AN EXISTENCE RESULT FOR THE MONGE PROBLEM IN $\mathbb{R}^n$ WITH NORM COST FUNCTIONS

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Abstract. We establish existence of solutions to the Monge problem in $\mathbb{R}^n$ with a norm cost function, assuming absolute continuity of the initial measure.

The loss in strict convexity of the unit ball implies that transport is possible along several directions. As in [4], we single out particular solutions to the Kantorovich relaxation with a secondary variational problem, which involves a strictly convex norm. We then define a map rearranging the mass within the rays, by a Sudakov-type argument with the disintegration technique in [8, 10, 17].

In the secondary variational problem the cost is also infinite valued and in general there is no Kantorovich potential. However, all the optimal transport plans share the same maximal transport rays. We derive also an expression for the transport density associated to these optimal plans.

Remark: The construction presently given in the preprint needs the further technical assumption that, with the notation of Section 3.4, the set of points $x \in \mathcal{T}_s$ whose secondary transport ray belongs to the relative border of the convex envelope of $\{y : \phi(x) - \phi(y) = \|y - x\|\}$ is μ-negligible.

Contents

1. Introduction

Topic of this paper is the existence of solutions to the Monge problem in $\mathbb{R}^n$ when the cost function is given by a norm $\|\cdot\|$, possibly asymmetric: given two Borel probability measures $\mu, \nu \in \mathscr{P}(\mathbb{R}^n)$, we study the minimization of the functional

$$\mathcal{I}_M(t) = \int_{\mathbb{R}^n} \|t(x) - x\| \, d\mu(x) \qquad \text{(MP1)}$$

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where t varies among the Borel maps $t : \mathbb{R}^n \to \mathbb{R}^n$ whose image measure of μ is ν. We exhibit a particular class of minimizers, selecting them with the additional optimality conditions in [4], under the natural assumption that μ is absolutely continuous w.r.t. the Lebesgue measure $\mathcal{L}^n$. This assumption is necessary, as shown in Section 8 of [5]. Roughly, the strategy is a reduction argument to a one dimensional transport problem, proving a regularity of disintegration along rays with the technique in [8, 10], and then [17].

Remark 1.1. The construction presently given in the preprint needs the further technical assumption that, with the notation of Section 3.4, the set of points $x \in \mathcal{T}_{\mathrm{s}}$ whose secondary transport ray belongs to the relative border of the convex envelope of $\{y : \phi(x) - \phi(y) = \|y - x\|\}$ is μ-negligible.

Before describing the work, we present a brief review of the literature, referring to [39] for a broad overview.

1.1. **An account on the literature.** The original Monge problem arose in 1781 for continuous masses μ, ν supported on compact, disjoint sets in dimension 2, 3 and with the cost defined by the Euclidean norm ([31]). Monge himself conjectured important features of the transport, such as, with the Euclidean norm, the facts that two transport rays may intersect only at endpoints and that the directions of the transported particles form a family of normals to some family of surfaces.
Investigated first in [6, 21], the problem was left apart for a long period.

A fundamental improvement in the understanding came with the relaxation of the problem in the space of probability measures ([26, 27]), consisting in the Kantorovich formulation. Instead of looking at maps in $\mathbb{R}^n$, ones considers the following minimization problem in the space $\Pi(\mu, \nu)$ of couplings between μ and ν: one deals with the minimization of the linear functional

$$\mathcal{I}_K(\pi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|y - x\| \, d\pi(x, y) \qquad \text{(KP1)}$$

among the *transport plans* π in the convex, w^*-compact set

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{M}^+ : \ p^x_\sharp \pi = \mu, \ p^y_\sharp \pi = \nu\},$$

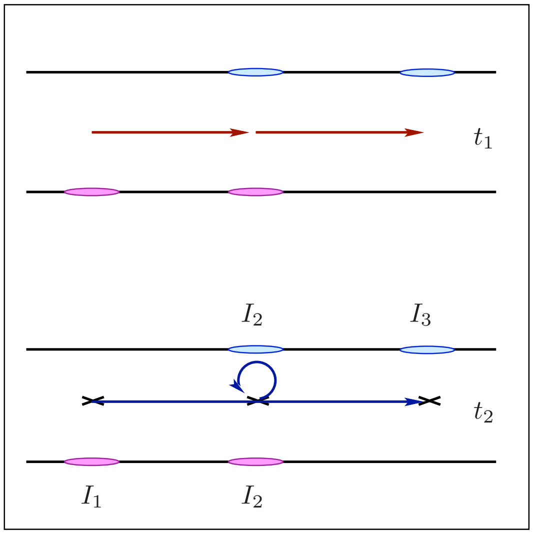
where p^x, p^y are respectively the projections on the first and on the second factor space of $\mathbb{R}^n \times \mathbb{R}^n$.
In particular, minimizers to (KP1) always exist by the direct method of Calculus of Variations. However, the formulation (KP1) is indeed a generalization of the model, allowing that mass at some point can be split to more destinations. Therefore, a priori the minimum value in (MP1) is higher than the one in (KP1), and the minimizers of the latter are not suitable for the former.

A standard approach to (MP1) consists in showing that at least one of the optimizers to (KP1) is concentrated on the graph of a function.
This is plainly effective when the cost is given by the squared Euclidean distance instead of $\|y - x\|$: by the uniform convexity there exists a unique optimizer π to (KP1) of the form $\pi = (\mathrm{Id}, \mathrm{Id} - \nabla\phi)_\sharp \mu$ for a quasiconvex function ϕ, the Kantorovich potential. Therefore, when $\mu \ll \mathcal{L}^n$ and $\nu \ll \mathcal{L}^n$, the optimal map is μ-a.e. defined by $x \mapsto x - \nabla\phi(x)$ and it is one-to-one ([13, 14, 28] are the first results, extended to uniformly convex functions of the distance e.g. in [33, 30, 25, 16]).

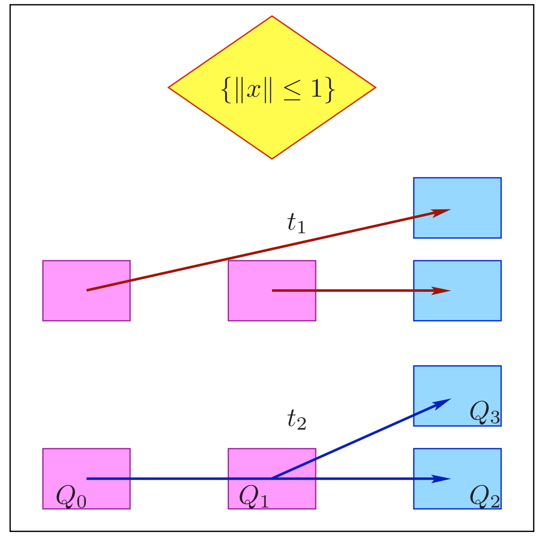
However, even in the case of the Euclidean norm, it is well known that this approach presents difficulties: at $\mathcal{L}^n$-a.e. point the Kantorovich potential fixes the direction of the transport, but not the precise point where the mass goes to. This is a feature of the problem, also in dimension one (see the example in Figure 1a). The data are not sufficient to determine a single transport map, since there is no uniqueness. Uniqueness can be recovered with the further requirement of monotonicity along transport rays ([24]).

The situation becomes even more complicated with a generic norm cost function, instead of the Euclidean one. The symmetry of the norm plays no role, but the loss in strict convexity of the unit ball is relevant, since the transport may not occur along lines and the direction of the transport can vary (see the example in Figure 1b).

The Euclidean case, and thus the one proposed by Monge, has been rigorously solved only around 2000 in [22, 38, 5, 15].



(a) One dimensional example. Let μ be the Lebesgue measure on $I_1 \cup I_2 \subset \mathbb{R}$ and ν the Lebesgue measure on $I_2 \cup I_3$. Both the maps t_1 translating I_1 to I_2, I_2 to I_3 and the map t_2 translating I_1 to I_3 and leaving I_2 fixed are optimal. Moreover, any convex combination of the two transport plans induced by t_1, t_2 is again a minimizer for (KP1), but clearly it is not induced by a map.



(b) Two dimensional example. The unit ball of $\|\cdot\|$ is given by the rhombus. Let μ be the Lebesgue measure $Q_0 \cup Q_1 \subset \mathbb{R}^2$ and ν the Lebesgue measure on $Q_2 \cup Q_3$. Both the maps t_1, t_2 translating one of the first two squares to one of the second to squares are optimal, and they transport mass in different directions.

Figure 1: Examples of non uniqueness of optimal transport maps with a generic norm.

Roughly, the approaches in the last three papers is at least partially based on a decomposition of the domain into one dimensional invariant regions for the transport, called transport rays. Due to the strict convexity of the unit ball, these regions are 1-dimensional convex sets. Due to regularity assumptions on the unit ball and a clever countable partition of the ambient space, it is moreover possible to reduce to the case where the directions of these segments is Lipschitz continuous. This, by Area or Coarea formula, allows to disintegrate the Lebesgue measure w.r.t. the partition in transport rays, obtaining absolutely continuous conditional probabilities on the one dimensional rays. In turn, this suffices to perform a reduction argument, that we also use in the present paper, which yields the thesis: indeed, one can fix within each ray an optimal transport map, uniquely defined imposing monotonicity within each ray. However, as in [10, 17, 18], we do not rely on any Lipschitz regularity of the vector field of directions.
This kind of approach was introduced already in 1976 by Sudakov ([37]), in the more generality of a possibly asymmetric norm — which actually is the case we are considering. However, its argument remains incomplete: a regularity property of the disintegration of the Lebesgue measure w.r.t. decompositions of the space into affine regions was not proved correctly, and, actually, stated in a form which does not hold ([2]). Indeed, there exists a compact subset of the unit square having measure 1 and made of disjoint segments, with Borel direction, such that the disintegration of the Lebesgue measure w.r.t. the partition in segments has atomic conditional measures ([29, 1]). The reduction argument described above requires instead absolutely continuous conditional measures, in order to solve then the one dimensional transport problems, and therefore a regularity of the partition in transport rays must be proved.
In the case of a strictly convex norm, where the affine regions reduce to lines, Sudakov argument was completed in [17]. In this paper we choose an alternative one dimensional decomposition selected by the additional variational principles, instead of the affine one considered by Sudakov.

The method in [22] is based on PDEs and they introduce the concept of transport density, widely studied since there — the very first works are [23, 2, 12, 24], in [34] one finds more references. Given a Kantorovich potential u for the transport problem between two absolutely continuous measures with compactly supported and smooth densities f^+, f^-, they define as transport density a nonnegative function a supported on the family of transport rays and satisfying

$$-\operatorname{div}(a\nabla u) = f^+ - f^-$$

in distributional sense. The above equation was present already in [7] with different motivation. It allows a generalization to measures, and an alternative definition introduced first in [11] for $\rho := a\mathcal{L}^n$ is given by the Radon measure defined on $A \in \mathcal{B}(\mathbb{R}^n)$ as

$$\rho(A) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{H}^1 \llcorner (A \cap [\![x, y]\!])\, d\pi(x, y), \tag{1.1}$$

where π is an optimal transport plan.

When the unit ball in not *strictly* convex, the results available are in the two-dimensional case, which is completely solved, and for crystalline norms ([4]). The strategy is to fix both the direction of the transport and the transport map by imposing additional optimality conditions, and then to carry out a Sudakov-type argument on the selected transports.

We follow the same strategy, and the disintegration technique available from [10], whose adaptation however is not really straightforward, allows to find the result for a generic norm.

A different proof of existence for general norms, which does not rely on disintegration of measures and is more concerned with the regularity of the transport density, is contemporary presented in [20], improving their argument for strictly convex norms in [19].

1.2. **Overview of the paper.** We present below an overview of the paper, where we establish existence of minimizers to the Monge problem in $\mathbb{R}^n$ with a norm cost function. We follow the selection argument in [4] in order to perform a one dimensional Sudakov-type decomposition of the space. We deal with the key difficulty of disintegrating the Lebesgue measure w.r.t. this decomposition, arising because of the more general norm, following the technique introduced in [8, 10] and then in [17] for this setting.

We state the problems and introduce some notations before summarizing the results.

Primary Transport Problem. *Consider the Monge-Kantorovich optimal transport problem*

$$\min_{\pi \in \Pi(\mu,\nu)} \mathcal{I}_K(\pi), \tag{1.2}$$

where

$$\mathcal{I}_K(\pi) := \int \|y - x\| \,\mathrm{d}\pi(x, y)$$

between two positive Radon measures μ, ν with the same total variation. We assume that $\mu \ll \mathcal{L}^n$ and, in order to avoid triviality, we suppose moreover that there exists a transport plan with finite cost.

Let the *optimal primary transport plans* be the family $\mathcal{O}_{\mathrm{p}} \subset \Pi(\mu, \nu)$ of minimizers to the primary problem. Let moreover ϕ be a Kantorovich potential, which is a function $\mathbb{R}^n \to \mathbb{R}$ such that

$$\phi(x) - \phi(y) \leq \|y - x\| \qquad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \tag{1.3a}$$

$$\phi(x) - \phi(y) = \|y - x\| \qquad \text{for } \pi\text{-a.e. } (x, y) \quad \forall \pi \in \mathcal{O}_{\mathrm{p}}. \tag{1.3b}$$

Define Γ_{p} as the $\|\cdot\|$-subdifferential of ϕ, which is the set $\partial^-\phi = \{(x, y) : \ \phi(x) - \phi(y) = \|y - x\|\}$.

Secondary Transport Problem. *Consider a strictly convex norm $|\cdot|$. Study*

$$\min_{\pi \in \mathcal{O}_{\mathrm{p}}} \int |y - x| \,\mathrm{d}\pi(x, y) = \min_{\pi \in \Pi(\mu,\nu)} \mathcal{I}^{\mathrm{s}}_K \tag{1.4}$$

where

$$\mathcal{I}^{\mathrm{s}}_K(\pi) := \int c_{\mathrm{s}}(x, y)\, d\pi(x, y)$$

and where the secondary cost function c_s is defined by

$$c_s(x,y) := |y-x|\mathbb{1}_{\Gamma_p}(x,y).$$

Let the *optimal secondary transport plans* be the family $\mathcal{O}_s \subset \Pi(\mu,\nu)$ of minimizers to (1.4).

We establish the following theorem concerning the existence of optimal transport maps for the secondary transport problem, and thus for the primary one. The proof will be given in Section 2.

Theorem. *The following statements concerning the primary and secondary transport problems hold.*

1: Monge Problem.

There exists a unique optimal secondary transport plan $\pi_s \in \mathcal{O}_s$ induced by a map t monotone along rays. More precisely, it is induced by a Borel map t such that

- *for all $x \in \mathbb{R}^n$ the outgoing ray $r^+(x) := \cup_{n\in\mathbb{N}} [\![x, t^n(x)]\!]$ is a segment;*
- *for all $x \in \mathbb{R}^n$ the map $t \restriction_{r^+(x)}$ is monotone;*
- *the graph of t is σ-compact.*

Then, the graph of r^+, of the secondary transport rays $r := r^+ \cup (r^+)^{-1}$ and of the vector field d vanishing where $t(x) = x$ and pointing otherwise towards $t(x) - x$ are σ-compact.

The optimal map t provides a solution to the primary Monge problem which may change with $|\cdot|$.

2. Secondary Transport Problem.

Every optimal secondary transport plan $\pi_s \in \mathcal{O}_s$ is concentrated on the c_s-monotone set $\mathrm{Graph}(r^+)$.

In particular, there exists a σ-compact subset $\mathtt{Q}$ of countably many hyperplanes such that the family of non trivial oriented segments defined by $\{r_\mathtt{q} := r(\mathtt{q})\}_{\mathtt{q}\in\mathtt{Q}}$ has the following properties:

- *$t(x) \neq x$ for all x in $\mathcal{T}_s := \cup_{\mathtt{q}\in\mathtt{Q}} r_\mathtt{q}$ which is not a terminal point of some $r_\mathtt{q}$;*
- *$t(x) = x$ for all x in $\mathcal{F} = \mathbb{R}^n \setminus \mathcal{T}_s$;*
- *$(\mu+\nu)$-a.e. fixed points of t are fixed points for every $\pi_s \in \mathcal{O}_s$;*
- *for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q} \in \mathtt{Q}$, $r_\mathtt{q}$ is an invariant sets for every $\pi_s \in \mathcal{O}_s$.*

3. Secondary Transport Density.

The transport density (1.1) associated to any optimal secondary transport plan $\pi_s \in \mathcal{O}_s$ is the unique solution $\rho = a\mathcal{L}^n$, $a \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, to the transport equations, for $Z \in \mathcal{B}(\mathtt{Q})$ arbitrary open sets,

$$\mathrm{div}(\mathbb{1}_{r(Z)} d\, \rho) = \mathbb{1}_{r(Z)}(\mu - \nu) \qquad\qquad \rho \in \mathcal{M}^+_{\mathrm{loc}}$$

whose density vanishes moving on the rays towards initial points, and out of $\mathcal{T}_s$. It may vary with $|\cdot|$.

We give in (2.12) two formulas for the transport density, in terms of either the disintegrations of μ and ν or the map t, following Section 8 of [10]. It satisfies a divergence formula on 'regular' subsets of $r(Z)$, for $Z \in \mathcal{B}(\mathtt{Q})$.

The above theorem is derived as a corollary of the regularity of the disintegration w.r.t. transport rays, stated below, and of the dimensional reduction arguments present in the literature.

The basic idea is the following. If we had two regions invariant for the transport, and it were possible to define an optimal transport map for the restricted measures within each region, then we would trivially get an optimal transport map solving our problem.

The same reasoning is applied to the continuous family of invariant sets $\{r_\mathtt{q}\}_{\mathtt{q}\in\mathtt{Q}}$ by means of the disintegration theorem. In order to solve the transport problem on each segment, one needs the absolute continuity of the initial measure to be transported on the segment. If that holds, the one dimensional solution is know and one shows that the maps on each segment provide together an optimal transport plan.

The issue of the following disintegration theorem is indeed to show that disintegrating the Lebesgue measure on the *secondary transport set* $\mathcal{T}_s = \cup_{\mathtt{q}\in\mathtt{Q}} r_\mathtt{q}$ one obtains conditional measures absolutely continuous w.r.t. the Lebesgue measure on the segments.

We state here below the Disintegration Theorem by itself, starting from a c_s-monotone set Γ and defining the transport rays related to Γ. In view of the application to the transport problem, we directly assume w.l.o.g. Γ σ-compact and contained in $\{c_s < +\infty\}$. Indeed, one obtains the statement of the theorem above choosing $\Gamma_{\rm s}$ as the graph of the multivalued function $x \to r^+_\Gamma(x)$ in Definition 2.2, whenever Γ is chosen as a $c_{\rm s}$-monotone, σ-compact carriage for any optimal secondary transport plan $\pi \in \mathcal{O}_{\rm s}$ contained in $\{c_{\rm s} < +\infty\}$.

The following construction of the transport rays is similar to the one in [4], and it is present in several other papers in terms of the Kantorovich potential of the transport problem. The definition of disintegration is recalled in Definition 2.10, while the one of $c_{\rm s}$-monotone set in Definition 2.1.

Definition 1.2. Given a set $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$, define the multivalued map $r : \mathbb{R}^n \to \mathbb{R}^n$ having graph

$$\Big\{(x,y) : \exists (x',y') \in \Gamma_{\rm s} \text{ such that } [\![x,y]\!] \subseteq [\![x',y']\!]\Big\}.$$

and the following equivalence relation in Graph(r):

$$(x,w) \overset{\rm s}{\sim} (y,z) \qquad \Longleftrightarrow \qquad r(x) \cap r(w) = r(y) \cap r(z).$$

The possible r.h.s. identify the equivalence classes, denote the family of those ones which are not singletons as the family $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ of transport rays relative to Γ.
The relative transport set is the union of the transport rays: $\mathcal{T}_{\rm s} = \cup_{\mathtt{q}\in\mathtt{Q}} r_{\mathtt{q}}$.
Define d as the multivalued vector field giving at each point $x \in \mathcal{T}_{\rm s}$ the unit direction from x to any point in $r(x)$, and vanishing elsewhere.

