limit $k \to \infty$ to complete the proof of (17). $\square$

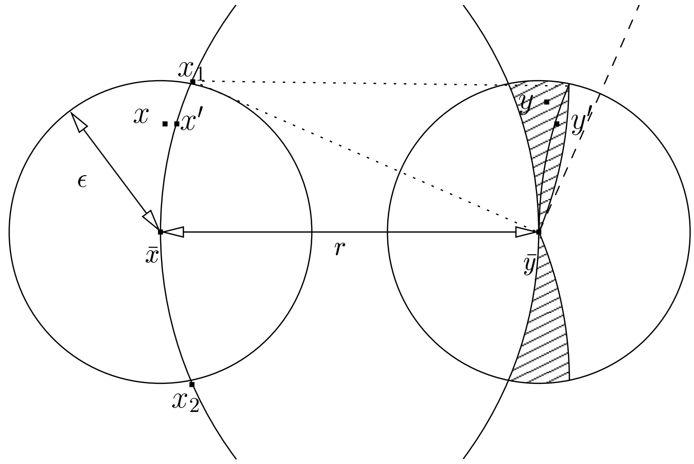

Figure 5: The point y lies in the shaded region.

Lemma 6.5. *Let $r > \varepsilon > 0$ be given and let $x, \bar{x}, y, \bar{y} \in \mathbb{R}^2$ be such that*

$$|\bar{x} - \bar{y}| = |x - y| = r, \quad |\bar{x} - y| \geq r, \quad |x - \bar{y}| \geq r, \quad |\bar{x} - x| \leq \varepsilon, \quad |\bar{y} - y| \leq \varepsilon.$$

Then

$$|\langle \bar{y} - y, \bar{x} - \bar{y} \rangle| \leq \frac{\varepsilon}{r} |\bar{y} - y| \, |\bar{x} - \bar{y}|.$$

Proof. Let x_1 and x_2 be the two intersections of the circle $\partial B_r(\bar{y})$ with the boundary of the convex hull of $B_\varepsilon(\bar{x}) \cup B_\varepsilon(\bar{y})$ (so that x_1 and x_2 have distance ε from the segment $[\bar{x}, \bar{y}]$, see Figure 5).

We claim that

$$y \in (\bar{B}_\varepsilon(\bar{y}) \setminus B_r(\bar{x})) \cap \overline{(B_r(x_1) \cup B_r(x_2))} \tag{18}$$

(i.e. y belongs to the shaded region of Figure 5). In fact, the hypotheses of the lemma being proven mean $y \in \bar{B}_\varepsilon(\bar{y}) \setminus B_r(\bar{x})$ and $x \in \bar{B}_\varepsilon(\bar{x}) \setminus B_r(\bar{y})$. Also we know that $|x - y| = r$. Let x' be the intersection of the segment $[x, y]$ with the circle $\partial B_r(\bar{y})$. Suppose that x' is closer to x_1 than x_2 (the other case is symmetric), which means that x' and x_1 belong to the same half-plane π^+ bounded by the line $(\bar{x}\bar{y})$ (for definiteness, we consider it to be the half-plane "above" this line). It is easy to observe that also y must belong to the same half-plane, because the set $\partial B_r(x) \cap (\bar{B}_\varepsilon(\bar{y}) \setminus B_r(\bar{x}))$ containing y, is contained in this half-plane.

Clearly $|x' - y| \leq r$ so we know that $y \in \bar{B}_r(x')$. Moreover, we observe that $|y - x_1| \leq |y - x'|$. In fact, both x_1 and x' belong to $\partial B_r(\bar{y})$ by construction, hence the triangle with vertices x_1, x' and $\bar{y}$ is isosceles, which implies that the axis of symmetry of the segment $[x', x_1]$ passes through $\bar{y}$ (being both the median and the height of the mentioned triangle). Hence y stays "above" this axis, since otherwise, minding $y \in \pi^+$ we would have that necessarily $y \in B_r(\bar{x})$ contrary to our assumptions.

