Fibered nonlinearities for $p(x)$-Laplace equations

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Abstract. In $\mathbb{R}^m \times \mathbb{R}^{n-m}$, endowed with coordinates $X = (x, y)$, we consider the PDE

$$-\mathrm{div}\left(\alpha(x)|\nabla u(X)|^{p(x)-2}\nabla u(X)\right) = f(x, u(X)).$$

We prove a geometric inequality and a symmetry result.

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1 Introduction

The purpose of this paper is to give some geometric results on the following problem:

$$-\mathrm{div}\left(\alpha(x)|\nabla u(X)|^{p(x)-2}\nabla u(X)\right) = f(x, u(X)) \qquad \text{in } \Omega, \tag{1.1}$$

where $f = f(x, u) \in L^\infty(\mathbb{R}^m \times \mathbb{R})$ is differentiable in u with $f_u \in L^\infty(\mathbb{R})$, $\alpha \in L^\infty(\mathbb{R}^m)$, with $\inf\limits_{\mathbb{R}^m} \alpha > 0$, $p \in L^\infty(\mathbb{R}^m)$, with $p(x) \geq 2$ for any $x \in \mathbb{R}^m$, and Ω is an open subset of $\mathbb{R}^n$.
Here, $u = u(X)$, with $X = (x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.
As well known, the operator in (1.1) comprises, as main example, the degenerate $p(x)$-Laplacian (and, in particular, the degenerate p-Laplacian).
The motivation of this paper is the following. In [15], it was asked whether or not the level sets of bounded, monotone, global solutions of

$$-\Delta u(X) = u(X) - u^3(X) \tag{1.2}$$

for $X \in \mathbb{R}^n$, are flat hyperplanes, at least when $n \leq 8$.
In spite of the marvelous progress performed in this direction (see, in particular, [43, 8, 31, 32, 7, 5, 46, 16]), part of the conjecture and many related problems are still unsolved (see [27]).
In [47], the following generalization of (1.2) was taken into account:

$$-\Delta u(X) = f(x, u(X)). \tag{1.3}$$

where, as above, the notation $X = (x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ is used.

We observe that when $f(x,u)$ does not depend on x, then (1.3) reduces to a usual semilinar equation, of which (1.2) represents the chief example.

When $f(x,u)$ depends on x, the dependence on the space variable of f changes only with respect to a subset of the variables, namely the nonlinearity takes no dependence on y.

In particular, for fixed $u \in \mathbb{R}$, we have that $f(x,u)$ is constant on the "vertical fibers" $\{x = c\}$, and for this the nonlinearity in (1.3) is called "fibered".

Moreover, the model in (1.3) was considered in [47] as a sort of interpolation between the classical semilinear equation in (1.2) and the boundary reactions PDEs of [11, 49], which are related to fractional power operators (see also [12]).

The purpose of this paper is to extend the results of [47] to degenerate operators of $p(x)$-Laplace type and thus replace (1.3) with the more general PDE in (1.1). Indeed, when $p(x)$ is identically equal to 2, (1.1) was dealt with in [47]. Here, further technical difficulties arises when $p(x) > 2$, due to the presence of a degenerate operator. To overcome these difficulties, the technique developed in [50] will turn out to be useful.

We recall that the $p(x)$-Laplace equations have recently become quite popular, in view of some important physical applications: see, for instance, [57, 34, 45, 18, 38].

Moreover, many analytical results related to the $p(x)$-Laplacian operator have been recently appeared: see, among the others, [13, 2, 3, 1, 19, 21, 20, 23, 4, 24, 54, 9, 6, 35, 36, 39, 55, 22, 30, 42, 41, 37, 56].

For us, a weak solution of (1.1) is a function u satisfying

$$\int_\Omega \alpha(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \xi \, dX = \int_\Omega f(x,u)\xi \, dX \tag{1.4}$$

for any $\xi \in C_0^\infty(\Omega)$.

In what follows, we always assume that

$$u \in C^1(\Omega) \cap C^2(\Omega \cap \{\nabla u \neq 0\}) \cap L^\infty(\Omega) \text{ and that } \nabla u \in L^\infty(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega). \tag{1.5}$$

We recall that these regularity assumptions are very mild, and automatically fulfilled in many cases of interest (see, for instance, [17, 53, 14] and the discussion after Theorem 1.1 in [26]).

In the sequel, we consider the map $\mathcal{B} : \mathbb{R}^m \times \left(\mathbb{R}^n \setminus \{0\}\right) \to \mathrm{Mat}(n \times n)$ given by

$$\mathcal{B}(x,\eta)_{ij} := \alpha(x)|\eta|^{p(x)-2}\left(\delta_{ij} + (p(x)-2)\frac{\eta_i\eta_j}{|\eta|^2}\right) \tag{1.6}$$

for any $1 \le i, j \le n$, where $\mathrm{Mat}(n \times n)$ denotes the space of square $(n \times n)$-matrices.

We also extend this definition by continuity, setting $\mathcal{B}(x,0)_{ij} := \alpha(x)\delta_{ij}$ when $p(x) = 2$ and $\mathcal{B}(x,0)_{ij} := 0$ when $p(x) > 2$.

We remark that

$$\frac{d}{d\varepsilon}\Big[\alpha(x)|\nabla u + \varepsilon\nabla\varphi|^{p(x)-2}(\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi\Big]_{\varepsilon=0} = < \mathcal{B}(x,\nabla u)\nabla\varphi, \nabla\varphi > \tag{1.7}$$

for any smooth test function φ, where $<,>$ denotes the standard scalar product in $\mathbb{R}^n$.