Theorem. *Let Γ be a σ-compact, $c_{\rm s}$-monotone subset of $\{c_{\rm s} < +\infty\}$. Define the relative σ-compact transport set $\mathcal{T}_{\rm s}$ and its covering by secondary transport rays $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ and d as in Definition 1.2.*

Then, the following strongly consistent disintegration w.r.t. the covering $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ holds:

$$\mathcal{L}^n \llcorner \mathcal{T}_{\rm s} = \int_{\mathtt{Q}} \big(\gamma\, \mathcal{H}^1 \llcorner r_{\mathtt{q}}\big)\, d\mathcal{H}^{n-1}(\mathtt{q}) \tag{1.5}$$

where $\gamma : \mathbb{R}^n \to \mathbb{R}^+ \setminus 0$ is a Borel function and $\mathtt{Q}$ is a σ-compact subset of countably many hyperplanes.

In particular, the set of endpoints $\mathcal{E}$ of the secondary transport rays $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ is $\mathcal{L}^n$-negligible and therefore $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ is a partition of $\mathcal{T}_{\rm s}$, removing an $\mathcal{L}^n$-negligible set.

Moreover, a Green-Gauss-type divergence formula holds on special subsets of the transport set for the vector field of secondary rays directions d.

The proof of the theorem will be given in Section 3, following [8, 10, 17].
A difficulty consists in the fact that in this approach the disintegration technique naturally involves a Hopf-Lax formula with a Kantorovich potential for the secondary transport problem, in order to define a more regular vector field of directions approximating the directions of the secondary transport rays — for which in our case no regularity is known more than Borel measurability.
However, since $c_{\rm s}$ is ∞-valued and only lower semicontinuous, in general there is no function $\phi_{\rm s}$ such that $\Gamma_{\rm s}$ is the $c_{\rm s}$-subdifferential of $\phi_{\rm s}$, which would correspond to a Kantorovich potential for the secondary transport problem — see Example 3.14.

The disintegration strategy consists basically in the following. One reduces the Lebesgue measure to special sets made of secondary rays transversal to an hyperplane H_0, by a countable partition. Let H_t denote an hyperplane at distance t from H_0, on the side where terminal points of secondary rays lye if $t > 0$, on the other otherwise. The special sets $\mathcal{Z}$ have to be chosen so that the following approximation holds: if we consider the subset $\mathcal{Z}'$ of those secondary rays intersecting an hyperplane H_t, the restriction of d to $\mathcal{Z}' \cap H_t$ is $\mathcal{H}^{n-1}$-a.e. the pointwise limit of a sequence of vector fields each pointing towards finitely many points of a dense sequence in $\mathcal{Z}' \cap H_s$, for any H_s different from H_t which is transversal to the rays in $\mathcal{Z}'$ (see Figure 2). This approximation by cones provides quantitative estimates of the Hausdorff $(n-1)$-dimensional measure of the intersection $\mathcal{Z} \cap H_t$ between the hyperplane H_t parallel to H_0 and the secondary rays of $\mathcal{Z}$. The transversality of the hyperplanes ensures that the membership to a secondary transport ray establishes a bijective correspondence between the points of these parallel sections. The estimates on the Hausdorff measure of $\mathcal{Z} \cap H_t$ yields that the $\mathcal{H}^{n-1}$-measure on each section is absolutely continuous w.r.t. the push forward measure, by the map above defined by the membership to a same

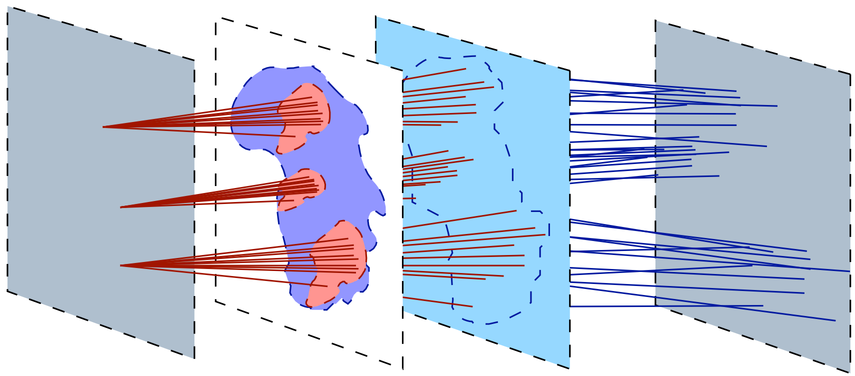


Figure 2: Cone approximation.

secondary transport ray, of the Hausdorff measure on a fixed intermediate section, say $Z_0 := \mathcal{Z} \cap H_0$. This leads to a justification of the following steps, with the notation of measure disintegrations

$$\begin{aligned}
\mathcal{L}^n(x) \llcorner \mathcal{Z} &= \int_{\mathbb{R}} (\mathcal{H}^{n-1}(x) \llcorner Z_t)\, dt && \text{by Fubini-Tonelli, where } Z_t = \mathcal{Z} \cap H_t \\
&\overset{x\in Z_t \leftrightarrow \mathsf{q}\in Z_0}{=} \int_{\mathbb{R}} \Big\{ \alpha(t,\mathsf{q})\, \mathcal{H}^{n-1}(\mathsf{q}) \llcorner Z_0 \Big\}\, dt && \text{by the push forward estimates on the sections} \\
&= \int_{Z_0} (\alpha(t,\mathsf{q})\, \mathcal{H}^1(t))\, d\mathcal{H}^{n-1}(\mathsf{q}) && \text{by Fubini-Tonelli, since } \alpha \text{ is regular enough} \\
&\overset{x=\mathsf{q}+t\mathsf{d}(x)}{=} \int_{Z_0} (\gamma(x)\, \mathcal{H}^1(x) \llcorner r_{\mathsf{q}})\, d\mathcal{H}^{n-1}(\mathsf{q}).
\end{aligned}$$

The new part of the construction, in Section 3, amounts thus in exhibiting a countable partition of the secondary transport set into model sets where one can prove the estimate on the push forward measure.

We first provide a secondary potential ϕ_s for a suitable restriction of the plan. Then, looking e.g. at the sections Z_0, Z_t, we choose the following approximating vector field of directions restricted to Z_t: the vector field defined by the correspondence

$$y \mapsto \operatorname*{arg\,min}_{x\in Z_0} \big[\phi(x) + \varepsilon\phi_s(x)\big] + \big[\|y-x\| + \varepsilon|y-x|\big].$$

It substitutes the cone approximation described above, due to the fact that it is almost a potential w.r.t. a strictly convex norm and therefore it can in turn be approximated by cone functions.
The secondary potential is exhibited on subsets of $\mathcal{T}_s$ such that, when partitioned into invariant sets for the primary transport problem, any two points can be connected by a coordinate cycle with finite cost, we will be more precise in Subsection 3.4.

The second effort is in order to provide a countable partition of $\mathcal{T}_s$ which reduces the disintegration problem to the model sets, and which is based on a partition of $\mathcal{T}$ into invariant sets for the primary problem — useful for the construction of the 'local' secondary potential ϕ_s. We partition first a subset of $\mathcal{T}_s$, and we show then that the set left apart with the partition has zero Lebesgue measure.

2. The Secondary Transport Problem

The secondary transport problem is a Monge-Kantorovich problem

$$\min_{\pi\in\Pi(\mu,\nu)} \int c_s(x,y)\, d\pi(x,y)$$

w.r.t. the secondary function

$$c_{\rm s}(x,y) = |y-x|\chi_{\{(x,y):\ \phi(x)-\phi(y)=|y-x|\}}(x,y).$$

This problem on one hand is more difficult than the primary one, since the secondary cost function is not continuous and takes the value $+\infty$. The advantage is that the strict convexity on regions which are invariant sets for the primary problem implies the known fact that any secondary optimal transport plan moves mass along rays.

We remind the following definition.

Definition 2.1. Given a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$, a subset Γ of $\mathbb{R}^n \times \mathbb{R}^n$ is *c-monotone* if for all finite number of points $\{(x_i, y_i)\}_{i=1,\dots,M}$ belonging to Γ one has the following inequality:

$$c(x_0,y_0)+\cdots+c(x_M,y_M) \le c(x_0,y_1)+\cdots+c(x_{M-1},y_M)+c(x_M,y_0).$$

Any optimal plan π for a Monge-Kantorovich problem in Polish spaces with a positive, analytic cost function c, and such that $\int c\, d\pi < \infty$, is concentrated on a c-monotone set (Lemma 5.2 in [9]). When the cost c is continuous, one is moreover allowed to take as this set the support of π. As a consequence, there exists a closed set $\Gamma \subset \{c < +\infty\}$ such that every optimal transport plan is supported on Γ: for any dense sequence $\{\pi_k\}_{k\in\mathbb{N}}$ in the convex, w*-closed set of optimal transport plans and one can take as Γ the support of the secondary optimal transport plan $\sum_{k\in\mathbb{N}} \pi_k/2^{-k}$.

This is no more true when the cost function is not continuous and possibly $+\infty$-valued, since the closure of a c-monotone set in general is not c-monotone. However, all the secondary optimal transport plans for $c_{\rm s}$ are concentrated on a common $c_{\rm s}$-monotone set, and therefore they share the same transport rays (Lemma 2.11).

2.1. **A universal family of secondary transport rays.** In the present subsection we associate to a $c_{\rm s}$-monotone set $\Gamma \subset \{c_{\rm s} < \infty\}$ the *secondary transport rays* and relative *initial* and *terminal points*, the *secondary rays directions*, the *secondary transport set* and the *secondary fixed set*.

The construction has the following meaning in terms of the secondary transport problem.
Consider any optimal secondary transport plan $\pi_{\rm s} \in \mathcal{O}_{\rm s}$, that we assume to have finite cost, and a $c_{\rm s}$-monotone carriage $\Gamma_{\rm s} \subset \{c_{\rm s} < +\infty\}$ (which is possible by Theorem 3.2 in [5]). Let $\pi_{\rm s} = \int_{\mathbb{R}^n} \pi_{\rm s}^x\, d\mu(x)$ be the strongly consistent disintegration of $\pi_{\rm s}$ w.r.t. the projection on the first variable $p^x : (x,y) \to x$.
The conditional measure $\pi_{\rm s}^x$ describes the transport taking place with $\pi_{\rm s}$ from the point x. Since $\pi_{\rm s}^x$ is concentrated on $\Gamma_{\rm s} \cap (\{x\} \times \mathbb{R}^n)$, then in order to describe the transport it is natural to associate the points in $p^y(\Gamma_{\rm s} \cap (\{x\} \times \mathbb{R}^n))$ to x, which are the possible destinations of the mass at x with $\pi_{\rm s}$.
However, a generic other plan $\pi'_{\rm s} \in \mathcal{O}_{\rm s}$ in general is not concentrated on the above $\Gamma_{\rm s}$, as discussed in the next subsection; therefore, if we want to associate to x all the possible destinations, not only with $\pi_{\rm s}$, but with every transport plan $\pi'_{\rm s} \in \mathcal{O}_{\rm s}$, it is natural to associate to x more points, precisely the set $r_\Gamma(x)$ in Definition 2.2. The universality of this sets will be prove in Lemma 2.11 of the next subsection.

Definitions and claims analogous to the ones in the present sections, also in terms of a $c_{\rm s}$-monotone function containing $\Gamma_{\rm s}$ in its subdifferential, are already present for example in [37], [22], [15], [4], [10], [17]. Having in mind the application to the secondary transport problem, and since, at this point, a secondary Kantorovich potential is not available, in the definitions below we are forced to start from $c_{\rm s}$-monotone sets as in [4].

Let Γ be a $c_{\rm s}$-monotone subset of $\{(x,y) :\ c_{\rm s}(x,y) < +\infty\}$.

Definition 2.2. The multivalued map $\mathbb{R}^n \ni x \mapsto r_\Gamma^+(x) \subset \mathbb{R}^n$ which gives the union of *secondary transport rays outgoing from* x is defined by the formula $\mathrm{Graph}(r_\Gamma^+) := G_\Gamma$ with

$$G_\Gamma := \left\{(x,y) :\ \exists m, \exists\{(x_i',y_i')\}_{i=1}^m \subset \Gamma_{\rm s} \text{ such that } [\![x,y]\!] \sqsubseteq \bigcup_{i=1,\dots,m} [\![x_i',y_i']\!]\right\}. \tag{2.1}$$

The map r giving the *secondary transport rays through* x identified by the formula $\mathrm{Graph}(r) = G_\Gamma \cup (G_\Gamma)^T$, where $(G_\Gamma)^T$ is the transpose of G_Γ.
When specifying otherwise which is a set Γ defining G_Γ, we will denote r_Γ simply as r.

Remark 2.3. As quickly discussed above, the ones of Definition 2.2 should be called more properly the *maximal secondary transport rays* associated to Γ, while the *secondary transport rays* $r_\Gamma'^{+}$, r'_Γ associated to Γ should be similarly defined from

$$G'_\Gamma := \Big\{(x,y) :\ \exists (x',y') \in \Gamma_{\mathrm{s}} \text{ such that } [\![x,y]\!] \sqsubseteq [\![x',y']\!]\Big\}. \tag{2.2}$$

Since we are mainly interested in the maximal secondary transport rays, we put them in evidence and we define the ones from (2.2) as the Γ-*secondary transport rays.*
Notice that $r_\Gamma^+(x) = \cup_{i\in\mathbb{N}}(r'^+)^i(x)$, which is the set $\cup_{i\in\mathbb{N}}K_i$ with $K_0 := \{x\}$ and $K_i := \cup r_\Gamma'^+(K_{i-1})$.

Definition 2.4. We define the following equivalence relation in $\mathrm{Graph}(r)$ (see Lemma 2.9 below):

$$(x,w) \overset{\mathrm{s}}{\sim} (y,z) \qquad \Longleftrightarrow \qquad r(x)\cap r(w) = r(y)\cap r(z). \tag{2.3}$$

The r.h.s. of (2.3) identifies the equivalence classes.
Secondary transport rays. A (nontrivial) secondary transport rays is each element in the r.h.s. above which do not reduce to a single point. We denote the family of secondary rays with $\{r_{\mathtt{q}}\}_{\mathtt{Q}}$.
Endpoints. The endpoints of a ray $r_{\mathtt{q}}$ are the points in its relative border. Let $\mathcal{E}$ be the set of endpoints of secondary transport rays.
Initial and terminal points. The terminal and initial points of a ray $r_{\mathtt{q}}$ are, when they exist, those $a, b \in r_{\mathtt{q}}$ such that, respectively, $r^+(b) = \{b\}$ and $(r^+)^{-1}(a) = \{a\}$. Denote with $\mathcal{E}^-$ the set of initial points and with $\mathcal{E}^+$ the set of terminal points of secondary transport rays.
Secondary transport set. The secondary transport set is the union of secondary transport rays:

$$\mathcal{T}_{\mathrm{s}} = \bigcup_{\mathtt{q}\in\mathtt{Q}} r_{\mathtt{q}} = p^x(\mathrm{Graph}(r_\Gamma)).$$

Fixed set. The fixed set is the complementary of the transport set.
Rays direction. The direction of a ray $r_{\mathtt{q}}$ is the unit vector pointing from the initial to the terminal point of $r_{\mathtt{q}}$. The multivalued vector field $d : \mathbb{R}^n \to \mathbf{S}^{n-1}$ of rays directions gives at each point $x \in \mathcal{T}_{\mathrm{s}}$ the direction of the rays through x and vanishes elsewhere.

Remark 2.5. The equivalence classes of (2.3) which do not reduce to point are of the form

$$r_{\mathtt{q}} \times r_{\mathtt{q}} = (r_{\mathtt{q}} \times \mathbb{R}^n) \cap \mathrm{Graph}(r),$$

where one should remove the two points $\{(e,e)\}$ for initial or terminal points e of $r_{\mathtt{q}}$ which belong also to other secondary transport rays. Therefore, the partition induced on $((\mathcal{T}_{\mathrm{s}}\setminus\mathcal{E})\times\mathbb{R}^n)\cap\mathrm{Graph}(r\lceil_{\mathcal{T}_{\mathrm{s}}})$ is precisely

$$\big\{(r_{\mathtt{q}}\times\mathbb{R}^n)\cap\mathrm{Graph}(r)\big\}_{\mathtt{q}\in\mathtt{Q}}.$$

Remark 2.6. Suppose Γ is compact. Then also the set in (2.2) must be compact: indeed, for every sequence $(x_k, y_k) \in \mathrm{Graph}(r'_\Gamma)$, $(x'_k, y'_k) \in \Gamma$ with $[\![x_k,y_k]\!] \subseteq [\![x'_k,y'_k]\!]$, by compactness of Γ up to subsequences $\{(x'_k,y'_k)\}_{k\in\mathbb{N}}$ converges to some point $(x',y') \in \Gamma$; therefore, $\{(x_k,y_k)\}_{k\in\mathbb{N}}$ must converge, up to subsequences, to a point (x,y): $[\![x,y]\!] \subseteq [\![x',y']\!]$, thus belonging to $\mathrm{Graph}(r'_\Gamma)$.
In particular, the set in (2.1) must be σ-compact, by Remark 2.3.

Clearly, a σ-compact Γ defines then σ-compacts G_Γ, G'_Γ.

As a consequence, when Γ is σ-compact the multivalued functions r, d and the function associating to each point in $\mathcal{T}_{\mathrm{s}}$ the initial and terminal points of the relative secondary transport rays, in the compactification of $\mathbb{R}^n$ with points at ∞, are Borel. As well, the secondary transport set, the set of initial points the set of terminal points are Borel. For detailed proofs see Lemma 2.2, Lemma 2.9 in [17] (taken from [10]).
In particular, in this case also the projection map $\mathtt{q} : \mathcal{T}_{\mathrm{s}} \to \mathtt{Q}$ has a σ-compact graph, by the explicit

construction that we will give of the quotient space, intersecting branches of rays with a transversal hyperplane.

Lemma 2.7. *Let (x, y) be such that $c_s(x, y) < \infty$. Then $c_s(x', y') < \infty$ for all $[\![x', y']\!] \sqsubset [\![x, y]\!]$. Moreover, if (z, w) is such that $c_s(z, w) < \infty$ and $[\![x, y]\!] \cap [\![z, w]\!] \neq \emptyset$, then $c_s(x', y') < \infty$ for all $[\![x', y']\!] \sqsubset [\![x, y]\!] \cup [\![z, w]\!]$.*

Proof. This is an immediate consequence of the inequality $\phi(x) - \phi(y) \leq \|y - x\|$ for all (x, y):

$$\phi(x) - \phi(x') + \phi(x') - \phi(y') + \phi(y') - \phi(y) = \phi(x) - \phi(y) = \|x' - x\| + \|y' - x'\| + \|y - y'\|$$

implies $\phi(x) - \phi(x') = \|x' - x\|$, $\phi(x') - \phi(y') = \|y' - x'\|$, $\phi(y') - \phi(y) = \|y - y'\|$.
The second claim is instead a consequence of the convexity of the norm: if $x' \in [\![x, y]\!] \cup [\![z, w]\!]$

$$\phi(x) - \phi(w) = \phi(x) - \phi(x') + \phi(x') - \phi(w) = \|x' - x\| + \|w - x'\| \leq \|w - x\|,$$

and therefore, since the opposite inequality holds for all (x, w), equality yields $c_s(x, w) < \infty$. Since $[\![x, y]\!] \cup [\![z, w]\!]$ is or $[\![x, y]\!]$, or $[\![z, w]\!]$, or $[\![x, w]\!]$, the first claim proves the second one. □

Lemma 2.8. *For any two points $(x', y'), (x'', y'') \in \Gamma$ and any x such that $[\![x', y']\!] \cap [\![x'', y'']\!] \ni x$, then or $x = x' = x''$ or $x = y' = y''$ or the two segments $[\![x', y']\!]$, $[\![x'', y'']\!]$ are aligned.*

Proof. Consider any x and any two points $(x', y'), (x'', y'') \in \Gamma$ such that $[\![x', y']\!] \cap [\![x'', y'']\!] \ni x$. Then we have

$$|y' - x'| + |y'' - x''| = |y' - x| + |x - x'| + |y'' - x| + |x - x''|. \tag{2.4}$$

On the other hand, the c_s-monotonicity inequality

$$|y' - x'| + |y'' - x''| = c_s(x', y') + c_s(x'', y'') \leq c_s(x'', y') + c_s(x', y'')$$

together with the strict convexity inequalities

$$\begin{aligned} c_s(x'', y') &\leq c_s(x'', x) + c_s(x, y') = |x - x''| + |y' - x| \\ c_s(x', y'') &\leq c_s(x', x) + c_s(x, y'') = |x - x'| + |y'' - x| \end{aligned} \tag{2.5}$$

implies

$$c_s(x', y') + c_s(x'', y'') \leq c_s(x'', x) + c_s(x, y') + c_s(x', x) + c_s(x, y''). \tag{2.6}$$

Since equality must old by (2.4), equality must hold also in (2.5): by the strict convexity of $|\cdot|$ defining c_s, if the segments $(\!|x', y'|\!)$ and $(\!|x'', y''|\!)$ are not parallel then either $x = x' = x''$ or $x = y' = y''$. □

Lemma 2.9. *Let Γ be a c_s-monotone subset of $\{c_s < \infty\}$.*
Then, the set $\mathrm{Graph}(r^+)$ is a c_s-monotone subset of $\{c_s < \infty\}$ and (2.3) is a partition of it.
Each set $r(x)$ is either a segment or a union of segments having x as a common either initial, or terminal point.

