Γ CONVERGENCE OF HAUSDORFF MEASURES

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Abstract: We study the dependence of the Hausdorff measure $\mathcal{H}^1_d$ on the distance d. We show that the uniform convergence of d_j to d is equivalent to the Γ convergence of $\mathcal{H}^1_{d_j}$ to $\mathcal{H}^1_d$ with respect to the Hausdorff convergence on compact connected subsets. We also consider the case when distances are replaced by semi-distances.

1. Introduction

In this paper we investigate the sensitivity of the Hausdorff measure $\mathcal{H}^1_d$, in a metric space Q endowed with a distance d, with respect to the distance d. More precisely, our analysis starts from the well known Gołab theorem, which states that the Hausdorff measure $\mathcal{H}^1_d$ is lower semicontinuous for the Hausdorff distance between sets, when we restrict ourselves to the class of subsets which are compact and connected. In other words,

$$\mathcal{H}^1_d(E) \le \liminf_{j\to+\infty} \mathcal{H}^1_d(E_j)$$

whenever E_j and E are compact, connected, and $E_j \to E$ in the Hausdorff convergence.

We show that the Gołab theorem is actually only a part of a more general result where we obtain that, always for the Hausdorff convergence on compact and connected subsets,

$$\mathcal{H}^1_{d_j} \to \mathcal{H}^1_d \text{ for the } \Gamma \text{ convergence}$$

whenever $d_j \to d$ uniformly as a sequence of functions of two variables in $Q \times Q$. Making explicit the meaning of Γ convergence, this gives:

i) for every $E_j \to E$ in the Hausdorff convergence, we have

$$\mathcal{H}^1_d(E) \le \liminf_{j\to+\infty} \mathcal{H}^1_{d_j}(E_j);$$

ii) for every E there exists $E_j \to E$ in the Hausdorff convergence, such that

$$\mathcal{H}^1_d(E) = \lim_{j\to+\infty} \mathcal{H}^1_{d_j}(E_j).$$

The Gołab theorem is then a particular case of i) when we take all the distances d_j equal to d. We actually show that the Γ convergence of $\mathcal{H}^1_{d_j}$ to $\mathcal{H}^1_d$ is equivalent to the uniform convergence of d_j to d, and this is consistent with the results of [4] where the uniform convergence of d_j to d is shown to be equivalent to several other convergences, among which the convergence (still in the Γ sense) of the length functionals defined on curves $\gamma : [0,1] \to Q$

$$L_d(\gamma) := \sup\left\{ \sum_{k=1}^{K} d(\gamma(t_{k-1}), \gamma(t_k)) \,\middle|\, 0 = t_0 \le t_1 \le \ldots \le t_K = 1 \right\}.$$

Some interesting facts occur if we try to remove the assumption that d_j and d are distances, by simply imposing that they are semi-distances, in the sense that the associated metrics may degenerate. In this case we are not able to characterize completely the Γ limit set function as a Hausdorff measure, and we prove only an inequality (see Theorem 4.2).

Finally, in the last section we present some examples and open problems, mainly dealing with the homogenization of distance functions.

2. Preliminaries

Let (Q, e) be a complete metric space, we think of a closed set $Q \subset \mathbb{R}^n$ together with the Euclidean distance $e : Q^2 \to \mathbb{R}$. We consider the set $\mathcal{C}(Q)$ of curves $\gamma : [0,1] \to Q$ that are Lipschitz continuous with respect to the underlying distance e.

Given a distance d on Q, we can define the corresponding d-diameter diam_d of a set A as

$$\operatorname{diam}_d(A) = \sup\{d(x,y) | x, y \in A\}.$$

The length L_d of a Lipschitz curve Γ is independent of the choice of the parametrization and we sometimes write $L_d(\Gamma) := L_d(\gamma)$. We call a distance *geodesic*, if

$$d(x,y) = \inf\left\{L_d(\gamma) | \gamma \in \mathcal{C}(Q), \gamma(0) = x, \gamma(1) = y\right\}.$$

A concept closely related to the length is that of the 1-dimensional Hausdorff measure $\mathcal{H}^1_d$ with respect to d,

$$\mathcal{H}^1_d(A) := \sup_{\delta > 0} \mathcal{H}^1_{d,\delta}(A),$$

where

$$\mathcal{H}^1_{d,\delta}(A) := \inf\left\{ \sum_{i \in I} \operatorname{diam}_d(A_i) \,\middle|\, \operatorname{diam}_d(A_i) < \delta, A \subset \bigcup_{i \in I} A_i \right\}.$$

We can define the previous quantities also in the case that d is only a semi-distance, that is, we only impose $d(x,y) \ge 0$ and not the strict inequality for $x \ne y$. Then $\mathcal{H}^1_d$ is a subadditive functional on $X :=$

$\{A \subset Q | A \text{ compact and connected}\}$. We endow X with the distance τ_e, the Hausdorff distance between sets,

$$\tau_e(A, B) := \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\}.$$

We will use the Theorem of Blaschke (see [7], theorem 3.16) which implies that every bounded sequence of compact non-empty sets has a τ_e-convergent subsequence with limit in X.

By $X_d \subset X$ we denote the set of maximal elements in X with respect to $\mathcal{H}^1_d$. An element A is called maximal if for every $X \ni A' \supset A$, $\mathcal{H}^1_d(A') = \mathcal{H}^1_d(A)$ implies $A' = A$. For a semi-distance d the quotient space $Q_d := Q/d$ endowed with d (evaluated on arbitrarily chosen representants) is a metric space.

Our work continues the investigations of [4], where the equivalence between uniform convergence of distances and Γ-convergence of the length-functionals was shown assuming a uniform bound of the form $d(x, y) \geq \alpha e(x, y)$ for some $\alpha > 0$. Our aim is to remove this restriction; on the contrary, we will still use an upper estimate of the form $d(x, y) \leq Ce(x, y)$.