We have therefore $|y - x_1| \leq |y - x'| \leq r$ which means that $y \in \bar{B}_r(x_1)$. If we also consider the symmetric case (namely, x and hence also y below the line $(\bar{x}, \bar{y})$ we find that $y \in \overline{B_r(x_1) \cup B_r(x_2)}$. This completes the proof of the claim (18).

To conclude the proof of the lemma, one can easily check that the region $R = \bar{B}_r(x_1) \cap \bar{B}_\varepsilon(\bar{y}) \setminus B_r(\bar{x})$ is contained in a cone with aperture angle $2\varepsilon/r$ centered in

$\bar{y}$ and perpendicular to $[\bar{x},\bar{y}]$. Therefore, if α stays for the angle between $\bar{x}-\bar{y}$ and $y-\bar{y}$, then $|\alpha-\pi/2|\leq\varepsilon/r$. Therefore,

$$|\cos\alpha| = |\sin(\alpha-\pi/2)|\leq\varepsilon/r,$$

which proves the lemma. □

Proposition 6.6. *Let* $r := F_M(\Sigma) > 0$. *Given* $x \in E_\Sigma$ *there exists a sequence* $y_k \in M$ *which converges to* $y \in M$ *such that* $y_k \neq y$, $|x-y| = r$, $B_r(y)\cap\Sigma=\emptyset$ *and* $\langle y_k - y, y-x\rangle/|y_k-y|\to 0$.

Proof. Let $r = F_M(\Sigma)$. Since x is not isolated in E_Σ, there exists a sequence $\{x_k\}\subset E_\Sigma$, $x_k\to x$. In view of Lemma 6.4, setting $y_k := \tau(x_k)\in M$, we get $|x_k-y_k| = r$ and $B_r(y_k)\cap\Sigma=\emptyset$. By extracting a subsequence we may suppose that y_k converges to some $y\in M$. Again according to Lemma 6.4 we have that $y_k\neq y$ for all sufficiently large k (otherwise $\tau^{-1}(y)$ would not be a finite set). Hence, we have $|y-x| = r$, $|y_k-x_k| = r$, $|y_k - x|\geq r$, $|y-x_k|\geq r$. Letting $\varepsilon_k = \max\{|y_k-y|,|x_k-x|\}$ we can apply Lemma 6.5 to deduce that

$$\frac{|\langle y_k-y, x-y\rangle|}{|y_k-y||x-y|}\leq\frac{\varepsilon_k}{r}\to 0 \quad \text{as } k\to\infty$$

which concludes the proof. □

Proposition 6.7. *Let* $\Sigma\in \mathrm{OPT}^*_\infty(M)$. *Then given an arbitrary point* $x\in S$, *there exists an* $\varepsilon>0$ *such that* $B_\varepsilon(x)\cap S$ *is either a diameter of* $B_\varepsilon(x)$ *or the union of three radii with relative angles of* 120 *degrees.*

Proof. Note that Σ is a continuous (even Lipschitz continuous) image of a unit interval by lemma 3.3, hence is locally connected by Hahn-Mazurkiewicz-Sierpiński theorem II.2 from [8, § 50]. Since $S\subset\Sigma$ is an open set, then it contains a connected open subset S_0 containing x. We may choose therefore an $\varepsilon>0$ small enough so that $B_\varepsilon(x)\cap S = B_\varepsilon(x)\cap S_0$.

Further, consider a $\rho>0$ such that $F_M(\Sigma\setminus B_\rho(x)) = F_M(\Sigma)$. We may consider $\varepsilon<\rho$ to be small enough so that $\Sigma\cap\partial B_\varepsilon(x)$ has only a finite number of points. Such an ε can be found, since otherwise, by the coarea formula, we would find that $\mathcal{H}^1(\Sigma\cap B_\rho(x))=\infty$.