Proof. We separate the three claims in the statement.

Claim 1: Finite cost. By the first claim in Lemma 2.7, $c_s(x, y) < +\infty$ for all $(x, y) \in \mathrm{Graph}(r')$ in (2.2). Moreover, since $r^+(x) = \cup_{i \in \mathbb{N}} (r'^+)^i(x)$, the second claim in Lemma 2.7 proves that also $\mathrm{Graph}(r^+)$ is contained in $c_s < \infty$

Claim 2: Cyclical monotonicity. Consider any M points $\{(x'_i, y'_i)\}_i \in \mathrm{Graph}(r'^+)$ in order to test the definition of c_s-monotonicity. Let $\{(x'_i, y'_i)\}_i$ be points of Γ such that $(\!|x_i, y_i|\!) \sqsubseteq (\!|x'_i, y'_i|\!)$. Then, since Γ is c_s-monotone

$$\sum |y'_i - x'_i| \leq \sum c_s(x'_{i+1}, y'_i),$$

where $x'_{M+1} = x'_1$, and we set also $x_{M+1} = x_1$. The triangular inequality yields

$$c_s(x'_{i+1}, y'_i) \leq |y'_i - y_i| + c_s(x_{i+1}, y_i) + |x_{i+1} - x'_{i+1}|.$$

Moreover, being x_i', x_i, y_i, y_i' all aligned, for fixed $i \in \{1, \dots, M\}$,

$$|y_i' - y_i| + |y_i - x_i| + |x_i - x_i'| = |y_i' - x_i'|;$$

substituting the last two expressions in the one above, after cancellations one has

$$\sum |y_i - x_i| \leq \sum c_{\mathrm{s}}(x_{i+1}, y_i).$$

Claim 3: Structure of the partition. It is almost immediate that (2.3) is an equivalence relation. Indeed, for every $(x,w), (y,z), (p,r) \in \mathrm{Graph}(\mathcal{r})$

- $(x,w) \overset{\mathrm{s}}{\sim} (x,w)$, being trivially $\mathcal{r}(x) \cap \mathcal{r}(w) = \mathcal{r}(x) \cap \mathcal{r}(w)$;
- $(x,w) \overset{\mathrm{s}}{\sim} (y,z)$ implies $(y,z) \overset{\mathrm{s}}{\sim} (x,w)$, since $\mathcal{r}(x) \cap \mathcal{r}(w) = \mathcal{r}(y) \cap \mathcal{r}(z)$ implies trivially $\mathcal{r}(y) \cap \mathcal{r}(z) = \mathcal{r}(x) \cap \mathcal{r}(w)$;
- if $(x,w) \overset{\mathrm{s}}{\sim} (y,z)$ and $(y,z) \overset{\mathrm{s}}{\sim} (p,r)$, then $(x,w) \overset{\mathrm{s}}{\sim} (p,r)$, by $\mathcal{r}(x) \cap \mathcal{r}(w) = \mathcal{r}(y) \cap \mathcal{r}(z) = \mathcal{r}(p) \cap \mathcal{r}(r)$.

By definition $\mathcal{r}'(x)$ is the union of those segments $[\![x', y']\!]$ containing x and such that $(x', y') \in \Gamma$. By Lemma 2.8, therefore, whenever $\mathcal{r}'(x) \neq \{x\}$, then $\mathcal{r}'(x)$, and thus $\mathcal{r}'^{+}$, is either a segment, or union of segments intersecting at the common endpoint x.
For every $y \in \mathcal{r}'(x)$ the set $\mathcal{r}'(x) \cup \mathcal{r}'(y)$ must again be a line or union of lines intersecting at the common endpoint x. Indeed, by definition of $\mathcal{r}'$ there exist $(x', y') \in \Gamma$ such that $[\![x, y]\!] \subset [\![x', y']\!]$ and moreover, again by Lemma 2.8, for all $(x'', y'') \in \Gamma$ such that $y \in [\![x'', y'']\!]$, the segment $[\![x'', y'']\!]$ must elongate $[\![x', y']\!] \subset \mathcal{r}'(x)$.
Therefore, $\mathcal{r}(x)$ is just constructed elongating the lines already present in $\mathcal{r}'(x)$, showing the thesis. □

2.2. **Statement of the disintegration w.r.t. secondary transport rays.** For the rest of Section 2 we temporarily assume some regularity of the disintegration of the Lebesgue measure on $\mathcal{T}_{\mathrm{s}}$ w.r.t. the covering by secondary transport rays: in the next subsections we will present some applications.

We remind first the definition of disintegration.

Definition 2.10. A *disintegration* of a measure $\lambda \in \mathcal{M}^+_{\mathrm{loc}}(X)$ on a Polish space X *consistent with a partition*, up to λ-negligible sets, $\{X_\alpha\}_{\alpha \in \mathsf{A}}$ of X: it is a family $\{\lambda_\alpha\}_{\alpha \in \mathsf{A}}$ of probability measures on X and a measure m on A such that

$$\textbf{1. } \forall E \in \Sigma, \quad \alpha \mapsto \lambda_\alpha(E) \text{ is } m\text{-measurable;} \tag{2.7a}$$

$$\textbf{2. } \lambda = \int \lambda_\alpha \, dm, \text{ i.e.}$$

$$\lambda(E \cap p^{-1}(F)) = \int_F \lambda_\alpha(E) \, dm(\alpha), \quad \forall E \in \Sigma,\ F\ m\text{-measurable.} \tag{2.7b}$$

The disintegration is *unique* if the measures λ_α are uniquely determined for m-a.e. $\alpha \in \mathsf{A}$.
The disintegration is *strongly consistent* if $\lambda_\alpha(X \setminus X_\alpha) = 0$ for m-a.e. $\alpha \in \mathsf{A}$.
The measures λ_α are also called conditional measures of λ w.r.t. m.

With an abuse of notation, we denote as disintegration $\lambda = \int \lambda_\alpha \, dm$ also any family with the properties in (2.7) even w.r.t. a *covering* $\{X_\alpha\}_{\alpha \in \mathsf{A}}$ of X which is not a partition.

Existence and uniqueness results are classical, a presentation is in [9]. The issue in the present work is a regularity property of the disintegration of the Lebesgue measure w.r.t. the partition into transport rays: in Section 3, Theorem 2.13, we will prove the following statement.

Theorem. *Let Γ be a σ-compact, $\mathtt{c}_{\mathrm{s}}$-monotone subset of $\{\mathtt{c}_{\mathrm{s}} < +\infty\}$. Define the relative σ-compact transport set $\mathcal{T}_{\mathrm{s}}$ and its covering by secondary transport rays $\{\mathcal{r}_{\mathsf{q}}\}_{\mathsf{q} \in \mathsf{Q}}$ as in Definition 2.4.*

Then, the following disintegration holds:

$$\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}} = \int_{\mathsf{Q}} \left(\gamma\, \mathcal{H}^1 \llcorner \mathcal{r}_{\mathsf{q}} \right) d\mathcal{H}^{n-1}(\mathsf{q}) \tag{2.8}$$

where $\gamma : \mathbb{R}^n \to \mathbb{R}^+ \setminus 0$ is a Borel function and Q is a σ-compact subset of countably many hyperplanes. In particular, the set of endpoints $\mathcal{E}$ of the secondary transport rays $\{\mathcal{r}_{\mathsf{q}}\}_{\mathsf{q} \in \mathsf{Q}}$ is $\mathcal{L}^n$-negligible.

2.3. **Solutions to the Monge secondary transport problem.** We present here the main application of the regularity of the disintegration developed the next section.

We show that for any optimal secondary transport plan $\pi \in \mathcal{O}_{\mathrm{s}}$ with finite cost and for any c_{s}-monotone set Γ_{s} which carries π_{s}, the set $G_{\Gamma_{\mathrm{s}}} = \mathrm{Graph}(r^+_{\Gamma_{\mathrm{s}}})$ defined in Section 2.1 must carry all the optimal secondary transport plans. This is basically the fact that every optimal secondary transport plan must move the mass within the same (maximal) transport rays, and there exists a maximal transport set. The transport set in unique, up to μ-negligible sets.
We solve then the secondary transport problem, and thus the primary one, by a reduction argument to dimension one, reducing more precisely to transport problems on the secondary transport rays (Theorem 2.13).

Lemma 2.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel function such that $\mu = f\mathcal{L}^n$.*
Fix any optimal secondary transport plan $\pi \in \mathcal{O}_{\mathrm{s}}$ and a c_{s}-monotone carriage Γ for π.

Claim 1. The disintegration (2.8) induces the following disintegration of $\mu \llcorner \mathcal{T}_{\mathrm{s}}$, $\nu \llcorner \mathcal{T}_{\mathrm{s}}$ w.r.t. the $\mathcal{L}^n$-partition of $\mathcal{T}_{\mathrm{s}}$ into secondary transport rays $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ of Definition 2.4

$$\mu \llcorner \mathcal{T}_{\mathrm{s}} = \int_{\mathtt{Q}} (f\gamma\mathcal{H}^1 \llcorner r_{\mathtt{q}})\, d\mathcal{H}^{n-1}(\mathtt{q}) \qquad \nu \llcorner \mathcal{T}_{\mathrm{s}} = \int_{\mathtt{Q}} \nu_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}) \tag{2.9a}$$

where

- *$\nu \llcorner (\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E}) = \int_{\mathtt{Q}} \nu_{\mathtt{q}} \llcorner (\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E})\, d\mathcal{H}^{n-1}(\mathtt{q})$ is a disintegration of ν w.r.t. the partition $\{r_{\mathtt{q}} \setminus \mathcal{E}\}_{\mathtt{q}\in\mathtt{Q}}$;*
- *when b is a terminal point of a secondary transport ray $r_{\mathtt{q}}$, then $\nu_{\mathtt{q}}(\{b\}) = \mu_{\mathtt{q}}(r_{\mathtt{q}}) - \nu_{\mathtt{q}}(\mathrm{ri}(r_{\mathtt{q}}))$;*
- *when a is an initial point of any secondary transport ray $r_{\mathtt{q}}$, then $\nu_{\mathtt{q}}(\{a\}) = 0$.*

Claim 2. For every $\hat\pi \in \mathcal{O}_{\mathrm{s}}$, the disintegration (2.8) induces the following disintegration of $\hat\pi \llcorner (\mathcal{T}_{\mathrm{s}} \times \mathcal{T}_{\mathrm{s}})$ w.r.t. the $(\mathcal{L}^n \times \nu)$-partition $\{r_{\mathtt{q}} \times \mathbb{R}^n\}_{\mathtt{q}\in\mathtt{Q}}$ introduced in (2.3):

$$\hat\pi \llcorner (\mathcal{T}_{\mathrm{s}} \times \mathbb{R}^n) = \int_{\mathtt{Q}} \hat\pi_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}) \tag{2.9b}$$

with $\hat\pi_{\mathtt{q}} \in \Pi(\mu_{\mathtt{q}} = f\gamma\mathcal{H}^1 \llcorner r_{\mathtt{q}}, \nu_{\mathtt{q}})$ optimal transport plan w.r.t. $c(x,y) = |y-x|$.

Claim 3. Each plan $\hat\pi \in \mathcal{O}_{\mathrm{s}}$ leaves each point of $\mathcal{F}$ fixed.

Claim 4. Every $\hat\pi \in \mathcal{O}_{\mathrm{s}}$ is concentrated on the c_{s}-monotone set $G_\Gamma = \mathrm{Graph}(r^+)$ in (2.2).

Remark 2.12. Every map $\pi \in \mathcal{O}_{\mathrm{s}}$ leaves μ-a.e. point of $\mathcal{F}$ fixed. However, if $(\mu \wedge \nu) \llcorner \mathcal{T}_{\mathrm{s}} \neq 0$ it may leave other points, belonging to the transport set $\mathcal{T}_{\mathrm{s}}$, fixed (consider the previous example in Figure 1a).

Proof. By the inner regularity of Radon measures, and since we are assuming that $\int c_{\mathrm{s}}\, d\pi < +\infty$, we can directly fix w.l.o.g. that Γ is a σ-compact subset of $\{c_{\mathrm{s}} < \infty\}$. Indeed, shrinking Γ the thesis becomes stronger.

Step 1: Disintegration of μ. It is an immediate consequence of (2.8) and $\mu = f\nu$.

Step 2: Disintegration of π. Since μ is the first marginal of π and the partition is of the form $\{(p^x)^{-1}(r_{\mathtt{q}})\}_{\mathtt{q}\in\mathtt{Q}}$, we can endow the quotient space, which is still $\mathtt{Q}$, with the same Borel σ-algebra and the same measure $\mathcal{H}^{n-1}$, obtaining then the strongly consistent disintegration

$$\pi = \int_{\mathtt{Q}} \tilde\pi_{\mathtt{q}}\, \mathcal{H}^{n-1}(\mathtt{q}).$$

For all $A \in \mathcal{B}(\mathcal{T}_{\mathrm{s}})$ and $B \in \mathcal{B}(\mathtt{Q})$, denoting with $\mathtt{q} : \mathcal{T}_{\mathrm{s}} \to \mathtt{Q}$ the quotient projection,

$$\int_A \mu_{\mathtt{q}}(B)\, d\mathcal{H}^{n-1}(\mathtt{q}) = \mu(B \cap \mathtt{q}^{-1}(A)) = \pi((B \cap \mathtt{q}^{-1}(A)) \times \mathbb{R}^n) = \int_A \tilde\pi_{\mathtt{q}}(B \times \mathbb{R}^n)\, d\mathcal{H}^{n-1}(\mathtt{q}).$$

Then, for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q}$ we have that $p^x_\sharp(\tilde\pi_{\mathtt{q}}) = \mu_{\mathtt{q}}$.

Step 3: Disintegration of ν. Let $\tilde\nu_{\mathtt{q}} := p^y \tilde\pi_{\mathtt{q}}$. Since for all $A \in \mathcal{B}(\mathcal{T}_{\mathrm{s}})$ and $B \in \mathcal{B}(\mathtt{Q})$

$$\nu(B \cap p^{-1}(A)) = \pi(\mathbb{R}^n \times (B \cap p^{-1}(A))) = \int_A \tilde\pi_{\mathtt{q}}(\mathbb{R}^n \times B)\, d\mathcal{H}^{n-1}(\mathtt{q}) = \int_A \tilde\nu_{\mathtt{q}}(B)\, d\mathcal{H}^{n-1}(\mathtt{q}),$$

we have that $\int_{\mathtt{Q}} \tilde\nu_{\mathtt{q}}\, d\mathcal{H}^n(\mathtt{q})$ is a disintegration of $\nu \llcorner \mathcal{T}_{\mathrm{s}}$ w.r.t. the covering $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$.

In particular, $\nu \llcorner (\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E}) = \int_{\mathtt{Q}} \tilde\nu_{\mathtt{q}} \llcorner (\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E})\, d\mathcal{H}^{n-1}(\mathtt{q})$ must be a disintegration of $\nu \llcorner (\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E})$ w.r.t. the partition of $\mathcal{T}_{\mathrm{s}} \setminus \mathcal{E}$ into $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$.

Moreover, having an absolutely continuous first marginal no mass can arrive to initial points. Indeed, whenever $(x,a) \in \Gamma$ necessarily $x = a$ and therefore

$$\int_{\mathtt{Q}} \tilde\nu_{\mathtt{q}}(\mathcal{E}^-)\, d\mathcal{H}^n(\mathtt{q}) = \nu(\mathcal{E}^-) = \pi(\mathbb{R}^n \times \mathcal{E}^-) \le \pi(\{(a,a)\}_{a\in\mathcal{E}^-}) \le \mu(\mathcal{E}^-) = 0,$$

which yields $\tilde\nu_{\mathtt{q}}(\mathcal{E}^-) = 0$ for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q}$.

Finally, the mass which arrives to a terminal point is the source mass on the rays ending there which has not been delivered to points within the rays. Indeed, for every compact $K \subset \mathcal{E}$, denoting with $p : \mathcal{T}_{\mathrm{s}} \to \mathtt{Q}$ the multivalued projection onto the quotient,

$$\mu(r(K)) = \pi(r(K) \times r(K)) = \pi(r(K) \times (r(K) \setminus K)) + \pi(r(K) \times K) = \nu(r(K) \setminus K) + \nu(K),$$

which implies

$$\int_{\mathtt{Q}} \nu_{\mathtt{q}}(K)\, d\mathcal{H}^{n-1}(\mathtt{q}) = \nu(K) = \mu(r(K)) - \nu(r(K)) = \int_{\mathtt{Q}} (\nu_{\mathtt{q}}(r(K)) - \mu_{\mathtt{q}}(r(K)))\, d\mathcal{H}^{n-1}(\mathtt{q}).$$

By the strong consistency, moreover, the first and the last side of the above equation can be rewritten as

$$\int_{\mathtt{Q}\cap p(K)} \tilde\nu_{\mathtt{q}}(b_{\mathtt{q}})\, d\mathcal{H}^{n-1}(\mathtt{q}) = \int_{\mathtt{Q}\cap p(K)} (\nu_{\mathtt{q}}(r_{\mathtt{q}}) - \mu_{\mathtt{q}}(r_{\mathtt{q}}))\, d\mathcal{H}^{n-1}(\mathtt{q}),$$

which proves that for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q}$ we have $\tilde\nu_{\mathtt{q}}(b_{\mathtt{q}}) = \nu_{\mathtt{q}}(r_{\mathtt{q}}) - \mu_{\mathtt{q}}(r_{\mathtt{q}})$ for the terminal point $b_{\mathtt{q}}$ of $r_{\mathtt{q}}$.

Step 4: Optimality of the 1*-dimensional transports* $\{\pi_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ *and normalization*. By strong consistency of the disintegration, for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q} \in \mathtt{Q}$ the transport $\tilde\pi_{\mathtt{q}} \in \Pi(\mu_{\mathtt{q}}, \tilde\nu_{\mathtt{q}})$ is concentrated on $\Gamma \cap (r_{\mathtt{q}} \times r_{\mathtt{q}})$ — in particular it is $|\cdot|$-monotone. Since $c(x,y) = |y-x|$ is a continuous and real cost function, then π must be optimal ([32], [35]).
One can then adjust the disintegration suitably replacing those $\tilde\pi_{\mathtt{q}}$, $\tilde\nu_{\mathtt{q}}$ which do not satisfy the properties in the statement without affecting the disintegration, since the exchange is made on a $\mathcal{H}^{n-1}$-negligible set of $\mathtt{q}$.

Step 5: A universal c_{s}*-monotone carriage*. Let $\pi^1 \in \mathcal{O}_{\mathrm{s}}$ and let Γ_1 be any c_{s}-monotone carriage of π_1. We want to show that any other secondary optimal transport plan π^2 is concentrated on G_{Γ_1}, where G_{Γ_1} is the graph of a multivalued function r_1^+ in Definition 2.2. By the arbitrariness of π^1, π^2 in $\mathcal{O}_{\mathrm{s}}$, this shows that the family of transport rays associated to any secondary optimal transport plan is universal, and that there exists a universal c_{s}-monotone carriage for all optimal secondary transport plans.
Substep 5.1: Reduction to dimension 1. Suppose the plan π considered above is the secondary optimal transport plan $\pi := (\pi^1 + \pi^2)/2$. In particular, we have $\pi^1(\Gamma^{\mathrm{c}}) = \pi^2(\Gamma^{\mathrm{c}}) \le \pi(\Gamma^{\mathrm{c}}) = 0$ and then π_1 and π_2 must be concentrated on Γ. Taking the intersection with Γ we directly assume w.l.o.g. that Γ_1 is a σ-compact subset of Γ: the set $\mathrm{Graph}_{\Gamma_1}(r^1)$ becomes possibly smaller.

