Second Order Variational Problems with Free Discontinuity and Free Gradient Discontinuity

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Introduction

In recent years many functionals depending on higher derivatives have been studied [12],[13],[14],[15], [16],[18],[5],[32],[23]: many related examples of variational principles for image segmentation have been focused [9],[38],[36], [34],[37],[15], [16], [17],[20],[21] and several models of elastic plastic plates [12], [13],[46] and elastic energies with damage at small scale were introduced and studied ([39], [40],[41], [24],[31],[33] and the contributed paper [22] in this book).

In this contribution we unify general properties of functionals depending on first and second derivatives together with free discontinuities and free gradient discontinuities. Our framework has an interest in itself, nevertheless we emphasize that the scheme includes many relevant applications to image processing and continuum mechanics.

More precisely we focus the functional $\mathcal{F}(v, \alpha, \beta, \gamma, \sigma, A)$ defined below and shortly denoted by $\mathcal{F}(v)$ or $\mathcal{F}(v, A)$ when the parameters $\alpha, \beta, \gamma, \sigma$ and/or the localization in A are prescribed. For any scalar function v defined in Ω and Borel set $A \subset \Omega$ we study:

(0.1)

$$\begin{aligned} \mathcal{F}(v, \alpha, \beta, \gamma, \sigma, A) \;=\; & \int_A \left(|\nabla^2 v|^2 + \sigma |\nabla v|^2 + G(x, v)\right) dx + \\ & + \alpha \mathcal{H}^{n-1}(S_v \cap A) + \beta \mathcal{H}^{n-1}((S_{\nabla v} \setminus S_v) \cap A) + \gamma \int_{S_{\nabla v} \cap A} |[\nabla v]| \, d\mathcal{H}^{n-1}. \end{aligned}$$

The following structural hypotheses are assumed in the definition (0.1) of the functional $\mathcal{F}$ and in the whole paper.

(0.2) $\quad \alpha, \sigma, \gamma \geq 0,\ \beta > 0,\ A \subset \Omega \subset \mathbf{R}^n$ open sets, $n \geq 1$, G Carathéodory.

We focus the two following main issues.

Problem 0.1. *Minimize $\mathcal{F}(v, 0, \beta, \gamma, \sigma, \Omega)$ among $v \in SBH(\Omega)$ under the assumptions (0.2) and (0.3)*

(0.3)
$$\alpha = 0,\ \ \beta > 0,\ \ \gamma > 0\ , \sigma \geq 0.$$

Definition and properties related to the space $SBH(\Omega)$ of Special Bounded Hessian Functions are collected in Section 1.

Problem 0.2. *Minimize $\mathcal{F}(v,\alpha,\beta,\gamma,\sigma,\Omega)$ among $v \in GSBV^2(\Omega)$ under the assumptions (0.2) and (0.4)*

$$0<\beta \le \alpha \le 2\beta\,, \qquad \gamma,\sigma \ge 0\,. \tag{0.4}$$

The set $GSBV^2(\Omega)$, which arises in a natural way when considering the finite energy set for Problem 0.2, is the subset of generalized special bounded variation functions whose approximate gradient belongs to the same class. Definition and properties related to the class $GSBV^2$ are given in Section 2.

Notice that, when $\alpha = 0$, then the minimization of $\mathcal{F}$ is not well-posed in $GSBV^2(\Omega)$; while the functional space $SBH(\Omega)$ is the natural setting: on this subject see also Remark 5.7

Some explicit examples of forcing term G allowed by our framework are given in Section 7.

Here we list the main results: existence of solutions for both Problems 0.1, 0.2 corresponding to weak form of nonhomogeneous Dirichlet boundary value problem or homogeneous Neumann boundary value problem; they are proved by the direct method in Calculus of Variations. For relevant explicit problems and a more general setting see Theorems 7.1, 7.2, 7.3, 7.4, 7.5, 7.6.

Theorem 0.3. *Assume (0.2),(0.4), $\exists w \in GSBV^2(\Omega)$ s.t. $\mathcal{F}(w,\alpha,\beta,\gamma,\sigma,\Omega)$ is finite and*

$$\begin{cases} \operatorname{spt} G(\cdot,s) \subset \overline{\Omega} \ \forall s \\ \exists c_3 \in \mathbf{R}, c_4 \in L^1(\mathbf{R}^n) \ : \ c_3>0,\ c_3|s|^2 - c_4 \le G(x,s)\ \forall s,\ \text{a.e.}\, x. \end{cases} \tag{0.5}$$

Then there is a minimizer of $\mathcal{F}(v,\alpha,\beta,\gamma,\sigma,\Omega)$ among v in $GSBV^2(\Omega)$ with finite energy.

Theorem 0.4. *Assume (0.2),(0.4),(0.5), $\exists w \in GSBV^2(\mathbf{R}^n)$ s.t. $\mathcal{F}(w,\alpha,\beta,\gamma,\sigma,\mathbf{R}^n)$ is finite. Then there is a minimizer of $\mathcal{F}(v,\alpha,\beta,\gamma,\sigma,\mathbf{R}^n)$ among $v \in GSBV^2(\mathbf{R}^n)$ such that $v = w$ in $\mathbf{R}^n \setminus \overline{\Omega}$ with finite energy.*

Theorem 0.5. *Assume (0.2),(0.3) and $w \in SBH(\mathbf{R}^n)$ s.t. $\mathcal{F}(w,0,\beta,\gamma,\sigma,\mathbf{R}^n)$ is finite and*

$$
(0.6)\quad \begin{cases} \Omega \textit{ bounded with } \ |\Omega| = |\overline{\Omega}| \ \ , \ \mathrm{spt}\, G(\cdot,s) \subset \overline{\Omega}\ \forall s\,, \\ \exists c_1 > 0, c_2 \in L^1(\mathbf{R}^n):\ G(x,s) \le c_1|s| + c_2(x) \text{ a.e.}x, \forall s, \textit{if } n=1,2, \\ \qquad\qquad\qquad G(x,s) \le c_1 |s|^{\frac{n}{n-2}} + c_2(x) \text{ a.e. } x,\ \forall s, \textit{if } n \ge 3, \\ \exists \lambda:\ 0 < \lambda < \dfrac{\gamma}{C_n(\Omega)} \quad \textit{and} \ \ \exists\, c_0 \in L^1(\mathbf{R}^n)\, s.t. \\ \qquad\qquad\qquad G(x,s) \ \ge\ c_0(x)\ -\ \lambda\, s\ , \quad \text{a.e. } x,\ \forall s, \end{cases}
$$

where $C_n(\Omega)$ and $K_n(\Omega)$ are constants such that

$$
C_n(\Omega) \ = \ \begin{cases} \frac{1}{2}|\overline{\Omega}|^2 & if\ \ n=1 \\ \frac{1}{4}|\overline{\Omega}| & if\ \ n=2 \\ |\overline{\Omega}|^{2/n}\, K_n(\Omega) & if\ \ n>2 \end{cases}
$$

$$
\|v\|_{L^{\frac{n}{n-2}}(\Omega)} \ \le\ K_n(\Omega)\, |D^2 v|_{T(\overline{\Omega})} \qquad \forall v \in SBH(\mathbf{R}^n) \ s.t.\ \mathrm{spt}\, v \subset \overline{\Omega}\,,\ \ n>2\,.
$$

Then there is a minimizer of $\mathcal{F}(v,0,\beta,\gamma,\sigma,\mathbf{R}^n)$ among $v \in SBH(\mathbf{R}^n)$ such that $v = w$ in $\mathbf{R}^n \setminus \overline{\Omega}$ with finite energy.

Theorem 0.6. *Assume (0.2),(0.3) and $\exists w \in SBH(\Omega)$ s.t. $\mathcal{F}(w,0,\beta,\gamma,\sigma,\Omega)$ is finite,*

$$
(0.7)\quad \begin{cases} \Omega \textit{ is } \mathcal{R} \textit{ regular (Definition 1.5)}, \quad \mathrm{spt}\, G(\cdot,s) \subset\ \overline{\Omega}\ \ \forall s\,, \\ \exists c_1 > 0, c_2 \in L^1(\Omega):\ G(x,s) \le c_1|s| + c_2(x) \ \text{a.e.}x,\ \forall s \ \textit{if } n=1,2 \\ \qquad\qquad\qquad G(x,s) \le c_1|s|^{\frac{n}{n-2}} + c_2(x) \ \ \text{a.e.}x,\ \forall s \ \ \textit{if } n \ge 3, \\ \exists \lambda:\ 0 < \lambda < \dfrac{\gamma}{\mathcal{C}_n(\Omega)} \quad \textit{and} \ \ \exists\, c_0 \in L^1(\Omega)\, s.t. \\ \qquad\qquad\qquad G(x,s) \ \ge\ c_0(x)\ -\ \lambda\, s\ , \quad \text{a.e.}x,\ \forall s, \end{cases}
$$

where, referring to the constant $\Gamma_n(\Omega)$ in Theorem 1.12, $\mathcal{C}_n(\Omega)$ is given by

$$
\mathcal{C}_n(\Omega) \ = \ \begin{cases} |\Omega|\, \Gamma_n(\Omega) & if\ \ n=1,2 \\ |\Omega|^{2/n}\, \Gamma_n(\Omega) & if\ \ n>2 \end{cases}
$$

and

$$
(0.8)\quad \int_\Omega G(x, v(x) + t + r\cdot x) = \int_\Omega G(x, v(x))\ \forall t \in \mathbf{R}, r \in \mathbf{R}^n,\ v \in SBH(\Omega).
$$

Then there is a minimizer of $\mathcal{F}(v, 0, \beta, \gamma, \sigma, \Omega)$ among v in $SBH(\Omega)$ with finite energy.

Regularity results (existence of strong solutions with essentially closed singular set) were proved by the authors [13, 16] in some relevant cases of Problems 0.1 and 0.2.

The typical difficulty of second order functionals in $GSBV^2(\Omega)$ amounts to the fact that, though truncating the competing functions is allowed in the finite energy space, nevertheless the truncation jeopardizes the estimates on distributional hessian D^2u since functional (0.1) does not provide any information on the L^1norm of the jump of ∇u (the absolutely continuous part of the distributional gradient Du) when $\gamma = 0$, in contrast with Lipschitz behavior of truncation in the space $BH(\Omega)$ of functions whose hessian matrix is a Radon measure (see [43],[44]).

Moreover, since discontinuous competing functions u are admissible in Problem 0.2, when $\sigma = 0$ and $\alpha > 0$ there is lack of compactness for sub-levels of the functional $\mathcal{F}$ even with respect to the a.e. convergence, as shown by the Example 2.5

In search for a finite energy space of $\mathcal{F}$ with $\sigma = 0,\ \alpha > 0$ where appropriate compactness and lower semicontinuity properties of minimizing sequences hold, an additional difficulty is that finiteness of (0.1) does not provide any estimate on Du or even on ∇u : more precisely $|\nabla u|$ may be essentially unbounded even if the energy (0.1) is arbitrarily small, as shown by the Example 2.5. In addition the fact that the competing functions u are allowed to be discontinuous immediately drives the analysis outside the framework of the space of functions with Special Bounded Hessian $SBH(\Omega)$ ([12]) where second order energies for elastic-perfectly plastic plates and rigid-plastic slabs have been successfully studied [11], [12], [13], [46].

One significant issue in Problems 0.1 and 0.2, is that $\mathcal{F}$ is an example of functional in Calculus of Variations whose computation is not reducible to one dimensional slices, as like as in the case of perimeter. This fact produces some technical obstacles, nevertheless $\mathcal{F}$ can be estimated through lower dimensional sections: in this way we deal with mixed derivatives in Section 5.

The proofs rely on the direct method of the Calculus of Variations and are split in the following steps: coerciveness (Theorems 1.15, 1.16), slicing properties of first and second derivatives (Section 3), an interpolation inequality (Theorem 4.3), a compactness theorem for the sublevels of the functional $\mathcal{F}$ (Theorems 4.4, 4.5) and a lower semicontinuity property with respect to a.e. convergence (Theorems 5.13, 5.15).

The case $\alpha = 0, \sigma = 0$ with the choice $G(x,v) = -g(x)v$ (of type dead load acting on the transverse displacement v), includes the elastic plastic plate with Neumann boundary conditions ([12],[13],[14],[39], [40],[41]) :

$$\mathcal{F}(v,0,\beta,0,0,\Omega) = \int_\Omega \left(|\nabla^2 v|^2 - g\,v\right) dx \; + \; \beta\mathcal{H}^{n-1}(S_{\nabla v}) \;, \tag{0.9}$$

On this subject see also Theorems 7.3, 7.4 about functional (7.3) where the Kirchhoff-Love elastic energy of the plate coupled with free plastic yield is studied.

The case $\alpha > 0, \sigma > 0$ is new, and includes a case interesting in image segmentation ([9],[36]): see functional (7.2) and related Theorem 7.2 .

The case $\alpha > 0, \sigma = 0$ includes the weak formulation of Blake & Zisserman functional ([15],[16],[5]), with the choice $G(x,v) = |v(x) - g(x)|^2$ (of kind penalization of quadratic correction), leads to

$$\begin{aligned} &\mathcal{F}(v,\alpha,\beta,0,0,\Omega) = \\ &= \int_\Omega \left(|\nabla^2 v|^2 + |v-g|^2\right) dx \, + \alpha\mathcal{H}^{n-1}(S_v) \; + \; \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v) \end{aligned} \tag{0.10}$$

which is the weak formulation of the Blake & Zisserman functional for image segmentation ([9],[34]) (called the thin plate surface under tension): a relaxed version of the following strong counterpart F defined as follows

$$\begin{aligned} F(K_0,K_1,u) \;=\; &\int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^2 + |u-g|^2\right)\, dy \\ &+ \alpha\mathcal{H}^{\,n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{\,n-1}((K_1\setminus K_0)\cap\Omega) \end{aligned} \tag{0.11}$$

to be minimized over closed sets K_0, K_1 and $u \in C^0(\Omega\setminus K_0)\cap C^2(\Omega\setminus(K_0\cup K_1))$ in order to achieve an optimal segmentation $K_0 \cup K_1$ of the noisy image with g intensity level on the picture in Ω (where K_0 is the set of jump points for u and $K_1 \setminus K_0$ is the set of crease points). About functional F in (0.11) among various results proved in [16] we recall the following statement.