In view of (1.7), it is natural to say that u is stable if

$$\int_\Omega < \mathcal{B}(x,\nabla u)\nabla\xi, \nabla\xi > - f_u(x,u)\xi^2 \, dX \ \ge \ 0 \tag{1.8}$$

for any $\xi \in C_0^\infty(\Omega)$.

The notion of stability given in (1.8) appears naturally in the calculus of variations setting and it is usually related to minimization and monotonicity properties. In particular, (1.7) and (1.8) state that the (formal) second variation of the energy functional associated to the equation has a sign (see, e.g., [44, 29, 5, 26] and Lemmata B.1 and B.2 here for further details).

The main results we prove are a geometric formula, of Poincaré-type, given in Theorem 1.1, and a symmetry result, given in Theorem 1.2.

For our geometric result, we need to recall the following notation. Fixed $x \in \mathbb{R}^m$ and $c \in \mathbb{R}$, we look at the level set

$$S := \{y \in \mathbb{R}^{n-m} \,:\, u(x,y) = c\}.$$

We will consider the regular points of S, that is, we define

$$L := \{y \in S \,:\, \nabla_y u(y,x) \neq 0\}.$$

Note that L depends on the $x \in \mathbb{R}^m$ that we have fixed at the beginning, though we do not keep explicit track of this in the notation. In the same way, S has to be thought as the level set of u on the slice selected by the fixed x.

Let ∇_L to be the tangential gradient along L, that is, for any $y_o \in L$ and any $G : \mathbb{R}^{n-m} \to \mathbb{R}$ smooth in the vicinity of y_o, we set

$$\nabla_L G(y_o) := \nabla_y G(y_o) - \left(\nabla_y G(y_o) \cdot \frac{\nabla_y u(x,y_o)}{|\nabla_y u(x,y_o)|} \right) \frac{\nabla_y u(x,y_o)}{|\nabla_y u(x,y_o)|}. \tag{1.9}$$

Since L is a smooth $(n-m-1)$-manifold, in virtue of the Implicit Function Theorem and (1.5), we can define the principal curvatures on it, denoted by

$$\kappa_1(x,y), \ldots, \kappa_{n-m-1}(x,y),$$

for any $y \in L$. We will then define the total curvature

$$\mathcal{K}(x,y) := \sqrt{\sum_{j=1}^{n-m-1} \big(\kappa_j(x,y)\big)^2}.$$

Here is the geometric formula we prove in this paper:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Assume that u is a stable weak solution of* (1.1) *in Ω under assumption* (1.5).

Then,

$$\begin{aligned} &\int_{\mathcal{R}} \alpha(x) |\nabla u|^{p(x)-2} \left(\mathcal{S} + \mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2 + \frac{(p(x)-2)}{|\nabla u|^2} \mathcal{T} \right) \phi^2 \\ &\qquad \le \int_\Omega |\nabla_y u|^2 < \mathcal{B}(x, \nabla u) \nabla\phi, \nabla\phi > \end{aligned} \tag{1.10}$$

for any $\phi \in C_0^\infty$, where

$$\mathcal{R} := \{(x,y) \in \Omega \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m} \,:\, \nabla_y u(x,y) \neq 0\}, \tag{1.11}$$

$$\mathcal{S} := -|\nabla_x |\nabla_y u||^2 + \sum_{i=1}^{m} \sum_{j=1}^{n-m} (u_{x_i y_j})^2 \qquad \textit{and} \tag{1.12}$$

$$\mathcal{T} := -(\nabla u \cdot \nabla |\nabla_y u|)^2 + \sum_{j=1}^{n-m} (\nabla u \cdot \nabla u_{y_j})^2. \tag{1.13}$$

Also

$$\mathcal{S},\ \mathcal{T} \ge 0 \ \textit{on}\ \mathcal{R} \tag{1.14}$$

and

$$\begin{aligned} &\mathcal{S}(X) = 0 \ \textit{at some}\ X \in \mathbb{R}^n \\ &\textit{if and only if}\ \nabla_y u_{x_i}(X)\ \textit{is parallel to}\ \nabla_y u(X) \\ &\textit{for any}\ i = 1, \ldots, m. \end{aligned} \tag{1.15}$$

The second result we present is a symmetry result:

Theorem 1.2. *Let u be a weak solution of* (1.1) *in whole $\mathbb{R}^n$ under assumption* (1.5) *(with $\Omega := \mathbb{R}^n$ in* (1.5)*).*
Suppose that

$$\partial_{y_1} u(X) > 0 \text{ for any } X \in \mathbb{R}^n, \tag{1.16}$$

and that there exists $C_o \geq 1$ in such a way that

$$\int_{B_R} \alpha(x) |\nabla u|^{p(x)} \, dX \leq C_o \, R^2, \tag{1.17}$$

for any $R \geq C_o$.
Then, there exist $\omega \in \mathrm{S}^{n-m-1}$ and $u_o : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = u_o(x, \omega \cdot y)$$

for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

For explicit conditions that imply the energy bound in (1.17), we refer to Appendix B here below.

We observe that Theorem 1.1 may be seen as a weighted Poincaré inequality. Namely, the L^2-norm of any test functions is bounded by the L^2-norm of its gradient, but these norms are taken with appropriate weights.

Remarkably, such weights have nice geometric meanings, which make Theorem 1.1 feasible for the application in Theorem 1.2, which is related to the problem posed in [15] settled for the PDE in (1.1) instead of the one in (1.2).

We recall that [51, 52] introduced a similar weighted Poincaré inequality in the classical uniformly elliptic semilinear framework. The idea of making use of Poincaré type inequalities on level sets to deduce suitable symmetries for the solutions was already in [25] and it has been also used in [10, 26].

For related Sobolev-Poincaré inequalities, see [28].