By the previous steps we can disintegrate π^1 and π^2 w.r.t. the common carriage Γ:

$$\mu = \int_{\mathtt{Q}} \mu_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}), \quad \nu = \int_{\mathtt{Q}} \nu_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}), \quad \pi^1 = \int_{\mathtt{Q}} \pi^1_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}), \quad \pi^2 = \int_{\mathtt{Q}} \pi^2_{\mathtt{q}}\, d\mathcal{H}^{n-1}(\mathtt{q}),$$

finding that $\pi^1_{\mathtt{q}}, \pi^2_{\mathtt{q}} \in \Pi(\mu_{\mathtt{q}}, \nu_{\mathtt{q}})$ are optimal transports on $r_{\mathtt{q}}$ w.r.t. $c(x,y) = |y-x|$.

The thesis reduces thus in showing that $\pi^1_{\mathtt{q}}$, $\pi^2_{\mathtt{q}}$ have basically the same family of transport rays. More precisely, we show that $\pi^2_{\mathtt{q}} \llcorner (r_{\mathtt{q}} \times \mathbb{R}^n)$ is concentrated on $\mathrm{Graph}(r^{1+} \restriction_{r_{\mathtt{q}}}) = \mathrm{Graph}_{\Gamma_1 \cap (r_{\mathtt{q}} \times \mathbb{R}^n)}(r^{1+}_{\mathtt{q}})$: this implies

$$\pi^2(G_{\Gamma_1}) = \int_{\mathsf{Q}} \pi^2_{\mathsf{q}}(G_{\Gamma_1})\, d\mathcal{H}^{n-1}(\mathsf{q}) = \int_{\mathsf{Q}} \pi^2_{\mathsf{q}}(\mathrm{Graph}(r^{1+}_{\mathsf{q}}))\, d\mathcal{H}^{n-1}(\mathsf{q}) = \int_{\mathsf{Q}} |\pi^2_{\mathsf{q}}|\, d\mathcal{H}^{n-1}(\mathsf{q}) = |\pi_2|.$$

We are therefore left with showing the thesis when the transports π^1, π^2 are 1-dimensional.
Substep 5.2: Solution of the 1*-dimensional problem.* Fix any two transport rays $r^1(x) \subset r(x)$ and assume w.l.o.g. that π transports mass on $r(x)$ from left to right. By Lemma 2.9 this is possible and also π^1, π^2 must transport mass from/to points of r in the same direction. Denote these two particular rays just as r^1, r. The thesis amounts in showing that $\pi^2 \llcorner (r^1 \times \mathbb{R}^n)$ is concentrated on $r^1 \times r^1$.
Define ℓ^-, ℓ^+ as the connected components of $\mathbb{R} \setminus r^1$, respectively on the left and on the right of r^1. Observe that, again by the fact that rays, in this case of r^1, may intersect only at points that are for both either terminal or initial (Lemma 2.9), ℓ^-, r and ℓ^+ are invariant sets for π^1:

$$\begin{aligned}
&\mu \llcorner \ell^+ = p^x_\sharp(\pi^1 \llcorner (\ell^+ \times \ell^+)) && p^y_\sharp(\pi^1 \llcorner (\ell^+ \times \ell^+)) = \nu \llcorner \ell^- && |\mu \llcorner \ell^+| = |\nu \llcorner \ell^-| \\
&\mu \llcorner r^1 = p^x_\sharp(\pi^1 \llcorner (r^1 \times r^1)) && p^y_\sharp(\pi^1 \llcorner (r^1 \times r^1)) =: \nu_{r^1} && |\mu \llcorner r^1| = |\nu_{r^1}| \\
&\mu \llcorner \ell^- = p^x_\sharp(\pi^1 \llcorner (\ell^- \times \ell^-) && p^y_\sharp(\pi^1 \llcorner (\ell^- \times \mathrm{clos}(\ell^-))) =: \nu^\ell && |\mu \llcorner \ell^-| = |\nu^\ell|.
\end{aligned}$$

The plan π^2 can't transport mass neither from ℓ^+ to $(\ell^+)^{\mathrm{c}}$, nor from r to ℓ^-, because this would contradict the direction of the transport on r. Therefore, the mass $\nu \llcorner \ell^-$ can arrive only from ℓ^- itself, and therefore necessarily π^2 transports $\mu \llcorner \ell^-$ to $\nu \llcorner \ell^+$. As a consequence, the mass ν_{r^1} can arrive only from r^1: therefore $\pi^2 \llcorner (r^1 \times \mathbb{R}^n) \in \Pi(\mu \llcorner r^1, \nu_{r^1})$ and it is concentrated on $r^1 \times r^1$.

We have incidentally shown that the transport set is an invariant set for all optimal secondary transport plans, being so each single secondary transport ray. Therefore, also the set of fixed points $\mathcal{F}$ is fixed for all secondary transport plans. Since the transport $(\mathrm{Id}, \mathrm{Id})_\sharp(\mu \llcorner \mathcal{F})$ has zero cost, μ-a.e. point of $\mathcal{F}$ must be fixed for all optimal secondary transport plan.

Step 6: Disintegration of a generic $\hat\pi \in \mathcal{O}_{\mathrm{s}}$. Consider now any other $\hat\pi \in \mathcal{O}_{\mathrm{s}}$. In Step 5 we have shown that $\hat\pi(G_\Gamma) = 1$, for the Γ in the statement. As a consequence, we can disintegrate $\hat\pi$ exactly as we did in Step 2 for π and the disintegration will then enjoy the consequent marginal and optimality properties. □

A second corollary of Lemma 2.11 is the solution to the secondary transport problem.

Theorem 2.13. *There exists a transport map solving the secondary problem.*
Each optimal secondary transport plan moves mass along the same lines, since there exists a c_{s}-monotone set Γ such that $\pi_{\mathrm{s}}(\Gamma) = 1$ for all $\pi_{\mathrm{s}} \in \mathcal{O}_{\mathrm{s}}$.
If we require that the transport is monotone along each ray, then the optimal transport map t is uniquely defined up to an $\mathcal{L}^n$-negligible set.

Proof. By Lemma 2.11, each optimal transport plan $\pi \in \mathcal{O}_{\mathrm{s}}$ is of the form

$$\pi = \int_{\mathsf{Q}} \pi_{\mathsf{q}}\, d\mathcal{H}^{n-1}(\mathsf{q}) + (\mathrm{Id}, \mathrm{Id})_\sharp(\mu \llcorner \mathcal{F})$$

with $\pi_{\mathsf{q}} \in \Pi(f\gamma\mathcal{H}^1 \llcorner r_{\mathsf{q}}), \nu_{\mathsf{q}})$ optimal transport plan w.r.t. the cost function $c(x, y) = |y - x|$.
We are left with 1-dimensional transport problems with absolutely continuous initial measure, on the secondary transport rays. Then, there exists a unique optimal monotone map t_{q} solving this 1-dimensional transport problem. A global map can be obtained placing these map side by side:

$$t \restriction_{r_{\mathsf{q}}} := t_{\mathsf{q}}.$$

To be more precise, t_{q} is uniquely defined only out of countably many points while t is not well defined at the common initial points of multiple secondary rays, but this is irrelevant since the set is μ-negligible. The Lebesgue measurability can be found in the analogue Th. 3.4 of [17], and it is based on the change of the change of variables which leads to the disintegration.

By the disintegration and the optimality of both π_s and each t_{q}, the secondary cost of the transport with t_s is the optimal one:

$$\begin{aligned}\int_{\mathbb{R}^{2n}} c_s(x,y)\,d\pi_s(x,y) &= \int_{\mathsf{Q}}\left\{\int_{r_{\mathsf{q}}\times\mathbb{R}^n} c_s(x,y)\,d\pi_{\mathsf{q}}(x,y)\right\}d\mathcal{H}^{n-1}(\mathsf{q})\\ &= \int_{\mathsf{Q}}\left\{\int_{r_{\mathsf{q}}} c_s(x,t(x))f(x)\,d\lambda_{\mathsf{q}}(x)\right\}d\mathcal{H}^{n-1}(\mathsf{q}) = \int_{\mathbb{R}^n}|t(x)-x|\,d\mu.\end{aligned}$$

□

Definition 2.14. We call t^{-1} the surjective multivalued function, monotone along each ray, whose graph contains the transpose of the graph of t. The Borel measurability can be deduced by observing that in Theorem 2.13 one can choose a representative of t with σ-compact graph, by the inner regularity of the Radon measure $(\mathrm{Id},t)_\sharp(\mu)$.
Let $\tilde{t}^{-1}$ be the single valued function whose graph is contained in the graph of t^{-1} and which is left continuous (and monotone nondeacreasing) on secondary transport rays.
Then $\nu(\llparenthesis a(x),x\rrparenthesis)=\mu(\llparenthesis a(x),\tilde{t}^{-1}(x)\rrparenthesis)$ and $\nu(\llparenthesis a(x),x\rrbracket)=\mu(\llparenthesis a(x),t^{-1}(x)\rrparenthesis)$, where $a(x)$ denotes formally the first endpoint of $r(x)$.

As explained in [5], [4], the above uniqueness theorem of the c_s-monotone and monotone along rays solution to (KP1) implies the following stability result.

The requirement of monotonicity along secondary transport rays in Theorem 2.13 is equivalent to impose the third additional optimality condition

$$(\mathrm{Id},t)_\sharp\mu = \operatorname*{arg\,min}_{\pi\in\mathcal{O}_s}\int\frac{|y-x|^2}{1+|y-x|}\,d\pi(x,y), \tag{2.10}$$

and the theorem states that $(\mathrm{Id},t)_\sharp\mu$ is the unique minimizer.

One can approximate (KP1) with the Monge-Kantorovich problem of the optimal transportation between μ, ν with the cost function

$$c_\varepsilon(x,y) := \|y-x\| + \varepsilon|x-y| + \varepsilon^2\frac{|y-x|^2}{1+|y-x|}.$$

By the strict convexity of c_ε, there exists a unique optimal transport plan π_ε and moreover it is induced by a transport map t_ε. As $\varepsilon\to 0$, by the general theory of Γ-convergence applied to the Kantorovich relaxation of the Monge functionals, the optimal plans $\pi_\varepsilon=(\mathrm{Id},t_\varepsilon)_\sharp\mu$ weakly converge to the solution of (KP1) defined in 2.10. As a consequence, one finds that $t_\varepsilon\to t$ in measure.

2.4. **Determination of the transport density.** As a further application of the disintegration theorem, we derive the expression of the transport density relative to optimal secondary transport plans π_s in terms of the conditional measures of μ, ν on $\mathcal{T}_s$ w.r.t. the $\mathcal{L}^n$-partition of $\mathcal{T}_s$ into secondary transport rays. In particular, one can see its absolute continuity. Moreover, even in this non-smooth setting, the density function w.r.t. $\mathcal{L}^n \llcorner \mathcal{T}_s$ vanishes approaching initial points along secondary transport rays. As know, the same property does not hold for the terminal points — see Example 2.17 below taken from [24]. Notice that the transport density depends on the choice of $|\cdot|$ — see Example 2.18.

Theorem 2.15. *Fix the notations in the statement of Lemma 2.11: let $\mathcal{T}_s$ be a universal transport set, let $f,\gamma:\mathbb{R}^n\to\mathbb{R}$ be Borel functions and let*

$$\mu\llcorner\mathcal{T}_s = \int_{\mathsf{Q}}\mu_{\mathsf{q}}\,d\mathcal{H}^{n-1}(\mathsf{q}) = \int_{\mathsf{Q}}(f\gamma\mathcal{H}^1\llcorner r_{\mathsf{q}})\,d\mathcal{H}^{n-1}(\mathsf{q}) \qquad \nu\llcorner\mathcal{T}_s = \int_{\mathsf{Q}}\nu_{\mathsf{q}}\,d\mathcal{H}^{n-1}(\mathsf{q}).$$

Denote formally the ray $r(x)$ as $\llparenthesis a(x),b(x)\rrparenthesis$, where a, b are $\mathcal{H}^n$-a.e. uniquely defined on $\mathcal{T}_s$ and possibly at infinity and let $\mathsf{q}:\mathcal{T}_s\to\mathsf{Q}$ be the Borel multivalued projection onto the quotient. Set $d=0$ where d is multivalued.

Then, a particular solution $\rho \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^n)$ to the transport equation

$$\operatorname{div}(\mathtt{d}\,\rho) = \mu - \nu \tag{2.11}$$

is given by the transport density associated to any optimal secondary transport plan $\pi_{\mathrm{s}} \in \mathcal{O}_{\mathrm{s}}$. Considering the Borel map $\tilde{t}^{-1} : \mathcal{T}_{\mathrm{s}} \to \mathcal{T}_{\mathrm{s}}$ in Definition 2.14, this transport density can be written as

$$\rho(x) = \frac{(\mu_{\mathtt{q}(x)} - \nu_{\mathtt{q}(x)})((\!| \mathtt{a}(x), x |\!))}{\gamma(x)} \mathcal{L}^n(x) \llcorner \mathcal{T}_{\mathrm{s}} = \left(\frac{\mathbb{1}_{\mathcal{T}_{\mathrm{s}}}(x)}{\gamma(x)} \int_{(\!|\tilde{t}^{-1}(x), x|\!)} f\gamma \, d\mathcal{H}^1 \right) \mathcal{L}^n(x). \tag{2.12}$$

Proof. The basic reasoning follows Section 8 in [10].

Step 1: Construction. Consider any measure $\lambda \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n)$. Since in (2.11) there is the coefficient $\mathtt{d}$ vanishing out of $\mathcal{T}_{\mathrm{s}}$, we directly normalize λ requiring $\lambda(\mathbb{R}^n \setminus \mathcal{T}_{\mathrm{s}}) = 0$.

Equation (2.11) implies the absolute continuity of λ w.r.t. $\mu - \nu$: for any $S \in \mathcal{B}(\mathbb{R}^n)$ with $\mathcal{L}^n(S) = 0$ one has

$$\int_{S'} \nabla\varphi \cdot \mathtt{d} \, d\lambda = 0 \qquad \forall S' \in \mathcal{B}(S), \, \forall \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n);$$

however, one can partition S' into countably many sets $\{S'_j\}_j$ such that there exists φ_j such that $\nabla\varphi_j \cdot \mathtt{d} > 0$ on S'_j, contradicting the above equality.
As a consequence, one can fix the disintegration $\lambda = \int_{\mathtt{Q}} \lambda_{\mathtt{q}} \, d\mathcal{H}^{n-1}(\mathtt{q})$ of λ w.r.t. the covering $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ of $\mathcal{T}_{\mathrm{s}}$ into universal secondary transport rays.

If one applies the disintegration w.r.t. $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ to the integral form of the transport equation

$$-\int_{\mathcal{T}_{\mathrm{s}}} \nabla\varphi(x) \cdot \mathtt{d}(x) \, d\lambda(x) = \int_{\mathcal{T}_{\mathrm{s}}} \varphi(x) \, d[\mu(x) - \nu(x)] \qquad \forall \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n), \tag{2.13}$$

one can see that *if the following equality holds for $\mathcal{H}^{n-1}$-a.e. $\mathtt{q}$ in $\mathtt{Q}$,*

$$-\int_{r_{\mathtt{q}}} (\nabla\varphi \cdot \mathtt{d}) \, d\lambda_{\mathtt{q}} = \int_{r_{\mathtt{q}}} \varphi \, d[\mu_{\mathtt{q}} - \nu_{\mathtt{q}}] \qquad \forall \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n) \tag{2.14}$$

then λ must be a solution to the transport equation (2.11). This condition is equivalent to require that λ solves the transport equations

$$\operatorname{div}(\mathbb{1}_{r(Z)} \mathtt{d}\,\rho) = \mathbb{1}_{r(Z)}(\mu - \nu) \qquad \forall \text{open set } Z \in \mathcal{B}(\mathtt{Q})$$

Notice that the function $\nabla\varphi \cdot \mathtt{d}$ is the derivative of φ in the direction of $r_{\mathtt{q}}$ and moreover, if we consider the line $\ell_{\mathtt{q}}$ containing $r_{\mathtt{q}}$ being $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, all the test functions $\varphi \in C^\infty_{\mathrm{c}}(\ell_{\mathtt{q}})$ are allowed. Equation (2.14) is than equivalent to the fact that the measure $(\mu_{\mathtt{q}} - \nu_{\mathtt{q}})$ on $\ell_{\mathtt{q}}$ is the distributional derivative of $\lambda_{\mathtt{q}}$ on $\ell_{\mathtt{q}}$: therefore, since $\mu_{\mathtt{q}}$ and $\nu_{\mathtt{q}}$ have the same mass, $\lambda_{\mathtt{q}}$ is absolutely continuous w.r.t. $\mathcal{H}^1 \llcorner r_{\mathtt{q}}$ with a $\mathrm{BV}_{\mathrm{loc}}(r_{\mathtt{q}})$ density function given by

$$c^-(\mathtt{q}) + (\mu_{\mathtt{q}} - \nu_{\mathtt{q}})((\!| \mathtt{a}(x), x |\!))$$

where $\mathtt{q} \mapsto c^-(\mathtt{q})$ must satisfy $\operatorname{div}(c^-(\mathtt{q}(x)\mathtt{d}(x)\mathcal{L}^n(x)) = 0$. The constant $c^-(\mathtt{q})$ is the limit value, on $r_{\mathtt{q}}$, of the density of $\lambda_{\mathtt{q}}$ towards the initial point of the ray $r_{\mathtt{q}}$. In particular, the expression of such a λ is

$$\lambda(x) = \int_{\mathtt{q}} \left\{ \left(c^-(\mathtt{q}) + (\mu_{\mathtt{q}} - \nu_{\mathtt{q}})((\!| \mathtt{a}(x), x |\!))\right) \mathcal{H}^1(x) \llcorner r_{\mathtt{q}} \right\} d\mathcal{H}^{n-1}(\mathtt{q}).$$

It makes no difference if we choose above $(\mu_{\mathtt{q}} - \nu_{\mathtt{q}})([\![\mathtt{a}(x), x]\!])$, since $\mu_{\mathtt{q}}$ is absolutely continuous and one can have $\nu_{\mathtt{q}}(\{x\}) > 0$ only for countably many points $\{x\}$; therefore, differing on a $\mathcal{H}^1$-negligible set, the two functions identify the same measure $\lambda_{\mathtt{q}}$.

Step 2: Existence. Consider the map

$$g(x)=\frac{(\mu_{\mathsf{q}(x)}-\nu_{\mathsf{q}(x)})((\!|a(x),x|\!))}{\gamma(x)}\mathbb{1}_{\mathcal{T}_{\mathrm{s}}}(x)=\frac{\int f\gamma\,d\mathcal{H}^1\llcorner(\!|\tilde{t}^{-1}(x),x|\!)}{\gamma(x)},$$

which is pointwise unambiguously defined when x is not an endpoint of a secondary ray, therefore, as shown in Corollary 3.18 as a consequence of the disintegration in the next section, at $\mathcal{L}^n$-a.e. x.