Theorem 0.7. *If* $n = 2$, $0 < \beta \le \alpha \le 2\beta$, *and* $g \in L^4_{loc}(\Omega) \cap L^2(\Omega)$, *then there is at least one triplet among* $K_0, K_1 \subset \mathbf{R}^2$ *Borel sets with* $K_0 \cup K_1$ *closed and* $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ *approximately continuous on* $\Omega \setminus K_0$ *minimizing functional (0.11) and having finite energy F. Moreover the sets* $K_0 \cap \Omega$ *and* $K_1 \cap \Omega$ *are* $(\mathcal{H}^1, 1)$ *rectifiable.*

The assumption $0 < \beta \le \alpha \le 2\beta$, is necessary for the l.s.c. of $\mathcal{F}$ ([15]) in the case $\alpha > 0$; actually in such case the terms dependent on α and β are l.s.c. as a whole and cannot be split. On the other hand the term dependent on γ is l.s.c. by itself (see Remarks 5.5, 5.9, 5.14), for any $\alpha \ge 0$.

We emphasize that the contribution of $\int |[\nabla v]| d\mathcal{H}^{n-1}$ in functional $\mathcal{F}$ has to be taken into account on the whole set $S_{\nabla v}$ and not on $S_{\nabla v} \setminus S_v$ to prevent the lack of lower semicontinuity even in the one dimensional case, as shown by the example in Remark 5.8.

Eventually we notice that in dimension 1 the weak solutions are also strong solutions for Problem 0.1 and Problem 0.2, since strong and weak formulation of functional $\mathcal{F}$ coincide when $n = 1$ and is given by:

$$\begin{aligned}\mathcal{F}^1(v_h, \alpha, \beta, \gamma, \sigma, A) \; &= \int_A (|\ddot{v}_h(x)|^2 + \sigma |\dot{v}_h(x)|^2 + G(x, v_h(x)))\, dx + \\ &+ \alpha\, \#(S_{v_h} \cap A) + \beta\, \#((S_{\dot{v}_h} \setminus S_{v_h}) \cap A) + \gamma \sum_{S_{\dot{v}_h} \cap A} |[\dot{v}_h]| \, .\end{aligned}$$

1. Special Bounded Hessian functions (SBH)

In this section we recall some properties of the functional setting suitable for functional $\mathcal{F}$ when $\alpha = 0$: say the class *SBH*.

Given a connected open subset $\Omega \subseteq \mathbf{R}^n$ $(n \ge 1)$ we define the class of real valued functions with special bounded hessian *SBH*(Ω) and we point out some of its properties.

For a given set $U \subset \mathbf{R}^n$ we denote by $\overline{U}$, ∂U its topological closure and boundary; moreover we denote by $\mathcal{H}^{n-1}(U)$ its $(n-1)$–dimensional Hausdorff measure (in particular $\mathcal{H}^0(U)$ is the counting measure also denoted by $\#(U)$) and by $\mathcal{L}^n(U)$ (or shortly $|U|$) its Lebesgue outer measure. We indicate by

$B_\rho(x)$ the open ball $\{y \in \mathbf{R}^n ; |y-x| < \rho\}$, and we set $B_\rho = B_\rho(0)$. If Ω , Ω' are open subsets in $\mathbf{R}^n$, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.

We introduce the following notations: $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ for every $a, b \in \mathbf{R}$; $M_{k,n}$ stands for $k \times n$ matrices ($k \geq 1$) and I for the identity in $M_{n,n}$; given two vectors $a = \{a_i\}$, $b = \{b_i\}$, we set $a \cdot b = \sum_i a_i b_i$, $(a \otimes b)_{ij} = a_i b_j$.

Let $v : \Omega \to \mathbf{R}^k$ be a Borel function; for $x \in \Omega$ and $z \in \widetilde{\mathbf{R}}^k = \mathbf{R}^k \cup \{\infty\}$ (the one point compactification of $\mathbf{R}^k$) we say, following [27], that z is the approximate limit of v at x, and we write

$$z = \operatorname{ap} \lim_{y \to x} v(y),$$

if

$$g(z) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} g(v(y)) dy}{|B_\rho|}$$

for every $g \in C^0(\widetilde{\mathbf{R}}^k)$.

The set

$$S_v = \{x \in \Omega ;\ \operatorname{ap} \lim_{y \to x} v(y) \quad \text{does not exist } \}$$

is a Borel set, of negligible Lebesgue measure (see e.g. [30], 2.9.13); for brevity's sake we denote by $\tilde{v} : \Omega \setminus S_v \to \widetilde{\mathbf{R}}^k$ the function

$$\tilde{v}(x) = \operatorname{ap} \lim_{y \to x} v(y).$$

Let $x \in \Omega \setminus S_v$ be such that $\tilde{v}(x) \in \mathbf{R}^k$; we say that v is approximately differentiable at x if there exists a $k \times n$ matrix $\nabla v(x)$ such that

$$\operatorname{ap} \lim_{y \to x} \frac{|v(y) - \tilde{v}(x) - \nabla v(x)(y-x)|}{|y-x|} = 0.$$

If v is a smooth function then ∇v is the jacobian matrix. In the following with the notation $|\nabla v|$ we mean the euclidean norm of ∇v.

If $p \in [1, +\infty]$, we denote by $L^p(\Omega, M_{k,n})$ and by $W^{1,p}(\Omega, M_{k,n})$ the Lebesgue and Sobolev spaces of functions with values in $M_{k,n}$, endowed with the usual norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ respectively. We denote by $\mathcal{M}(\Omega, M_{k,n})$ the space of the bounded measures on Ω with values in $M_{k,n}$ and by $|\cdot|_T$ the total variation of a measure of $\mathcal{M}(\Omega, M_{k,n})$, i.e.

$$|\mu|_T = \sup\left\{\int_\Omega \sum_{ij} \varphi_{ij} d\mu_{ij} \ ; \varphi_{ij} \in C_0^0(\Omega), \ \sum_{ij} \varphi_{ij}^2 \le 1 \text{ in } \Omega\right\}.$$

If A is any open set then $|\mu|_{T(A)}$ is defined in the same way with $\varphi_{ij} \in C_0^0(A)$ and we define a Borel measure $|\mu|$ setting for every Borel set $B \subset \Omega$

$$|\mu|_{T(B)} = \inf\left\{|\mu|_{T(A)}; \ B \subset A, \ A \text{ open }\right\}.$$

We recall the definition of the space of functions of bounded variation in Ω with values in $\mathbf{R}^k$:

$$BV(\Omega, \mathbf{R}^k) = \{v \in L^1(\Omega, \mathbf{R}^k); Dv \in \mathcal{M}(\Omega, M_{k,n})\}$$

where $Dv = \{D_j v_i\}_{i=1,..k,j=1,..n}$ denotes the distributional derivatives of v. For every $v \in BV(\Omega, \mathbf{R}^k)$ the following properties hold:

1) $\widetilde{v}(x) \in \mathbf{R}^k$ for $\mathcal{H}^{n-1}$–almost all $x \in \Omega \setminus S_v$ (see [47], 5.9.6);

2) S_v is countably $(\mathcal{H}^{n-1}; n-1)$ rectifiable (see [47], 5.9.6);

3) ∇v exists a.e. on Ω and coincides with the Radon–Nikodym derivative of Dv with respect to the Lebesgue measure (see [30], 4.5.9(26));

4) for $\mathcal{H}^{n-1}$ almost all $x \in S_v$ there exist $\nu = \nu_v(x) \in \partial B_1$, $v^+(x), v^-(x) \in \mathbf{R}^k$ (outer and inner trace, respectively, of v at x in the direction ν) such that (see [47], 5.14.3)

$$\lim_{\rho\to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y\cdot\nu > 0\}} |v(y) - v^+(x)|\, dy = 0,$$

$$\lim_{\rho\to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y\cdot\nu < 0\}} |v(y) - v^-(x)|\, dy = 0,$$

and also

$$|Dv|_T \ge \int_\Omega |\nabla v|\, dx + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^{n-1} \tag{1.1}$$

(see [30], 4.5.9(15)). In recent papers (see [26, 27]), for studying some free discontinuity problems, a class of functions with special bounded variation has been considered. Such functions are characterized by a property stronger than (1.1), as shown in the following definition.

Definition 1.1. *$SBV(\Omega, \mathbf{R}^k)$ denotes the class of all functions $v \in BV(\Omega, \mathbf{R}^k)$ such that*

$$|Dv|_T = \int_\Omega |\nabla v|\, dx + \int_{S_v} |[v]|\, d\mathcal{H}^{n-1}, \tag{1.2}$$

where $[v] = v^+ - v^-$.

By the previous definition it follows as in [3], Proposition 3.1, that $v \in SBV(\Omega, \mathbf{R}^k)$ if and only if $v \in BV(\Omega, \mathbf{R}^k)$ and

$$Dv = \nabla v\, dx + \left(v^+ - v^-\right) \otimes \nu_v\, d\mathcal{H}^{n-1} \llcorner S_v,$$

where $\mathcal{H}^{n-1} \llcorner S_v(B) = \mathcal{H}^{n-1}(B \cap S_v)$ for any Borel set B.
In dimension 1 we denote by $v' = Dv$ and $\dot{v} = \nabla v$.

In the theory of elastic–perfectly plastic plates developed by F. Demengel in [28, 29] the space $BH(\Omega)$ of the functions with bounded hessian in Ω has been introduced. Namely

$$BH(\Omega) = \{v \in W^{1,1}(\Omega); D^2 v \in \mathcal{M}(\Omega, M_{n,n})\} = \{v \in L^1(\Omega); Dv \in BV(\Omega, \mathbf{R}^n)\},$$

where $D^2 v$ denotes the distributional hessian of v. The space $BH(\Omega)$ is endowed with the norm

$$\|v\|_{BH(\Omega)} = \|v\|_{L^1(\Omega)} + \|Dv\|_{L^1(\Omega)} + |D^2 v|_T,$$

and it is the dual of a Banach space.

In the paper [12] we introduced the space $SBH(\Omega)$ of the functions with special bounded hessian to deal with elastic plates with free plastic yield lines.

Definition 1.2. *We define $SBH(\Omega)$ as the class of all functions $v \in L^1(\Omega)$ such that $Dv \in SBV(\Omega, \mathbf{R}^n)$.*

Remark 1.3 - By definition $SBH(\Omega)$ is a closed subspace of $BH(\Omega)$ with respect to the strong norm, while it is not closed with respect to the w^*–$BH(\Omega)$ topology.

The following properties can be deduced immediately for a function $v \in SBH(\Omega)$:

1) the distributional derivative of v is absolutely continuous with respect to $\mathcal{L}^n$, hence we have for every $v \in SBH(\Omega)$

$$\nabla v \ = \ Dv \ , \qquad \nabla Dv \ = \ \nabla^2 v \ , \qquad \nabla \cdot Dv \ = \ \Delta^a v,$$

where $\Delta^a v$ is the Radon–Nikodym derivative with respect to $\mathcal{L}^n$ of the distributional Laplacian Δv (in the following we shall use always the notations in the right hand side of the previous equalities);

2) $S_{Dv} = \bigcup_{i=1}^{n} S_{D_i v}$;

3) $|D^2 v|_T = \int_\Omega |\nabla^2 v| \, dx + \int_{S_{Dv}} |[Dv]| \, d\mathcal{H}^{n-1}$,

where $[Dv] = (Dv)^+ - (Dv)^-$.

Now we list some embedding results for the space $BH(\Omega)$ which follow immediately by theorems proved in [28, 29].

Theorem 1.4. *Let $\Omega \subset \mathbf{R}^n$ $(n > 1)$ be a bounded open set with the exterior cone property. Then*

$$BH(\Omega) \subset W^{1,q}(\Omega) \tag{1.3}$$

with continuous embedding if $q \le \frac{n}{n-1}$; the embedding is compact if $q < \frac{n}{n-1}$. In particular

$$BH(\Omega) \subset L^q(\Omega) \tag{1.4}$$

for $q \le \frac{n}{n-2}$ (compactly when the inequality is strict) if $n > 2$; for any $q \ge 1$ (compactly for finite q) if $n = 2$. If $\Omega \subset \mathbf{R}$ is a bounded interval then $BH(\Omega)$ is compactly embedded in $C^{0,1}(\Omega)$.

Since we want to consider both the case of smooth domains and polygonal ones, we introduce the following definition.

Definition 1.5. *We say that a set $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ satisfies the property $\mathcal{R}$ if it is a bounded connected open set and*

Ω is strongly Lipschitz and $\partial\Omega$ is the union of finitely many C^2 curves, if $n = 2$,

Ω is C^2 uniformly regular, if $n > 2$ (see e.g. [1], 4.5, 4.6).

Theorem 1.6. *Let $\Omega \subset \mathbf{R}^2$ be an open set. If $v \in BH(\Omega)$ has compact support in Ω then*

$$\|v\|_{L^\infty(\Omega)} \leq \frac{1}{4}|D^2 v|_{T(\Omega)} \ . \tag{1.5}$$

If Ω satisfies property $\mathcal{R}$ then

$$BH(\Omega) \subset C^0(\overline{\Omega}). \tag{1.6}$$

(See Theorem 3.3 and Remark 3.2 in [28]).

The following extension theorem holds (see Theorem 2.2, Remark 2.1 in [28]).

Theorem 1.7. *Let $\Omega \subset \mathbf{R}^n$ with property $\mathcal{R}$ and let $\overline{x} \in \Omega$. Then there is a constant $M_n = M_n(\Omega) > 0$ and a linear continuous map $\Pi : BH(\Omega) \to BH(\mathbf{R}^n)$ such that*

$$\Pi\, v \ = \ v \ \ \text{a.e. } \textit{in } \Omega \qquad \operatorname{spt}(\Pi v) \subset B_t(\bar{x}) \ \ \textit{where } t = 2\,\operatorname{diam}\Omega$$

$$\|\Pi v\|_{BH(\mathbf{R}^n)} \ \leq \ M_n\, \|v\|_{BH(\Omega)} \ ,$$
$$\Pi\left(W^{2,1}(\Omega)\right) \subset W^{2,1}(\mathbf{R}^n) \ .$$

Remark 1.8 - If Ω does not satisfy property $\mathcal{R}$ both theorems 1.6, 1.7 may fail: take, for instance, $v(x,y) = \log\sqrt{x^2+y^2}$ and $\Omega = \{(x,y) : \ 0 < x < 1, \ |y| < x^2\}$. Then $v\big|_\Omega$ belongs to $BH(\Omega)$ but it is unbounded. Of course $v\big|_{B_1} \notin BH(B_1)$.

In the space $BH(\Omega)$ the following trace theorem holds (see [28], Appendix).

Theorem 1.9. *Let $\Omega \subset \mathbf{R}^n$ with the property $\mathcal{R}$. Two bounded linear maps exist*

$$\gamma_0 : BH(\Omega) \to W^{1,1}(\partial\Omega) \qquad \gamma_1 : BH(\Omega) \to L^1(\partial\Omega)$$

such that

$$\gamma_0(v) = v\Big|_{\partial\Omega}, \qquad \gamma_1(v) = \frac{\partial v}{\partial N}\Big|_{\partial\Omega}$$

for every $v \in C^2(\overline{\Omega})$, where N is the outward normal to $\partial\Omega$. Moreover γ_1 is onto.