Lemma 2.1. *Let d be a semi-distance on Q and γ a d-continuous curve connecting x and y. Then*

$$\mathcal{H}^1_d(\gamma([0, 1])) \geq d(x, y).$$

Proof. Given $\varepsilon > 0$ and $\delta > 0$, let A_i be a covering of $A := \gamma([0, 1])$ with $\text{diam}(A_i) < \delta$ such that $\sum_{i \in I} \text{diam}(A_i) \leq \mathcal{H}^1_d(A) + \varepsilon$. Given $t_k \in [0, 1)$ we find $i(k)$ such that $\gamma(t_{k-1}) \in A_{i(k)}$, and define $t_{k+1} := \sup\{t | \gamma(t) \in A_{i(k)}\}$. We set $t_0 = 0$ and repeat the process until $t_K = 1$. Then $d(\gamma(t_k), \gamma(t_{k-1})) \leq \text{diam}(A_{i(k)})$, and we find

$$\begin{aligned} \mathcal{H}^1_d(A) + \varepsilon &\geq \sum_{i \in I} \text{diam}(A_i) \geq \sum_{k=1}^{K} \text{diam}(A_{i(k)}) \\ &\geq \sum_{k=1}^{K} d(\gamma(t_k), \gamma(t_{k-1})) \geq d(x, y). \end{aligned}$$

Since ε was arbitrary we have proved the claim. □

The second part of the next lemma is Theorem 4.4.1 of [2].

Lemma 2.2. *For every semi-distance d we have $\mathcal{H}^1_d \leq L_d$ on $\mathcal{C}(Q)$. Moreover, if d is a distance and $\gamma \in \mathcal{C}(Q)$ is a curve without self-intersections, then*

$$\mathcal{H}^1_d(\gamma([0, 1])) = L_d(\gamma).$$

Proof. For the first part we can assume that $L_d(\gamma) < \infty$. For $\delta > 0$ arbitrary, we will construct a covering of $\gamma([0,1])$ with balls of radius δ. Choose $0 = t_0 < ... < t_k < ... < t_K = 1$ such that $L_d(\gamma([t_{k-1}, t_k])) \le \delta$. This is possible, since $L_d(\gamma([0,1]))$ is finite. By definition of L_d we have

$$L_d(\gamma) = \sum_{k=1}^{K} L_d(\gamma|_{[t_{k-1},t_k]}) =: \sum_{k=1}^{K} l_k.$$

For every k we choose $\bar{t}_k \in (t_{k-1}, t_k)$ such that $L_d(\gamma|_{[t_{k-1},\bar{t}_k]}) = \frac{1}{2} l_k$. Such $\bar{t}_k$ exists, it is not necessarily uniquely defined. We set $U_k := B^d_{l_k/2}(\gamma(\bar{t}_k))$. Then $\gamma([0,1]) \subset \bigcup_{k=1}^{K} U_k$, since any $x \in \gamma([t_{k-1}, \bar{t}_k])$ satisfies $d(x, \gamma(\bar{t}_k)) \le L_d(\gamma([t_{k-1}, \bar{t}_k])) \le l_k/2$, and the same for $x \in \gamma([\bar{t}_k, t_k])$. Since $\mathrm{diam}_d(U_k) \le l_k \le \delta$,

$$\mathcal{H}^1_{d,\delta}(\gamma([0,1])) \le \sum_{k=1}^{K} l_k = L_d(\gamma)$$

and, taking the supremum with respect to $\delta > 0$, the inequality $\mathcal{H}^1_d \le L_d$ follows.

We now show $\mathcal{H}^1_d \ge L_d$ in the case that d is a distance and γ is free of self-intersections. Let $\varepsilon > 0$ be arbitrary and $0 = t_0 < \ldots < t_K = 1$ such that

$$L_d(\gamma) \le \sum_{k=1}^{K} d(\gamma(t_k), \gamma(t_{k-1})) + \varepsilon.$$

If d is a distance, the Hausdorff measure is additive and therefore

$$\begin{aligned}\mathcal{H}^1_d(\gamma([0,1])) &= \sum_{k=1}^{K} \mathcal{H}^1_d(\gamma((t_{k-1}, t_k))) \overset{L.2.1}{\ge} \sum_{k=1}^{K} d(\gamma(t_k), \gamma(t_{k-1})) \\ &\ge L_d(\gamma) - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we found the result. Note that in the case of a semi-distance d we do not have the additivity of the Hausdorff measure that we used in the last line. $\square$

It is easy to construct curves γ with self-intersections such that

$$\mathcal{H}^1_e(\gamma([0,1])) < L_e(\gamma).$$

Every curve that repeats part of itself has this property. Less obvious is an example of a semi-distance d and a curve without self-intersections such that length and 1-dimensional Hausdorff-measure do not coincide.

Example 2.3. We consider the semi-distance d on $\bar{B}_1(0) \subset \mathbb{R}^2$, $d(x, y) = ||x| - |y||$, and the curve $\gamma(t) = -1 + 2t$ connecting $(-1, 0)$

with $(1,0)$. Then

$$\mathcal{H}^1_d(\gamma([0,1])) = 1 < 2 = L_d(\gamma).$$

The above equalities follow from the fact that the length of the segment AB is the sum of the lengths of AC and CB, while $\mathcal{H}^1_d(AB) = \mathcal{H}^1_d(AC)$, since every covering of AC with closed sets automatically covers AB. Note that in this example the set $[0,1] \times \{0\}$ is not measurable for $\mathcal{H}^1_d$ in the sense of Carathéodory.

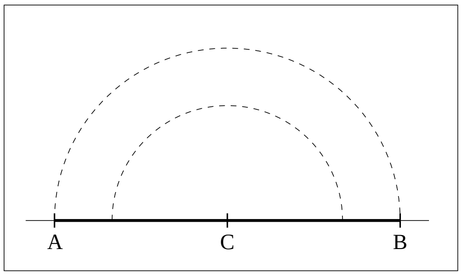

FIGURE 1. Illustration of the semi-distance of example 2.3

The above example motivates the use of curves without d-self-intersections in our forthcoming constructions. Another aspect will concern the continuity of the curves. We will see in example 2.5 that the use of e-continuous curves is too restrictive in the case of semi-distances. Note that for a semi-distance d, a d-continuous curve satisfies $d(\gamma(t), \gamma(s)) \to 0$ for $s \to t$, and need not to be e-continuous.