Define the multivalued function $\ell : x\to [\![x,t(x)]\!]$, which has σ-compact graph.
By the disintegration theorem stated in Section 2.2, a disintegration of $(\mu\otimes\mathcal{L}^n)\llcorner\,\mathrm{Graph}(\ell)$ w.r.t. the $(\mu\otimes\mathcal{L}^n)$-partition given by $\{(r_{\mathsf{q}},y)\}_{\mathsf{q}\in\mathsf{Q},y\in\mathbb{R}^n}$ is provided by

$$\begin{aligned}(\mu\otimes\mathcal{L}^n(x,y))\llcorner\,\mathrm{Graph}(\ell)&=\int_{\mathsf{Q}}d\mathcal{H}^{n-1}(\mathsf{q})\Big\{f(x)\gamma(x)\,(d\mathcal{H}^1\otimes\mathcal{L}^n(x,y))\llcorner(\mathrm{Graph}(\ell\restriction_{r_{\mathsf{q}}}))\Big\}\\&=\int_{\mathsf{Q}}d\mathcal{H}^{n-1}(\mathsf{q})\int_{\mathbb{R}^n}d\mathcal{L}^n(y)\Big\{f\gamma\,d\mathcal{H}^1\llcorner(r_{\mathsf{q}}\cap\ell^{-1}(y)\Big\}.\end{aligned}$$

Therefore, we get the $(\mathcal{H}^{n-1}\otimes\mathcal{L}^n)$-measurability of

$$u:(\mathsf{q},y)\mapsto\int_{\ell^{-1}(y)}f\gamma\,d\mathcal{H}^{n-1}\llcorner\,r_{\mathsf{q}}.$$

Since g is just a rewriting of the composite map

$$x\mapsto(\mathsf{q}(x),x)\mapsto\int_{\ell^{-1}(x)}f\gamma\,d\mathcal{H}^{n-1}\llcorner\,r_{\mathsf{q}(x)}\mapsto\frac{\int_{\ell^{-1}(x)}f\gamma\,d\mathcal{H}^{n-1}\llcorner\,r_{\mathsf{q}(x)}}{\gamma(x)},$$

we proved the Lebesgue measurability of g.

Define the nonnegative measure $\rho:=g\mathcal{L}^n$. The classical Disintegration Theorem yields the disintegration

$$\rho=g\mathcal{L}^n=\int_{\mathsf{Q}}\Big\{(\mu_{\mathsf{q}(x)}-\nu_{\mathsf{q}(x)})((\!|a(x),x|\!))\mathcal{H}^1(x)\llcorner\,r_{\mathsf{q}}\Big\}\,d\mathcal{H}^n(\mathsf{q}).$$

Therefore, one can see that the nonnegative function g is locally integrable and the fact that ρ is indeed a distributional solution of (2.11) follows from Step 1.

Step 3: Identification with the transport density. We show now that the measure ρ in (2.12) is precisely the transport density relative to any optimal secondary transport plan $\pi_{\mathrm{s}}\in\mathcal{O}_{\mathrm{s}}$. Indeed,

$$\begin{aligned}\rho(A)&:=\int_{\mathbb{R}^n\times\mathbb{R}^n}\mathcal{H}^1\llcorner(A\cap[\![x,y]\!])\,d\pi_{\mathrm{s}}(x,y)\\&=\int_{\Gamma_{\mathrm{s}}}\mathcal{H}^1\llcorner(A\cap[\![x,y]\!])\,d\pi_{\mathrm{s}}(x,y)\\&=\int_{\mathsf{Q}}\Big\{\int_{r_{\mathsf{q}}\times r_{\mathsf{q}}}\mathcal{H}^1\llcorner(A\cap[\![x,y]\!])\,d\pi_{\mathsf{q}}(x,y)\Big\}\,d\mathcal{H}^{n-1}(\mathsf{q}).\end{aligned}$$

The inner integral can be rewritten as

$$\begin{aligned}&\int_{r_{\mathsf{q}}\times r_{\mathsf{q}}\times r_{\mathsf{q}}}\mathbb{1}_A(w)\mathbb{1}_{[\![x,y]\!]}(w)\,d\mathcal{H}^1(w)\otimes\pi_{\mathsf{q}}(x,y)\\&=\int_{r_{\mathsf{q}}\times r_{\mathsf{q}}\times r_{\mathsf{q}}}\mathbb{1}_A(w)\mathbb{1}_{([\![a(w),w]\!]\times[\![w,b(x)]\!])}(x,y)\,d\mathcal{H}^1(w)\otimes\pi_{\mathsf{q}}(x,y)\\&=\int_{r_{\mathsf{q}}\cap A}\pi_{\mathsf{q}}(([\![a(w),w]\!]\times[\![w,b(w)]\!]))\,d\mathcal{H}^1(w)\\&=\int_{r_{\mathsf{q}}\cap A}\big(\mu_{\mathsf{q}}-\nu_{\mathsf{q}}\big)([\![a(w),w]\!])\,d\mathcal{H}^1(w).\end{aligned}$$

Therefore, continuing from above we get

$$\begin{aligned}\rho(A) &= \int_{\mathtt{Q}} \left\{ \int_{r_{\mathtt{q}}\times r_{\mathtt{q}}} \mathcal{H}^1 \llcorner (A \cap [\![x,y]\!])\, d\pi_{\mathtt{q}}(x,y) \right\} d\mathcal{H}^{n-1}(\mathtt{q}) \\ &= \int_{\mathtt{Q}} \left\{ \int_{r_{\mathtt{q}}\cap A} (\mu_{\mathtt{q}} - \nu_{\mathtt{q}})([\![a(w), w]\!])\, d\mathcal{H}^1(w) \right\} d\mathcal{H}^{n-1}(\mathtt{q}) \\ &= \int_{\mathcal{T}_{\mathrm{s}}\cap A} \frac{(\mu_{\mathtt{q}} - \nu_{\mathtt{q}})([\![a(x), x]\!])}{\gamma(x)}\, d\mathcal{L}^n(x).\end{aligned}$$

□

Remark 2.16. We anticipate from the next section that condition (2.14) is equivalent to the requirement that the divergence formula (3.11) holds for all $\mathcal{K}$ suitable in Corollary 3.19.

Example 2.17 (Taken from [24]). Consider in $\mathbb{R}^2$ the measures $\mu = 2\mathcal{L}^2 \llcorner \mathbf{B}_1$ and $\nu = \frac{1}{2|x|^{3/2}}\mathcal{L}^2\mathcal{B}_1$, where $|\cdot|$ here denotes the Euclidean norm. A Kantorovich potential is provided by $|x|$. The transport density is $\rho = (|x|^{-\frac{1}{2}} - |x|)\mathcal{L}^2 \llcorner \mathbf{B}_1$. While vanishing towards $\partial\mathbf{B}_1$, the density of ρ blows up towards the origin. Concentrating ν at the origin, the density would be instead $\rho = -|x|^2 \llcorner \mathbf{B}_1$.

Example 2.18. Consider in $\mathbb{R}^2$ the norm $\|(x,y)\| = |x| + |y|$ and the measures $\mu = \mathcal{L}^2 \llcorner ((Q + (1,1)) \cup (Q + (-1,-1))$, $\nu = \mathcal{L}^2 \llcorner ((Q + (-1,1)) \cup (Q + (1,-1))$, where Q is the square with diameter 1. The maps translating the squares horizontally and vertically have different transport density and they can be selected choosing different strictly convex norms $|\cdot|$.

3. The Disintegration w.r.t. Secondary Transport Rays

In the present section we focus on the secondary transport set $\mathcal{T}_{\mathrm{s}}$ associated to a c_{s}-monotone set Γ_{s} lying in the $\|\cdot\|$-subdifferential of the primary potential ϕ. We set

$$\mathcal{T}_{\mathrm{s}} = \left\{ z \in [\![x,y]\!] :\ (x,y) \in \Gamma_{\mathrm{s}} \right\}, \qquad c_{\mathrm{s}}(x,y) = |y-x|\chi_{\{\phi(x)-\phi(y)=\|y-x\|\}}(x,y).$$

We want to determine the disintegration of $\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}}$ w.r.t. the covering by secondary rays $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ defined in Section 2.1. We are directly supposing, by Lemma 2.9 w.l.o.g. ,that Γ_{s} coincides with $G_{\Gamma_{\mathrm{s}}}$ in Definition 2.2, so that maximal transport rays (2.1) and transport rays (2.2) coincide.

We have recalled in Lemma 2.9 that each $r_{\mathtt{q}}$ is a convex 1-dimensional set, since the strict triangular inequality holds for the cost c_{s}, and we defined from Γ_{s} the multivalued function r which associates to $x \in \mathcal{T}_{\mathrm{s}}$ the union of those rays $r_{\mathtt{q}}$ containing x — see Definition 2.4.

We will follow basically the disintegration strategy presented in [10]. Before adapting the problem-dependent steps of the technique to this setting, we explain that strategy. We skip some technical computations, referring to the precise statements in [17].

Preliminary simplification of the setting. Since our aim is to derive a disintegration of the Lebesgue measure on $\mathcal{T}_{\mathrm{s}}$, w.l.o.g. we can directly make the following assumptions.

1: Remove an $\mathcal{L}^n$-negligible set. Cut off from $\mathcal{T}_{\mathrm{s}}$ itself a Lebesgue negligible set, in particular the Borel set of points where the Lipschitz primary potential ϕ is not differentiable.

Since whenever the primary potential ϕ is differentiable at two points belonging to a same primary ray, then ϕ is differentiable also on their convex combinations (see Lemma 3.8), this corresponds to shortening the rays $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ and neglecting some of them, but without affecting the Lebesgue measure on their union $\mathcal{T}_{\mathrm{s}}$.

2: Restrict to a compact set. Restrict the attention to each element of a countable covering of compact transport sets $\{\mathcal{T}_{\mathrm{s}}^j\}_{j\in\mathbb{N}}$ such that $\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}}^j$ increases to the original $\mathcal{L}_{\mathrm{s}}^n \llcorner \mathcal{T}_{\mathrm{s}}$. Again, each secondary ray of the transport sets constituting the partition is part of a secondary ray of the original transpor set, and we can require that two distinct secondary ray of $\mathcal{T}_{\mathrm{s}}^j$ come from distinct rays of $\mathcal{T}_{\mathrm{s}}$.

3: Notation. We are allowed to think such new compact $\mathcal{T}'_{\mathrm{s}}$ defined by the new compact set

$$\Gamma'_{\mathrm{s}} = \{(x,y) \in \mathcal{T}'_{\mathrm{s}} \times \mathcal{T}'_{\mathrm{s}} : x \overset{\mathrm{s}}{\sim} y\} = \mathrm{Graph}(r) \cap \mathcal{T}'_{\mathrm{s}} \times \mathcal{T}'_{\mathrm{s}}.$$

In the following, we directly rename Γ'_{s}, $\mathcal{T}'_{\mathrm{s}}$ and the new r', $\{r'_{\mathsf{q}' \in \mathsf{Q}'}\}$ corresponding to Γ'_{s} as Γ_{s}, $\mathcal{T}_{\mathrm{s}}$, r, $\{r_{\mathsf{q}}\}_{\mathsf{q} \in \mathsf{Q}}$.

Study on model sets. *1: Definition.* Consider first the problem of disintegrating the Lebesgue measure on a model set $\mathcal{Z}$ made of secondary rays transversal to a fixed hyperplane and intersecting it in a point in their relative interior. If we fix, up to an affine change of variables, the hyperplane $H_0 = \{x \cdot \mathrm{e} = 0\}$, then

$$\mathcal{Z} = \bigcup \Big\{ r_{\mathsf{q}} : \ 0 \in \mathrm{ri}(\pi_{\langle \mathrm{e} \rangle}(r_{\mathsf{q}})) \Big\}.$$

We call sets of this kind *sheaf sets*.
2: Parameterization providing the isomorphism. The rays constituting $\mathcal{Z}$ can be parameterized by their intersection with H_0, denote this set by $Z_0 := \mathcal{Z} \cap H_0$. More generally, each point $x \in \mathcal{Z}$ is determined by its projection $t = x \cdot \mathrm{e}$ on e and by any point $z \in Z_0$ where any secondary ray through x intersects H_0; if x is not an endpoint, then z is uniquely determined.
As a consequence, the set $\mathcal{Z}$ is the image of the map $(t,z) \mapsto \sigma(t,z)$ which moves the point z within its ray up to the point with projection t on e,

$$\sigma(t,z) = z + t\,d(z) = r(z) \cap H_t \qquad \text{where } H_t := \{x \cdot \mathrm{e} = t\},$$

and which is defined on the compact subset of $\mathbb{R}^n$

$$\mathsf{Z} = \Big\{ (t,z) : \ z \in Z_0, t \in \pi_{\langle \mathrm{e} \rangle}(r(z)) \Big\}.$$

As a consequence of the area estimate below, we will derive as an essential step toward the disintegration the fact that σ provides an isomorphism between $(\mathsf{Z}, \mathscr{L}(\mathsf{Z}))$ and $(\mathcal{Z}, \mathscr{L}(\mathcal{Z}))$.
3: Area estimate. Suppose one can control the push forward of the Hausdorff measure on the sections orthogonal to e with the estimates

$$\mathcal{H}^{n-1}(\sigma(t,S)) \leq \left(\frac{t-h^-}{s-h^-}\right)^{n-1} \mathcal{H}^{n-1}(\sigma(s,S)) \qquad \text{for } h^- < s \leq t \leq h^+\text{: } [h^-\mathrm{e}, h^+\mathrm{e}] + S \subset \mathsf{Z} \tag{3.1a}$$

$$\mathcal{H}^{n-1}(\sigma(s,S)) \leq \left(\frac{h^+-t}{h^+-s}\right)^{n-1} \mathcal{H}^{n-1}(\sigma(t,S)) \qquad \text{for } h^- \leq s \leq t < h^+\text{: } [h^-\mathrm{e}, h^+\mathrm{e}] + S \subset \mathsf{Z}. \tag{3.1b}$$

Notice that these, with equality, are the estimates one would have if the secondary rays were rays of a cone with center on H_{h^-} for (3.1a) and on H_{h^+} for (3.1b); this is the key of their derivation, that we discuss later.
4: Density function. Then, considering either s or t equal to 0, (3.1a) or (3.1b) depending on whether t is positive or negative ensures that the measure $\mathcal{H}^n \llcorner \sigma(t, Z_0)$ is absolutely continuous w.r.t. $\sigma^t_\sharp(\mathcal{H}^n \llcorner Z_0)$: let

$$\beta(t,z) : \mathsf{Z} \to \mathbb{R}^+$$

be the function which at each time t gives the Radon-Nicodym derivative $\beta(t,\cdot)$ of $\mathcal{H}^n \llcorner \sigma(t,Z_0)$ w.r.t. $\sigma^t_\sharp(\mathcal{H}^n \llcorner Z_0)$. Considering also the other estimate in (3.1) one finds that $\beta(t,z)$ is strictly positive and admits a representative which is Lipschitz continuous w.r.t. t, measurable in (t,z) and, together, one finds also estimates on β and $\partial_t\beta/\beta$, which are locally integrable (see Corollary 2.19 in [17]).
5: Disintegration. As a consequence of the above absolute continuity estimate on the sections, the measure $\mathcal{L}^n \llcorner \mathcal{Z}$ is absolutely continuous w.r.t. the push forward measure $\sigma_\sharp(\mathcal{L}^n \llcorner \mathsf{Z})$ and the Radon-Nicodym derivative is precisely $\beta(t,z)$. Indeed, extending β as 0 on $(\langle \mathrm{e} \rangle + Z_0) \setminus \mathsf{Z}$, for each function $\varphi : \mathcal{Z} \to \mathbb{R}$

either positive or integrable one has

$$\begin{aligned}\int_{\mathcal{Z}} \varphi(x)\, d\mathcal{H}^n(x) &= \int_{-\infty}^{+\infty} \left\{ \int_{\sigma(t,Z_0)} \varphi(y)\, d\mathcal{H}^{n-1}(y) \right\} dt && \text{by Fubini, slicing with } \{H_t\}_{t\in\mathbb{R}} \\ &= \int_{-\infty}^{+\infty} \left\{ \int_{Z_0} \varphi(\sigma(t,z))\beta(t,z)\, d\mathcal{H}^{n-1}(z) \right\} dt && \text{by definition of } \beta.\end{aligned}$$

Since $\beta(t,x) > 0$ whenever $\sigma(t,z) \in \mathcal{Z}$, if φ is the indicator function of a $\mathcal{L}^n$-negligible set $N \subset \mathcal{Z}$ the equality above vanishes and tells that $\mathcal{H}^{n-1}(\sigma(t, N \cap H_0)) = 0$ for $\mathcal{H}^1$-a.e. t.

Therefore, σ carries Lebesgue measurable functions on $\mathcal{Z}$ back to Lebesgue measurable functions on Z: *σ is an isomorphism between the measure spaces $(\mathsf{Z}, \mathscr{L}(\mathsf{Z}))$ and $(\mathcal{Z}, \mathscr{L}(\mathcal{Z}))$.*

The last integral is then the integral of the function $(t,z) \mapsto \varphi(\sigma(t,z))\beta(t,z)$ on the product space:

$$\int_{\langle \mathrm{e}\rangle + Z_0} \varphi(\sigma(t,z))\beta(t,z)\, dt \otimes d\mathcal{H}^{n-1}(z).$$

One can finally apply Fubini-Tonelli theorem in order to get the disintegration of $\mathcal{L}^n \llcorner \mathcal{Z}$ w.r.t. the covering defined by the membership to secondary rays: defining out of the endpoints the density function $\gamma(x) = \beta(\sigma^{-1}(x))\mathbb{1}_{\mathcal{Z}}(x)$

$$\begin{aligned}\int_{\mathcal{Z}} \varphi(x)\, d\mathcal{H}^n(x) &= \int_{\mathsf{Z}} \varphi(\sigma(t,z))\beta(t,z)\, dt \otimes d\mathcal{H}^{n-1}(z) \\ &= \int_{Z_0} \left\{ \int_{r(z)} \varphi(x)\beta(x\cdot \mathrm{e}, z)\, d\mathcal{H}^1(x) \right\} d\mathcal{H}^{n-1}(z) \\ &= \int_{Z_0} \left\{ \int_{r(z)} \varphi(x)\gamma(x)\, d\mathcal{H}^1(x) \right\} d\mathcal{H}^{n-1}(z).\end{aligned}$$

Global result. *1: Disintegration.* Come back now to the original set $\mathcal{T}_{\mathrm{s}}$, which is the increasing union of the compacts $\{\mathcal{T}_{\mathrm{s}}^m\}_{m\in\mathbb{N}}$, up to an $\mathcal{L}^n$-negligible set that we suppose w.l.o.g. to be empty.

One covers $\mathcal{T}_{\mathrm{s}}$ with a countable family of sheaf sets $\{\mathcal{Z}_\ell\}_{\ell\in\mathbb{N}}$ whose elements possibly overlap on a set having $\mathcal{H}^1$-negligible intersection with every secondary ray, i.e. we require

$$\mathcal{H}^1 \llcorner r_{\mathtt{q}}(N) = 0 \qquad \text{for all } \mathtt{q} \in \mathtt{Q} \text{ and all } i \neq j$$

for $N = \cup_{i\neq j} \mathcal{Z}_i \cap \mathcal{Z}_j$ (the construction is similar to Lemma 2.6 in [17]). Since we have an increasing sequence of transport sets, we can choose the covering in such a way that $\{\mathcal{Z}_\ell^m := \mathcal{Z}_\ell \cap \mathcal{T}_{\mathrm{s}}^m\}_{m\in\mathbb{N}}$ is a sequence of compact sheaf sets increasing with m. Clearly, $\{\mathcal{Z}_\ell^m\}_{\ell\in\mathbb{N}}$ covers $\mathcal{T}_{\mathrm{s}}^m$.
By construction then there exists an hyperplane H_ℓ orthogonal to e_ℓ and transversal to the relative interior of secondary rays in $\mathcal{Z}_\ell^m$, for all m. Let $Z_\ell := H_\ell \cap \mathcal{Z}_\ell$. By Lemma 2.9 the family $\{Z_\ell\}_{\ell\in\mathbb{N}}$ is disjoint family in bijective correspondence with $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$. We identify then $\mathtt{Q}$ with $\cup_\ell Z_\ell$.