Remark 1.10 - We notice explicitely that $C^\infty(\Omega)$ is neither dense in $BH(\Omega)$ nor in $SBH(\Omega)$ with respect to the strong topology; nevertheless, if Ω is strongly Lipschitz, the density holds true with respect to the intermediate topology associated to the distance

$$d_2(u,v) \;=\; \|u-v\|_{L^1(\Omega)} + \left| \int_\Omega |D^2u| - \int_\Omega |D^2v| \; \right|,$$

as in the case of $BV(\Omega)$ and $SBV(\Omega)$ with

$$d_1(u,v) \;=\; \|u-v\|_{L^1(\Omega)} + \left| \int_\Omega |Du| - \int_\Omega |Dv| \; \right|$$

(see [45], III, 2.8 and I, 1.3, where the result is obtained by mollification of the trivial extension).

From now on we denote by $\mathcal{P}_1(\Omega)$ the space of the affine functions. We define the linear map $p : BH(\Omega) \to \mathcal{P}_1(\Omega)$ by

$$(pv)(x) \;=\; v_\Omega + (\nabla v)_\Omega \cdot (x - x_\Omega) \qquad \forall v \in BH(\Omega) \tag{1.7}$$

where

$$v_\Omega = |\Omega|^{-1} \int_\Omega v\,dx, \quad (\nabla v)_\Omega = |\Omega|^{-1} \int_\Omega \nabla v\,dx, \quad x_\Omega = |\Omega|^{-1} \int_\Omega x\,dx.$$

Obviously

$$(1.8) \qquad pv \;=\; v \qquad \forall v \in \mathcal{P}_1,$$

$$(1.9) \qquad p(v - pv) \;=\; 0 \qquad \forall v \in BH(\Omega).$$

Now we can prove a Poincaré type inequality.

Theorem 1.11. *Assume $\Omega \subset \mathbf{R}^n$ with the property $\mathcal{R}$. Then there are constants $\mathcal{K}_n = \mathcal{K}_n(\Omega) > 0$, $S_n = 1 + \sqrt{n}\mathcal{K}_n\,(1 + \mathcal{K}_n)$ such that*

$$(1.10) \qquad \|v\|_{L^1(\Omega)} \;\le\; \mathcal{K}_n |Dv|_{T(\Omega)} \qquad \forall v \in BV(\Omega) \text{ with } \int_\Omega v\,dx = 0,$$

$$(1.11) \qquad \|v - pv\|_{BH(\Omega)} \;\le\; S_n\,|D^2 v|_{T(\Omega)} \qquad \forall v \in BH(\Omega).$$

PROOF – Take $v_h \in C^\infty(\Omega)$, such that, referring to remark 1.10,

$$d_1(v_h, v) \to 0, \qquad \int_\Omega v_h\,dx = 0,$$

then the well known Poincaré inequality in $W^{1,1}(\Omega)$ gives the existence of a constant $\mathcal{K}_n = \mathcal{K}_n(\Omega)$ such that

$$\|v_h\|_{L^1(\Omega)} \;\le\; \mathcal{K}_n \|Dv_h\|_{L^1(\Omega)} \;=\; \mathcal{K}_n\,|Dv_h|_{T(\Omega)} \qquad \forall h \in \mathbf{N}$$

and since $|Dv_h|_{T(\Omega)} \to |Dv|_{T(\Omega)}$, we get (1.10).

Then, by choosing $v_h \in C^\infty(\Omega)$ such that $d_2(v_h, v) \to 0$ and by using (1.9), (1.10), we get

$$\begin{aligned}
&\|v_h - pv_h\|_{BH(\Omega)} \;=\; \|v_h - pv_h\|_{W^{2,1}(\Omega)} \\
&= \; \|v_h - pv_h\|_{L^1(\Omega)} + \|D\,(v_h - pv_h)\,\|_{L^1(\Omega,\mathbf{R}^n)} + \|D^2 v_h\|_{L^1(\Omega, M_{n,n})} \\
&\le (1 + \mathcal{K}_n)\;\|D(v_h - pv_h)\|_{L^1(\Omega,\mathbf{R}^n)} + \|D^2 v_h\|_{L^1(\Omega,M_{n,n})} \\
&\le (1 + \sqrt{n}\mathcal{K}_n(1 + \mathcal{K}_n))\;\left\|D^2 v_h\right\|_{L^1(\Omega,M_{n,n})} \\
&= S_n\;\left|D^2 v_h\right|_{T(\Omega)}.
\end{aligned}$$

Taking the limit as $h \to +\infty$, inequality (1.11) follows. q.e.d.

By summarizing Theorems 1.4, 1.6, 1.7 and 1.11, and by direct computation in the case $n = 1$, we get

Theorem 1.12. *For any set $\Omega \subset \mathbf{R}^n$ with property $\mathcal{R}$, there is a constant $\Gamma_n(\Omega) > 0$, such that for every $v \in BH(\Omega)$*

$$\|v - pv\|_{L^{\frac{n}{n-2}}(\Omega)} \leq \Gamma_n(\Omega)\,|D^2 v|_{T(\Omega)} \qquad \text{if } n > 2,$$

$$\|v - pv\|_{L^\infty(\Omega)} \leq \Gamma_n(\Omega)|D^2 v|_{T(\Omega)} \qquad \text{if } n = 1, 2.$$

More precisely $\Gamma_2(\Omega) = \frac{1}{4} M_2(\Omega) S_2(\Omega)$ for $n = 2$ and $\Gamma_1(\Omega) = |\Omega|$ for $n = 1$.

Moreover, for any open set $\Omega \subset \mathbf{R}$ we have

$$\|v\|_{L^\infty(\mathbf{R})} \leq \frac{1}{2}|\overline{\Omega}||v''|_{T(\overline{\Omega})} \qquad \forall v \in SBH(\mathbf{R}) \;:\; \operatorname{spt} v \subset \overline{\Omega}\,.$$

In the next theorem we point out the structure of the singular part of the hessian matrix of a function $v \in SBH(\Omega)$. On this subject we refer also to [7] and [2].

Theorem 1.13. *Let $\Omega \subset \mathbf{R}^n$ be an open set and $v \in SBH(\Omega)$. Then*

1) $$(D^2 v)^s = [Dv] \otimes \nu \, d\mathcal{H}^{n-1} \llcorner S_{Dv} = \left[\frac{\partial v}{\partial \nu}\right] \nu \otimes \nu \, d\mathcal{H}^{n-1} \llcorner S_{Dv},$$

2) $$\left|(D^2 v)^s\right|_{T(\Omega)} = \int_{S_{Dv}} |[Dv]| \, d\mathcal{H}^{n-1} = \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| \, d\mathcal{H}^{n-1},$$

3) $$\left|(D^2 v)^s\right|_{T(\Omega)} = |\Delta^s v|_{T(\Omega)}\,,$$

where $\nu = \nu_v$, $\frac{\partial v}{\partial \nu} = \nu \cdot Dv$, $(D^2 v)^s$ and $\Delta^s v$ denote respectively the singular part of the distributional hessian and laplacian of v with respect to $\mathcal{L}^n$.

Proof – By the definition of $SBH(\Omega)$ we have $Dv \in SBV(\Omega, \mathbf{R}^m)$ so that the first equalities in 1) and in 2) immediately follow. Even we get

$$[Dv] = \left[\frac{\partial v}{\partial \nu}\right] \nu.$$

Since the singular part of the hessian matrix is rank one and symmetric we get

$$\left|(D^2v)^s\right|_{T(\Omega)} = \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathcal{H}^{n-1} = \int_{S_{Dv}} \left|\sum_{i=1}^{n} [D_i v]\,\nu_i\right| d\mathcal{H}^{n-1} = |\Delta^s v|_{T(\Omega)}$$

and the proof is achieved. q.e.d.

We state two basic results on *SBV* functions, that will be applied to the gradients of a minimizing sequence for the functional (0.1). Theorem 1.14 is a compactness property and Theorem 1.17 is related to semicontinuity. Theorems 1.15 and 1.16 are coerciveness results.

Theorem 1.14. *Let $\Omega \subset \mathbf{R}^n$ be an open set with property $\mathcal{R}$. Let $\phi : [0,+\infty[\to [0,+\infty]$ be a convex, non decreasing function satisfying the condition*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty,$$

let a, b be strictly positive constants and let $\{z_h\}_{h\in\mathbf{N}}$ be a sequence of functions in $SBV(\Omega,\mathbf{R}^k)$ such that

$$\int_\Omega z_h\,dx = 0 \qquad \forall h \in \mathbf{N},$$

$$\sup_{h\in\mathbf{N}} \left\{ \int_\Omega \phi(|\nabla z_h|)\,dx + \int_{S_{z_h}} \left(a + b|{z_h}^+ - {z_h}^-|\right)\, d\mathcal{H}^{n-1} \right\} < +\infty.$$

Then there is a function $z \in SBV(\Omega,\mathbf{R}^k)$ and a subsequence $\{z_{h_i}\}_{i\in\mathbf{N}}$ such that

1) $z_{h_i} \to z$ strongly in $L^1(\Omega,\mathbf{R}^k)$;

2) $\nabla z_{h_i} \rightharpoonup \nabla z$ weakly in $L^1(\Omega, M_{k,n})$;

3) $Dz_{h_i} - \nabla z_{h_i} dx = ({z_{h_i}}^+ - {z_{h_i}}^-) \otimes \nu\, d\mathcal{H}^{n-1} \llcorner S_{z_{h_i}}$ converges weakly in $\mathcal{M}(\Omega, M_{k,n})$ to $Dz - \nabla z dx = (z^+ - z^-) \otimes \nu\, d\mathcal{H}^{n-1} \llcorner S_z$;*

4) $\int_\Omega z\,dx = 0$.

Proof – We can assume $\phi(s) \geq cs - d$ for every $s \in [0, +\infty[$, so that there exists a constant C' such that

$$|Dz_h|_{T(\Omega)} \leq C'$$

and, by Theorem 1.11,

$$\|z_h\|_{L^1} \leq C' \ .$$

Hence there exists a subsequence, still denoted by z_h, and a function $z \in BV(\Omega, \mathbf{R}^k)$ such that $z_h \to z$ in $L^1(\Omega, \mathbf{R}^k)$. Let $t > 0$ and denote by z_h^t the vector valued function whose components are $(z_h^t)_j = ((z_h)_j \wedge t) \vee (-t)$ $(j = 1, \ldots, k)$. Then for every t we have $z_h^t \to z^t$ in $L^1(\Omega, \mathbf{R}^k)$ and

$$\sup_h \left\{ \|z_h^t\|_{L^\infty} + \int_\Omega \phi(|\nabla z_h^t|)dx + \mathcal{H}^{n-1}\Big(S_{z_h^t}\Big) \right\} < +\infty,$$

so by Theorem 2.1 of [4] there exists a subsequence such that $z_{h_i}^t \to z^t \in SBV(\Omega, \mathbf{R}^k)$ in $L^1(\Omega, \mathbf{R}^k)$. Since $z \in BV(\Omega, \mathbf{R}^k)$ and $z^t \in SBV(\Omega, \mathbf{R}^k)$ for every $t > 0$, then we obtain $z \in SBV(\Omega, \mathbf{R}^k)$.

The second assertion is proved in [4], Theorem 2.2. The third assertion follows by difference and the fourth one is trivial. q.e.d.

Theorem 1.15. *Assume (0.2),(0.3)(0.6) and fix $w \in SBH(\mathbf{R}^n)$ such that the energy $\mathcal{F}(w, 0, \beta, \gamma, \sigma, \mathbf{R}^n)$ is finite. Then $(\gamma - \lambda C_n(\Omega)) > 0$ and*

$$\begin{aligned} \mathcal{F}(v, 0, \beta, \gamma, \sigma, \mathbf{R}^n) \ &\geq \ (\gamma - \lambda C_n(\Omega))\, |D^2 v|_{T(\overline{\Omega})} + \ \delta_n(\Omega, w) \\ &\qquad \forall v \in SBH(\mathbf{R}^n) \ s.t. \operatorname{spt}(v - w) \subset \overline{\Omega} \end{aligned} \tag{1.12}$$

where

$$\delta_n(\Omega, w) = \mathcal{F}\big(w, \mathbf{R}^n \setminus \overline{\Omega}\big) + \|c_0\|_{L^1(\overline{\Omega})} - \frac{\gamma^2}{4}|\overline{\Omega}| - \lambda \tau_n(\Omega, w) - \lambda C_n |D^2 w|_{T(\overline{\Omega})},$$

$$\tau_1 = \tau_2 = |\overline{\Omega}|\, \|w\|_{L^\infty(\overline{\Omega})} \ , \quad \tau_n = |\overline{\Omega}|^{2/n}\, \|w\|_{L^{n/(n-2)}(\overline{\Omega})} \ \textit{if } n > 2 \ .$$

Proof – Since $\alpha = 0$

$$\begin{aligned}
\mathcal{F}(v,\mathbf{R}^n) \;=\; & \int_{\mathbf{R}^n} \left(|\nabla^2 v|^2 + \sigma|\nabla v|^2\right)\, dx + \int_{\mathbf{R}^n} G(x,v)\, dx + \\
& + \beta\mathcal{H}^{n-1}((S_{\nabla v}\setminus S_v)) + \gamma\int_{S_{\nabla v}} |[\nabla v]|\; d\mathcal{H}^{\,n-1} = \\
= & \int_{\mathbf{R}^n} \left(|\nabla^2 v|^2 + \sigma|\nabla v|^2\right)\, dx + \int_{\overline{\Omega}} G(x,v)\, dx + \\
& + \beta\mathcal{H}^{n-1}((S_{\nabla v}\setminus S_v)) + \gamma\int_{S_{\nabla v}} |[\nabla v]\,||\, d\mathcal{H}^{\,n-1} \geq \\
\geq & \mathcal{F}\big(w,\mathbf{R}^n\setminus\overline{\Omega}\big) + \|\nabla^2 v\|^2_{L^2(\overline{\Omega})} + \gamma\textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]| + \|c_0\|_{L^1(\overline{\Omega})} - \lambda\textstyle\int_{\overline{\Omega}} v
\end{aligned} \tag{1.13}$$