Definition 2.4 (d-curves and trees)**.** *We say that a curve γ is free of d-self-intersections if $d(\gamma(t), \gamma(s)) = 0, s < t$ implies $d(\gamma(t), \gamma(\tau)) = 0$ for every $\tau \in [s,t]$. We say that γ is a d-curve if it is d-continuous and free of d-self-intersections.*

A tree in a metric space (Y,d) is a d-compact set such that two points $x, y \in Y$ are joined by a unique d-rectifiable path.

Up to constant parts, every d-curve γ can be interpreted as an injective continuous curve $\tilde{\gamma}$ in $(Q/d, d)$, and, vice versa, every injective continuous curves $\tilde{\gamma}$ in $(Q/d, d)$ has a representative γ which is a d-curve.

In example 2.3 the set $E := \gamma([0,1])$ contains no curve connecting $(-1,0)$ with $(1,0)$ whose length is bounded by the d-Hausdorff measure of E. But in this example E is not a maximal set for the semi-distance d, since $E' := \bar{B}_1$ contains E and has the same 1-dimensional d-Hausdorff-measure. In contrast, the set Σ in the next example is a maximal set.

Example 2.5. On $[-1,0] \times [-1,1]$ we consider the subset $\Sigma := \{(x, \sin(1/x)) | x \in [-1,0)\} \cup (\{0\} \times [-1,1])$, and define the semi-distance d as $d(x,y) := \min\{e(x,y), e(x,\Sigma) + e(y,\Sigma)\}$. Then Σ is maximal with respect to d and its d-Hausdorff measure vanishes (in particular it is finite). But there is no e-continuous curve that connects $(x_1, x_2) \in \Sigma$, $x_1 < 0$, with $(0,0) \in \Sigma$.

In this example $\gamma(t) = (x_1, x_2)$ for every $t < 1$ and $\gamma(1) = (0,0)$, is a d-curve connecting the two points in Σ.

Lemma 2.6 (d-curves and d-trees)**.** *For every d-curve γ we have*

$$\mathcal{H}^1_d(\gamma([0,1])) = L_d(\gamma).$$

Moreover, every tree $E = \bigcup_k E_k \subset Q/d$ composed of continuous injective curves E_k in Q/d has a representative $\bar{E} = \bigcup_k \bar{E}_k \subset Q$ consisting of d-curves $\bar{E}_k$ such that

$$\mathcal{H}^1_d(\bar{E}) = \sum_k \mathcal{H}^1_d(\bar{E}_k).$$

Proof. We note that every d-curve γ has a corresponding curve $\tilde{\gamma}$ in $Q_d = (Q/d, d)$ that is free of self-intersections. We find

$$\mathcal{H}^1_d(\gamma([0,1])) = \mathcal{H}^1_d(\tilde{\gamma}([0,1])) = L_d(\tilde{\gamma}) = L_d(\gamma)$$

by the second part of lemma 2.2. In the first and last equality we use that the definitions of L_d and of $\mathcal{H}^1_d$ yield the same values for γ and $\tilde{\gamma}$.

In order to choose representatives in a unique way, we select a family of points $\bar{Q} \subset Q$ such that $\bar{Q} \ni x \mapsto \tilde{x} = \{q \in Q | d(q,x) = 0\} \in Q/d$ is bijective. Given a tree E we can represent every $E_k \subset Q/d$ by a d-curve $\bar{E}_k \subset \bar{Q}$ and define $\bar{E} = \bigcup_k \bar{E}_k$. By the additivity of the Hausdorff-measure for a distance we find

$$\mathcal{H}^1_d(\bar{E}) = \mathcal{H}^1_d(E) = \sum_k \mathcal{H}^1_d(E_k) = \sum_k \mathcal{H}^1_d(\bar{E}_k),$$

which was the claim. ☐

We next quote a theorem that is classical for subsets of $\mathbb{R}^n$ (see e.g. [7], theorem 3.12), and is shown for metric spaces in [2], theorem 4.4.4.

Theorem 2.7. *If E is a complete metric space and $C \subset E$ is a closed connected set such that $\mathcal{H}^1_d(C) < \infty$, then C is compact and connected by injective rectifiable curves.*

We will use the following corollary of this theorem.

Corollary 2.8. *Let d be a semi-distance on Q satisfying $d(x,y) \leq Ce(x,y)$ for every $x, y \in Q$, and let A be closed and connected with* *respect to e with $\mathcal{H}^1_d(A) < \infty$. Then the d-closure of A is d-path-connected, i.e. for x, y in the d-closure of A exists a d-curve connecting x and y.*

Proof. We consider $\tilde{A} \subset (Q/d, d)$, the d-closure of the points $\{\tilde{x} \in Q/d | \tilde{x} = \{q \in Q : d(x, q) = 0\}, x \in A\}$. Then $\tilde{A}$ is closed and connected in $(Q/d, d)$ with $\mathcal{H}^1_d(\tilde{A}) = \mathcal{H}^1_d(A) < \infty$. Theorem 2.7 yields the existence of a d-rectifiable curve $\tilde{\gamma}$ without self-intersections connecting x and y in $\tilde{A}$. We can represent $\tilde{\gamma}$ with a d-curve γ in the d-closure of A connecting x and y. □

3. On families of semi-distances converging to a distance

The following lemma is similar to lemma 3.17 of [7] in a metric space; the idea stems from [6]. Note that the result is in general wrong for semi-distances d.

Lemma 3.1. *Let (Q, d) be a metric space, $F \subset Q$ a tree, $\mathcal{H}^1_d(F) < \infty$. Then for every $\delta > 0$ there exists a family $(F_i)_{i=1,\dots,k}$ such that $F = \bigcup_{i=1}^k F_i$ and*
(a) $\sum_i \operatorname{diam}_d(F_i) \le \sum_i \mathcal{H}^1_d(F_i) = \mathcal{H}^1_d(F)$.
(b) $\operatorname{diam}_d(F_i) \le \delta$ *for every* i.
(c) $k \le 3\delta^{-1}\mathcal{H}^1_d(F) + 1$.