We show now that, *if one achieves the disintegration on each $\mathcal{Z}_\ell^m$*, one obtains then a disintegration of $\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}}$. Indeed, if γ_ℓ^m is the density function relative to $\mathcal{Z}_\ell^m$ defined above, and extended as 0 when $x \notin \mathcal{Z}_\ell$, by subadditivity of the measures

$$\begin{aligned}\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}}^m \leq \sum_{\ell\in\mathbb{N}} \mathcal{L}^n \llcorner \mathcal{Z}_\ell^m &= \sum_{\ell\in\mathbb{N}} \int_{Z_\ell} \left(\gamma_\ell^m\, \mathcal{H}^1 \llcorner r_{\mathtt{q}} \right) d\mathcal{H}^{n-1}(\mathtt{q}) \\ &= \int_{\mathtt{Q} = \cup_{\ell\in\mathbb{N}} Z_\ell} \left(\sum_{\ell\in\mathbb{N}} \gamma_\ell^m\, \mathcal{H}^1 \llcorner r_{\mathtt{q}} \right) d\mathcal{H}^{n-1}(\mathtt{q}).\end{aligned} \tag{3.2}$$

However, being $\gamma_\ell^m \mathcal{H}^1 \llcorner r_{\mathtt{q}}(N) = 0$ for all ℓ, by the disintegration on $\mathcal{Z}_\ell^m$ one has $\mathcal{L}^n \llcorner \mathcal{Z}_\ell^m(N) = 0$ for all ℓ and therefore the inequality in (3.2) is indeed an equality.
By the monotone convergence theorem, already use to exchange integral and series in the last step, we

can take the limit as m increases obtaining

$$\mathcal{L}^n \llcorner \mathcal{T}_{\mathrm{s}}(x) = \int_{\mathtt{Q}=\cup_{\ell\in\mathbb{N}} Z_\ell} \Big(\gamma(x)\, \mathcal{H}^1 \llcorner r_{\mathtt{q}}(x) \Big)\, d\mathcal{H}^{n-1}(\mathtt{q}),$$

where $\gamma = \lim_{m\to\infty} \sum_\ell \gamma_\ell^m = \sup_{m,\ell} \gamma_\ell^m$. This proves the disintegration in secondary transport rays.

What is left is therefore to exhibit a covering of $\mathcal{T}_{\mathrm{s}}$, up to an $\mathcal{L}^n$-negligible set, into countably many sheaf sets $\{\mathcal{Z}_\ell\}_{\ell\in\mathbb{N}}$ containing distinct secondary rays and such that (3.1) holds, and this is the problem-dependent part of the technique. This is what we outline here and we develop in the present section.

Actual reduction to the model case. *1: Derivation of the estimate* (3.1). This is a difficulty of this problem, since the natural way from [10] is to prove (3.1) on any sheaf set by a Hopf-Lax formula involving the Kantorovich potential for the secondary transport problem. However, this point requires care, being the secondary cost also infinite valued.
2: Construction of a partition in model sets. We avoid the problem choosing a partition into special sheaf sets $\mathcal{W}$ where we are able to exhibit a secondary Kantorovich potential. This relies both on a partition of $\mathcal{T}$ into invariant sets and the requirement the within each invariant set $\Gamma_{\mathrm{s}} \cap \mathcal{W} \times \mathbb{R}^n$ two points are connected by a cycle with finite secondary cost, as already considered in [9].

Outline of the section. The plan of the present section is the following:

- Subsection 3.1: we partition $\mathcal{T}_{\mathrm{s}}$ into sheaf sets $\{\mathcal{W}_\ell\}_\ell$, and a residual set $\mathcal{N}$.
- Subsection 3.3: we partition each model set $\mathcal{W}$ into invariant sets for the transport.
- Subsection 3.4: we construct a secondary Kantorovich potential on each model set $\mathcal{W}$.
- Subsection 3.5: we disintegrate $\mathcal{L}^n \llcorner \mathcal{W}$ w.r.t. the membership to secondary transport rays.
- Subsection 3.2: we prove that the residual set $\mathcal{N}$ is $\mathcal{L}^n$-negligible.

3.1. **Partition of the secondary transport set into model sets.** In this section we define sets where we will construct a secondary potential. We give then a partition of the secondary transport set $\mathcal{T}_{\mathrm{s}}$ into model sets, up to a residual set $\mathcal{N}$ which will turn out to be $\mathcal{L}^n$-negligible (see Subsection 3.2).

Definition 3.1 (Model Set $\mathcal{W}^k$). Let V^k be a k-dimensional vector space of $\mathbb{R}^n$, $q \in \mathbb{R}^n$ and $\rho > 0$. Define the model set

$$\mathcal{W}^k_{q+B_\rho,V^k} = \bigcup_x r(x)$$

where x varies among the points in $\mathcal{T}_{\mathrm{s}}$ with the following properties:

- $r(x) \cap (q + B_\rho) \neq \emptyset$;
- $\dim(\operatorname{conv} \mathcal{R}(x)) = k$;
- $\pi_{V^k}(q + B_\rho) \subset \pi_{V^k}(\mathcal{R}(x))$.

Definition 3.2. Let $\mathcal{N}$ be the set of points $x \in \mathcal{T}_{\mathrm{s}}$ whose secondary transport ray $r(x)$ belongs to the relative border of the convex envelope of $\mathcal{R}(x)$.

We recall from Section 2 that $\overset{\mathrm{s}}{\sim}$ is indeed an equivalence relation on $\mathcal{T}_{\mathrm{s}}$ out of the endpoints and the Borel regularity of $\mathcal{T}$, $\mathcal{T}_{\mathrm{s}}$ and of the maps $r(x)$, $d(x)$, related to the secondary transport set, and $\mathcal{R}(x)$, $\mathcal{D}(x)$, of primary rays and directions.

Lemma 3.3. *The functions r, $\mathcal{R}$, $\mathcal{D}$ are Borel multivalued functions. The functions d are Borel. The primary and secondary transport sets $\mathcal{T}$, $\mathcal{T}_{\mathrm{s}}$ and the set of endpoints $\mathcal{E}^-$, $\mathcal{E}^+$ are Borel. The function associating the initial and terminal points of the relative rays are also Bore.*

Proof. One can see that the graphs of $\mathcal{R}$ and $\mathcal{D}$ are σ-compact exactly as in Lemma 2.2 of [17] (taken from [10]). A similar argument holds for r, since Γ_{s} is σ-compact, and for d. For a, b one can therefore repeat the construction for Lemma 2.9 in [17]. □

Lemma 3.4. *Each model set is Borel. Moreover, there is a partition of $\mathcal{T}_{\mathrm{s}} \setminus \mathcal{N}$ into countably many model sets $\{\mathcal{W}^k_{q+B_\rho,V^k}\}_{q,\rho,V^k}$.*

Proof. One can identify k regions $\mathcal{T}^k$ in $\mathcal{T}$, according to the dimension of the convex envelope of $\mathcal{R}(x)$. It is not difficult to see that the family of points x such that the orthogonal projection of $\mathrm{conv}(\mathcal{R}(x))$ on a fixed k-plane contains a ball is closed. Moreover, if $\mathrm{conv}(\mathcal{R}(x))$ is k-dimensional, and $\{V_\ell\}_\ell$ is a dense sequence in $\mathbf{G}(k,n)$, then the projection of $\mathrm{conv}(\mathcal{R}(x))$ on some $\{V_\ell\}_\ell$ must contain a ball. Therefore, one can see that each one of these regions is Borel.
The first condition defining the model set $\mathcal{W}^k_{q+B_\rho,V}$ correspond in selecting those x which belong to $\mathscr{r}^{-1}(q+B_\rho)$: this set is Borel, since so is $\mathscr{r}$ (Lemma 3.3).
The third condition instead selects those x in the intersection of $\mathcal{R}^{-1}(\pi_V^{-1}(q_k))$ when q_k varies in a sequence dense in $q+\partial B_\rho$: this set is Borel, since so is $\mathcal{R}$ (Lemma 3.3).
This proves that each model set is Borel.

Consider now a sequence $\{q_h\}_h$ dense in $\mathbb{R}^n$, $\{\rho_i\}_i$ dense in $\{\rho>0\}$ and $\{V^k_j\}_j$ dense in $G(k,n)$. Then the sequence $\{\mathcal{W}^k_{q_h+B_{\rho_i},V^k_j}\}_{hijk}$ is a countable covering of $\mathcal{T}_{\mathrm{s}}\setminus\mathcal{N}$. Indeed, suppose the relative interior of a secondary transport ray $\mathscr{r}(x)$ belongs to the relative interior of $\mathrm{conv}(\mathcal{R}(x))$, which we assume to be k-dimensional; let V^k_j be such that $\mathcal{L}^k(\pi_{V^k_j}(\mathrm{conv}(\mathcal{R}(x))))>0$. There is a ball $q_h+B_{\rho_i}$ of $\mathbb{R}^n$ containing x and whose intersection with $\mathrm{conv}(\mathcal{R}(x))$ lies in the relative interior of $\mathrm{conv}(\mathcal{R}(x))$ itself: this implies that $x\in\mathcal{W}_{q_h+B_{\rho_i},V^k_j}$.
One can finally refine this covering and extract a countable partition, in a standard way. □

3.2. **Negligibility of border points.** In this section we show by a density argument that the set $\mathcal{N}$ left apart by the partition of Subsection 3.1 is $\mathcal{L}^n$-negligible.
The basic idea is that when moving points from $\mathcal{N}$ to suitable primary direction, one falls in the complementary of $\mathcal{N}$, because of how $\mathcal{N}$ is defined. Therefore, showing an upper bound for those point moved in the complementary of $\mathcal{N}$, one ends in finding that $\mathcal{N}$ itself must be negligible.

In order to find those directions, let us focus on the structure of $\mathcal{N}\setminus\mathcal{E}$; indeed, proving the disintegration theorem we will show that $\mathcal{H}^n(\mathcal{E})=0$. Since at each point $x\in\mathcal{N}\setminus\mathcal{E}$ there exists a primary ray going through x, the convexity of the norm implies that whenever $\dim(\mathrm{conv}\,\mathcal{R}(x))=k$ then there is a convex k-dimensional subset of $\mathcal{R}(x)$ itself which contains x in the relative border. Notice that we are left with the case $k>1$.

As a consequence, arguing as in Lemma 3.4 one can cover $\mathcal{N}$ by countably many Borel subsets $\mathcal{A}^k_{q+B_\rho,V^k}$ where

- $\dim(\mathrm{conv}\,\mathcal{R}(x))=k$
- $\pi_{V^k}(x+q+B_\rho)\subset\pi_{V^k}(\mathcal{R}(x))$
- $\inf_{d\in\mathcal{D}(x)}\|\pi_{V^k}(d)\|\geq 1/\sqrt{2}$

with parameters $q\in\mathbb{R}^n$, $\rho>0$, $V^k\in\mathbf{G}(k,n)$ varying in a dense, countable family.
Therefore, by subadditivity of the measures and performing an affine change of variables, the thesis reduces w.l.o.g. to show that $\mathcal{L}^n(\mathcal{A})=0$ for the Borel set $\mathcal{A}$ of those $x\in\mathcal{N}$ satisfying

$$\dim(\mathrm{conv}\,\mathcal{R}(x))=k \tag{3.3a}$$

$$\pi_V(x+2\mathrm{e}+\mathbf{B}_1)\subset\pi_V(\mathcal{R}(x)) \tag{3.3b}$$

$$\inf_{d\in\mathcal{D}(x)}\|\pi_V(d)\|\geq 1/\sqrt{2} \tag{3.3c}$$

Lemma 3.5. *The set $\mathcal{A}$ is Lebesgue negligible.*

Proof. Let us assume $\mathcal{L}^n(\mathcal{A})>0$, contradicting the thesis, and consider a Lebesgue point of $\mathcal{A}$, suppose w.l.o.g. the origin.

Fix any $\varepsilon > 0$ small enough. Since 0 is a density point, for every $0 < r < \bar{r}(\varepsilon) < 1$ there exists a set $T \subset [0, r\mathrm{e}]$ with $\mathcal{H}^1(T) > (1-\varepsilon)r$ and such that

$$\mathcal{H}^{n-1}\big(\mathcal{A} \cap \pi^{-1}_{\langle \mathrm{e} \rangle}(\mathrm{t}) \cap [0,r]^n\big) \geq (1-\varepsilon)r^{n-1} \qquad \text{for all } \mathrm{t} \in T. \tag{3.4}$$

One can then choose two points $\mathrm{t} \in T$, $\mathrm{s} := \mathrm{t} + \lambda r\mathrm{e} \in T$ with $0 \leq \lambda \leq \varepsilon$.

Define the possibly multivalued map

$$x \mapsto \tau^{h\mathrm{e}}(x) := x + h(\langle \mathcal{D}(x) \rangle \cap \pi_V^{-1}(\mathrm{e}))$$

which moves each point $x \in \mathcal{A}$ along primary transport rays having projects on V parallel to e.
By condition (3.3b), for all $0 \leq h \leq 3$ and $x \in \mathcal{A}$ this map is well defined and moreover $\tau^{h\mathrm{e}}(x)$ belongs to the relative interior of a k-dimensional convex subset of $\mathcal{R}(x)$, which implies, as $\dim(\mathrm{conv}(\mathcal{R}(x)) = k$ by (3.3a), that $\tau^{h\mathrm{e}}(x)$ belongs to the relative interior of $\mathrm{conv}(\mathcal{R}(\tau^{h\mathrm{e}}(x)))$ itself.
Then, by Definition 3.2 of $\mathcal{N}$ for all $0 < h \leq 3$ the set $\tau^{h\mathrm{e}}(\mathcal{A})$ is in the complementary of $\mathcal{N}$.

Moreover, for all $0 \leq h \leq 3$ we now prove the push forward estimate

$$\mathcal{H}^{n-1}\big(\tau^{h\mathrm{e}}(S)\big) \geq \left(\frac{3-h}{3}\right)^{n-k} \mathcal{H}^{n-1}(S) \qquad \forall S \subset \mathcal{A} \cap \pi^{-1}_{\langle \mathrm{e} \rangle}(\mathrm{t}). \tag{3.5}$$

Slicing with the $(n-k+1)$-planes parallel to $\pi_V^{-1}(\langle \mathrm{e} \rangle)$, by Fubini theorem it is enough to prove the inequality for $S \subset \mathcal{A} \cap \pi_V^{-1}(\mathrm{t})$ and for the outer measure $\mathcal{H}^{n-k}$.
Choose now dense sequence of points $\{b_j\}_{j \in \mathbb{N}} \subset \tau^{3\mathrm{e}}(S)$ and consider the cones defining

$$\phi_i(x) = \min_{j=1,\dots,i} \Big\{ \phi(b_j) + \|x - b_j\| \Big\};$$

they establish a correspondence between $x \in S$ and those b_j such that

$$\phi_i(x) = \phi(b_j) + \|x - b_j\|.$$

Where more $\{b_j\}$ correspond to an x, associate to x the first $y_{\bar{i}}$ in our ordering $\{y_i\}$ and neglect the rays from the other y_i different from $y_{\bar{i}}$. Define the possibly multivalued function $\mathscr{D}_i$ associating to each $x \in S$ the directions, normalized by the requirement that they have projection on V equal to e, towards the relative b_j. Being the union of finitely many cone directions whose overlapping has been removed, and therefore are disjoint, one has the estimate on the approximation $\mathcal{H}^{n-k}\big(\tau^{h\mathscr{D}_i}(S)\big) \geq \big(\frac{3-h}{3}\big)^{n-k}\mathcal{H}^{n-1}(S)$, and by the u.s.c. of the Haudorff measure on compact sets one can pass to the limit as in Lemma 3.16, as the limit of $\mathscr{D}_i(S)$ is contained in $\mathcal{D}(S)$.

In particular, (3.5) implies that

$$\mathcal{H}^{n-1}\bigg(\tau^{\lambda r\mathrm{e}}(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0,r]^n\bigg) \geq \left(\frac{3-\lambda}{3}\right)^{n-k} \mathcal{H}^{n-1}\bigg(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0,r]^n\bigg). \tag{3.6}$$

Furthermore, condition (3.3c) implies that $\|x + 2\varepsilon r\mathrm{e} - \tau^{\mathrm{t}+2\varepsilon r\mathrm{e}}(x)\| \leq \lambda r$ for each $x \in \mathcal{A}$. Moving points from $\pi_V^{-1}(\mathrm{t}) \cap [0,r]^n$ by means of the map $\tau^{\mathrm{t}+\lambda r\mathrm{e}}$, they can therefore reach only the square $\pi_V^{-1}(\mathrm{s}) \cap [-\lambda r, (1+\lambda)r]^n$. Notice that for ε small, since our proof is needed for $n \geq 3$ and $k \geq 1$,

$$\mathcal{H}^{n-k}([-\lambda r, (1+\lambda)r]^n) \setminus [0,r]^n) = (1+2\lambda)^{n-k} r^{n-k} - r^{n-k} \leq 2(n-k)\lambda r^{n-k} + o(\lambda) < n2^n \varepsilon r^{n-k}.$$

As a consequence, the portion which exceeds $\pi_V^{-1}(\mathrm{s}) \cap [0,r]^n$ can be estimated as follows:

$$\begin{aligned} \mathcal{H}^{n-k}\big(\tau^{\lambda r\mathrm{e}}\big(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0,r]^n\big) \cap [0,r]^n\big) &\geq \mathcal{H}^{n-k}\big(\tau^{\lambda r\mathrm{e}}\big(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0,r]^n\big)\big) - n2^n\varepsilon r^{n-k} \\ &\overset{(3.6)}{\geq} \left(\frac{3-\lambda}{3}\right)^{n-k} \mathcal{H}^{n-1}\bigg(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0,r]^n\bigg) - n2^n\varepsilon r^{n-k}. \end{aligned}$$

As the argument of the l.h.s. is in the complementary of $\mathcal{A}$, being in $\tau^{\lambda r\mathrm{e}}(\mathcal{A})$, the last inequality shows the impossibility that both t and $\mathrm{s} = \mathrm{t} + \lambda r\mathrm{e}$ belong to T, providing the absurd. $\square$

3.3. **Partition of the primary transport set into invariant sets.** In this subsection we focus on some properties of the primary transport set $\mathcal{T}$, neglecting the singular points $\mathcal{S}$ of the primary potential ϕ. We partition $\mathcal{T} \setminus \mathcal{S}$ into invariant sets for the transport, meaning that every transport plan moves the mass from any point x to other points which must belong to the same class as x, if not to $\mathcal{S}$. With a strictly convex norm, this would be the familiar partition in transport rays.

Lemma 3.6. *The following relation holds in $\mathcal{T} \setminus \mathcal{S}$: $-\nabla\phi(x) \in \bigcap_{d\in\mathcal{D}(x)} \delta^*(d)$.*

Proof. By assumption, at each $x \in \mathcal{T} \setminus \mathcal{S}$ there is at least a direction $d \in \mathcal{D}(x)$ where ϕ decreases linearly. Therefore, the derivative of ϕ along d must be in $-\delta(d)$: from the Lipschitz condition, for small $t > 0$

$$\forall \ell : \ \|\ell\| = 1 \qquad \phi(x) - \phi(x + t\ell) \le t + o(t) \qquad \Longleftrightarrow \qquad -\nabla\phi(x) \cdot \ell \le 1,$$

and moreover

$$\phi(x) - \phi(x + td) = t + o(t) \qquad \Longleftrightarrow \qquad -\nabla\phi(x) \cdot d = 1.$$

By definition this means that $-\nabla\phi(x) \in \delta^*(d)$.
In the same way, one can see that $-\partial^+\phi(x) \subseteq \delta^*(d)$ if d is an outgoing direction from x, while $-\partial^-\phi(y) \subseteq \delta^*(d)$ if d is an incoming direction to y. □

Corollary 3.7. *Associate at each point of $\mathcal{T} \setminus \mathcal{S}$ the face $\mathcal{F}(x) = \delta(-\nabla\phi(x))$. Then $\mathcal{D}(x) \subset \mathcal{F}(x)$.*

In general, $\mathcal{D}(x)$ is smaller than $\mathcal{F}(x)$. Moreover, $\bigcap_{d\in\mathcal{D}(x)} \delta^*(d)$ in general is not single valued. Nevertheless, given two points of differentiability on a same ray, the gradient of ϕ must coincide.

Lemma 3.8. *Consider $x, y \in \mathcal{T}$ such that $\phi(x) - \phi(y) = \|y - x\|$. Then*

$$\partial^-\phi(y) \subseteq \partial^-\phi(x) \qquad \text{and} \qquad \partial^+\phi(y) \supseteq \partial^+\phi(x).$$

In particular, if $x, y \in \mathcal{T} \setminus \mathcal{S}$ then $\nabla\phi(x) = \nabla\phi(y)$ and the whole segment $[\![x, y]\!]$ is contained in $\mathcal{T} \setminus \mathcal{S}$.