If $n = 1$, by Young inequality and applying the last inequality in Theorem 1.12 to $(v - w)$, we get

$$\begin{aligned}
& \|\nabla^2 v\|^2_{L^2(\overline{\Omega})} + \gamma\textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]| + \|c_0\|_{L^1(\overline{\Omega})} - \lambda\textstyle\int_{\overline{\Omega}} v\, dx = \\
= \; & \|\ddot v\|^2_{L^2(\overline{\Omega})} + \gamma\textstyle\sum_{S_{\dot v}\cap\overline{\Omega}} |[\dot v]| + \|c_0\|_{L^1(\overline{\Omega})} - \lambda\textstyle\int_{\overline{\Omega}} v\, dx \geq \\
\geq \; & \gamma\left(\|\ddot v\|_{L^1(\overline{\Omega})} + \textstyle\sum_{S_{\dot v}\cap\overline{\Omega}} |[\dot v]|\right) - \lambda\,|\overline{\Omega}|\,\|v - w\|_{L^\infty(\overline{\Omega})} + \\
& + \left(\|c_0\|_{L^1(\overline{\Omega})} - \tfrac{\gamma^2}{4}|\overline{\Omega}| - \lambda|\overline{\Omega}|\|w\|_{L^\infty(\overline{\Omega})}\right) \geq \\
\geq \; & \left(\gamma - \tfrac{1}{2}\lambda|\overline{\Omega}|^2\right)|v''|_{T(\overline{\Omega})} + \delta_1(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}\setminus\overline{\Omega}\big) = \\
= \; & (\gamma - \lambda C_1(\Omega))\,|v''|_{T(\overline{\Omega})} + \delta_1(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}\setminus\overline{\Omega}\big)\ .
\end{aligned} \tag{1.14}$$

If $n = 2$, by (1.5) of Theorem 1.6, we get

$$\begin{aligned}
& \|\nabla^2 v\|^2_{L^2(\overline{\Omega})} + \gamma\textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]| + \|c_0\|_{L^1(\overline{\Omega})} - \lambda\textstyle\int_{\overline{\Omega}} v\, dx \geq \\
\geq \; & \gamma\left(\|\nabla^2 v\|_{L^1(\overline{\Omega})} + \textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]| d\mathcal{H}^{\,n-1}\right) - \lambda\,|\overline{\Omega}|\,\|v - w\|_{L^\infty(\overline{\Omega})} + \\
& + \left(\|c_0\|_{L^1(\overline{\Omega})} - \tfrac{\gamma^2}{4}|\overline{\Omega}| - \lambda|\overline{\Omega}|\,\|w\|_{L^\infty(\overline{\Omega})}\right) \geq \\
\geq \; & \left(\gamma - \tfrac{\lambda|\overline{\Omega}|}{4}\right)|D^2 v|_{T(\overline{\Omega})} + \delta_2(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}^2\setminus\overline{\Omega}\big) = \\
= \; & (\gamma - \lambda C_2(\Omega))\,|D^2 v|_{T(\overline{\Omega})} + \delta_2(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}^2\setminus\overline{\Omega}\big)\ .
\end{aligned} \tag{1.15}$$

If $n > 2$, by Theorem 1.4 we get

$$\begin{aligned}
& \|\nabla^2 v\|^2_{L^2(\overline{\Omega})} + \gamma\textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]| + \|c_0\|_{L^1(\overline{\Omega})} - \lambda\textstyle\int_{\overline{\Omega}} v\, dx \geq \\
\geq \; & \gamma\left(\|\nabla^2 v\|_{L^1(\overline{\Omega})} + \textstyle\int_{S_{\nabla v}\cap\overline{\Omega}} |[\nabla v]|\right) - \lambda|\overline{\Omega}|^{2/n}\|v - w\|_{L^{\frac{n}{n-2}}(\overline{\Omega})} + \\
& + \left(\|c_0\|_{L^1(\overline{\Omega})} - \tfrac{\gamma^2}{4}|\overline{\Omega}| - \lambda|\overline{\Omega}|^{2/n}\|w\|_{L^{\frac{n}{n-2}}(\Omega)}\right) \geq \\
\geq \; & \left(\gamma - \lambda\,|\overline{\Omega}|^{2/n} K_n(\Omega)\right)|D^2 v|_{T(\overline{\Omega})} + \delta_n(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}^n\setminus\overline{\Omega}\big) = \\
= \; & (\gamma - \lambda C_n(\Omega))\,|D^2 v|_{T(\overline{\Omega})} + \delta_n(\Omega,w) - \mathcal{F}\big(w,\mathbf{R}^n\setminus\overline{\Omega}\big)\ .
\end{aligned} \tag{1.16}$$

q.e.d.

Theorem 1.16. *Assume (0.2),(0.3)(0.7),(0.8). Then* $(\gamma - \lambda\mathcal{C}_n(\Omega)) > 0$ *and*

$$\mathcal{F}(v,0,\beta,\gamma,\sigma,\Omega)) \geq (\gamma - \lambda\mathcal{C}_n(\Omega))\,|D^2v|_{T(\Omega)} + \Big(\|c_0\|_{L^1(\Omega)} - \tfrac{\gamma^2}{4}|\Omega|\Big) \qquad \forall v \in SBH(\Omega)\ . \tag{1.17}$$

Proof – Since $\mathcal{F}(v,\Omega) = \mathcal{F}(v-p(v),\Omega)$, the proof goes as like as the one of the previous Theorem, except the fact that, by handling Theorem 1.12, we estimate $\|v-p(v)\|_{L^\infty(\Omega)}$ (and $\|v-p(v)\|_{L^{n/(n-2)}(\Omega)}$) instead of the corresponding norms of $v-w$. q.e.d.

Theorem 1.17. *Let* $\{z_h\}_{h\in\mathbf{N}}$ *be a sequence in* $SBV(\Omega,\mathbf{R}^k)$*. Assume that* $\{z_h\}_{h\in\mathbf{N}}$ *converges in measure to* z *and that* $\{\nabla z_h\}_{h\in\mathbf{N}}$ *is weakly compact in* $L^1(A,M_{k,n})$ *for every open set* $A \subset\subset \Omega$*. Moreover let* $\theta : [0,+\infty[\to [1,+\infty[$ *be a concave, non decreasing function. Then*

$$\int_{S_z} \theta(|z^+ - z^-|)\,d\mathcal{H}^{n-1} \leq \liminf_{h\to+\infty} \int_{S_{z_h}} \theta(|{z_h}^+ - {z_h}^-|)\,d\mathcal{H}^{n-1}.$$

Proof – See Theorem 3.7 in [4]. q.e.d.

Lemma 1.18. *Hypotheses (0.2) and, either (0.6) or (0.7),(0.8) together, entail the sequential weak l.s.c. in* $BH(\Omega)$ *of the following functional*

$$v \longrightarrow \int_\Omega G(x,v(x))\,dx\ .$$

Proof – It follows by Fatou Lemma. q.e.d.

2. The class $GSBV^2$

In this section we recall some properties of the functional setting suitable for functional $\mathcal{F}$ when $\alpha > 0$: say the class $GSBV^2$.

Definition 2.1. *For* $\Omega \subset \mathbf{R}^n$ *open set*

$$GBV(\Omega) = \big\{v : \Omega \to \mathbf{R} \text{ Borel function}; -k \vee v \wedge k \in BV_{loc}(\Omega)\ \forall k \in \mathbf{N}\big\},$$

$$GSBV(\Omega) = \big\{v : \Omega \to \mathbf{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbf{N}\big\}.$$

Lemma 2.2. *Let $v \in GSBV(\Omega)$. Then*

1) S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable;

2) ∇v exists a.e. in Ω.

Proof – See [4], proposition 1.3 and 1.4. q.e.d.

Definition 2.1 has a very simple interpretation in dimension one, as it is shown in the following lemma (whose proof is trivial): Lemma 2.3 below shows that in dimension one strong formulation and weak formulation of Problem 0.2 coincide and it will be used in the proof of the interpolation theorem.

Lemma 2.3. *Let $a, b \in \mathbf{R}$ and $v \in GSBV(a, b)$. If $\int_a^b |\dot{v}(t)| dt + \mathcal{H}^0(S_v) < +\infty$ then $v \in SBV(a, b)$. If $\int_a^b |\dot{v}(t)|^2 dt < +\infty$ and $\mathcal{H}^0(S_v) = 0$ then $v \in H^{1,2}(a, b)$.*

Now we may define the following function spaces related to several boundedness properties of second derivatives, to avoid confusion with different notations.

Definition 2.4. *For any open bounded $\Omega \subset \mathbf{R}^n$ we set*

$$SBH(\Omega) = \left\{ v \in W^{1,1}(\Omega),\ Dv \in [SBV(\Omega)]^n \right\}$$

$$SBV^2(\Omega) = \left\{ v \in SBV(\Omega),\ \nabla v \in [SBV(\Omega)]^n \right\}$$

$$GSBV^2(\Omega) = \left\{ v \in GSBV(\Omega),\ \nabla v \in [GSBV(\Omega)]^n \right\}$$

For functions in $SBH(\Omega)$, $SBV^2(\Omega)$ or $GSBV^2(\Omega)$ we use the notation $\nabla^2_{ij} v = \nabla_j(\nabla_i v)$, $\nabla^2_i v = \nabla_i(\nabla_i v)$ and, in the one dimensional case, $\ddot{v} = (\dot{v})\dot{}$. Notice that $Dv = \nabla v$ in $SBH(\Omega)$, $Dv \neq \nabla v$ in $SBV^2(\Omega), GSBV^2(\Omega)$. Moreover we set

$$S_{\nabla v} = \cup_{i=1}^n S_{\nabla_i v}.$$

We notice that in dimension $n > 1$ the sublevels $\{ v \in SBV^2(\Omega),\ \mathcal{F}(v) \leq c \}$ of the functional $\mathcal{F}$ with the choice $G(x, s) = s^2$, $\gamma = \sigma = 0$, are not compact with respect to a.e. convergence as shown by the following example.

Example 2.5 - We choice $G(x,v) = |v|^2$, $y = (y_1, y_2) \in \mathbf{R}^2$ and $h \in \mathbf{N}, h \geq 4$ set $r_h = 2^{-h-1}$ and

$$v_h(y_1,y_2) = \begin{cases} \dfrac{4^k}{k}\left(y_1 - \frac{1}{k}\right) & \text{if } y \in B_{r_k}\left(\left(\frac{1}{k},0\right)\right), k = 4,\dots,h \\ \\ 0 & \text{if } y \notin \cup_{k=4}^{h} B_{r_k}\left(\left(\frac{1}{k},0\right)\right). \end{cases}$$

We have, for $\sigma = \gamma = 0$,

$$\mathcal{F}(v_h, \mathbf{R}^2) = \int_{\mathbf{R}^2} |v_h|^2 dy + \alpha \mathcal{H}^1\left(S_{v_h}\right) = \frac{\pi}{64}\sum_{k=4}^{h}\frac{1}{k^2} + \alpha\pi\sum_{k=4}^{h}\frac{1}{2^k} \leq c < +\infty.$$

The sequence converges in $L^2(\mathbf{R}^2)$ to the function v_0 given by

$$v_0(y) = v_0(y_1,y_2) = \begin{cases} \frac{4^k}{k}\left(y_1 - \frac{1}{k}\right) & \text{if } y \in B_{r_k}\left(\left(\frac{1}{k},0\right)\right),\ k = 4,\dots \\ \\ 0 & \text{if } y \notin \bigcup_{k=4}^{\infty} B_{r_k}\left(\left(\frac{1}{k},0\right)\right). \end{cases}$$

We see that ∇v_0 is not $L^1_{loc}(\mathbf{R}^2)$, hence $v_0 \notin SBV^2_{loc}(\mathbf{R}^2)$ and even $v_0 \notin SBV_{loc}(\mathbf{R}^2)$.

Lemma 2.6. *Hypotheses (0.2) and (0.5) entail the sequential lower semicontinuity with respect to strong L^p convergence $p \in [1,2)$ of the map*

$$v \longrightarrow \int_\Omega G(x, v(x))\, dx\ .$$

PROOF – It follows from Fatou Lemma. q.e.d.

3. Slicing properties and second derivatives in $GSBV^2$

In this section, for any fixed direction in $\mathbf{R}^n$, we define the fibers of Ω along this direction and recall some properties related to the decomposition of derivatives for the traces on the fibers of functions defined on the whole Ω.

Let $\nu \in \partial B_1$ and define the orthogonal projection on the hyperplane perpendicular to ν :

$$\pi_\nu : \mathbf{R}^n \to \mathbf{R}^n$$

$$\pi_\nu(y) = y - (y \cdot \nu)\nu.$$

For every subset $E \subset \mathbf{R}^n$ we define, for every $x \in \pi_\nu(E)$,

$$E_x^\nu = \{t \in \mathbf{R};\ x + t\nu \in E\},$$

for every $v \in GSBV(\Omega)$ we define for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_\nu(E)$,

$$v_x^\nu(t) = v(x + t\nu) \quad \text{(defined for a.e. } \ t \in E_x^\nu).$$

We prove some slicing properties in the case $\nu = \mathbf{e}_i$ and $\Omega = (0,1)^n$; for brevity's sake in this case we set $\pi_i = \pi_{\mathbf{e}_i}$ and we omit the label ν in the previous definitions.

Theorem 3.1. *Let $\Omega = (0,1)^n$ and $v \in GBV(\Omega)$. Then for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega)$ $(i = 1, \dots, n)$*

(i) $v_x \in GBV(0,1)$,

(ii) $\dot{v}_x(t) = \nabla_i v(x + t\mathbf{e}_i)$ *for a.e.* $t \in (0,1)$,

(iii) $S_{v_x} = (S_v)_x$.

Moreover

(iv) $v \in GSBV(\Omega)$ *iff* $v_x \in GSBV(0,1)$ *for* $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega), \forall i$,

(v) *if* $v \in SBV(\Omega, \mathbf{R}\ )$ *then* $v_x \in SBV(0,1)$ *for* $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega), \forall i$.

PROOF – See Theorem 2 of [15]. q.e.d.

Theorem 3.2. *Let $\Omega = (0,1)^n$ and $v \in GSBV^2(\Omega)$. Then for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega)$ $(i = 1, \dots, n)$*

(i) $\ddot{v}_x(t) = \nabla_i^2 v(x + t\mathbf{e}_i)$ *a.e.* $t \in (0,1)$,

(ii) $S_{\dot{v}_x} \subseteq (S_{\nabla v})_x$,

(iii) $(\dot{v}_x)^+(t) = (\nabla_i v)^+(x + t\mathbf{e}_i), \quad (\dot{v}_x)^-(t) = (\nabla_i v)^-(x + t\mathbf{e}_i) \quad \forall t \in (0,1).$

PROOF – See Theorem 3 of [15]. q.e.d.

Theorem 3.3. *For every $\mathcal{H}^{n-1}$ measurable set $E \subset \mathbf{R}^n$, $(\mathcal{H}^{n-1}, n-1)$ rectifiable and for every $\nu \in \partial B_1$,*

$$\int_{\pi_\nu(E)} \mathcal{H}^0(E_x)\, d\mathcal{H}^{n-1}(x) = \int_E |\nu \cdot \nu_E(y)| d\mathcal{H}^{n-1}(y) \leq \mathcal{H}^{n-1}(E),$$

where $\nu_E(y)$ is an approximate unit normal to E at y.