We remark that results analogous to Theorems 1.1 and 1.2 hold, with the same proofs we present in this paper, even for slightly more general degenerate operators. For example, the arguments we perform here also work when (1.1) is replaced by

$$-\mathrm{div}\Big(a\big(x, |\nabla u(X)|\big) \nabla u(X) \Big) = f(x, u(X)),$$

with $0 \leq a \in L^\infty(\mathbb{R}^m \times [0, +\infty))$, $\inf_{x \in \mathbb{R}^m} a(x, t) > 0$ for any $t > 0$ and $0 \leq a_t \in L^\infty(\mathbb{R}^m \times [0, +\infty))$.

The rest of the paper is devoted to the proofs of Theorems 1.1 and 1.2, which will be given in Sections 2 and 3 respectively. The paper ends with an Appendix, which contains some auxiliary lemmata, some comments on when conditions (1.16) and (1.17) are satisfied, and explicit examples of smooth, global, bounded solutions of (1.1).

2 Proof of Theorem 1.1

By (1.6), we have that

$$\begin{aligned}
&\int_\Omega \alpha(x) |\nabla u|^{p(x)-2} \nabla u \cdot \Psi_{y_j} \\
&= -\int_\Omega \left(\alpha(x) |\nabla u|^{p(x)-2} \nabla u_{y_j} \cdot \Psi + (p(x) - 2) \alpha(x) |\nabla u|^{p(x)-2} \frac{\nabla u \cdot \nabla u_{y_j}}{|\nabla u|^2} \nabla u \cdot \Psi \right) \\
&= -\int_\Omega < \mathcal{B}(x, \nabla u) \nabla u_{y_j}, \Psi > .
\end{aligned} \tag{2.1}$$

for any $j = 1, \dots, n-m$ and any $\Psi \in C_0^\infty(\Omega, \mathbb{R}^{n-m})$.
The use of (1.4) and (2.1) with $\Psi := \nabla\psi$ yields

$$\begin{aligned}\int_\Omega f_u(x,u)u_{y_j}\psi &= \int_\Omega (f(x,u))_{y_j}\psi = -\int_\Omega f(x,u)\psi_{y_j} = -\int_\Omega \alpha(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi_{y_j} \\ &= \int_\Omega < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla\psi > \end{aligned} \tag{2.2}$$

for any $j = 1, \dots, n-m$ and any $\psi \in C_0^\infty(\Omega)$.
Actually,

$$\text{(2.2) holds for any } \psi \in W_0^{1,2}(\Omega). \tag{2.3}$$

To prove (2.3), we perform a density argument (which may be skipped by the expert reader). Namely, we take K to be a compact subset of Ω, $\psi \in W_0^{1,2}(K)$ and a sequence $\psi_\varepsilon \in C_0^\infty(K)$ approaching ψ in the $W^{1,2}$-norm.
We observe that, from (1.5), there exists $C_K \geq 1$ such that

$$\sup_{X\in K} |f_u(x,u(X))| + |\nabla_y u(X)| + |\mathcal{B}(x,\nabla u(X))| \leq C_K. \tag{2.4}$$

Furthermore, $\mathcal{B}$ is nonnegative definite.
Consequently, by Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\int_\Omega < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla(\psi-\psi_\varepsilon) >\right| \\ &\leq \sqrt{\int_K < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla u_{y_j} >}\sqrt{\int_K < \mathcal{B}(x,\nabla u)\nabla(\psi-\psi_\varepsilon), \nabla(\psi-\psi_\varepsilon) >} \\ &\leq C_K^2\sqrt{|K|}\,\|\nabla(\psi-\psi_\varepsilon)\|_{L^2(K)}.\end{aligned} \tag{2.5}$$

Moreover,

$$\left|\int_\Omega f_u(x,u)u_{y_j}(\psi-\psi_\varepsilon)\right| \leq C_K^2\int_K |\psi-\psi_\varepsilon| \leq C_K^2\sqrt{|K|}\,\|\psi-\psi_\varepsilon\|_{L^2(K)}. \tag{2.6}$$

Then, (2.3) plainly follows from (2.5) and (2.6).
We also claim that

$$\text{(1.8) holds for any } \xi \in W_0^{1,2}(\Omega). \tag{2.7}$$

The proof of (2.7) is analogous to the one of (2.3) and its reading may be omitted by the expert readers. The details of the proof of (2.7) consist in taking a compact subset K of Ω, a function $\xi \in W_0^{1,2}(K)$, and a sequence $\xi_\varepsilon \in C_0^\infty(K)$ which approaches ξ in the $W^{1,2}$-norm.
Then, using (2.4) once more,

$$\begin{aligned}&\left|\int_\Omega \Big(< \mathcal{B}(x,\nabla u)\nabla\xi, \nabla\xi > - < \mathcal{B}(x,\nabla u)\nabla\xi_\varepsilon, \nabla\xi_\varepsilon > \Big)\right| \\ &\qquad + \left|\int_\Omega \big(f_u(x,u)(\xi^2-\xi_\varepsilon^2)\big)\right| \\ \leq\ &\left|\int_K \Big(< \mathcal{B}(x,\nabla u)\nabla(\xi-\xi_\varepsilon), \nabla\xi > + < \mathcal{B}(x,\nabla u)\nabla\xi_\varepsilon, \nabla(\xi-\xi_\varepsilon) > \Big)\right| \\ &\qquad + C_K\int_K \big(|\xi+\xi_\varepsilon||\xi-\xi_\varepsilon|\big) \\ \leq\ & 4C_K\big(1+\|\xi\|_{W^{1,2}(K)}\big)\|\xi-\xi_\varepsilon\|_{W^{1,2}(K)},\end{aligned}$$

for ε small, and this proves (2.7).