Proof. The proof follows that of Falconer [7]; we repeat it for completeness. Note that the proof does not work if d is only a semi-distance since we use the additivity of the Hausdorff measure and the second part of lemma 2.2.

In the case $\operatorname{diam}_d(F) \le \delta$ we do not have to decompose F. Assume therefore $\operatorname{diam}_d(F) > \delta$. Let x be any point of F, and let m denote the supremum of the d-path-distance of points in F from x; then by lemma 2.2 $m \le \mathcal{H}^1_d(F) < \infty$, while $m > \delta/2$ because of $\operatorname{diam}_d(F) \ge \delta$. Let y be a point of F at d-path-distance greater than $m - \delta/6$ from x, and let z be the point on the unique path joining x and y at d-path-distance $m - \delta/2$ from x. The point z determines a dissection of F into two subtrees F_1 and F' with z as their only commen point, where F_1 consists of those points of F whose joins to x pass through z. Every point of F_1 is within d-path-distance $\delta/2$ of z, so $\operatorname{diam}_d(F_1) \le \delta$. By the second part of lemma 2.2, $\mathcal{H}^1_d(F_1)$ is greater than the d-path-distance from y to z, which is at least $(m - \delta/6) - (m - \delta/2) = \delta/3$.

If $\operatorname{diam}_d(F') > \delta$ we repeat this process with the tree F' to break off a subtree F_2, and so on, until we are left with a tree F_k of diameter, at most, δ. Parts (a) and (b) of the conclusion are immmediate using the subadditivity of $\mathcal{H}^1$, while (c) follows from (a) since $\delta/3 < \mathcal{H}^1_d(F_i)$ for all $i = 1, \dots, k-1$. □

In the proof of the next lemma we use the following elementary equivalence for a locally compact space (Q, e) and a distance on Q with $d \leq Ce$.

$$(x_k)_k \; e-\text{bounded}, \; d(x_k, x) \to 0 \iff e(x_k, x) \to 0. \tag{3.1}$$

The implication $' \Leftarrow'$ is immediate by $d \leq Ce$. For the converse implication we note that by local compactness of (Q, e) the sequence x_k has an e-convergent subsequence, $x_{k(j)} \to \bar{x} \in Q$. By the triangle inequality $d(x, \bar{x}) \leq d(x, x_{k(j)}) + d(x_{k(j)}, \bar{x}) \leq d(x, x_{k(j)}) + Ce(x_{k(j)}, \bar{x}) \to 0$. Since d is a distance, we infer $x = \bar{x}$. The limit of subsequences is unique, therefore the whole sequence x_k converges to x.

Lemma 3.2. *Let $d \leq Ce$ be a distance and d_j a sequence of geodesic semi-distances converging uniformly to d. Then for every e-bounded and d-continuous curve γ there exists a sequence of curves γ_j with $\gamma_j([0,1]) \to \gamma([0,1])$ in τ_e and such that*

$$\limsup_{j\to\infty} L_{d_j}(\gamma_j) \leq L_d(\gamma). \tag{3.2}$$

Proof. We can assume $L_d(\gamma) < \infty$. We set $\varepsilon = 1/N$ and choose for $K = K(N)$ time instances $0 = t_0 < \ldots < t_K = 1$ such that $L_d(\gamma|_{(t_{k-1},t_k)}) \leq \varepsilon$ for every $k \leq K$. For two points $x = x_k = \gamma(t_{k-1})$ and $y = y_k = \gamma(t_k)$ we find an index $J(k)$ such that $d_j(x, y) \leq d(x, y) + \frac{\varepsilon}{2K}$ for every $j \geq J(k)$. Since d_j is geodesic, we find an e-continuous curve $\gamma_{k,j}$ connecting x and y with

$$L_{d_j}(\gamma_{k,j}) \leq d_j(x, y) + \frac{\varepsilon}{2K}.$$

Connecting the curves $(\gamma_{k,j})_k$ to a single curve $\bar{\gamma}_j$ we find

$$\begin{aligned} L_{d_j}(\bar{\gamma}_j) &= \sum_{k=1}^{K} L_{d_j}(\gamma_{k,j}) \leq \sum_{k=1}^{K} \left[d_j(x_k, y_k) + \frac{\varepsilon}{2K} \right] \\ &\leq \sum_{k=1}^{K} \left[d(x_k, y_k) + \frac{\varepsilon}{K} \right] \leq L_d(\gamma) + \varepsilon \qquad \forall j \geq J_0(K). \end{aligned}$$

Here we set $J_0(K) = \max\{J(k) : k = 1, \ldots, K\}$. In the above construction the index $J_0(K) = J_0(K(N))$ and the curves $\bar{\gamma}_j = \gamma_j^N$ both depend on N. We now construct the final sequence γ_j by setting $\gamma_j := \bar{\gamma}_j^N$ for every $j = \max\{N, J_0(K(N))\}, \ldots, \max\{N+1, J_0(K(N+1))\} - 1$ in step N. Note that the set of j on the right hand side may be empty. The curves γ_j are chosen arbitrary for indices $j \leq J_0(K(1))$. We find for arbitrary $\varepsilon = 1/N$ an index $j_0 = J_0(K(N))$ such that $L_{d_j}(\gamma_j) \leq L_d(\gamma) + \varepsilon$ for all $j \geq j_0$, and therefore (3.2).

It remains to show that $\Gamma_j := \gamma_j([0,1]) \to \Gamma := \gamma([0,1])$ in τ_e. By the e-connectedness of Γ_j it suffices to show that

$$\Gamma = \{x | x \text{ is an } e\text{-limit point of } (\Gamma_j)_j\}. \tag{3.3}$$

Indeed, choose R large enough and a ball B^e_R to have $\Gamma \subset B^e_R$. The e-compact family $\Gamma_j \cap \bar{B}^e_{2R}$ has some τ_e-limit Γ^R by the theorem of Blaschke. (3.3) implies $\Gamma^R = \Gamma$. We conclude that from some index j_0 on, $\Gamma_j \cap (\bar{B}^e_{2R} \setminus B^e_R) = \emptyset$. By e-connectedness we have $\Gamma_j \subset \bar{B}^e_R$ for every $j \geq j_0$. Then $\Gamma_j \to \Gamma$ in τ_e.