PROOF – See [30], 3.2.22. q.e.d.

Remark 3.4 - The definition of $GSBV^2(\Omega)$ and the properties proved in Theorems 3.1, 3.2, 3.3 are invariant under translations and rotations of the coordinates (see [30], 3.1.4), hence the results obtained hold for sections of v in any direction.

4. Compactness in $GSBV^2$ and in *SBH*

In this section we show compactness properties of sublevels of the functional. Many proofs could be simplified in the case $\sigma > 0$, anyway all along this section we assume the more general case $\sigma \geq 0$.

At first we show a compactness property, under assumption (0.4), valid only for 1 dimensional open sets which are finite union of intervals; then we deduce a compactness property valid for any bounded open set in any dimension n (hence also for any 1 dimensional open set). The case of assumption (0.3) is much easy.

Theorem 4.1. *Assume $A \subset \mathbf{R}$ be an open set which is a finite union of open intervals and there are $c_3 > 0,\ c_4 \in L^1(A)$ s.t. $G(x,s) > c_3|s|^2 - c_4,\ a.e.\, x$, fix the parameters such that ,*

$$\alpha > 0,\ \beta > 0,\ \sigma \geq 0,\ \gamma \geq 0.$$

Assume that, for any $h \in \mathbf{N}$, $v_h \in \mathcal{W}^2(A)$ the set of piece-wise H^2 functions possibly with a finite number of jumps, and

$$\sup_{h \in \mathbf{N}} \mathcal{F}^1(v_h, \alpha, \beta, \gamma, \sigma, A) < \infty$$

where

(4.1)
$$\mathcal{F}^1(v_h,\alpha,\beta,\gamma,\sigma,A) \;=\; \int_A (|\ddot v_h(x)|^2+\sigma|\dot v_h(x)|^2+G(x,v_h(x)))\,dx+ \\ +\alpha\,\#(S_{v_h}\cap A)+\beta\,\#((S_{\dot v_h}\setminus S_{v_h})\cap A)+\gamma\sum_{S_{\dot v_h}\cap A}|[\dot v_h]|\,.$$

Then there are v_{h_k} *and* $v_\infty\in\mathcal{W}^2(A)$ *such that*

$$\begin{array}{lll} v_{h_k} & \longrightarrow\; v_\infty & \text{in } L^p(A),\quad p\in[1,2), \\ \dot v_{h_k} & \longrightarrow\; \dot v_\infty & \text{a.e. in } A. \\ \ddot v_{h_k} & \rightharpoonup\; \ddot v_\infty & \text{weakly in } L^2(A). \end{array} \tag{4.2}$$

Proof – The finiteness of the functional entails $\#(S_{v_h}\cup S_{\dot v_h})\le K<\infty$. If $K=0$ the statement is trivial. It is not restrictive to assume that the cardinality is definitively constant and equal to K. This leads to consider the K sequences $\{y^i_k\}_k,\ i=1,...,K$ of singular points. Since $A=\cup_M A_M$ with $A_M=A\cap(-M,+M)$, by Bolzano-Weierstrass Theorem, up to successive extraction of subsequences, without relabelling, we find a finite number of open intervals $I^i\subset A_M,\ i=1,...,m(M)\le K$ s.t. $y^i_k\in I^i,\ 1\le i\le m(M)$.

By summarizing we find a finite number ($K(\infty)=\sup_M m(M)\le K$) of significative intervals which contain all the singular points. Then it is not restrictive to examine just one open interval and assume $A=(a,b)$.

Choose a minimal partition $x^j_h,\ j=1,...,s$ of the interval (a,b) such that

$$x^0_h=a,\quad x^{s+1}_h=b,\quad x^j_h<x^{j+1}_h,\quad \mathop{\cup}_{j=1}^{s}\left\{x^j_h\right\} \;=\; S_{v_h}\cup S_{\dot v_h} \tag{4.3}$$

Then there are subsequences $\{x^j_h\}_h,\ j=1,...,s$, such that, without relabelling,

$$\begin{array}{l} x^j_h \longrightarrow x^i \qquad i=1,...,r\le s, \qquad j=r_i,...,r_{i+1}-1, \\ \qquad 1\le r_i\le r_{i+1}\le s, \qquad x^i<x^{i+1}, \qquad x^0=a,\ x^{r+1}=b\,. \end{array} \tag{4.4}$$

There is $\delta>0$ such that the intervals $[x^i-\delta,x^i+\delta]$, $i=1,...,r$ are pair-wise disjoint. By the interpolation inequality (Lemma 4.10 in [1])

$$\int_0^l |\dot v|^2\,dt \;\le\; 162\,\max(l^2,l^{-2})\left(\int_0^l |\ddot v|^2+|v|^2\,dt\right) \qquad \forall v\in H^2(0,l) \tag{4.5}$$

for all ϵ s.t. $0 < \epsilon < \delta$, by choosing $l = \min_i(x^{i+1} - x^i - 2\delta)$ in (4.5), we get the existence of $v \in H^2(x^i + \epsilon, x^{i+1} - \epsilon)$ s.t. up to subsequences

(I) $v_h \longrightarrow v$ strongly in $H^1(x^i + \epsilon, x^{i+1} - \epsilon)$ and uniformly in $(x^i + \epsilon, x^{i+1} - \epsilon)$
(II) $\dot{v}_h \longrightarrow v'$ uniformly in $(x^i + \epsilon, x^{i+1} - \epsilon)$
(III) $\ddot{v}_h \longrightarrow v''$ weakly in $L^2(x^i + \epsilon, x^{i+1} - \epsilon)$.

Moreover, by the l.s.c. of the L^2 norm we get

$$\int_{x^i+\epsilon}^{x^{i+1}-\epsilon} |v''|^2\, dt \ \le\ \liminf_h \int_{x^i+\epsilon}^{x^{i+1}-\epsilon} |\ddot{v}_h|^2\, dt \ \le\ \sup_h \mathcal{F}^1(v_h, A) \ \le\ C \ <\ +\infty.$$

Hence by a diagonal argument we can define $v_\infty \in H^2_{loc}(x^i, x^{i+1})$ s.t. $v_{h_k} \to v_\infty$ point-wise in (x^i, x^{i+1}) and strongly in $H^1_{loc}(x^i, x^{i+1})$

$$(IV) \qquad \dot{v}_h \longrightarrow \dot{v} = v' \text{ point-wise in } (x^i, x^{i+1}) \qquad i = 0, ..., r$$

and $\ddot{v}_{h_k} \rightharpoonup v''_\infty$ weakly in $L^2_{loc}(x^i, x^{i+1})$. By Fatou Lemma $v_\infty \in L^2(x^i, x^{i+1})$ and by the previous inequality as $\varepsilon \to 0$

$$\int_{x^i}^{x^{i+1}} |v''_\infty|^2 dt \le C < +\infty.$$

Hence $\ddot{v}_{h_k} \rightharpoonup v''_\infty$ weakly in $L^2(x^i, x^{i+1})$ and, by the interpolation inequality (4.5), $v_\infty \in H^2(x^i, x^{i+1})$. By the arbitrariness of i, $v_\infty \in \mathcal{W}^2(A)$ and the thesis follows. q.e.d.

In order to face the multidimensional case we introduce suitable technical tools.

Truncations are allowed in SBH and $GSBV^2$. Anyway the truncation has the effect of creating an increase of energy in the length of $S_{\nabla v}$ and in the jump of ∇v, moreover this amount cannot be estimated for general choices of parameters and data. To overcome the increase of the singular part of the energy due to truncations we introduce the sequence of functions (φ_k) as follows in order to mimic the truncation with the composition with them. We set

$$(4.6) \qquad \begin{cases} \varphi_k \in C^2(\mathbf{R}), & 0 \le \varphi'_k \le 1, \\ \varphi_k(t) = t & \forall t \in [-k+1, k-1] \\ |\varphi_k(t)| = k & \forall |t| > k+1. \end{cases}$$

Remark 4.2 - The property v belongs to $GSBV(\Omega)$ is clearly equivalent to requiring that

$$\varphi_k \circ v \in SBV_{\mathrm{loc}}(\Omega) \quad \forall k \in \mathbf{N}$$

(see [4], section 1).

We want a weak estimate of the first gradient: to overcome the lack of a suitable interpolation inequality we exploit the inequality in n dimensional cubes proved in [15] which allows a control of the first gradient even when $\gamma = \sigma = 0$ and without assuming any information on the topology of the complement of the singular set.

Theorem 4.3. *([15],Th.6) Let Q be a cube with edges of lenght l and parallel to the axes, $v \in GSBV^2(Q)$ and $k \in \mathbf{N}$. Then for every $i = 1, \dots, n$ the following interpolation inequality holds true*

$$\int_Q |\nabla_i(\varphi_k \circ v)| dy \le 2k \left(\mathcal{H}^{n-1}(S_v \cup S_{\nabla v}) + l^{n-1}\right) + \frac{2}{3} l^{\frac{n}{2}+1} \left(\int_Q |\nabla^2 v|^2 dy\right)^{\frac{1}{2}}.$$

The previous interpolation inequality is essential to deal with the case $\alpha > 0$ which is faced in $GSBV^2$ while it is not necessary in the case $\alpha = 0$ which is studied in SBH.

Theorem 4.4. *Assume $\Omega \subset \mathbf{R}^n$ be a bounded open set, $n \ge 1$, $\alpha > 0$, $\beta > 0$, $\sigma \ge 0$, $\gamma \ge 0$, (0.2),(0.4),(0.5) and v_h in $GSBV^2(\Omega)$ be such that*

$$\sup_h \mathcal{F}\left(v_h, \alpha, \beta, \gamma, \sigma, \Omega\right) < \infty.$$

Then there are $v_\infty \in GSBV^2(\Omega) \cap L^2(\Omega)$ and a subsequence v_{h_m} such that

$v_{h_m} \to v_\infty$ a.e. *in* Ω *and* w-$L^2(\Omega)$
$v_{h_m} \to v_\infty$ *strongly in* $L^p(\Omega)$ *for every* $p \in [1, 2)$
$\nabla v_{h_m} \to \nabla v_\infty$ a.e. *in* Ω
$\nabla^2 v_{h_m} \rightharpoonup \nabla^2 v_\infty$ w-$[L^2(\Omega)]^{n \times n}$.

PROOF – Since $\sup_h \|v_h\|_{L^2} < \infty$, there exist a subsequence that we still denote by (v_h) and $v_\infty \in L^2(\Omega)$ such that

$$\lim_h v_h = v_\infty \qquad w - L^2(\Omega). \tag{4.7}$$

Fix $k \in \mathbf{N}$ and let $\varphi_k \in C^2(\mathbf{R})$ as in (4.6). Then we have

$$\|\varphi_k \circ v_h\|_{L^1} \leq \|v_h\|_{L^2} |\Omega|^{1/2} \qquad \forall h \in \mathbf{N}$$

and, by the interpolation Theorem 4.3, for every $i = 1, \dots, n$ and for every cube $Q \subset \Omega$ with edges parallel to the axes of lenght l, we have

$$\sup_h \int_Q |\nabla_i(\varphi_k \circ v_h)| dy \leq c(k, l) < +\infty.$$

By the definition of $SBV(\Omega)$ and by the previous inequalities, for every $\Omega' \subset\subset \Omega$ we obtain

$$\begin{aligned} \sup_h \|\varphi_k \circ v_h\|_{BV(\Omega')} &= \sup_h \left(\int_{\Omega'} |D(\varphi_k \circ v_h)| + \int_{\Omega'} |\varphi_k \circ v_h| dy\right) \leq \\ &\sup_h \left(\int_{\Omega'} (|\nabla(\varphi_k \circ v_h)| + |\varphi_k \circ v_h|)\, dy + 2k\mathcal{H}^{n-1}(S_{v_h})\right) < +\infty, \end{aligned} \tag{4.8}$$

hence, by the compactness theorem in $BV_{loc}(\Omega)$, by the equiboundedness in $L^\infty(\Omega)$ with respect to h of $(\varphi_k \circ v_h)$ and by a diagonalization argument, there exist a subsequence (that we do not relabel) and $w_k \in BV_{loc}(\Omega)$ such that

$$\lim_h \varphi_k \circ v_h = w_k \qquad \text{a.e. and in } L^p(\Omega) \quad 1 \leq p < +\infty, \forall k \in \mathbf{N}. \tag{4.9}$$

Set $E_k = \{|w_k| \leq k - 1\}$, then by (4.7)

$$\lim_h v_h \chi_{E_k} = v_\infty \chi_{E_k} \quad w - L^2(\Omega),$$

by the properties of φ_k and by (4.9)

$$\lim_h v_h \chi_{E_k} = \lim_h (\varphi_k \circ v_h) \chi_{E_k} = w_k \chi_{E_k} \qquad \text{a.e. in } \Omega,$$

hence, since $|\varphi_k \circ v_h| \le k$ a.e. in Ω, we get

$$w_k = v_\infty \quad \text{a.e. in } E_k \ \forall k \in \mathbf{N}.$$

Since $E_k = \{|v_\infty| \le k-1\}$ and $v_\infty \in L^2(\Omega)$, we have $\lim_k |\Omega \setminus E_k| = 0$ and then

$$\lim_h v_h = v_\infty \qquad \text{a.e. in } \Omega, \tag{4.10}$$

hence for every i and for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega)$ we have

$$(v_h)_x \to (v_\infty)_x \qquad \text{a.e. in } \Omega_x. \tag{4.11}$$

By (4.7), (4.10) and the boundedness of Ω we get $v_h \to v_\infty$ strongly in $L^p(\Omega)$ for $p \in [1,2)$. By (4.8), (4.9) and (4.10), $\varphi_k \circ v_\infty = w_k \in BV_{loc}(\Omega)$ $\forall k$, hence $v_\infty \in GBV(\Omega)$. By using Fatou's Lemma, Fubini's Theorem, and Theorems 3.1, 3.2, 3.3, 4.1, 4.3 and Remark 4.2, for every i, we get

$$\int_{\pi_i(\Omega)} \liminf_h \left(\int_{\Omega_x} |(\ddot{v}_h)_x|^2 dt + \mathcal{H}^0(S_{(v_h)_x} \cup S_{(\dot{v}_h)_x}) \right) dx \le$$
$$\liminf_h \int_{\pi_i(\Omega)} \left(\int_{\Omega_x} |(\ddot{v}_h)_x|^2 dt + \mathcal{H}^0(S_{(v_h)_x} \cup S_{(\dot{v}_h)_x}) \right) dx \le$$
$$\liminf_h \left(\int_\Omega |\nabla_i^2 v_h|^2 dx + \mathcal{H}^{n-1}(S_{v_h} \cup S_{\nabla_i v_h}) \right) < +\infty$$

by the assumption, so that for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega)$ we have

$$\liminf_h \left(\int_{\Omega_x} \left(|(\ddot{v}_h)_x|^2 + |(v_h)_x|^2\right) dt + \mathcal{H}^0(S_{(v_h)_x} \cup S_{(\dot{v}_h)_x}) \right) < +\infty, \tag{4.12}$$

hence for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_i(\Omega)$, by Theorem 3.3, (4.11) and the limit uniqueness, we get $(v_\infty)_x \in GSBV^2(\Omega_x)$. By the same argument for every $i = 1, \dots, n$, and by Theorem 3.1 we obtain $v_\infty \in GSBV(\Omega)$ and also $\nabla v_\infty(x) \in \mathbf{R}^n$ for a.e. $x \in \Omega$. The property just stated allows us to use the same argument in the proof of Theorem 2.1 in [3] for the sequence $(\nabla_i v_h)$, hence there exists $f_i \in GSBV(\Omega)$ such that, up to a subsequence,

$$\nabla_i v_h \to f_i \qquad \text{a.e. in } \Omega$$

$$\nabla^2_{ij} v_h \rightharpoonup \nabla_j f_i \qquad w - L^2(\Omega).$$

By (4.11) and the slicing argument above we know that

$$\nabla_i v_h \to \nabla_i v_\infty \qquad \text{a.e. in } \Omega$$

hence $\nabla_i v_\infty = f_i \in GSBV(\Omega)$ and then $v_\infty \in GSBV^2(\Omega)$. q.e.d.