From (1.5) and (2.3), we may take $\psi := u_{y_j}\phi^2$ in (2.2), where $\phi \in C_0^\infty(\Omega)$.
So, we obtain

$$\begin{aligned} 0 &= \int_\Omega \left[< \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 + < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla \phi^2 > u_{y_j} \right] \\ &= \int_\Omega f_u(x,u) u_{y_j}^2 \phi^2. \end{aligned} \tag{2.8}$$

Now, we notice that, by (1.5) and Stampacchia's Theorem (see, e.g., Theorem 6.19 in [40]),

$$\begin{aligned} &\nabla |\nabla_y u| = 0 = \nabla u_{y_j} \\ &\text{for a.e. } x \in \mathbb{R}^m \text{ and a.e. } y \in \mathbb{R}^{n-m} \text{ such that } \nabla_y u(x,y) = 0. \end{aligned} \tag{2.9}$$

By (1.11), (2.8) and (2.9), we obtain

$$0 = \int_{\mathcal{R}} \left[< \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 + < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla \phi^2 > u_{y_j} \right] + \int_\Omega f_u(x,u) u_{y_j}^2 \phi^2.$$

We now sum over $j = 1, ..., n$ to get (dropping, for short, the dependences of $\mathcal{B}$) and we obtain

$$-\int_{\mathcal{R}} \left[\sum_{j=1}^n < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 - \frac{1}{2} < \mathcal{B}\nabla |\nabla_y u|^2, \nabla \phi^2 > \right] = \int_\Omega f_u(x,u) |\nabla_y u|^2 \phi^2. \tag{2.10}$$

Now, we recall (2.7) and we choose $\xi := |\nabla_y u|\phi$ in (1.8), obtaining

$$\begin{aligned} 0 \le \int_{\mathcal{R}} \Big[& < \mathcal{B}\nabla |\nabla_y u|, \nabla |\nabla_y u| > \phi^2 + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 \\ & + 2 < \mathcal{B}\nabla |\nabla_y u|, \nabla \phi > |\nabla_y u| \phi \Big] + \int_\Omega f_u(x,u) |\nabla_y u| \phi^2, \end{aligned}$$

where (2.9) has been used once more.
This and (2.10) imply that

$$0 \le \int_{\mathcal{R}} \Big[< \mathcal{B}\nabla |\nabla_y u|, \nabla |\nabla_y u| > \phi^2 + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 - \sum_{j=1}^n < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 \Big]. \tag{2.11}$$

By using (1.6) and (2.11), we are lead to the following inequality:

$$\begin{aligned} 0 \le \int_{\mathcal{R}} \Big\{ & \alpha(x) |\nabla u|^{p(x)-2} \phi^2 \Big[|\nabla |\nabla_y u||^2 - \sum_{j=1}^{n-m} |\nabla u_{y_j}|^2 \Big] + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 \\ & + \frac{(p(x)-2)\alpha(x)|\nabla u|^{p(x)-2}\phi^2}{|\nabla u|^2} \Big[(\nabla u \cdot \nabla |\nabla_y u|)^2 - \sum_{j=1}^{n-m} (\nabla u \cdot \nabla u_{y_j})^2 \Big] \Big\}. \end{aligned} \tag{2.12}$$

We denote $\mathcal{S}$ and $\mathcal{T}$ as in (1.12) and (1.13).
We also set

$$\mathcal{U} := \big|\nabla |\nabla_y u|\big|^2 - \sum_{j=1}^{n-m} |\nabla u_{y_j}|^2.$$

Making use of formula (2.1) of [51], we have that, on $\mathcal{R}$,

$$\mathcal{U} + \mathcal{S} = \big|\nabla_y |\nabla_y u|\big|^2 - \sum_{i,j=1}^{n-m} (u_{y_i y_j})^2 = -(\mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2).$$

Accordingly, (2.12) becomes

$$0 \le \int_{\mathcal{R}} \Big\{ \alpha(x)|\nabla u|^{p(x)-2}\phi^2 \Big(-\mathcal{S} - (\mathcal{K}^2|\nabla_y u|^2 + |\nabla_L|\nabla_y u||^2) \Big) - \frac{(p(x)-2)\alpha(x)|\nabla u|^{p(x)-2}}{|\nabla u|^2}\mathcal{T}\phi^2 + <\mathcal{B}\nabla\phi, \nabla\phi> |\nabla_y u|^2 \Big\},$$

and this gives (1.10).
Furthermore, if we set

$$\zeta_j := \nabla u \cdot \nabla u_{y_j} \qquad \text{for } j = 1, \dots, n-m,$$

and

$$\zeta := (\zeta_1, \dots, \zeta_{n-m}) \in \mathbb{R}^{n-m},$$

we have that, on $\mathcal{R}$,

$$\begin{aligned} -\mathcal{T} &= \left(\sum_{\ell=1}^{n} \partial_\ell u \partial_\ell |\nabla_y u| \right)^2 - |\xi|^2 = \left(\sum_{\ell=1}^{n} \partial_\ell u \frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y \partial_\ell u \right)^2 - |\xi|^2 \\ &= \left(\frac{\nabla_y u}{|\nabla_y u|} \cdot \xi \right)^2 - |\xi|^2 \le 0, \end{aligned} \tag{2.13}$$

thanks to Cauchy-Schwarz inequality.
Analogously, for any $i = 1, \dots, m$, on $\mathcal{R}$,

$$\big|\partial_{x_i} |\nabla_y u|\big| = \left| \frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y u_{x_i} \right| \le |\nabla_y u_{x_i}| = \sqrt{\sum_{j=1}^{n-m} (u_{x_i y_j})^2}, \tag{2.14}$$

and

$$\text{equality holds in (2.14) if and only if } \nabla_y u_{x_i} \text{ is parallel to } \nabla_y u. \tag{2.15}$$