We prove the inclusion '$\subset$' in (3.3). $L_d(\gamma|_{(t_{k-1},t_k)}) \leq \varepsilon$ for every k implies that every point $x \in \Gamma$ is a d-limit point of $(x_k)_k = (\gamma(t_{k-1}))_k$. By $x_k \in \Gamma_j$, x is also d-limit point of Γ_j. Since Γ is e-bounded we can use (3.1) to see that x is also an e-limit point of Γ_j.

We prove now the inclusion '$\supset$' in (3.3). Assume that $x \in Q$ is an e-limiting point of Γ_j with $x \notin \Gamma$. Then $d(x, \Gamma) > 0$, since d is a distance. Since x is an e-limit point it is also a d-limit point, and for every j_0 we find $j \geq j_0$ such that $d(x, \Gamma_j) < d(x, \Gamma)/2$. We choose $j_0 = J_0(K(N))$ with N large enough to have $\varepsilon_N = 1/N < d(x, \Gamma)/4$. The distance of x to Γ_j is attained at some piece $\gamma_{k,j} := \gamma|_{[t_{k-1},t_k]}$ for some $k = k(j)$, that is, $d(x, \gamma_{k,j}([0,1])) < d(x, \Gamma)/2$. Since $L_{d_j}(\gamma_{k,j}) \leq d(x_k, y_k) + \varepsilon/K \leq \varepsilon(1 + 1/K)$ we find

$$\begin{aligned} d(x, \Gamma) \leq d(x, x_k) &\leq d(x, \gamma_{k,j}([0,1])) + L_{d_j}(\gamma_{k,j}) \\ &< d(x, \Gamma)/2 + d(x, \Gamma)/2, \end{aligned}$$

which is a contradiction. □

Theorem 3.3. *Let (Q, d) be a metric space and $d_j \leq Ce$ a family of geodesic semi-distances on Q with $d_j \to d$ uniformly. Then*

$$\mathcal{H}^1_{d_j} \xrightarrow{\Gamma} \mathcal{H}^1_d$$

as functionals on (X, τ_e).

Proof. We have to verify the following two properties.

$$\forall E_j \to E \text{ in } (X, \tau_e): \ \mathcal{H}^1_d(E) \leq \liminf_{j \to \infty} \mathcal{H}^1_{d_j}(E_j), \tag{3.4}$$

$$\exists E_j \to E \text{ in } (X, \tau_e): \ \mathcal{H}^1_d(E) \geq \limsup_{j \to \infty} \mathcal{H}^1_{d_j}(E_j). \tag{3.5}$$

The idea to prove (3.4) is to repeat the proof of Falconer [7], theorem 3.18, where lower semicontinuity of $\mathcal{H}^1_e$ is shown, i.e. the theorem of Gołab

$$\mathcal{H}^1_e(E) \leq \liminf_{j \to \infty} \mathcal{H}^1_e(E_j).$$

The idea to prove (3.5) is to follow the proof of theorem 3.1 of [4], where the corresponding result is shown for the length functional. We cannot apply this result, since $d \geq \alpha e$ for some $\alpha > 0$ was used there.

Proof of (3.4), not using that d is a distance. Without loss of generality we can assume $\mathcal{H}^1_{d_j}(E_j) \leq C_0 < \infty$ for every j. We can take the d_j-closure of E_j without changing $\mathcal{H}^1_{d_j}(E_j)$ and the d_j-connectedness. We can therefore assume that every E_j is d_j-closed. We choose finite subsets $S_j \subset E_j$ such that $S_j \to E$ with respect to τ_e, and d_j-continuous trees F_j connecting the points S_j/d_j within E_j/d_j (using theorem 2.7). Using lemma 3.1, we can write $F_j = \bigcup_{i=1}^k F_{ji}$ with k independent of j and

$$\operatorname{diam}_{d_j}(F_{ji}) \leq \delta \quad \forall i, j, \tag{3.6}$$

$$\sum_{i=1}^{k} \operatorname{diam}_{d_j}(F_{ji}) \leq \mathcal{H}^1_{d_j}(F_j). \tag{3.7}$$

We next choose representatives $\bar{F}_{ji} \subset Q$ consisting of d_j-curves and set

$$\hat{F}_{ji} := \bar{F}_{ji} \cup \left\{ x \in S_j | d_j(x, \bar{F}_{ji}) = 0 \right\},$$

which does not alter property (3.6) or (3.7). In this way we find that $\hat{F}_j := \bigcup_i \hat{F}_{ji} \to E$ in τ_e.

We then find subsequences $j_l \to \infty$ such that $\hat{F}_{ji} \to H_i$ in τ_e for some set H_i. We next calculate a bound for the d-diameter of H_i. For every $x, y \in H_i$ there exist $x_j, y_j \in \hat{F}_{ji}$ such that $x_j \to x$ and $y_j \to y$ with respect to e. By uniform convergence $d_j \to d$ we find

$$d(x, y) = \lim_{j\to\infty} d_j(x_j, y_j) \leq \liminf_{j\to\infty} \operatorname{diam}_{d_j}(\hat{F}_{ji}). \tag{3.8}$$

In particular we find $\operatorname{diam}_d H_i \leq \delta$ using (3.6). Then the family H_i provides a covering of E with sets of d-diameter bounded by δ, so that

$$\begin{aligned}
\mathcal{H}^1_{d,\delta}(E) \leq \sum_{i=1}^{k} \operatorname{diam}_d H_i &\overset{(3.8)}{\leq} \liminf_{j\to\infty} \sum_{i=1}^{k} \operatorname{diam}_{d_j}(\hat{F}_{ji}) \overset{(3.7)}{\leq} \liminf_{j\to\infty} \mathcal{H}^1_{d_j}(\hat{F}_j) \\
&\overset{(\bar{F}_j \subset E_j)}{\leq} \liminf_{j\to\infty} \mathcal{H}^1_{d_j}(E_j).
\end{aligned}$$

Since $\delta > 0$ was arbitrary, this shows inequality (3.4).