Proof. Denote with v any direction with $\|v\| = 1$. Then, for all $\ell^* \in \partial^-\phi(y)$ and small $t > 0$

$$\begin{aligned} o(t) &\ge \phi(y + tv) - \phi(y) - t\ell^* \cdot v && \text{since } \ell^* \in \partial^-\phi(y) \\ &\ge \phi(x + tv) - \|x - y\| - \phi(y) - t\ell^* \cdot v && \text{1-Lipschitz condition} \\ &= \phi(x + tv) - \phi(x) - t\ell^* \cdot v && \text{since } y \in \mathcal{R}(x) \end{aligned}$$

By the arbitrariness of v, this means exactly that $\ell^* \in \partial^-\phi(x)$. The inverse inclusion for $\partial^+\phi$ is similar. Therefore, if both $\partial^-\phi(x) = \partial^+\phi(x)$ and $\partial^-\phi(y) = \partial^+\phi(x)$, they must coincide. Finally, for every $w \in (\!|x, y|\!)$ the inclusions $\partial^-\phi(y) \subseteq \partial^-\phi(w) \subseteq \partial^-\phi(x)$ and $\partial^+\phi(x) \subseteq \partial^+\phi(w) \subseteq \partial^+\phi(y)$ show the differentiability at w. □

Consider the following partition of the transport set $\mathcal{T} \setminus \mathcal{S}$. Two points x, y are equivalent, $x \sim y$, if

$$\Big\{ y \in x + \langle \mathcal{F}(x) \rangle, \qquad \phi(y) = \phi(x) + \nabla\phi(x) \cdot (y - x), \qquad \nabla\phi(y) = \nabla\phi(x) \Big\} \tag{3.7}$$

where $\mathcal{F}(x)$ denotes the face $\delta(-\nabla\phi(x))$ and $\langle \cdot \rangle$ denotes the linear span.

Lemma 3.9. *Equation 3.7 defines a partition of $\mathcal{T} \setminus \mathcal{S}$.*

Proof. Clearly $x \sim x$. Suppose $x \sim y$. Then $\nabla\phi(y) = \nabla\phi(x)$, therefore $\mathcal{F}(y) = \delta(-\nabla\phi(y)) = \delta(-\nabla\phi(x)) = \mathcal{F}(x)$. As a consequence,

$$x \in y - \langle \mathcal{F}(y) \rangle = y + \langle \mathcal{F}(y) \rangle, \qquad \phi(x) = \phi(y) - \nabla\phi(x) \cdot (y - x) = \phi(y) + \nabla\phi(x) \cdot (x - y).$$

Therefore $y \sim x$. Consider now $x \sim y$, $y \sim z$. Then, as above one has $\nabla\phi(x) = \nabla\phi(y) = \nabla\phi(z)$, thus $\mathcal{F}(x) = \mathcal{F}(y) = \mathcal{F}(z)$ and, by linearity,

$$z \in y + \langle F(y) \rangle = x + \langle F(x) \rangle, \qquad \phi(z) = \phi(y) + \nabla\phi(y) \cdot (z - y) = \phi(x) + \nabla\phi(x) \cdot (z - x).$$

Therefore $x \sim z$. □

The quotient space can be parameterized by a subset of $\partial B^* \times \mathbb{R} \times \mathtt{A}(N)$, where $\mathtt{A}(N)$ denotes the family of affine subspaces of $\mathbb{R}^n$; the distance between two affine subspaces is the distance of the relative orthogonal projection onto them, as linear operators. The quotient map is given by

$$p: \ x \quad \longrightarrow \quad \ell = -\nabla\phi(x), \ \alpha = \phi(x) + \ell \cdot x, \ A = x + \langle \delta\big(\ell\big) \rangle.$$

In particular, it is a subset of a Polish space.

Lemma 3.10. *The graph of the projection p onto the quotient is Borel measurable.*

Proof. Consider the sub-differential of the convex function ϕ. In particular, it is an upper semicontinuous function (Theorem 1.1, [3]). Therefore, the counterimage of the σ-compact set of ℓ such that $\dim \delta(\ell) = k$ is a σ-compact set, for $k = 1, \dots, N$. Remove the set $\mathcal{S}$ of points where the Lipschitz potential ϕ is not differentiable, $\mathcal{L}^n$-negligible and Borel, and consider the remaining part of one of these sets where $\dim \mathcal{F}(x)$ is constant. Here, the sub-differential is exactly the differential.

Consider a sequence $z_n \to z$, with z_n, z belonging to it. Then $-\ell_n := \nabla\phi(z_n) \to \nabla\phi(z) =: -\ell$, and therefore also $\alpha_n := \phi(z_n) - \ell \cdot z_n \to \phi(z) - \ell \cdot z =: \alpha$. Finally, by upper semicontinuity, $\langle\delta(\ell)\rangle \subset \lim_n \langle \delta(\ell_n)\rangle$ — but equality holds since we fixed a set where they have the same dimension. Therefore, $A(z_n) := z_n + \langle\delta(\ell_n)\rangle \to z + \langle\delta(\ell)\rangle =: A(z)$. This proves the continuity of p on each of these sets. $\square$

Lemma 3.11. *Each primary ray lies in at most one equivalence class.*

Proof. The classes partition $\mathcal{T} \setminus \mathcal{S}$. Suppose than there is a ray from x to y, with $x, y \in \mathcal{T} \setminus \mathcal{S}$: by definition $\phi(x) = \phi(y) + \|y - x\|$. In Lemma 3.8 we have shown that $\nabla\phi(x) = \nabla\phi(y)$. Therefore, by definition $\mathcal{F}(x) = \mathcal{F}(y)$. Since $y - x \in \langle\mathcal{F}(x)\rangle$ (Corollary 3.7), this shows that $x \sim y$, being $\|y - x\| = \nabla\phi(x) \cdot (y - x)$. $\square$

Assume that both the measures are absolutely continuous. Then Lemma 3.11 ensures that we have a partition in invariant sets. When ν is singular, instead, some mass can be transported to $\mathcal{S}$. We chose to remove it for the following reason: consider a point x of non differentiability which belongs to two primary rays. In general, each of these rays can have points, different from x where ϕ differentiable. However, in general the points of the two rays do not belong to the same equivalence class, and therefore x would belong to two classes.

Remark 3.12. The primary potential ϕ is affine on each equivalence class. Indeed, consider $x \sim y$, with projection $\mathtt{q} = (\ell, \alpha, A)$. If A is k-dimensional, then $y = x + t_1 d_1 + \dots + t_k d_k$ with $d_i \in \delta(\ell)$. The equality $\phi(y) + \ell \cdot y = \alpha = \phi(x) + \ell \cdot x$ implies

$$\phi(x) = \phi(y) + \ell \cdot \big(y - x\big) = \phi(y) + t_1 + \dots + t_k. \tag{3.8}$$

For $x, y \in \mathcal{T} \setminus \mathcal{S}$, one has

$$\phi(x) = \phi(y) + \|y - x\| \qquad \Longleftrightarrow \qquad x \sim y, \quad y - x \in \mathbb{R}^+\mathcal{F}(x), \tag{3.9}$$

where $\mathcal{F}(x) = \delta(\ell)$, if $p(x) = (\ell, \alpha, A) = \mathtt{q}$.

Implication $\Leftarrow$ is a direct consequence of (3.8), let us show converse. Since ϕ is differentiable at both x, y by assumption, the existence of a primary ray from x to y implies $\nabla\phi(x) = \nabla\phi(y) = -\ell$ (Lemma 3.8), $y - x \in \delta(\ell)$ (Corollary 3.7) and $x \sim y$ (Lemma 3.11).

3.4. **Construction of a local secondary potential.** In this subsection we give a function ϕ_{s} such that $\Gamma_{\mathrm{s}} \cap \mathcal{W}^2$ is in the c_{s}-subdifferential of ϕ_{s}, where $\mathcal{W}$ is a model set as in Definition 3.1,

$$c_{\mathrm{s}}(x, y) = \begin{cases} |y - x| & \text{if } \phi(x) - \phi(y) = \|y - x\| \\ +\infty & \text{if } \phi(x) - \phi(y) < \|y - x\| \end{cases}$$

and Γ_{s} is a compact, c_{s} monotone subset of $\{c_{\mathrm{s}} < \infty\}$ such that whenever $(x, y) \in \Gamma_{\mathrm{s}}$, then ϕ is differentiable at both x and y.

Before the proof, we sketch the construction.
The restriction of $\Gamma_{\rm s}$ to each invariant class, $\Gamma_{\rm s}\cap(p^{-1}(\mathtt{q})\times p^{-1}(\mathtt{q}))$ is $c_{\mathtt{q}}$-monotone w.r.t. the cost function

$$c_{\mathtt{q}}(x,y)=\begin{cases}|y-x| & y-x\in\mathbb{R}^+\delta(\ell)\\ +\infty & \text{otherwise}\end{cases}\qquad \text{where } \mathtt{q}=(\ell,\alpha,A).$$

This is basically a consequence of Remark 3.12.
If we restrict moreover to points on the model set, and we eventually add to $\Gamma_{\rm s}$ the points which are in relation by the membership to a secondary ray, then each two points in $\Gamma_{\rm s}\cap(\mathcal{W}\cap p^{-1}(\mathtt{q}))\times(\mathcal{W}\cap p^{-1}(\mathtt{q}))$ are joined by a coordinate cycle with finite $c_{\mathtt{q}}$-cost, meaning that for each two points (x,y), (x',y') in $\mathrm{Graph}(r)\cap((\mathcal{W}\cap p^{-1}(\mathtt{q}))\times(\mathcal{W}\cap p^{-1}(\mathtt{q})))$ there exists a finite number of points $(x_i,y_i)\in\mathrm{Graph}(r)\cap((\mathcal{W}\cap p^{-1}(\mathtt{q}))\times(\mathcal{W}\cap p^{-1}(\mathtt{q})))$, two of which coinciding with (x,y), (x',y'), such that

$$\sum_{i=1}^{M}c_{\rm s}(x_{i+1},y_i)<\infty.$$

Therefore we show that for each $\mathtt{q}$ there exists a function $\phi^{\mathtt{q}}$, $-\operatorname{diam}(\Gamma)\le\phi^{\mathtt{q}}\le 0$, whose $c_{\mathtt{q}}$-subdifferential contains the $c_{\mathtt{q}}$-monotone set $\Gamma_{\rm s}\cap(\mathcal{W}\cap p^{-1}(\mathtt{q}))^2$.
Finally, the crosswise structure

$$\Gamma_{\rm s}\cap p^{-1}(\mathtt{q})^2=\Gamma_{\rm s}\cap(p^{-1}(\mathtt{q})\times\mathbb{R}^n)=\Gamma_{\rm s}\cap(\mathbb{R}^n\times p^{-1}(\mathtt{q}))$$

ensures that the analytic function $\phi:p^x(\Gamma_{\rm s})\cup p^y(\Gamma_{\rm s})\to\mathbb{R}^-$ defined by

$$\phi_{\rm s}=\inf_{\mathtt{q}}\left\{\phi^{\mathtt{q}}\,\chi_{\Gamma_{\rm s}\cap\mathcal{W}^2\cap p^{-1}(\mathtt{q})^2}\right\}$$

contains in its $c_{\rm s}$-subdifferential precisely the $c_{\mathtt{q}}$-subdifferentials of the functions $\phi^{\mathtt{q}}$, and thus $\Gamma_{\rm s}\cap\mathcal{W}\times\mathcal{W}$.

In the proof of Lemma 3.13 below, the above argument is presented in a single step, giving directly $\phi_{\rm s}$. We show before the easy fact that enlarging $\Gamma_{\rm s}$ in order to contain the points in relation by the membership to secondary rays we obtain still a $c_{\rm s}$-monotone set. We recall that the graph of the multivalued function r is a $c_{\rm s}$-monotone subsets of $\mathbb{R}^{2n}$, and that $c_{\rm s}$-monotonicity is not affected by the union of points on the diagonal.

Lemma 3.13. *Consider a model set $\mathcal{W}$ as in Definition 3.1. Then, the function*

$$\phi_{\rm s}^{\mathcal{W}}(x):=\inf_{M}\inf_{\substack{x_i,y_i\in\mathcal{W}\\ y_i\in r^+(x_i),\ x_{M+1}:=x}}\left\{\sum_{i=1}^{M}\big[c_{\rm s}(x_{i+1},y_i)-|y_i-x_i|\big]\right\}\tag{3.10}$$

is analytic and $-\operatorname{diam}(\Gamma_{\rm s})\le\phi_{\rm s}\le 0$. Its $c_{\rm s}$-subdifferential contains $\mathrm{Graph}(r)\cap(\mathcal{W}\times\mathcal{W})$.

Proof. The coAnalyticity follows by the general theory of analytic functions ([36], Chap. 4).
An upper bound for $\phi_{\rm s}$ is quite trivial: for each $x\in\mathcal{W}$, by choosing $x_1=y_1=x$ we estimate from above $\phi_{\rm s}(x)\le 0$. The lower bound is more involved, and is achieved on each single class.
Consider any admissible choice of M points (x_i,y_i) for evaluating $\phi(x)$. We can assume that $c_{\rm s}(x_{i+1},y_i)$ is finite, since we are interested in the infimum. Therefore, being

$$\phi(x_{i+1})=\phi(y_i)+\|y_i-x_{i+1}\|,$$

Remark 3.12 ensures that $x_{i+1}\sim y_i$: since moreover $x_i\sim y_i$, we deduce that all the points x_i, y_i must belong to the same class as x. By definition of the model set (Definition 3.1), one can choose a point y in B and on the secondary ray from x: we will have, again by Definition 3.1 and Corollary 3.7, that $y-x_1\in\delta(-\nabla\phi(x_1))$. Moreover, $y\sim x\sim x_1$: Remark 3.12 ensures then that $c_{\rm s}(x_1,y)<\infty$. Since $\mathrm{Graph}(r^+)$ is $c_{\rm s}$-monotone, the $c_{\rm s}$-monotonicity of $\{(x_1,y_1),\dots,(x_n,y_n),(x,y)\}$ (Lemma 2.9) implies

$$\sum_{i=1}^{M}\big[c_{\rm s}(x_{i+1},y_i)-|y_i-x_i|\big]+c_{\rm s}(x_1,y)-|y-x|\ge 0,$$

from which one deduces the lower bound $\phi(x) \geq |y-x| - \mathrm{diam}(\Gamma_{\mathrm{s}})$.

We want to establish now that ϕ_{s} is really a secondary potential. Evaluate $\phi_{\mathrm{s}}(x)$ with paths arriving at y, and extended with the other point (y,y): taking the infimum among these paths

$$\phi_{\mathrm{s}}(x) \leq \phi_{\mathrm{s}}(y) + c_{\mathrm{s}}(x,y).$$

Finally, consider $(x,y) \in \mathrm{Graph}(r) \cap \mathcal{W}^2$. Evaluating $\phi_{\mathrm{s}}(y)$ with paths having as the last point $(x_M, y_M) = (x,y)$, and taking the infimum among these paths, one finds

$$\phi_{\mathrm{s}}(y) \leq \phi_{\mathrm{s}}(x) - |y-x| \qquad \Longleftrightarrow \qquad \phi_{\mathrm{s}}(x) - \phi_{\mathrm{s}}(y) \geq |y-x|.$$

This shows that $\phi_{\mathrm{s}}(x) - \phi_{\mathrm{s}}(y) = |y-x|$ whenever there is a secondary ray from $x \in \mathcal{W}$ to $y \in \mathcal{W}$. □

The following example shows that the above definition of secondary potential does not work on a whole class: therefore it was important to reduce the construction on special sets.

Example 3.14. Consider the optimal transport problem in $\mathbb{R}^2$ with the cost function given by

$$(x,y) = ((x_1,y_1),(x_2,y_2)) \mapsto c(x,y) = |y-x| \mathbb{1}_{\{y_2 \geq y_1, x_2 \geq x_1\}}$$

and between the finite measures (see Figure 3)

$$\mu = \sum_{i=1}^{\infty} \sum_{j=0}^{4/h_i} \mathcal{L}^2 \llcorner \mathbf{B}_{h_i/12}(w_{ij}) + \mathcal{L}^2 \llcorner \mathbf{B}_{1/24}(w_\infty) \qquad \nu = \sum_{i=1}^{\infty} \sum_{j=0}^{4/h_i} \mathcal{L}^2 \llcorner \mathbf{B}_{h_i/12}(z_{ij}) + \mathcal{L}^2 \llcorner \mathbf{B}_{1/24}(z_\infty)$$

where $h_i = 2^{-i-1}$, $w_\infty = (-1.5, 0)$, $z_\infty = (-1.5, 1)$ and

$$\begin{cases} w_{10} = (0,\, 0) \\ w_{i0} = (-\sum_{k=1}^{i-1} 2h_k,\, 0) \\ w_{ij} = w_{i0} + (-h_i,\, jh_i/4) \quad j = 0, \dots, 4/h_i \end{cases} \qquad \begin{cases} z_{i0} = w_{i0} + (0,1) \\ z_{ij} = w_{ij} + (h_i/2,\, 0) \quad j = 0, \dots, 4/h_i. \end{cases}$$

The only plan π with finite cost translates each w_{ik} to z_{ik}, and therefore it is the optimal one. However, no c-monotone carriage Γ of π is contained in the c-subdifferential of a c-monotone function ϕ, which by definition would satisfy

$$\phi(x) - \phi(y) \leq c(x,y)\ \forall (x,y) \in \mathbb{R}^n \qquad \text{and} \qquad \phi(x) - \phi(y) = c(x,y)\ \forall (x,y) \in \Gamma.$$

Indeed, suppose the contrary. Then, applying repeatedly the maximal growth equality and the Lipschitz inequality, we find

$$\begin{aligned} \phi(w_{(i+1)0}) &\leq \phi(w_{i0}) - 1 + \frac{h_i}{2} + \frac{h_i}{2}\left(\frac{\sqrt{5}}{2} - 1\right) \cdot \frac{4}{h_i} + \frac{3h_i}{2} \\ &= \phi(w_{i0}) + \sqrt{5} - 3 + 2h_i. \end{aligned}$$

We find therefore that $\phi(w_{(i+1)0}) \to -\infty$ for $i \to \infty$, as well as every other $\phi(w_{(i+1)j})$. This implies that every potential ϕ which is finite on $\mathbf{B}_{h_1/12}(w_{10})$ must be $-\infty$ on $\mathbf{B}_{1/24}(w_\infty)$: for all i,j

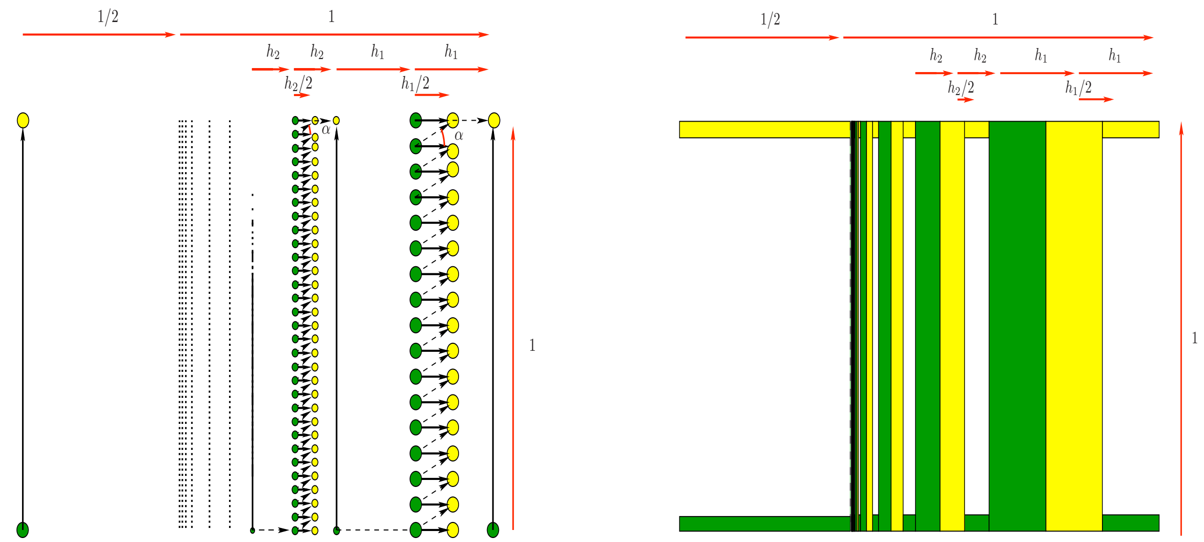
$$\phi(w_\infty) \leq \phi(w_{(i+1)j}) + |\phi(w_{(i+1)j}) - \phi(w_\infty)|,$$

which implies $\phi(w_\infty) = -\infty$.