Therefore, from (2.14),

$$\begin{aligned} -\mathcal{S} &= |\nabla_x |\nabla_y u||^2 - \sum_{i=1}^{m} \sum_{j=1}^{n-m} (u_{x_i y_j})^2 \\ &= \sum_{i=1}^{m} \big(\partial_{x_i} |\nabla_y u|\big)^2 - \sum_{i=1}^{m} \sum_{j=1}^{n-m} (u_{x_i y_j})^2 \le 0. \end{aligned}$$

This, (2.13) and (2.15) give (1.14) and (1.15), thus completing the proof of Theorem 1.1. ■

3 Proof of Theorem 1.2

From (1.16) and Lemma B.2, we have that u is stable. Therefore, the assumptions of Theorem 1.1 are implied by the ones of Theorem 1.2.
Given $\rho_1 \le \rho_2$, we define

$$\mathcal{A}_{\rho_1,\rho_2} := \{ X \in \mathbb{R}^n \, : \, |X| \in [\rho_1, \rho_2] \}. \tag{3.1}$$

From (1.17) and Lemma A.2, applied here with

$$h(X) := \alpha(x)|\nabla u|^{p(x)},$$

we obtain

$$\int_{\mathcal{A}_{\sqrt{R},R}} \frac{\alpha(x)|\nabla u|^{p(x)}}{|X|^2} \le C_1 \log R \tag{3.2}$$

for a suitable $C_1 > 0$, if R is big.
Now we define

$$\phi_R(X) := \begin{cases} \log R & \text{if } |X| \le \sqrt{R}, \\ 2\log\big(R/|X|\big) & \text{if } \sqrt{R} < |X| < R, \\ 0 & \text{if } |X| \ge R \end{cases}$$

and we observe that

$$|\nabla \phi_R| \le \frac{C_2\, \chi_{\mathcal{A}_{\sqrt{R},R}}}{|X|},$$

for a suitable $C_2 > 0$.
Moreover, employing (1.6) and Cauchy-Schwarz inequality,

$$\big| < \mathcal{B}(x, \nabla u(x))w, w > \big| \ \le\ \alpha(x)\,(p(x)-1)\,|\nabla u(x)|^{p(x)-2}|w|^2 \qquad \text{for all } w \in \mathbb{R}^n.$$

Thus, plugging ϕ_R in (1.10) and recalling (1.14), we see that

$$\begin{aligned} &(\log R)^2 \int_{B_{\sqrt{R}} \bigcap \mathcal{R}} \Big[\alpha(x)|\nabla u|^{p(x)-2}\left(\mathcal{S} + \mathcal{K}^2|\nabla_y u|^2 + |\nabla_L|\nabla_y u||^2\right)\Big] \\ &\qquad \le\ C_3 \int_{\mathcal{A}_{\sqrt{R},R}} \frac{\alpha(x)|\nabla u|^{p(x)-2}|\nabla_y u|^2}{|X|^2} \end{aligned}$$

for large R.
Hence, we divide by $(\log R)^2$, we use (3.2) and we send $R \to +\infty$. In this way, we obtain that $\mathcal{S}$, $\mathcal{K}$ and $\big|\nabla_L|\nabla_y u|\big|$ vanish identically on $\mathcal{R}$.
Then, by Lemma 2.11 of [26] (applied to the function $y \mapsto u(x, y)$, for any fixed $x \in \mathbb{R}^m$), we obtain that there exist $\omega : \mathbb{R}^m \to \mathrm{S}^{n-m-1}$ and $u_o : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that $u(x, y) = u_o(x, \omega(x) \cdot y)$ for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.
From (1.15) and Lemma A.1, we deduce that ω is constant, and this ends the proof of Theorem 1.2. ■

Appendices

A Auxiliary lemmata

Lemma A.1. *Let $u \in C^2(\mathbb{R}^n)$, with*

$$\big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \ :\ \nabla_y u(x, y) = 0\big\} = \emptyset. \tag{A.1}$$

Let also $\omega : \mathbb{R}^m \to \mathrm{S}^{n-m-1}$ and $u_o : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$.
Suppose that

$$u(x, y) = u_o(x, \omega(x) \cdot y) \tag{A.2}$$

for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.
Assume also that

$$\nabla_y u_i \textit{ is parallel to } \nabla_y u \tag{A.3}$$

for any $i = 1, \ldots, m$ and any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n-m}$.
Then, ω is constant.

Proof. To start, we claim that

$$\nabla_y u(x, y) \text{ is parallel to } \omega(x). \tag{A.4}$$

To check this, we let $\eta(x) \in \mathrm{S}^{n-m-1}$ be orthogonal to $\omega(x)$ and we use (A.2) to get that

$$u(x, y + t\eta(x)) = u_o(x, \omega(x) \cdot y).$$

Therefore, by differentiating with respect to t,

$$\nabla_y u(x, y) \cdot \eta(x) = 0.$$

This proves (A.4).
From (A.4), we now write

$$\nabla_y u(x, y) = c(x, y)\omega(x), \tag{A.5}$$

for some $c(x, y) \in \mathbb{R}$.
In fact, from (A.1) and (A.5),

$$c(x, y) \neq 0 \text{ for all } (x, y) \in \mathbb{R}^n. \tag{A.6}$$

Also, from (A.5),

$$\text{the map } (x, y) \mapsto c(x, y)\omega(x) \text{ belongs to } C^1(\mathbb{R}^n). \tag{A.7}$$