Proof of (3.5). We can assume $\mathcal{H}^1_d(E) < \infty$. Since d is a distance, the e-closed set E is also d-closed: Assume $E \ni a_n \to a$ with respect to d. Then by compactness of E a subsequence satisfies $a_{n(k)} \to \bar{a} \in E$ with respect to e. By $d \leq Ce$ the point $\bar{a}$ is also a d-limit and therefore $\bar{a} = a$ is in E.

We choose finite sets of points S_i such that $\bigcup S_i$ is dense in E with respect to τ_e, and connect the points of S_i with d-curves E_{ik} in E using corollary 2.8. This yields a sequence of trees $E \supset E_i := \bigcup_k E_{ik} \to E$ in τ_e. Note that $E_i \subset E$ implies that all E_{ik} are e-bounded. By lemma 3.2, for every E_{ik} we find approximating curves E_{ik}^j converging to E_{ik} in τ_e for $j \to \infty$, and such that

$$L_d(E_{ik}) \geq \limsup_{j\to\infty} L_{d_j}(E_{ik}^j).$$

By lemma 2.2, L_d and $\mathcal{H}_d^1$ coincide on E_{ik} and, setting $\tilde{E}_i^j := \bigcup_k E_{ik}^j$, we find

$$\begin{aligned}
\mathcal{H}_d^1(E) &\overset{E_i\subset E}{\geq} \mathcal{H}_d^1(E_i) = \sum_k \mathcal{H}_d^1(E_{ik}) = \sum_k L_d(E_{ik}) \\
&\geq \sum_k \limsup_{j\to\infty} L_{d_j}(E_{ik}^j) \overset{L.2.2}{\geq} \sum_k \limsup_{j\to\infty} \mathcal{H}_{d_j}^1(E_{ik}^j) \\
&\geq \limsup_{j\to\infty} \mathcal{H}_{d_j}^1(\tilde{E}_i^j),
\end{aligned}$$

using subadditivity of $\mathcal{H}_{d_j}^1$. For every $i > 0$ we find $j = j(i)$ such that $\tilde{E}_i^j$ is close to E_i in the Hausdorff distance τ_e, and $\mathcal{H}_{d_j}^1(\tilde{E}_i^{j(i)}) \leq \mathcal{H}_d^1(E) + \frac{1}{i}$. Then $\tilde{E}_i := \tilde{E}_i^{j(i)} \to E$ for $i \to \infty$ and

$$\limsup_{i\to\infty} \mathcal{H}_{d_j}^1(\tilde{E}_i) \leq \mathcal{H}_d^1(E)$$

as required. □

The theorem above allows us to analyze the periodic and the stochastic homogenization of microscopic, disconnected obstacles or conductors. Indeed, in these examples, even if the approximating structures give raise to semi-distances, the limit turns out to be a distance (see section 6).

4. The case of a limiting semi-distance.

Lemma 4.1. *Let d be a semi-distance, Γ the image of a d-continuous curve, and d_j a sequence of geodesic semi-distances converging uniformly to d. Then there exists a sequence of e-continuous curves Γ_j with the same end-points as Γ satisfying*

$$\limsup_{j\to\infty} L_{d_j}(\Gamma_j) \leq L_d(\Gamma).$$

Proof. The proof is obtained by following step by step the proof of lemma 3.2. The fact that d is a distance is used there in order to show that the curves Γ_j converge to Γ in τ_e. For a semi-distance d this is not necessarily true. □

Theorem 4.2. *Assume that Q is e-compact. Let $d_j \to d$ uniformly, with $d_j \leq Ce$ and d semi-distances on Q, and assume that all d_j are geodesic. Then, for some $F : X \to \mathbb{R}$,*

$$\mathcal{H}^1_{d_j} \xrightarrow{\Gamma} F \geq \mathcal{H}^1_d$$

as functionals on the space (X, τ_e). Moreover, on the subset X_d of d-maximal sets we have

$$F = \mathcal{H}^1_d \quad \text{on } X_d.$$

Proof. The inequality $F \geq \mathcal{H}^1_d$ on X, that is (3.4), is shown in theorem 3.3. Concerning inequality (3.5), it remains to find, for every $E \in X_d$, a sequence $F_l \to E$ with $\limsup_l \mathcal{H}^1_{d_l}(F_l) \leq \mathcal{H}^1_d(E)$; we can assume $\mathcal{H}^1_d(E) < \infty$. We consider the d-closure $\bar{E}$ of E, choose an e-dense family of finite sets $S_i \subset \bar{E}$, and connect S_i/d in $\bar{E}/d$ using theorem 2.7. Lemma 2.6 yields the existence of an e-dense family of trees E_i made up of d-curves E_{ik}. Approximating every curve E_{ik} as in lemma 4.1 we obtain

$$\mathcal{H}^1_d(E_i) \overset{L.2.6}{=} \sum_{k=1}^{K(i)} L_d(E_{ik}) \overset{L.4.1}{\geq} \sum_{k=1}^{K(i)} \limsup_{j\to\infty} L_{d_j}(E^j_{ik}).$$

A subsequence of the family $E^j_i := \bigcup_{k=1}^{K(i)} E^j_{ik}$ converges (for $j \to \infty$) in τ_e to an e-connected set $\hat{E}_i \supset S_i$. For a subsequence $i \to \infty$ and some $\hat{E} \subset Q$ we also find $\hat{E}_i \to \hat{E}$ for $i \to \infty$ in τ_e, and $\hat{E}$ is again connected in (Q, e). To conclude the proof it remains to show that $\hat{E} = E$. Since $(S_i)_i$ is dense in E we have $\hat{E} \supset E$ and so

$$\begin{aligned}
\mathcal{H}^1_d(E) = \mathcal{H}^1_d(\bar{E}) &\overset{\bar{E}\supset E_i}{\geq} \mathcal{H}^1_d(E_i) \geq \sum_{k=1}^{K(i)} \limsup_{j\to\infty} L_{d_j}(E^j_{ik}) \\
&\overset{L.2.2}{\geq} \sum_{k=1}^{K(i)} \limsup_{j\to\infty} \mathcal{H}^1_{d_j}(E^j_{ik}) \geq \liminf_{j\to\infty} \sum_{k=1}^{K(i)} \mathcal{H}^1_{d_j}(E^j_{ik}) \\
&\geq \liminf_{j\to\infty} \mathcal{H}^1_{d_j}(E^j_i) \overset{(3.4)}{\geq} \mathcal{H}^1_d(\hat{E}_i).
\end{aligned}$$