Eventually we give two compactness properties valid under the assumption (0.3). The proof is a straightforward consequence of Theorem 1.14, and Rellich-Kondratiev Theorem when $\sigma > 0$.

Theorem 4.5. *Assume $\Omega \subset \mathbf{R}^n$ be a bounded open set, (0.2), (0.3) and (0.6). Let v_h in $SBH(\mathbf{R}^n)$ with $\operatorname{spt} v_h \subset \overline{\Omega}$ and*

$$\sup_h \mathcal{F}(v_h, 0, \beta, \gamma, \sigma, \mathbf{R}^n) < \infty.$$

Then there are $v_\infty \in SBH(\mathbf{R}^n)$ and a subsequence v_{h_m} such that

$v_{h_m} \to v_\infty$ *a.e. in* $\mathbf{R}^n$
$v_{h_m} \to v_\infty$ *in* $L^q(\mathbf{R}^n)$ *for every* $q \in [1, \frac{n}{n-2})$ *if* $n \geq 3$ *(any* q *if* $n = 1, 2$*)*
$Dv_{h_m} \to Dv_\infty$ *in* $L^1(\mathbf{R}^n)$
$\nabla^2 v_{h_m} \rightharpoonup \nabla^2 v_\infty$ w-$[L^2(\mathbf{R}^n)]^{n\times n}$.

PROOF – By Theorem 1.15 we get a bound for $|D^2 v_h|_{T(\mathbf{R}^n)}$. By Theorem 1.14, applied to Dv_h, and by Theorem 1.4, applied in a ball containing $\overline{\Omega}$, the thesis follows. q.e.d.

Theorem 4.6. *Assume $\Omega \subset \mathbf{R}^n$ with the property $\mathcal{R}$, (0.2), (0.3), (0.7), (0.8). Let v_h in $SBH(\Omega)$ be such that*

$$\sup_h \mathcal{F}(v_h, 0, \beta, \gamma, \sigma, \Omega) < \infty.$$

Then there are $v_\infty \in SBH(\Omega)$ and a subsequence v_{h_m} such that

$$
\begin{aligned}
&v_{h_m} - pv_{h_m} \to v_\infty \qquad \text{a.e. } in\ \Omega, \\
&v_{h_m} - pv_{h_m} \to v_\infty \qquad in\ L^q(\Omega) \\
&\qquad\qquad\qquad\qquad for\ every\ q \in [1, \tfrac{n}{n-2})\ if\ n \geq 3\ \ (any\ q\ if\ n = 1,2), \\
&D(v_{h_m} - pv_{h_m}) \to Dv_\infty \qquad in\ L^1(\Omega), \\
&\nabla^2 v_{h_m} \rightharpoonup \nabla^2 v_\infty \qquad \text{w-}[L^2(\Omega)]^{n\times n}.
\end{aligned}
$$

PROOF – By Theorem 1.16 we get a bound for $|D^2 v_h|_{T(\Omega)}$. By Theorem 1.14, applied to $D(v_h - pv_h)$, and by Theorems 1.11 and 1.4, the thesis follows.
q.e.d.

Remark 4.7 - So far we have not proved that $\mathcal{F}(v_\infty)$ is finite even in the case $\gamma > 0$, where v_∞ is defined in the thesis of Theorems 4.4, 4.5 and 4.6. The finiteness of $\mathcal{F}(v_\infty)$ will follow from the semicontinuity properties in the next section.

5. Lower semicontinuity

We recall that the commutation of operators $\nabla_i\ \nabla_j$ is true in *SBH* but fails in $GSBV^2$ (see [32]). In particular the functional (0.1) cannot be sliced, due to the presence of mixed derivatives in $\nabla^2 v$. Hence in such case the difficult task is showing the lower semi continuity of length terms, integral jump contribution and the square of mixed derivatives $\nabla_{ij} v$. Several steps are required in order to achieve this l.s.c. property.

The idea is to estimate the part of the functional which cannot be sliced, by mean of a covering method and exploiting the 1-dimensional lower semicontinuity in every tile of the covering.

Before stating the lower semicontinuity theorem, we prove some lemmas. The first one allows us to deal with the singular part of the mixed derivatives which cannot be estimated by simply considering any fixed one dimensional fiber.

Definition 5.1. *For every $(\mathcal{H}^{n-1}, n-1)$ rectifiable set $\Sigma \subset \Omega$ we denote by $\nu_\Sigma(y)$ the approximate unit normal to Σ at y, defined for $\mathcal{H}^{n-1}$* a.e. *$y \in \Sigma$.*

We remark that, up to the orientation, for every $v \in GSBV^2(\Omega)$

$$\nu_\Sigma = \nu_{S_v} \qquad \text{if } \Sigma \subset S_v$$

$$\nu_\Sigma = \nu_{S_{\nabla v}} \qquad \text{if } \Sigma \subset S_{\nabla v}.$$

For every fixed $x \in \Sigma$ and $\nu \in \partial B_1$, we can make a conventional choice of $\nu_\Sigma(x)$ satisfying $\nu_\Sigma(x) \cdot \nu \geq 0$. Such choice induces a unique orientation in every portion of Σ where $\nu_\Sigma \cdot \nu \neq 0$.

Lemma 5.2. *Let $v \in GSBV^2(\Omega)$. Fix $i \in \{1, \ldots, n\}$. Choose $\nu, \eta \in \partial B_1$, and $\Sigma \subset S_{\nabla_i v} \setminus S_v$ be a Borel set such that*

(i) $$\nu \neq \eta, \qquad \nu \cdot \eta > \frac{3}{4},$$

(ii) $$\nu \cdot \nu_\Sigma(y) > \frac{3}{4}, \qquad \eta \cdot \nu_\Sigma(y) > \frac{3}{4}, \qquad \mathcal{H}^{n-1} \text{ a.e. } y \in \Sigma,$$

(iii) $\{\nu, \eta, \mathbf{e}_i\}$ *linearly dependent.*

Then there are two disjoint Borel sets Σ_0, Σ_1 such that $\mathcal{H}^{n-1}(\Sigma \setminus (\Sigma_0 \cup \Sigma_1)) = 0$ and

$$\forall y \in \Sigma_0 \ \exists x \in \pi_\nu(\Sigma_0), \exists t \in S_{(\dot{v}_x^\nu)} : y = x + t\nu$$

$$\forall y \in \Sigma_1 \ \exists z \in \pi_\eta(\Sigma_1), \exists \tau \in S_{(\dot{v}_z^\eta)} : y = z + \tau\eta.$$

Proof – Due to the assumptions (i) and (ii) the orientations induced on ν_Σ by ν and η coincide on Σ and this will be understood in the following.

By assumption (ii) and by Theorem 3.3, if $N \subset \Sigma$ then

$$\mathcal{H}^{n-1}(N) = 0 \quad \text{iff} \quad \mathcal{H}^{n-1}(\pi_\nu(N)) = 0 \quad \text{iff} \quad \mathcal{H}^{n-1}(\pi_\eta(N)) = 0.$$

Then by Theorem 3.1(ii) and Theorem 3.3 for a.e. $y \in \Sigma$, setting $x = \pi_\nu(y)$ and $y = x + t\nu$ we have

$$t \in S_{(\dot{v}_x^\nu)} \ \text{ iff } \ (\nabla v \cdot \nu)^+(y) \neq (\nabla v \cdot \nu)^-(y),$$

hence we can choose

$$\Sigma_0 = \{y \in \Sigma; (\nabla v \cdot \nu)^+(y) \neq (\nabla v \cdot \nu)^-(y) \ \}.$$

On the other hand, if we set

$$\Sigma_1 = \{y \in \Sigma; (\nabla v \cdot \nu)^+(y) = (\nabla v \cdot \nu)^-(y) \ \},$$

since $\Sigma \subset S_{\nabla_i v} \setminus S_v$ then

$$(\nabla_i v)^+(y) = (\nabla v \cdot \mathbf{e}_i)^+(y) \neq (\nabla v \cdot \mathbf{e}_i)^-(y) = (\nabla_i v)^-(y) \qquad \mathcal{H}^{n-1} \text{ a.e. } y \in \Sigma,$$

hence by the assumptions (i) and (iii) we get

$$(\nabla v \cdot \eta)^+(y) \neq (\nabla v \cdot \eta)^-(y) \qquad \mathcal{H}^{n-1} \text{ a.e. } y \in \Sigma_1.$$

The thesis follows since, by setting $z = \pi_\eta(y), y = z + \tau\eta$, we have

$$\tau \in S_{(\dot{v}_z^\eta)} \text{ iff } (\nabla v \cdot \eta)^+(y) \neq (\nabla v \cdot \eta)^-(y).$$

q.e.d.

Now we show the lower semicontinuity in the one dimensional case; then we will deduce the lower semicontinuity for $n \geq 1$.

We would like to exploit the one dimensional compactness of Theorem 4.1 but the convergence (I)(II)(III) hold only in $(x_h^i + \varepsilon, x_h^{i+1} - \varepsilon)$, say the interval endpoints may depend on h. Nevertheless, taking into account of possible collapse of several sequences, the convergence denoted by (I)(II)(III) in its proof actually holds up to the endpoints of the interval (as in the case of (IV)), say in (x^i, x^{i+1}) in a suitable sense: by handling translations as stated in the following lemma.

Lemma 5.3. *(Translations) With the notation defined in the proof of Theorem 4.1, we assume that* $\lim_h x_h^j = x^i$, $j = r_i, ..., r_{i+1} - 1$ *and* x^i *is not limit of other sequences. By setting* $E_+^i = [x^i, x^i + \delta]$, $E_-^i = [x^i - \delta, x^i]$, *we deduce that convergence denoted by (I)(II)(III)(IV) in the proof of Theorem 4.1 hold in intervals of length* $\delta > 0$ *for both*

$$\begin{aligned} &v_h(x - x^i + x_h^{r_i}) \longrightarrow v(x) && x \text{ in } E_-^i \\ &v_h(x - x^i + x_h^{r_{i+1}-1}) \longrightarrow v(x) && x \text{ in } E_+^i \end{aligned} \tag{5.1}$$

Hence

$$(5.2)\qquad \begin{aligned} v(x^i_+) &= \lim_h \; v_h\left((x_h^{r_{i+1}-1})_+\right) \\ v(x^i_-) &= \lim_h \; v_h\left((x_h^{r_i})_-\right) \\ \dot v(x^i_+) &= \lim_h \; \dot v_h\left((x_h^{r_{i+1}-1})_+\right) \\ \dot v(x^i_-) &= \lim_h \; \dot v_h\left((x_h^{r_i})_-\right) \end{aligned}$$

PROOF – It is a consequence of continuity of translations in L^2 applied to $\dot v, \ddot v$ and the uniform continuity of H^2 functions in 1 dimension. q.e.d.

We now exploit the fact that, if $\|\ddot v_h\|_{L^2}$ is kept uniformly bounded, then the contribution to the variation of $\dot v_h$ due to the absolutely continuous part of v_h'' is vanishing as the length vanishes. More precisely we have:

$$(5.3)\qquad \begin{aligned} |\dot w(x+\epsilon) - \dot w(x)| \;&\leq\; \sqrt{\varepsilon}\, \|\ddot w\|_{L^2(x,x+\delta)} \\ &\forall x \in A,\ \forall \epsilon \in (0,\delta],\ \forall\, w \in H^2(x,x+\delta) \end{aligned}$$

Theorem 5.4. *(1D Lower semicontinuity in SBV^2)*

Assume $A \subset \mathbf{R}$ is an open interval, (0.2),(0.4) hold true. Let v_∞, v_h, $\dot v_\infty$, $\dot v_h \in SBV(A)$ $(h \in \mathbf{N})$, such that

$$\sup_{h\in\mathbf{N}} \mathcal{F}^1(v_h,\alpha,\beta,\gamma,\sigma,A) < \infty \quad \textit{and} \quad v_h \to v_\infty \qquad \textit{a.e. in } A$$

where

$$(5.4)\qquad \begin{aligned} \mathcal{F}^1(v_h,\alpha,\beta,\gamma,\sigma,A) \;&=\; \int_A (|\ddot v_h(x)|^2 + \sigma|\dot v_h|^2 + G(x,v_h(x)))\,dx + \\ &+ \alpha\,\#(S_{v_h}\cap A) + \beta\,\#((S_{\dot v_h}\setminus S_{v_h})\cap A) + \gamma \sum_{S_{\dot v_h}\cap A} |[\dot v_h]| \,. \end{aligned}$$

Then

$$\mathcal{F}^1(v_\infty,\alpha,\beta,\gamma,\sigma,A) \leq \liminf_h \mathcal{F}^1(v_h,\alpha,\beta,\gamma,\sigma,A).$$

Proof – The absolutely continuous part of the functional $\int_A (|\ddot{v}_h(x)|^2 + \sigma|\dot{v}_h|^2 + G(x, v_h(x)))\,dx$ is sequentially l.s.c. (see Lemma 2.6), so we have only to show that

$$
\begin{aligned}
&\alpha\#(S_{v_\infty}) + \beta\#(S_{\dot{v}_\infty} \setminus S_{v_\infty}) + \gamma \sum_{S_{\dot{v}_\infty}} |[\dot{v}_\infty]| \le \\
&\le \liminf_h \left(\alpha\#(S_{v_h}) + \beta\#(S_{\dot{v}_h} \setminus S_{v_h}) + \gamma \sum_{S_{\dot{v}_h}} |[\dot{v}_h]| \right) .
\end{aligned}
\tag{5.5}
$$

By a selection argument identical to the one at the beginning of the proof of Theorem 5.1 it is not restrictive to assume, by localization, that there is at most one point in $S_{v_\infty} \cup S_{\dot{v}_\infty}$.