Hence,

$$\Big(c(x, y)\omega(x)\Big)_i = \nabla_y u_i(x, y), \tag{A.8}$$

for any $1 \leq i \leq n$.
Since

$$c^2(x, y) = \big(c(x, y)\omega(x)\big) \cdot \big(c(x, y)\omega(x)\big),$$

we deduce from (A.7) that $c^2 \in C^1(\mathbb{R}^n)$.
Thus, from (A.6),

$$c \in C^1(\mathbb{R}^n). \tag{A.9}$$

This, (A.5) and (A.6) imply that

$$\omega \in C^1(\mathbb{R}^m). \tag{A.10}$$

So,

$$0 = \left(\frac{1}{2}\right)_i = \left(\frac{\omega(x) \cdot \omega(x)}{2}\right)_i = \omega_i(x) \cdot \omega(x), \tag{A.11}$$

for any $1 \leq i \leq m$.
Furthermore, by (A.5), (A.3) and (A.4), we have that

$$\Big(c(x, y)\omega(x)\Big)_i = \Big(\nabla_y u(x, y)\Big)_i = \nabla u_i(x, y) = k^{(i)}(x, y)\omega(x), \tag{A.12}$$

for some $k^{(i)}(x, y) \in \mathbb{R}$.
Then, making use of (A.11) twice, we deduce from (A.12) that

$$0 = k^{(i)}(x, y)\omega(x) \cdot \omega_i(x) = \Big(c(x, y)\omega(x)\Big)_i \cdot \omega_i(x) = c(x, y)\omega_i(x) \cdot \omega_i(x) = c(x, y)|\omega_i(x)|^2,$$

for any $1 \leq i \leq m$.
Consequently, from (A.6), we conclude that $\omega_i(x) = 0$ for any $1 \leq i \leq m$. $\square$

We remark that the result in Lemma A.1 is, in general, false without condition (A.1). To see this, let us consider the following example. Let $m = 1$, $n = 3$, $\tau \in C^\infty(\mathbb{R})$, with $\tau(x) = 0$ for any $x \in [-1, 1]$ and $\tau(x) > 0$ for any $x \in \mathbb{R} \setminus [-1, 1]$.
Let also $\omega \in C^\infty(\mathbb{R}, \mathrm{S}^1)$ be such that $\omega(x) = (1, 0)$ for any $x \leq -1/2$ and $\omega(x) = (0, 1)$ for any $x \geq 1/2$.

Let $\gamma \in C^\infty(\mathbb{R})$, and set

$$u_o(x,r) := \tau(x)\gamma(r), \qquad \text{for any } (x,r) \in \mathbb{R}\times\mathbb{R}, \text{ and}$$
$$u(x,y) := \tau(x)\gamma(\omega(x)\cdot y), \qquad \text{for any } (x,y) \in \mathbb{R}\times\mathbb{R}^2.$$

Then, (A.2) holds true.
Moreover,

$$\nabla_y u(x,y) = \gamma'(\omega(x)\cdot y)\tau(x)\,\omega(x). \tag{A.13}$$

We also observe that

$$\begin{aligned}\partial_x\big(\tau(x)\omega(x)\big) &= \begin{cases} (0,0) & \text{if } x\in(-1,1),\\ \tau'(x)\,(1,0) & \text{if } x \le -1,\\ \tau'(x)\,(0,1) & \text{if } x\ge 1\end{cases}\\ &= \tau'(x)\,\omega(x).\end{aligned}$$

As a consequence,

$$\begin{aligned}\nabla_y u_1(x,y) &= \gamma'(\omega(x)\cdot y)\tau'(x)\,\omega(x) + \gamma''(\omega(x)\cdot y)(\omega'(x)\cdot y)\tau(x)\,\omega(x)\\ &= \Big(\gamma'(\omega(x)\cdot y)\tau'(x) + \gamma''(\omega(x)\cdot y)(\omega'(x)\cdot y)\tau(x)\Big)\,\omega(x).\end{aligned}$$

That is, $\nabla_y u_1$ is parallel to ω and so, by (A.13), we have that (A.3) holds true.
But (A.1) and the claim of Lemma A.1 are not satisfied.

Lemma A.2. *Let the notation in* (3.1) *hold.*
Let $R>0$ and $h: B_R\subset\mathbb{R}^n\to\mathbb{R}$ be a nonnegative measurable function.
For any $\rho\in(0,R)$, let

$$\eta(\rho) := 2\int_{B_\rho} h(X)\,dX.$$

Then,

$$\int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{|X|^2}\,dX \le \int_{\sqrt{R}}^R t^{-3}\eta(t)\,dt + \frac{\eta(R)}{R^2}.$$

Proof. The argument we give here is a modification of the ones on page 24 of [48] and page 403 of [33].
By Fubini's Theorem,

$$\begin{aligned}\int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{|X|^2}\,dX &= \int_{\mathcal{A}_{\sqrt{R},R}} h(X)\left(\int_{|X|}^R 2t^{-3}\,dt + R^{-2}\right)dX\\ &= 2\int_{\sqrt{R}}^R\int_{\mathcal{A}_{\sqrt{R},t}} t^{-3}h(X)\,dX\,dt + R^{-2}\int_{\mathcal{A}_{\sqrt{R},R}} h(X)\,dX\\ &\le \int_{\sqrt{R}}^R t^{-3}\eta(t)\,dt + R^{-2}\eta(R).\end{aligned}$$

□

B Motivating assumptions (1.8) and (1.17)

For $t_0\in\mathbb{R}$ fixed, we set

$$F(x,t) := \int_{t_0}^t f(x,s)\,ds. \tag{B.1}$$

Given an open set $\Omega \subseteq \mathbb{R}^n$, we define

$$\mathcal{E}_\Omega(v) := \int_\Omega \frac{\alpha(x)|\nabla u(X)|^{p(x)}}{p(x)} - F(x, u(X))\, dX.$$

It is well known that u is a local minimizer if for any bounded open set $U \subset \Omega$ we have $\mathcal{E}_U(u)$ is well-defined and finite, and

$$\mathcal{E}_U(u)(u+\phi) \geq \mathcal{E}_U(u)$$

for any $\phi \in C_0^\infty(U)$.