Applying (3.4) again for the trivial sequence $d_j = d$ (theorem of Gołab) we find

$$\mathcal{H}^1_d(E) \geq \mathcal{H}^1_d(\hat{E}).$$

Since E was maximal we infer $E = \hat{E}$ and therefore $E^{j(i)}_i \to E$ for some appropriate subsequence $j(i)$. □

One might ask if the stronger result $\mathcal{H}^1_{d_j} \to F = \mathcal{H}^1_d$ in the above theorem is true. This is not the case: in subsection 6.3 we provide an example with $F \neq \mathcal{H}^1_d$.

5. The converse implication: limits of geodesic distances

In this section we assume again the e-compactness of Q. Our aim is to show the converse implication of theorem 4.2: The Γ-convergence of the Hausdorff measures implies the uniform convergence of the distance functions.

Lemma 5.1. *Let $d_j \to d$ be a uniformly convergent sequence of geodesic semi-distances satisfying $d_j \leq Ce$ and, in the sense of Γ-convergence,*

$$\mathcal{H}^1_{d_j} \to F \geq \mathcal{H}^1_d$$

with equality on the d-maximal subsets of Q. Then

$$\begin{aligned} d(x,y) &= \inf\{\mathcal{H}^1_d(\gamma([0,1]))|\gamma\ d - \textit{curve connecting } x \textit{ and } y\} \\ &= \inf\{\mathcal{H}^1_d(A)|A \in X,\ x,y \in A\} \\ &= \lim_{j\to\infty} \inf\{\mathcal{H}^1_{d_j}(A)|A \in X,\ x,y \in A\}. \end{aligned} \tag{5.1}$$

Proof. We have the following chain of inequalities using lemma 2.1 for d-curves in the first line (in the second line we use that for an e-connected set A each two of its points can be connected with a d-curve in the d-closure of A):

$$\begin{aligned} d(x,y) &\overset{L.2.1}{\leq} \inf\left\{\mathcal{H}^1_d(\gamma([0,1]))|\gamma\ d - \text{curve connecting } x \text{ and } y\right\} \\ &\overset{Cor.2.8}{\leq} \inf\{\mathcal{H}^1_d(A)|A \in X,\ x,y \in A\} \\ &\leq \inf\{F(A)|A \in X,\ x,y \in A\} \\ &\overset{(\Gamma)}{=} \lim_{j\to\infty} \inf\{\mathcal{H}^1_{d_j}(A)|A \in X,\ x,y \in A\} \\ &= \lim_{j\to\infty} d_j(x,y) = d(x,y). \end{aligned}$$

In the equality (Γ) we used the Γ-convergence of $\mathcal{H}^1_{d_j}$ to F and the compactness of the space (X, τ_e). We then conclude that equality holds in all lines. □

Remark 5.2. Note that in the case of a distance d the infimum in the second line is actually a minimum (see [2]). This is not true for a semi-distance (see example 2.5).

Proposition 5.3. *Let $d_j \le Ce$ be a family of geodesic semi-distances such that for some semi-distance d*

$$\mathcal{H}^1_{d_j} \xrightarrow{\Gamma} F \geq \mathcal{H}^1_d$$

with equality on the d-maximal subsets of Q. Then $d_j \to d$ uniformly.

Proof. The family $d_j : Q^2 \to \mathbb{R}$ is uniformly continuous and we find a subsequence $d_j \to \tilde{d}$ uniformly for some $\tilde{d}$. It suffices to show that $\tilde{d} = d$. We have, by theorem 4.2,

$$\mathcal{H}^1_{d_j} \to F \geq \mathcal{H}^1_{\tilde{d}}$$

with equality on the $\tilde{d}$-maximal subsets of Q. We apply lemma 5.1 to d and $\tilde{d}$ and find

$$d(x,y) = \lim_{j\to\infty} \inf\{\mathcal{H}^1_{d_j}(A) | A \in X,\ x,y \in A\} = \tilde{d}(x,y).$$

Therefore the two distance functions coincide, as required. □

6. Examples

6.1. **Periodic homogenization.** We consider periodically distributed 'obstacles' and study shortest paths between points. We define the length of a curve as the euclidean length of that part of the curve that does not hit an obstacle (see figure 2). This problem appears in the homogenization of a stationary problem in fluid mechanics, where surface tension forces the liquid to minimize the perimeter of the occupied area (here the wetting energy of the obstacles is neglected). The corresponding variational problem is given by (P) (compare [8] for a nonstationary analysis).

(P) Given a convex macroscopic domain $Q \subset \mathbb{R}^2$ we choose a closed set $B \subset (0,1)^2$ and define $S^\varepsilon := \varepsilon(\mathbb{Z}^2 + B) \cap Q$. We are interested in paths Γ between points $x, y \in Q$ that minimize the functional

$$F_\varepsilon(\Gamma) := \mathcal{H}^1_e(\Gamma \setminus S^\varepsilon).$$

Problem (P) defines a geodesic semi-distance d_ε given by

$$d_\varepsilon(x,y) := \inf\{F_\varepsilon(\Gamma) | \Gamma \in \mathcal{C}(Q), x, y \in \Gamma\}.$$

The distances d_ε have a uniform limit d, and theorem 3.3 implies that the functionals F_ε Γ-converge to the Hausdorff-measure $\mathcal{H}^1_d$.