Moreover, if there is no such point, then $v_\infty \in H^2(A)$ and for $h > \tilde{h}$, $v_h \in H^2$, and the l.s.c. in A is straightforward.

Else there is exactly one point $\overline{x} \in S_{v_\infty} \cup S_{\dot{v}_\infty}$: in this case, up to subsequences, $v_h \notin H^2(A)$, $\forall h$ (otherwise $\|v_h\|_{H^2}$ is bounded which is a contradiction). Hence there are only two cases to be examined.

A) If there is exactly one point $\overline{x} \in S_{v_\infty}$ then, either there is an approximating sequence of jumps (and possibly something more in $S_{v_h} \cup S_{\dot{v}_h}$) or two approximating sequences of creases (and possibly something more in $S_{v_h} \cup S_{\dot{v}_h}$). This because one jump cannot be approximated by only one sequence of creases, without blow-up of the energy. In fact $\mathcal{F}^1(v_h)$ bounded together the absence of jumps and $\#S_{\dot{v}_h} = 1$ leads to a continuous limit v_∞.

B) If there is exactly one point $\overline{x} \in (S_{\dot{v}_\infty} \setminus S_{v_\infty})$ then such $\overline{x}$ can be the limit of one or more (finite number, uniformly bounded in h) points in $S_{v_h} \cup S_{\dot{v}_h}$. In any of the above cases, by $\beta \le \alpha \le 2\beta$ we get

$$
\alpha\#(S_{v_\infty}) + \beta\#(S_{\dot{v}_\infty} \setminus S_{v_\infty}) \le \liminf_h \left(\alpha\#(S_{v_h}) + \beta\#(S_{\dot{v}_h} \setminus S_{v_h}) \right)
\tag{5.6}
$$

In both cases A) B) it is not restrictive to assume that v_h has a fixed number M of points x_h^j, in $S_{v_h} \cup S_{\dot{v}_h}$ $h \in \mathbf{N}$, $j = 1, ..., M$, say there are neither jump nor crease points of v_h in $A \setminus \cup_j \{x_h^j\}$ and $x_h^j \to \overline{x}$, for all j, then $v_\infty \in H^2(A \setminus \{\overline{x}\})$.

By Lemma 5.3, (5.3), triangular inequality, by taking into account that

$\#(S_{v_\infty} \cup S_{\dot v_\infty}) = 1$ and that M is uniformly bounded in h , we get:

$$\gamma \sum_{S_{\dot v_\infty}} |[\dot v_\infty]| \;=\; \gamma |[\dot v_\infty(\overline{x})]| \;=$$

$$= \gamma\, |\dot v_\infty(\overline{x}_+) - \dot v_\infty(\overline{x}_-)| \;=\; \gamma \lim_h \; |\dot v_h(x_h^M{}_+) - \dot v_h(x_h^1{}_-)| \;\le$$

$$\le \gamma \lim_h \left(\sum_{j=1}^{M} |\dot v_h(x^j_{h+}) - \dot v_h(x^j_{h-})| + \sum_{j=1}^{M-1} \left|\dot v_h(x_h^{j+1}{}_-) - \dot v_h(x^j_{h+})\right| \right) \le$$

(5.7)
$$\le \gamma \liminf_h \left(\sum_{j=1}^{M} \left|[\dot v_h(x^j_h)]\right| + \sum_{j=1}^{M-1} \sqrt{x_h^{j+1} - x_h^j}\, \|\ddot v_h\|_{L^2(x^j_h, x^{j+1}_h)} \right) \le$$

$$\le \gamma \liminf_h \left(\sum_{j=1}^{M} \left|[\dot v_h(x^j_h)]\right| + (M-1)\sqrt{x_h^M - x_h^1}\; \|\ddot v_h\|_{L^2(a,b)} \right) =$$

$$= \gamma \liminf_h \;\left(\sum_{S_{\dot v_h}} \; |[\dot v_h]| \right) \;.$$

Inequalities (5.6) and (5.7) together give (5.5). q.e.d.

Remark 5.5 - We emphasize that, when $\gamma > 0$, the last three terms in (5.4) cannot be arbitrarily split if we want to mantain the l.s.c. with respect to the a.e. convergence. Nevertheless both

(5.8)
$$\alpha\, \#(S_v \cap A) + \beta\, \#((S_{\dot v} \setminus S_v) \cap A)$$

(5.9)
$$\gamma \sum_{S_{\dot v} \cap A} |[\dot v]|$$

are l.s.c. in $SBV^2(a,b)$. The two terms in (5.8) are not separately l.s.c. ([15]). The absolutely continuous terms (the main term $\int_A |\ddot v|^2 dx$, and lower order perturbations, $\sigma \displaystyle\int_A |\dot v|^2\, dx$ and $\displaystyle\int_A G(x, v(x))\, dx$) are each one separately l.s.c.in $SBV^2(a,b)$.

Theorem 5.6. *(1D Lower semicontinuity in SBH)*

Assume $A \subset \mathbf{R}$ is an open interval, (0.2),(0.3) hold true . Fix the parameters such that

$$\alpha = 0,\ \ \beta > 0,\ \ \gamma > 0,\ \ \sigma \geq 0$$

so that, referring to (5.4), $\mathcal{F}^1$ becomes

$$\begin{aligned} \mathcal{F}^1(v, 0, \beta, \gamma, \sigma, A) \ &= \int_A (|\ddot{v}|^2 + \sigma|\dot{v}|^2 + G(x, v(x)))\, dx + \\ &\qquad + \beta\, \#((S_{\dot{v}}) \cap A) + \gamma \sum_{S_{\dot{v}} \cap A} |[\dot{v}]|\,. \end{aligned} \tag{5.10}$$

Let $v_h, v_\infty \in SBH(A)$, $h \in \mathbf{N}$, such that

$$\sup_{h \in \mathbf{N}} \mathcal{F}^1(v_h, 0, \beta, \gamma, \sigma, A) < \infty \ \ \text{and} \quad v_h \to v_\infty \qquad \text{a.e. in } A$$

Then

$$\mathcal{F}^1(v_\infty, 0, \beta, \gamma, \sigma, A) \leq \liminf_h \mathcal{F}^1(v_h, 0, \beta, \gamma, \sigma, A).$$

Proof – The statement can be proved as like as the previous one: there are fewer and simpler cases than before.

Notice that one can also deduce the thesis by [4] as in [12] since the term with σ does not affect the argument. q.e.d.

Remark 5.7 - When $\alpha = 0$ we are forced to restrict the set of admissible functions to *SBH* since the minimization of $\mathcal{F}^1(\cdot, 0, \beta, \gamma, \sigma, A)$ is not well posed in SBV^2 due to lack of compactness.

From another viewpoint we notice that minimizing $\mathcal{F}^1(\cdot, \infty, \beta, \gamma, \sigma, A)$ over $GSBV^2(\Omega)$ with $\gamma > 0$, leads to finite energy class of admissible functions contained in $SBH(A)$, hence it is equivalent to the minimization of $\mathcal{F}(\cdot, 0, \beta, \gamma, \sigma, A)$ over $SBH(A)$.

Remark 5.8 - We emphasize that, when $\gamma > 0$, for both functionals (5.4) and (5.10), it is essential to take into account the contribution of $[\dot{v}]$ on the whole set $S_{\dot{v}}$ and not on $S_{\dot{v}} \setminus S_v$ since the functional

$$\widetilde{\mathcal{F}}^1(v) = \int_A (|\ddot{v}(x)|^2 + \sigma|\dot{v}|^2 + G(x, v(x)))\, dx +$$

$$+\alpha\,\#(S_v\cap A)+\beta\,\#((S_{\dot v}\setminus S_v)\cap A)+\gamma\sum_{(S_{\dot v}\setminus S_v)\cap A}|[\dot v]|$$

is not lower semicontinuous, as shown by the counterexample

$$A=(-1,1)\quad\text{and}\quad v_h(x)=\frac{1}{h}\,\mathrm{sign}(x)+M|x|,\quad x\in A,$$

which entails $v_h\to v_\infty=M|x|$, hence

$$\widetilde{\mathcal{F}}^1(v_\infty)\le\liminf_h\widetilde{\mathcal{F}}^1(v_h)$$

if and only if $\beta+2\gamma M\le\alpha$ for every M, which contradicts $\gamma>0$.

Remark 5.9 - We emphasize that the last two terms in (5.10) actually are separately l.s.c. with respect to the a.e. convergence.

Now we face the l.s.c in the n dimensional case, hence we are forced to consider the wider class $GSBV^2$. In the 1d case the finiteness of energy automatically restricted such class to SBV^2.

Lemma 5.10. *Assume* $0<\beta\le\alpha\le 2\beta$, $\gamma\ge 0$, $\sigma\ge 0$. *Let* $0<\epsilon<\frac{1}{4}$, $\nu\in\partial B_1$, $i\in\{1,\dots,n\}$, $v_\infty,v_h\in GSBV^2(\Omega)$ $(h\in\mathbf{N})$, $A\subset\Omega$ *open set and let* $\Sigma\subset A\cap(S_{\nabla_i v_\infty}\setminus S_{v_\infty})$ *be a Borel set such that*

$$\nu\cdot\nu_\Sigma(y)>1-\frac{\epsilon}{2}\quad\mathcal{H}^{n-1}\text{-a.e. } y\in\Sigma$$

$$v_h\to v_\infty\qquad\text{a.e. in }\Omega$$

$$\sup_{h\in\mathbf{N}}\mathcal{F}(v_h,A)<+\infty.$$

Then

$$(1-\epsilon)\left(\beta\mathcal{H}^{n-1}(\Sigma)\;+\;\gamma\int_\Sigma|[\nabla v]|d\mathcal{H}^{n-1}\right)\le\liminf_h\mathcal{F}(v_h,\alpha,\beta,\gamma,\sigma,A).$$

where

$$\mathcal{F}(v_h,\alpha,\beta,\gamma,\sigma,A)\;=\;\textstyle\int_A(|\nabla^2 v_h|^2+\sigma|\nabla v_h|^2+G(x,v_h))\,dx\,+$$

$$\alpha\mathcal{H}^{n-1}(S_{v_h}\cap A)+\beta\mathcal{H}^{n-1}\left((S_{\nabla v_h}\setminus S_{v_h})\cap A)\right)+\gamma\textstyle\int_{S_{\nabla v_h}\cap A}|[\nabla v_h]|d\mathcal{H}^{n-1}$$

Proof – Let $\eta \in \partial B_1$ be such that $0 < |\nu - \eta| < \frac{\epsilon}{2}$ and $\{\nu, \eta, \mathbf{e}_i\}$ linearly dependent, hence

$$\eta \cdot \nu_\Sigma(y) > 1 - \epsilon \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \Sigma.$$

Choose $\Sigma_0, \Sigma_1 \subset \Sigma$ disjoint Borel sets as in Lemma 5.2 and $K_j \subset \Sigma_j$ $(j = 0, 1)$ compact sets. Let A_0, A_1 be disjoint open sets such that $K_j \subset A_j \subset A$ $(j = 0, 1)$. We can also assume that A_0 (resp. A_1) is a finite union of cubes with edges parallel or orthogonal to ν (resp. η). Then by $|[\nabla v]| = |[\nabla_i v]|$ on $S_{\nabla_i v}$, by Theorems 3.3 and 5.4, Fatou's Theorem, Fubini's Theorem and again Theorem 3.3 we get for any K_0, K_1 as above

$$(1-\varepsilon)\left(\beta\mathcal{H}^{n-1}(K_0) + \beta\mathcal{H}^{n-1}(K_1) + \gamma\int_{K_0\cup K_1} |[\nabla v]| d\mathcal{H}^{n-1}\right) =$$
$$= (1-\varepsilon)\int_{K_0\cup K_1}\left(\beta + \gamma\left|\left[\frac{\partial v}{\partial \nu_\Sigma}\right]\right|\right)\nu_\Sigma\cdot\nu_\Sigma\, d\mathcal{H}^{n-1} \le$$
$$\le \int_{K_0}\left(\beta + \gamma\left|\left[\frac{\partial v}{\partial \nu}\right]\right|\right)\nu_\Sigma\cdot\nu\, d\mathcal{H}^{n-1} + \int_{K_1}\left(\beta + \gamma\left|\left[\frac{\partial v}{\partial \eta}\right]\right|\right)\nu_\Sigma\cdot\eta\, d\mathcal{H}^{n-1} =$$
$$\int_{\pi_\nu(K_0)} \beta\mathcal{H}^0\left((K_0)^\nu_x\right) + \gamma\left|\left[(v^\nu_x)'\right]\right| d\mathcal{H}^{n-1}(x) + \int_{\pi_\eta(K_1)} \beta\mathcal{H}^0\left((K_1)^\eta_z\right) + \gamma\left|\left[(v^\eta_z)'\right]\right| d\mathcal{H}^{n-1}(z) \le$$
$$\int_{\pi_\nu(K_0)} \liminf_h \mathcal{F}^1\left((v_h)^\nu_x, (A_0)^\nu_x\right) d\mathcal{H}^{n-1}(x) +$$
$$+ \int_{\pi_\eta(K_1)} \liminf_h \mathcal{F}^1\left((v_h)^\eta_z, (A_1)^\eta_z\right) d\mathcal{H}^{n-1}(z) \le$$
$$\liminf_h \Big(\int_{\pi_\nu(K_0)} \mathcal{F}^1\left((v_h)^\nu_x, (A_0)^\nu_x\right) d\mathcal{H}^{n-1}(x) +$$
$$+ \int_{\pi_\eta(K_1)} \mathcal{F}_1\left((v_h)^\eta_z, (A_1)^\eta_z\right) d\mathcal{H}^{n-1}(z)\Big) \le$$
$$\liminf_h \left(\mathcal{F}(v_h, A_0) + \mathcal{F}(v_h, A_1)\right) \le \liminf_h \mathcal{F}(v_h, A).$$

By the arbitrariness of the compact sets K_0, K_1 and by the regularity of $\mathcal{H}^{n-1}$ the thesis follows. q.e.d.