Lemma B.1. *Let u be a local minimizer in some domain Ω. Then u satisfies* (1.4) *and* (1.8).

Proof. We compute the first and second variation of $\mathcal{E}_\Omega$ with U a bounded open subset of Ω. We have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathcal{E}_U(u+\varepsilon\phi)\Big|_{\epsilon=0} \\ &= \int_\Omega \alpha(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi - f(x,u)\phi\, dX \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{d^2}{d\epsilon^2}\,\mathcal{E}_U(u+\epsilon\phi)\big|_{\epsilon=0} \\ &= \int_\Omega < \mathcal{B}(x,\nabla u)\nabla\phi, \nabla\phi > - f_u(x,u)\phi^2\, dX, \end{aligned}$$

due to (1.7). □

We now recall that monotonicity in one direction implies stability:

Lemma B.2. *Let u be a weak solution of* (1.1) *in Ω and suppose that $\partial_{y_1} u > 0$ in Ω. Then, u is stable, that is* (1.8) *holds.*

Proof. Fix $\xi \in C_0^\infty(\Omega)$. In view of (2.3), we may use (2.2) for $j=1$ and $\psi := \dfrac{\xi^2}{u_{y_1}} \in W_0^{1,2}(\Omega)$.
This yields that

$$\begin{aligned} & \int_\Omega f_u(x,u)\xi^2\, dX \\ =\ & \int_\Omega f_u(x,u)u_{y_1}\psi\, dX \\ =\ & \int_\Omega \Big[\frac{2\xi}{u_{y_1}} < \mathcal{B}(x,\nabla u)\nabla u_{y_1}, \nabla\xi > - \frac{\xi^2}{(u_{y_1})^2} < \mathcal{B}(x,\nabla u)\nabla u_{y_1}, \nabla u_{y_1} > \Big]\, dX \\ \leq\ & \int_\Omega < \mathcal{B}(x,\nabla u)\nabla\xi, \nabla\xi > \, dX, \end{aligned}$$

where in the last equation we used that

$$2 < \mathcal{B}(x,\nabla u)v, w > \ \leq\ < \mathcal{B}(x,\nabla u)v, v > + < \mathcal{B}(x,\nabla u)w, w >, \qquad \forall v, w \in \mathbb{R}^n.$$ □

We now give a sufficient condition for (1.17) to hold:

Lemma B.3. *Let $t_o := -1$ in* (B.1).
Assume that $F(x,t) \le 0$ for any $x \in \mathbb{R}^m$ and any $t \in \mathbb{R}$, $F(x,-1) = F(x,+1) = 0$, and

$$\sup_{\substack{x\in\mathbb{R}^m\\ |t|\le 1}} \big|F(x,t)\big| < +\infty. \tag{B.2}$$

Let $u \in W^{1,\infty}(\mathbb{R}^n,[-1,1])$ be a local minimum in the whole $\mathbb{R}^n$.
Then, there exists $C > 0$ such that

$$\int_{B_R} \alpha(x)|\nabla u(X)|^{p(x)}\,dX \le CR^{n-1}, \tag{B.3}$$

for any $R > 1$.
In particular, if also $n \le 3$, then (1.17) *holds.*

Proof. We take $R > 1$, $h \in C^\infty(B_R)$, with $h = -1$ in B_{R-1}, $h = 1$ on ∂B_R and $|\nabla h| \le 4$, and we set $v(x) := \min\{u(x), h(x)\}$.
Then, since u is minimal, we have that

$$\begin{aligned}
\inf_{x\in\mathbb{R}^m} \frac{1}{p(x)} \int_{B_R} \alpha(x)|\nabla u|^{p(x)}\,dX &\le \mathcal{E}_{B_R}(u) \le \mathcal{E}_{B_R}(v)\\
&= \int_{B_R\setminus B_{R-1}} \left(\frac{1}{p(x)}\alpha(x)|\nabla v|^{p(x)} - F(x,v)\right)\,dX\\
&\le \int_{B_R\setminus B_{R-1}} \left[\sup_{x\in\mathbb{R}^m}\frac{1}{p(x)}\sup_{x\in\mathbb{R}^m}\alpha(x)\Big(|\nabla u|^{p(x)} + |\nabla h|^{p(x)}\Big) + \sup_{\mathbb{R}^m\times[-1,1]}|F|\right]\,dX,
\end{aligned}$$

which implies (B.3). ∎

We would like to remark that the nonlinearities of the type in (1.2) satisfy the assumptions of Lemma B.3. The following is another criterion for obtaining (1.17):

Lemma B.4. *Suppose that $p(x) = p$ is constant.*
Let u be a bounded weak solution of (1.1) *in the whole $\mathbb{R}^n$.*
Let

$$I := \big[-\|u\|_{L^\infty(\mathbb{R}^m)}, \|u\|_{L^\infty(\mathbb{R}^m)}\big].$$

Suppose that there exist $C_0 > 0$ and $\sigma \in [1,2]$ such that

$$\int_{B_R\subset\mathbb{R}^m} \left[\sup_{r\in I}|f(x,r)|\right]\,dx \le C_0 R^{m-\sigma}, \tag{B.4}$$

for any $R \ge C_0$.
Then, there exists $C_1 > 0$ for which

$$\int_{B_R\subset\mathbb{R}^n} \alpha(x)|\nabla u(X)|^p\,dX \le C_1 R^{n-\sigma}, \tag{B.5}$$

for any $R \ge C_1$.
In particular, (1.17) *holds*

(P1) either if $n \le 3$ and $f(x,r) = 0$ for any $(x,r) = (x_1,\dots,x_m,r) \in \mathbb{R}^m\times\mathbb{R}$ such that $|x_1| \ge C_2$,

(P2) or if $m \ge 2$, $n \le 4$ and $f(x,r) = 0$ for any $(x,r) = (x_1,\dots,x_m,r) \in \mathbb{R}^m\times\mathbb{R}$ such that $|x_1| + |x_2| \ge C_2$,

for some $C_2 > 0$.