Remark 3.15 (Figure 3b). Consider $\mathbb{R}^2$ and let the square with endpoints at the points $\{(\pm 1, \pm 1)\}$ be the unit ball of $\|\cdot\|$. Then Example 3.14 above can be modified in order to show that there exists no secondary Kantorovich potential, for every choice (that will be unique up to constants) of the primary potential ϕ.

Indeed, replace as follows the old regions with new ones, on which we put a suitable multiple of the Lebesgue measure for leaving the total mass of μ ν constant after each replacement:

- the ball at $w_{i,0}$, for $i \in \mathbb{N}$, with the rectangle R_{i0} having edges $(-\sum_{k=1}^{i-1} 2h_k, 0)$, $(-\sum_{k=1}^{i-1} 2h_k, h_i/12)$, $(-\sum_{k=1}^{i-1} 2h_k - h_i/2, 0)$, $(-\sum_{k=1}^{i-1} 2h_k - h_i/2, h_i/12)$,
- the ball at $z_{i,0}$, for $i \in \mathbb{N}$, with $R_{i0} + (0, 1 - h_i/12)$,



(a) Example 3.14: Comparing the values on the balls on the path above, one finds that the secondary potential must be $-\infty$ on the two balls on the left side.

(b) Remark 3.15: this example is almost the same as in Example 3.14, but groups of balls are replaced with rectangles so that the primary Kantorovich potential is unique up to constants on the support of μ and ν.

Figure 3: Counterexample to the existence of a Kantorovich potential for the secondary transport problem

- all the ones at $\{w_{ij}\}_{j=1}^{4/h_i}$ with a single rectangle R_i with edges $(-\sum_{k=1}^{i-1} 2h_k - h_i, 0)$, $(-\sum_{k=1}^{i-1} 2h_k - h_i, 1)$, $(-\sum_{k=1}^{i-1} 2h_k - h_i/2, 0)$, $(-\sum_{k=1}^{i-1} 2h_k - h_i/2, 1)$,
- all the ones at $\{z_{ij}\}_{j=1}^{4/h_i}$ with $R_i + (h_i/2, 0)$,
- the ball at w_∞ with the rectangle R_∞ with edges $(-3/2, 0)$, $(-1, 0)$, $(-1, 1/24)$, $(-3/2, 1/24)$,
- the ball at z_∞ with $R_\infty + (0, 1 - 1/24)$.

The support of $\mu + \nu$ now is as in Figure 3b, and restricted to there the Kantorovich primary potentials must be of the form $\|x - (-3/2, 0)\| - \kappa$, with $\kappa \in \mathbb{R}$. The computation in Example 3.14 show the non existence of a function ϕ_s such that

$$
\begin{aligned}
&\phi_s(x) - \phi_s(y) \leq |y - x| && \forall x, y \in R \\
&\phi_s(x) - \phi_s(y) = |y - x| && \forall (x, y) \in \Gamma_s
\end{aligned}
$$

where Γ_s is the support of the unique optimal transport plan between μ and ν and $|\cdot|$ is the Euclidean norm.

One can generalize it to higher dimension.

3.5. **Approximation lemma and disintegration.** In order to disintegrate the Lebesgue measure on model sets $\mathcal{W}$ of Definition 3.1, we adopt the strategy in [10]. A difference is that here the vector field of secondary rays directions $\mathtt{d}$ is chosen according to a different selection principle, since it must solve a transport problem: it is selected by a secondary variational problem as in [4].

As described at the introduction to Section 3, the last step of the disintegration technique consists in proving the fundamental regularity estimate (3.1). In particular, we are left with a subset of a model set $\mathcal{W}$ which has the form $\mathcal{K} = \sigma^{[h^-,h^+]}(S)$ for any $S \subset Z$, h^-, h^+ such that $[h^-\mathrm{e}, h^+\mathrm{e}] + S \subset \mathsf{Z}$; in a more explicit way,

$$\mathcal{K} = \big\{\sigma^t(z) : \ z \in S, \ t \in [h^-, h^+]\big\}, \qquad \sigma^t(z) = z + t\frac{\mathsf{d}(z)}{\mathsf{d}(z)\cdot \mathrm{e}}$$

where S is a subset of $\{x \cdot \mathrm{e} = 0\}$ such that $\mathsf{r}(z)\cdot \mathrm{e} \supset [h^-, h^+]$.
We have to show the push forward inequality

$$\mathcal{H}^{n-1}(\sigma^t(S)) \leq \left(\frac{t-h^-}{s-h^-}\right)^{n-1} \mathcal{H}^{n-1}(\sigma^s(S)) \qquad \text{for } h^- < s \leq t \leq h^+.$$

The estimate corresponding to the above one in [10] is proven is proven in Lemma 4.7, for the strictly convex case, and then in Lemma 5.7 for the general case. We follow here a similar limiting argument, providing on $\sigma^s(S)$ a suitable approximation by vector fields d_ε which satisfy the regularity estimate and converge pointwise to $\mathsf{d} \lceil_{\sigma^s(S)}$. Due to the different limit, the vector field approximating d should be chosen differently from [10].
Here comes the role of the secondary potential: it detects the secondary rays through a Hopf-Lax formula, and it consequently allows to view the vector field of secondary directions as pointwise limit of approximating directions which satisfy the regularity estimate.

The approximation lemma is the following.

Lemma 3.16. *Fix $h^- < s \leq t \leq h^+$. Then there exist a sequence of vector fields $\{\mathsf{d}_\varepsilon\}_\varepsilon$ on $Z_t = \mathcal{K} \cap \{x \cdot \mathrm{e} = t\}$ such that*

- *they converge pointwise to d on Z_t;*
- *they satisfy the push forward regularity estimate: for all compact $S_t \subset Z_t$*

$$\mathcal{H}^{n-1}(S_t) \leq \left(\frac{t-h^-}{s-h^-}\right)^{N-1} \mathcal{H}^{n-1}(\sigma^{s-t}_{\mathsf{d}_\varepsilon} S_t).$$

In a symmetric way, there exists a similar vector field on $Z_s = \mathcal{K} \cap \{x \cdot \mathrm{e} = s\}$, if $h^- \leq s \leq t < h^+$.

Proof. For y in Z_t denote by $\mathfrak{x}(y)$ the points on $Z_{h^-} = \mathcal{K} \cap \{x \cdot \mathrm{e}_1 = h^-\}$ belonging to the secondary ray through y; in particular, if y is an endpoint, then $\mathfrak{x}$ can be multivalued. Let ϕ_{s} be the secondary potential constructed in 3.13, which is by definition continuous on secondary transport rays of $\mathcal{K}$. Since we are working up to countable partitions in compact sets, by Lusin theorem one can assume that ϕ_{s} is continuous on $\mathcal{K}$.

Defining the functions

$$f(x, y) := \phi(y) - \phi(x) + \|y - x\|,$$

$$f'(x, y) := \begin{cases} \phi_{\mathrm{s}}(y) - \phi_{\mathrm{s}}(x) + |y - x| & \text{if } \phi(x) - \phi(y) = \|y - x\| \\ +\infty & \text{otherwise} \end{cases}$$

then

$$\mathfrak{x}(y) = \operatorname*{arg\,min}_{\tilde{x} \in Z_{h^-}} f'(\tilde{x}, y) \subset \operatorname*{arg\,min}_{\tilde{x} \in Z_{h^-}} f(\tilde{x}, y),$$

Consider moreover the functionals

$$f_\varepsilon(x, y) := \phi(y) - \phi(x) + \|y - x\| + \varepsilon\big[\phi_{\mathrm{s}}(y) - \phi_{\mathrm{s}}(x) + |y - x|\big],$$

$$f'_\varepsilon(x, y) := \frac{\phi(y) + \|y - x\| - \phi(x)}{\varepsilon} + \phi_{\mathrm{s}}(y) - \phi_{\mathrm{s}}(x) + |y - x|,$$

and define an approximating vector field of unit directions $\{\mathsf{d}_\varepsilon\}_\varepsilon$ from y to $\mathfrak{x}_\varepsilon(y)$ by the formula

$$\mathfrak{x}_\varepsilon(y) := \operatorname*{arg\,min}_{\tilde{x} \in \mathrm{clos}(Z_{h^-})} f_\varepsilon(\tilde{x}, y).$$

Being on a bounded set, it is not difficult to observe that for all fixed $y \in Z_t$

$$f(\cdot, y) = \Gamma - \lim_{\varepsilon \to 0} f_\varepsilon(\cdot, y) \qquad\qquad f'(\cdot, y) = \Gamma - \lim_{\varepsilon \to 0} f'_\varepsilon(\cdot, y).$$

Indeed, $f_\varepsilon(x,y) - f(x,y) = \varepsilon(\phi_{\rm s}(y) - \phi_{\rm s}(x) + |y-x|)$ is uniformly bounded by $2\,\mathrm{diam}(\Gamma_{\rm s})$, while $f'_\varepsilon(\cdot,y)$ increases pointwise to the continuous function $f'(\cdot,y)$.
Then, from basic facts about Γ-convergence (see precisely recalls in Section 4 of [5]),

$$\limsup_{\varepsilon\to 0} \mathfrak{r}_\varepsilon(y) = \limsup_{\varepsilon\to 0} \operatorname*{arg\,min}_{\tilde{x}\in Z_{h^-}} f_\varepsilon(\tilde{x},y) \subseteq \operatorname*{arg\,min}_{\tilde{x}\in Z_{h^-}} f'(\tilde{x},y) = \mathfrak{r}(y).$$

This proves that on Z_{h^-} any pointwise limit of the vector field $\{d_\varepsilon(y)\}_\varepsilon$ belongs $d(y)$.

Call $\phi_\varepsilon(y) := \phi(y) + \varepsilon\phi_{\rm s}(y)$, and $\|\cdot\|_\varepsilon := \|\cdot\| + \varepsilon|\cdot|$. Notice that $\|\cdot\|_\varepsilon$ is a strictly convex norm, and

$$\mathfrak{r}_\varepsilon(y) = \arg\min\left(\phi_\varepsilon(y) - \phi_\varepsilon(x) + \|y-x\|_\varepsilon\right).$$

Then the approximation with countably many disjoint cones of Example 2.12 of [17] (already present in [10]) shows that $d_\varepsilon \lceil_K$ satisfies the push forward regularity estimate, following the argument of Lemma 2.13 itself, and by compactness of the sections one can pass to the limit, up to subsequences, by the upper semicontinuity of the Hausdorff measure.

In a symmetric way, replace minimums with maximums and subtract, instead of adding, the norms to the potentials. One finds then a vector field $\{d_\varepsilon(x)\}$ on Z_s converging to $d(x)$ as $\varepsilon\to 0$, satisfying

$$\mathcal{H}^{n-1}(S_s) \le \left(\frac{s-h^-}{t-h^-}\right)^{n-1} \mathcal{H}^{n-1}(\sigma^{t-s}_{d_\varepsilon} S_s) \qquad \text{for all compact } S_t \subset Z_t. \qquad \square$$

With the new approximation provided by Lemma 3.16, one can now follow the strategy explained in the introduction of the section, at Page 18, and the accurate analogous computations are given from Lemma 2.13 to Lemma 2.21 in [17].

One arrives then at the following disintegration result.

Theorem 3.17. *The following disintegration of $\mathcal{L}^n \llcorner \mathcal{T}_{\rm s}$ strongly consistent with the covering $\{r_{\mathtt{q}}\}_{\mathtt{q}\in\mathtt{Q}}$ holds:*

$$\mathcal{L}^n \llcorner \mathcal{T}_{\rm s} = \int_{\mathtt{Q}} (\gamma\, d\mathcal{H}^1 \llcorner r_{\mathtt{q}})\, d\mathcal{H}^{n-1}(\mathtt{q})$$

where $\gamma : \mathbb{R}^n \to \mathbb{R}$ is strictly positive and $\mathtt{Q}$ is identified with a σ-compact subset of countably many hyperplanes. Moreover, $\gamma\lceil_{r_{\mathtt{q}}}$ is locally Lipschitz continuous.

Corollary 3.18. *The set of endpoints $\mathcal{E}$ is $\mathcal{L}^n$-negligible.*

Proof. If one applies the disintegration formula to the Borel set $\mathcal{E}$, it is an immediate consequence of $\mathcal{H}^1(r_{\mathtt{q}}\cap\mathcal{E}) = \mathcal{H}^1(\mathrm{rb}(r_{\mathtt{q}})) = 0$. $\square$

The Disintegration Theorm 3.17 and the estimates in Lemma 3.16 yield moreover a regularity property for the divergence of the vector field of rays directions, as stated in Corollary 3.19 below.
One can find different proofs in Section 2.4 of [17] and Section 5.1 of [18]. The first argument was presented in [10].

Corollary 3.19. *Consider the notations defined in the introduction to the present section, Pages 18-21, and define $d_{\rm e} = \frac{d(z)}{d(z)\cdot {\rm e}_i}$ on the sets $\{\mathcal{Z}_i\}_{i\in\mathbb{N}}$ of a covering of the secondary transport set.*
The divergence of the vector field $d_{{\rm e}_i}$ of rays direction is a series of Radon measures. On sets $\mathcal{K}$ of the form

$$\mathcal{K} = \left\{\sigma^t(z):\ z\in S_i,\ t\in[h^-,h^+]\right\} \qquad \text{where } \sigma^t(z) = z + t d_{{\rm e}_i}(z)$$

the following divergence formula holds: for every $\varphi\in C^1_{\rm c}(\mathbb{R}^n)$

$$\langle \mathrm{div}\, d_{{\rm e}_i}, \varphi\rangle = \int_{\mathcal{K}} \varphi(x)(\mathrm{div}\, d_{{\rm e}_i})_{\rm a.c.}(x)\, d\mathcal{L}^n(x) - \int_{\partial\mathcal{K}} \varphi(x)\, d_{{\rm e}_i}(x)\cdot\hat{n}(x)\, d\mathcal{H}^{n-1}(x), \tag{3.11}$$

where $\mathfrak{d}\mathcal{K}$, the border of $\mathcal{K}$ transversal to $\mathpzc{d}_{\mathrm{e}_i}$, is defined as $\mathcal{K} \cap H_{h^+} \cup \mathcal{K} \cap H_{h^-}$, $\hat{n}(x)$ is the vector field of unit directions normal to $H_{h^+} \cup H_{h^-}$ in the outer direction and

$$(\operatorname{div} \mathpzc{d}_{\mathrm{e}_i})_{\mathrm{a.c.}}(x) = \frac{\partial_t \beta(t = \pi_{\langle \mathrm{e}_i \rangle}(x), x - x \cdot \mathrm{e}_i\, \mathpzc{d}_{\mathrm{e}_i}(x))}{\beta(\pi_{\langle \mathrm{e}_i \rangle}(x), x - x \cdot \mathrm{e}_i\, \mathpzc{d}_{\mathrm{e}_i}(x))}.$$

4. Notations

We group here some symbols one can find in the text. Standard ones are on the left, on the right the ones we defined.

$\|\cdot\|$	a norm on $\mathbb{R}^n$
$\lvert\cdot\rvert$	a norm on $\mathbb{R}^n$ whose unit ball is strictly convex
$\mathbf{B}_r(x)$	$\{x \in \mathbb{R}^n :\ \lvert x\rvert \leq r\}$
$\llparenthesis a, b \rrparenthesis$	the segment from a to b without endpoints
$[\![a, b]\!]$	the segment from a to b without endpoints
$\mathrm{rb}(\ell)$	the relative boundary of a segment ℓ consists in the endpoints
$\mathrm{ri}(\ell)$	the relative interior of a segment ℓ is $\ell \setminus \mathrm{rb}(\ell)$
$\subseteq$	the set inclusion
$\sqsubseteq$	the inclusion preserving the orientation
X	any subset of $\mathbb{R}^n$
$(X)^{\mathrm{c}}$	the complementary of X
$\mathrm{clos}(X)$	the closure of X in $\mathbb{R}^n$
p^x	the map $p^x : \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto x \in \mathbb{R}^n$
p^y	the map $p^x : \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto y \in \mathbb{R}^n$
χ_X	$\chi_X(x) = 1$ if $x \in X$, $+\infty$ otherwise
$\mathbb{1}_X$	$\mathbb{1}_X(x) = 1$ if $x \in X$, 0 otherwise
$\mathcal{B}(X)$	Borel σ-algebra of X
$\mathscr{L}(X)$	Lebesgue σ-algebra of X
$\mathcal{M}^+_{\mathrm{loc}}(X)$	positive and finite Radon measures on X
$\mathcal{M}^+(X)$	measures in $\mathcal{M}^+_{\mathrm{loc}}(X)$ with finite total variation
$\mathscr{P}(X)$	Borel probability measures on X
$\mathcal{L}^n(X)$	Lebesgue (outer) measure of X
$\mathcal{H}^k(X)$	k-dimensional Hausdorff (outer) measure of X
$\eta \ll \xi$	if $\eta, \xi \in \mathcal{M}^+_{\mathrm{loc}}(X)$, η is absolutely continuous w.r.t. ξ if $\forall S \in \mathcal{B}(X)$ $\xi(S) = 0$ implies $\eta(S) = 0$
$g_\sharp$	push forward with a map g
∂^-	subdifferential
∂^+	superdifferential
$\partial^-_c \varphi$	$\{(x, y) :\ \varphi(x) - \varphi(y) = c(x, y)\}$

μ, ν	$\mu, \nu \in \mathcal{M}^+(\mathbb{R}^n)$ fixed, with $\mu \ll \mathcal{L}^n$
$\Pi(\mu, \nu)$	$\{\pi \in \mathcal{M}^+(\mathbb{R}^n \times \mathbb{R}^n) :\ p^x_\sharp(\pi) = \mu,\ p^y_\sharp(\pi) = \nu\}$
$\mathcal{O}_{\mathrm{p}}$	see Page 4
$\mathcal{O}_{\mathrm{s}}$	see Page 4
ϕ	primary Kantorovich potential, (1.3)
$\mathcal{T}$	primary transport set: $\cup\{[\![x, y]\!] :\ \phi(x) - \phi(y) = \|y - x\|\}$
c_{s}	secondary cost function: $c_{\mathrm{s}}(x, y) = \lvert y - x\rvert \mathbb{1}_{\{\phi(x) - \phi(y) = \|y - x\|\}}(x, y)$
G_Γ	see Definition 2.2 at Page 8
$\mathpzc{r}^+_\Gamma$	see Definition 2.2 at Page 8
$\mathpzc{r}_\Gamma$	see Definition 2.2 at Page 8
G'_Γ	see (2.2) at Page 9
$\mathpzc{r}'^+_\Gamma$	see (2.2) at Page 9
$\mathpzc{r}'_\Gamma$	see Remark 2.3
$\mathpzc{r}_{\mathtt{q}}$, $\mathtt{q}$, $\mathtt{Q}$	see Definition 2.4, Section 2.2, Lemma 2.11
$\mathpzc{d}$	see Definition 2.4, Section 2.2, Lemma 2.11
$\mathcal{T}_{\mathrm{s}}$	see Definition 2.4, Section 2.2, Lemma 2.11
$\mathcal{E}^-$	see Definition 2.4, Section 2.2
$\mathcal{E}^+$	see Definition 2.4, Section 2.2
$\mathcal{E}$	see Definition 2.4, Section 2.2
$\mathcal{F}$	see Definition 2.4, Lemma 2.11
Γ_{s}	a c_{s}-monotone set, usually σ-compact and contained in $\{c_{\mathrm{s}} < +\infty\}$
ϕ_{s}	local secondary potential (Subsection 3.4)
$\mathcal{Z}$	see Page 19
Z	see Page 19
α, β, γ	see Page 19, 19
$\mathcal{W}$	see Section 3.1
ρ	transport density, see (2.12)

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