We claim that $\mathcal{H}^1(S_0)$ is minimal with respect to all compact connected sets S which contain $S_0\cap\partial B_\varepsilon(x)$. In fact let S be such a set, and consider $\Sigma' = \Sigma\setminus S_0\cup S$. Then $\Sigma'\supset\Sigma\setminus B_\rho(x)$ and hence $F_M(\Sigma')\leq F_M(\Sigma\setminus B_\rho(x)) = F_M(\Sigma)$. Being $\Sigma\in \mathrm{OPT}^*_\infty(M)$ we deduce that $\mathcal{H}^1(\Sigma)\leq\mathcal{H}^1(\Sigma')$ which means that $\mathcal{H}^1(S_0)\leq\mathcal{H}^1(S)$.

The above proven claim means that S_0 is a locally minimal network in the sense of [7], and hence theorem 2.1 from [7, Chapter III] immediately gives the conclusion. □

7 Final considerations

We point out that Theorem 6.2 is useful mainly when M is a 1-dimensional set. However we will show by means of the example below, that in some cases one can reduce the problem with a given datum M to the problem with datum ∂M.

Example 1. Let $M := \partial B_R(0)$ and consider a minimizer $\Sigma\in\mathrm{OPT}_\infty(M)$ with $F_M(\Sigma) = r$. Clearly, if $r\geq 1$, we have a trivial solution $\Sigma=\{0\}$. Otherwise we consider the partitioning $\Sigma = E\cup X\cup S$ defined in the previous section. Theorem 6.2 then says that the set E is contained in the circle $\partial B_r(0)$. Also Σ contains no closed loop, hence not all the circle ∂B_r is contained in Σ. It is easy to see that to every

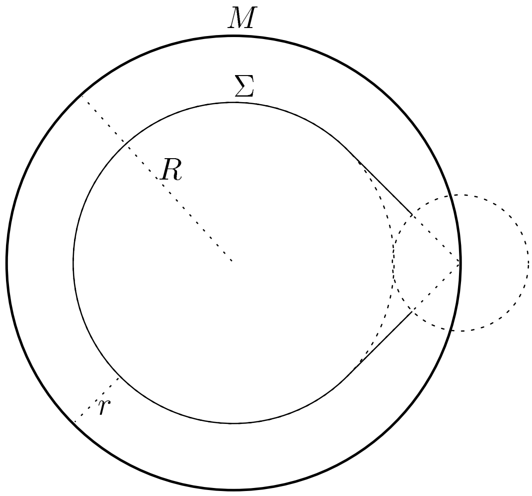

Figure 6: The conjectured minimizer Σ when M is a circle.

connected component of $\partial B_r(0) \setminus E$ at least two points of X must correspond. We expect the minimizer to be the one represented in Figure 6. In this example the set E is an arc of circle with distance r from M, the discrete set X is the union of the two endpoints and the minimal network S is the union of the two line segments connecting X to E.

Notice also that if this is the solution when $M = \partial B_R(0)$, then for $r \geq R/2$ this is also the solution when $M = \bar{B}_R(0)$. In fact, for this particular set Σ we have $F_{\bar{B}_R(0)}(\Sigma) = \max\{r, R-r\}$, while in general one obviously has $F_{\bar{B}_R(0)} \geq F_{\partial B_R(0)}$ being $\partial B_R(0) \subset \bar{B}_R(0)$.

It seems also worth mentioning that when M is a regular 1-dimensional set, Theorem 6.2 seems to be not so far from a regularity theorem for minimizers Σ. In fact, we notice that the set S_Σ is the union of segments and a negligible number of triple points, while the regularity of E_Σ is strongly related to that of M, and X_Σ is a negligible set. However, there is a gap in proving the generic regularity result for the whole Σ. The problem is to understand how the set S_Σ touches the set E_Σ and what happens when the points of X_Σ accumulate near a point of E_Σ.

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