Proof. The last claim plainly follows from (B.5) (taking $\sigma := 1$ in case (P1) holds and $\sigma := 2$ in case (P2) holds).

Let us now prove (B.5).

For this, we define

$$M := 1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|a\|_{L^\infty(\mathbb{R}^m)} + \sup_{\substack{x\in\mathbb{R}^m \\ |r|\le\|u\|_{L^\infty(\mathbb{R}^n)}}} |f(x,r)|.$$

We take $R \geq \max\{C_0, 1\}$ and we choose $\tau \in C_0^\infty(B_{2R}, [0,1])$, with $\tau = 1$ in B_R and $|\nabla\tau| \leq 4/R$.

We also observe that, by a scaled Young inequality,

$$\begin{aligned} &Mp\alpha(x)\tau^{p-1}|\nabla u|^{p-1}\,|\nabla\tau| = \big((\alpha(x))^{(p-1)/p}\tau^{p-1}|\nabla u|^{p-1}\big)\,\big(Mp(\alpha(x))^{1/p}|\nabla\tau|\big) \\ &\quad\leq \frac{1}{2}\big((\alpha(x))^{(p-1)/p}\tau^{p-1}|\nabla u|^{p-1}\big)^{p/(p-1)} + C_3\big((\alpha(x))^{1/p}|\nabla\tau|\big)^p \\ &\quad= \frac{1}{2}\alpha(x)\tau^p|\nabla u|^p + C_3\alpha(x)|\nabla\tau|^p, \end{aligned} \tag{B.6}$$

for a suitable $C_3 > 0$.

Then, using (1.4) and (B.6),

$$\begin{aligned} &\int_{B_{2R}} \alpha(x)\tau^p|\nabla u|^p\,dX \\ =\;& \int_{B_{2R}} \alpha(x)|\nabla u|^{p-2}\nabla u\cdot\nabla(\tau^p u) - p\alpha(x)u\tau^{p-1}|\nabla u|^{p-2}\nabla u\cdot\nabla\tau\,dX \\ \leq\;& \int_{B_{2R}} \big|f(x,u)\tau^p u\big| + Mp\alpha(x)\tau^{p-1}|\nabla u|^{p-1}\,|\nabla\tau|\,dX \\ \leq\;& M\int_{B_{2R}} \Big[\sup_{r\in I}|f(x,r)|\Big]\,dX \\ &\quad+\frac{1}{2}\int_{B_{2R}} \alpha(x)\tau^p|\nabla u|^p\,dX \\ &\quad+C_3\int_{B_{2R}} \alpha(x)|\nabla\tau|^p\,dX. \end{aligned}$$

This and (B.4) give that

$$\begin{aligned} \frac{1}{2}\int\limits_{B_R\subset\mathbb{R}^n} \alpha(x)|\nabla u|^p\,dX \;&\leq\; \frac{1}{2}\int\limits_{B_{2R}\subset\mathbb{R}^n} \alpha(x)\tau^p|\nabla u|^p\,dX \\ &\leq\; M\int\limits_{B_{2R}\subset\mathbb{R}^{n-m}} \left\{\int\limits_{B_{2R}\subset\mathbb{R}^m} \Big[\sup_{r\in I}|f(x,r)|\Big]\,dx\right\}dy \\ &\qquad+C_3\int\limits_{B_{2R}\subset\mathbb{R}^n} \alpha(x)|\nabla\tau|^p\,dX \\ &\leq\; C_0M\int\limits_{B_{2R}\subset\mathbb{R}^{n-m}} R^{m-\sigma}\,dy + C_4\int\limits_{B_R\subset\mathbb{R}^n} \frac{1}{R^p}\,dX \\ &=\; C_5R^{m-\sigma}R^{n-m} + C_6R^{n-p}, \end{aligned}$$

for suitable C_4, C_5, $C_6 > 0$.

This completes the proof of (B.5). $\square$

C An explicit example

We would like to point out that it is very easy to construct global, bounded, smooth solutions of (1.1). For this, we take $\beta \in C^\infty(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, with

$$\inf_{\mathbb{R}^m} \beta > 0. \tag{C.1}$$

Let also $\gamma \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume that γ is strictly increasing and let Γ its inverse, that is

$$\Gamma\big(\gamma(t)\big) = t \qquad \text{for any } t \in \mathbb{R}. \tag{C.2}$$

We fix $\omega \in \mathrm{S}^{n-m-1}$, and define

$$u(x,y) := \beta(x)\gamma(\omega \cdot y).$$

We also define $g : \mathbb{R}^m \times \mathbb{R}$ to be

$$g(x, \omega \cdot y) := -\mathrm{div}\,\big(\alpha(x)|\nabla u(X)|^{p(x)-2}\nabla u(X)\big).$$

Also, for any $x \in \mathbb{R}^m$ and any $r \in \mathbb{R}$, we set

$$f(x,r) := g\Big(x, \Gamma\big(r/\beta(x)\big)\Big).$$

Notice that this definition is well posed, due to (C.1).
Then, recalling (C.2), it is easy to check that u is a solution of (1.1).

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