Large obstacles: If the obstacles are large enough we find for $a_1 := \mathrm{dist}(B, (1,0) + B)$ and $a_2 := \mathrm{dist}(B, (0,1) + B)$ that $\mathrm{dist}(B, (k_1, k_2) + B) \geq k_1 a_1 + k_2 a_2$ for all positive integers k_1, k_2. In this case one calculates that d_ε converges uniformly to the 'Manhattan-distance' $d(x,y) = a_1|x_1 - y_1| + a_2|x_2 - y_2|$. Theorem 3.3 yields $\mathcal{H}^1_{d_\varepsilon} \to \mathcal{H}^1_d$ in the sense of Γ-convergence. In particular, for a family Γ_ε of minimizers of

F_ε, a subsequence of Γ_ε converges to a minimizer of $\mathcal{H}^1_d$. Note that for x, y with $x_1 \neq y_1$ and $x_2 \neq y_2$, not every sequence of minimizers has the property to stay away from planes parallel (but not identical) to the straight connection of x and y. This relates to a property studied in [5] for distances.

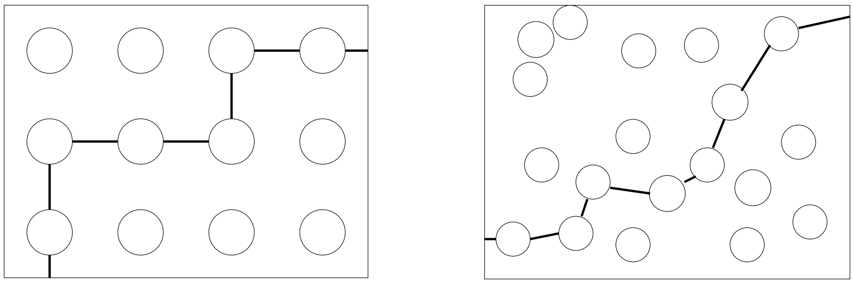

FIGURE 2. Shortest paths in a periodic and in a stochastic geometry

Obstacles of arbitrary shape: In the general case the limiting distance function can be expressed by

$$d(x,y) = \inf\left\{ \int_0^1 \Phi(\gamma'(t))\ dt \,\middle|\, \gamma \in \mathcal{C}(Q), \gamma(0) = x, \gamma(1) = y \right\}.$$

Here the Finsler function Φ is obtained via the characteristic function a of $\mathbb{R}^2 \setminus (\mathbb{Z}^2 + B)$ as (see, for instance, [1], [3])

$$\Phi(z) = \lim_{T\to\infty} \left[\inf\left\{ \frac{1}{T} \int_0^1 a(\gamma)|\gamma'| \,\middle|\, \gamma \in \mathcal{C}(\mathbb{R}^2), \gamma(0) = 0, \gamma(1) = Tz \right\} \right].$$

We conjecture that, independently of the obstacle B, the periodic homogenization never yields an isotropic limiting distance function.

6.2. **Stochastic homogenization.** A way to obtain an isotropic limiting distance function is to consider a random distribution of obstacles. Let obstacles of size 1 be distributed stochastically on $\mathbb{R}^2$ to define $S \subset \mathbb{R}^2$. We assume that the probability measure on the space of configurations is invariant under translations and rotations. We study problem (P) on the set $\varepsilon S \cap Q$. In this setting we expect that for almost every realization of obstacles the distances d_ε converge to an isotropic and translation invariant limit, that is, $d_\varepsilon \to d$ uniformly for $d = \alpha e$, $\alpha \in \mathbb{R}$, and e the Euclidean distance. By theorem 4.2 this information suffices to see that the functionals F_ε converge in the sense of Γ convergence to the Hausdorff measure $\alpha \mathcal{H}^1_e$. In particular, for

$\alpha > 0$, every minimizing sequence Γ_ε converges to a straight line, the Euclidean geodesic.

6.3. **Homogenization with macroscopic obstacles.** On $Q = [0,\pi] \times [-1,1]$ we consider the set

$$\Sigma^j := \{(x, \sin(jx)) | x \in [0,\pi]\},$$

and the semi-distances d_j defined as $d_j(x,y) := \min\{e(x,y), e(x,\Sigma^j) + e(y,\Sigma^j)\}$. In this setting shortest paths are either straight lines or shortest paths from x and y to Σ^j, connected by a curve contained in Σ^j. The limiting semi-distance is for $S = [0,\pi] \times [0,1]$ given by $d(x,y) = \min\{e(x,y), e(x,S) + e(y,S)\}$.

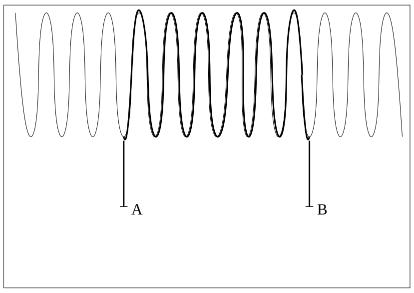

FIGURE 3. A shortest path in the presence of a macroscopic obstacle

We study shortest paths A_j connecting $x = (0,-1/3)$ with $y = (1,-1/3)$ for large j. For a subsequence, A_j converges in τ_e to some set A that minimizes F. Because of $F = \mathcal{H}^1_d$ on X_d we know that the d-closure $\bar{A}$ of A is a minimizer of $\mathcal{H}^1_d$ and therefore $\bar{A} = (\{0\} \times [-1/3, 0]) \cup S \cup (\{1\} \times [-1/3, 0])$.

Note that in this example the path $P := (\{0\} \times [-1/3,0]) \cup ([0,1] \times \{0\}) \cup (\{1\} \times [-1/3,0])$ satisfies $F(P) = 5/3$ and is not a minimizer of F. This yields an example for $F \neq \mathcal{H}^1_d$, since $\mathcal{H}^1_d(P) = 2/3$. We see that the knowledge of the Γ-limit F of the functionals $\mathcal{H}^1_{d_j}$ includes indeed more information than $\mathcal{H}^1_d$: Knowing only $\mathcal{H}^1_d$ we can say that limits of minimizing sequences are contained in $P \cup S$, but we can not exclude that the limit is P.

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