Lemma 5.11. *Assume $0 < \beta \le \alpha \le 2\beta$, $\gamma \ge 0$, $\sigma \ge 0$. Let $0 < \epsilon < \frac{1}{4}$, $\nu \in \partial B_1$, $i \in \{1, \dots, n\}$, $v_\infty, v_h \in GSBV^2(\Omega)$ ($h \in \mathbf{N}$), $A \subset \Omega$ open set and let $\Sigma \subset A \cap (S_{\nabla_i v_\infty} \cap S_{v_\infty})$ be a Borel set such that*

$$\nu \cdot \nu_\Sigma(y) > 1 - \frac{\epsilon}{2} \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \Sigma$$

$$v_h \to v_\infty \quad \text{a.e. } in\ \Omega$$

$$\sup_{h\in\mathbf{N}} \mathcal{F}(v_h, A) < +\infty.$$

Then

$$(1-\epsilon)\left(\alpha\mathcal{H}^{n-1}(\Sigma) + \gamma\int_\Sigma |[\nabla v]|d\mathcal{H}^{n-1}\right) \le \liminf_h \mathcal{F}(v_h,\alpha,\beta,\gamma,\sigma,A).$$

Proof – Let $\eta \in \partial B_1$ be such that $0 < |\nu-\eta| < \frac{\epsilon}{2}$ and $\{\nu,\eta,\mathbf{e}_i\}$ linearly dependent, hence

$$\eta\cdot\nu_\Sigma(y) > 1-\epsilon \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \Sigma.$$

Choose $\Sigma_0, \Sigma_1 \subset \Sigma$ disjoint Borel sets as in Lemma 5.2 and $K_j \subset \Sigma_j$ $(j=0,1)$ compact sets. Let A_0, A_1 be disjoint open sets such that $K_j \subset A_j \subset A$ $(j=0,1)$. We can also assume that A_0 (resp. A_1) is a finite union of cubes with edges parallel or orthogonal to ν (resp. η). Then by Theorems 3.3 and 5.4, Fatou's Theorem, Fubini's Theorem and again Theorem 3.3 we get for any K_0, K_1 as above

$$\begin{aligned}
&(1-\varepsilon)\left(\alpha\mathcal{H}^{n-1}(K_0)+\alpha\mathcal{H}^{n-1}(K_1)+\gamma\int_{K_0\cup K_1}|[\nabla v]|d\mathcal{H}^{n-1}\right)=\\
&=\ (1-\varepsilon)\int_{K_0\cup K_1}\left(\alpha+\gamma\left|\left[\frac{\partial v}{\partial \nu_\Sigma}\right]\right|\right)\nu_\Sigma\cdot\nu_\Sigma\, d\mathcal{H}^{n-1}\le\\
&\le\int_{K_0}\left(\alpha+\gamma\left|\left[\frac{\partial v}{\partial \nu}\right]\right|\right)\nu_\Sigma\cdot\nu\, d\mathcal{H}^{n-1}+\int_{K_1}\left(\alpha+\gamma\left|\left[\frac{\partial v}{\partial \eta}\right]\right|\right)\nu_\Sigma\cdot\eta\, d\mathcal{H}^{n-1}=\\
&\int_{\pi_\nu(K_0)}\alpha\mathcal{H}^0\left((K_0)^\nu_x\right)+\gamma\left|\left[(\dot{v^\nu_x})\right]\right|d\mathcal{H}^{n-1}(x)+\\
&\qquad\qquad+\int_{\pi_\eta(K_1)}\alpha\mathcal{H}^0\left((K_1)^\eta_z\right)+\gamma\left|\left[(\dot{v^\eta_z})\right]\right|d\mathcal{H}^{n-1}(z)\le\\
&\int_{\pi_\nu(K_0)}\liminf_h \mathcal{F}^1\left((v_h)^\nu_x,(A_0)^\nu_x\right)d\mathcal{H}^{n-1}(x)+\\
&\qquad\qquad+\int_{\pi_\eta(K_1)}\liminf_h \mathcal{F}^1\left((v_h)^\eta_z,(A_1)^\eta_z\right)d\mathcal{H}^{n-1}(z)\le\\
&\liminf_h\Big(\int_{\pi_\nu(K_0)}\mathcal{F}^1\left((v_h)^\nu_x,(A_0)^\nu_x\right)d\mathcal{H}^{n-1}(x)+\\
&\qquad\qquad+\int_{\pi_\eta(K_1)}\mathcal{F}_1\left((v_h)^\eta_z,(A_1)^\eta_z\right)d\mathcal{H}^{n-1}(z)\Big)\le\\
&\liminf_h\left(\mathcal{F}(v_h,A_0)+\mathcal{F}(v_h,A_1)\right)\le\liminf_h\mathcal{F}(v_h,A).
\end{aligned}$$

By the arbitrariness of the compact sets K_0, K_1 and by the regularity of $\mathcal{H}^{n-1}$ the thesis follows. q.e.d.

Lemma 5.12. *Assume* $0 < \beta \leq \alpha \leq 2\beta$, $\gamma \geq 0$, $\sigma \geq 0$.
Let $0 < \epsilon < \frac{1}{4}$, $\nu \in \partial B_1$, $v_\infty, v_h \in GSBV^2(\Omega)$ $(h \in \mathbf{N})$, $U \subset \Omega$ *open set and let* $T \subset U \cap (S_{v_\infty} \setminus S_{\nabla v_\infty})$ *be a Borel set such that*

$$\nu \cdot \nu_T(y) > 1 - \frac{\epsilon}{2} \quad \mathcal{H}^{n-1}\text{-a.e. } y \in T$$

$$v_h \to v_\infty \quad \text{a.e. in } \Omega$$

$$\sup_{h \in \mathbf{N}} \mathcal{F}(v_h, \alpha, \beta, \gamma, \sigma, U) < +\infty.$$

Then

$$(1-\epsilon)\alpha\mathcal{H}^{n-1}(T) \leq \liminf_h \mathcal{F}(v_h, \alpha, \beta, \gamma, \sigma, U).$$

Proof – We can argue as in the proof of the previous lemma in a simpler way since it is not necessary to split T into two sets due to the fact that we can use Theorem 3.1(iii) instead of Theorem 3.2(ii) and Lemma 5.2.

Fix a compact set $K \subset T$. We can choose an open set A which is a finite union of cubes with edges parallel or orthogonal to ν_T such that $K \subset A \subset U$. Then by Theorem 3.3 and 5.4, Fatou's Theorem, Fubini's Theorem and again Theorem 3.3 we get

$$(1-\varepsilon)\,\alpha\,\mathcal{H}^{n-1}(K) = (1-\varepsilon)\int_K \alpha\,\nu_T\cdot\nu_T\,d\mathcal{H}^{n-1} \leq \int_K \alpha\,\nu_T\cdot\nu\,d\mathcal{H}^{n-1} =$$
$$\alpha\int_{\pi_\nu(K)} \mathcal{H}^0\left((K)^\nu_x\right)d\mathcal{H}^{n-1}(x) \leq \int_{\pi_\nu(K)} \liminf_h \mathcal{F}^1\left((v_h)^\nu_x, (A)^\nu_x\right)d\mathcal{H}^{n-1}(x) \leq$$
$$\liminf_h \int_{\pi_\nu(K)} \mathcal{F}^1\left((v_h)^\nu_x, (A)^\nu_x\right)d\mathcal{H}^{n-1}(x) \leq \liminf_h \mathcal{F}(v_h, A).$$

By the arbitrariness of the compact set K and by the regularity of $\mathcal{H}^{n-1}$ the thesis follows. q.e.d.

We use the previous results to show the lower semicontinuity of $\mathcal{F}$.

Theorem 5.13. *Assume $\Omega \subset \mathbf{R}^n$ bounded open set, (0.2),(0.4),(0.5) hold true. Let $v_\infty, v_h \in GSBV^2(\Omega)$ $(h \in \mathbf{N})$, such that*

$$\sup_{h\in\mathbf{N}} \mathcal{F}(v_h,\alpha,\beta,\gamma,\sigma,\Omega) \;<\; \infty \text{ and } v_h \to v_\infty \qquad \text{a.e. in } \Omega.$$

Then

$$\mathcal{F}(v_\infty,\alpha,\beta,\gamma,\sigma,\Omega) \;\le\; \liminf_h \mathcal{F}(v_h,\alpha,\beta,\gamma,\sigma,\Omega).$$

Proof – Fix $0<\epsilon<\frac{1}{4}$. Let $\{\nu_k\}$ be a finite set in ∂B_1 such that for every $\nu \in \partial B_1$ there exists k such that $|\nu-\nu_k|<\frac{\epsilon}{4}$. We can find a finite family of Borel sets $\{\Sigma_j; j=1,\ldots,p+q+r\}$ such that

$$\begin{aligned}
&\Sigma_j \subset (S_{v_\infty}\setminus S_{\nabla v_\infty}) \ \text{ for } j=1,\ldots,p,\\
&\Sigma_j \subset (S_{v_\infty}\cap S_{\nabla v_\infty}) \ \text{ for } j=p+1,\ldots,p+q,\\
&\Sigma_j \subset (S_{\nabla v_\infty}\setminus S_{v_\infty}) \ \text{ for } j=p+q+1,\ldots,p+q+r,
\end{aligned}$$

$$\Sigma_j\cap\Sigma_i=\emptyset \ \text{ for } j\ne i, \quad \mathcal{H}^{n-1}((S_{v_\infty}\cup S_{\nabla v_\infty})\setminus \cup_{j=1}^{p+q+r}\Sigma_j)=0$$

$$\forall j\ \exists k_j:\ \nu_{S_{v_\infty}\cup S_{\nabla v_\infty}}(y)\cdot\nu_{k_j} > 1-\frac{\epsilon}{2} \qquad \mathcal{H}^{n-1}-\text{a.e. } y\in\Sigma_j\,.$$

By the regularity of $\mathcal{H}^{n-1}$ there exist disjoint compact sets $K_j\subset\Sigma_j$ such that

$$\mathcal{H}^{n-1}\big((S_{v_\infty}\cup S_{\nabla v_\infty})\setminus\cup_{j=1}^{p+q+r}K_j\big)<\epsilon$$

and there exist pairwise disjoint open sets $\{A_j; j=0,1,\ldots,p+q+r\}$ such that $K_j\subset A_j$ for $j=1,\ldots,p+q+r$, and

$$\int_{\Omega\setminus A_0}(|\nabla^2 v_\infty|^2+\sigma|\nabla v_\infty|^2+G(x,v_\infty(x)))dx<\epsilon.$$

From Lemmas 5.10, 5.11, 5.12 we get

$$\alpha\mathcal{H}^{n-1}(S_{v_\infty}\setminus S_{\nabla v_\infty})+\alpha\mathcal{H}^{n-1}(S_{v_\infty}\cap S_{\nabla v_\infty})+\beta\mathcal{H}^{n-1}(S_{\nabla v_\infty}\setminus S_{v_\infty})+$$

$$+\gamma\int_{S_{\nabla v_\infty}\setminus S_{v_\infty}}|[\nabla v_\infty]|\,d\mathcal{H}^{n-1} + \gamma\int_{S_{v_\infty}\cap S_{\nabla v_\infty}}|[\nabla v_\infty]|\,d\mathcal{H}^{n-1} \;\le$$

$$\varepsilon(\alpha+\beta+\gamma)+\alpha\mathcal{H}^{n-1}(\bigcup_{j=1}^{p}K_j)+\alpha\mathcal{H}^{n-1}(\bigcup_{j=p+1}^{p+q}K_j)+\beta\mathcal{H}^{n-1}(\bigcup_{j=p+q+1}^{p+q+r}K_j)+$$

$$+\gamma \sum_{j=p+1}^{p+q+r} \int_{K_j} |[\nabla v_\infty]|\, d\mathcal{H}^{n-1} \ \le$$

$$\varepsilon(\alpha+\beta+\gamma) + \frac{1}{1-\varepsilon} \sum_{j=1}^{p+q+r} \liminf_h \mathcal{F}(v_h, A_j).$$

By the compactness Theorem 4.4, by Lemma 2.6 and by the lower semicontinuity of quadratic forms with respect to weak convergence in L^2

$$\begin{aligned}&\int_\Omega \left(|\nabla^2 v_\infty|^2 + \sigma\,|\nabla v_\infty|^2 + G(x, v_\infty(x))\right)\, dx \le \\ &\le \varepsilon + \liminf_h \int_{A_0} \left(|\nabla^2 v_h|^2 + \sigma\,|\nabla v_h|^2 + G(x, v_h(x))\right)\, dx.\end{aligned}$$

Since $\{A_j\}$ are pairwise disjoint, summing the previous inequalities and taking into account the arbitrariness of ϵ the thesis follows. q.e.d.

Remark 5.14 - We emphasize that, when $\gamma > 0$, the last three terms in $\mathcal{F}$ cannot be arbitrarily split if we want to keep the l.s.c. with respect to the a.e. convergence (which holds true for their sum). Nevertheless both

$$\alpha\mathcal{H}^{n-1}(S_v) + \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v) \tag{5.11}$$

and

$$\gamma \int_{S_{\nabla v}\cap A} |[\nabla v]| \tag{5.12}$$

are l.s.c.in $GSBV^2(\Omega)$ with respect to the a.e. convergence.
The absolutely continuous terms $\int_A |\nabla v|^2\, dx$ and $\int_A G(x, v(x)))\, dx$ are lower order perturbations, each one separately l.s.c. in $GSBV^2(A)$, by Lemma 2.6.

Theorem 5.15. *Assume $\Omega \subset \mathbf{R}^n$ bounded open set, (0.2),(0.3) and, either (0.6) or (0.7),(0.8) together. Let $v_\infty, v_h \in SBH(\Omega)$ ($h \in \mathbf{N}$), such that*

$$\sup_{h\in\mathbf{N}} \mathcal{F}(v_h, 0, \beta, \gamma, \sigma, \Omega) \ < \ \infty \qquad \textit{and} \qquad v_h \to v_\infty \quad \text{a.e. } \textit{in } \Omega.$$

Then

$$\mathcal{F}(v_\infty, 0, \beta, \gamma, \sigma, \Omega) \ \le \ \liminf_h \mathcal{F}(v_h, 0, \beta, \gamma, \sigma, \Omega).$$

Proof – By the compactness Theorems 4.5 and 4.6 and by Lemma 1.18 it follows that the absolutely continuous part of $\mathcal{F}$ is l.s.c., hence the thesis follows from Theorem 1.17. q.e.d.

6. Existence of minimizers

Here we show the proofs of main Theorems.

Proof of Theorem 0.3 – By (0.5) and the properties of w there is a minimizing sequence v_h s.t. $\mathcal{F}(v_h,\Omega) \in \mathbf{R}$.
By Theorem 4.4 we find $v \in GSBV^2(\Omega)$ s.t. $v_h \to v$ $a.e.x \in \Omega$.
By Theorem 5.13 and (0.5) we get

$$-\infty < -\|c_4\|_{L^1} \leq \mathcal{F}(v,\Omega) \leq \inf \mathcal{F}(v_h,\Omega) < +\infty \ .$$

q.e.d.

Proof of Theorem 0.4 – By (0.5) and the properties of w there is a minimizing sequence v_h s.t. $\mathcal{F}(v_h,\mathbf{R}^n) \in \mathbf{R}$.
By Theorem 4.4 we find $v \in GSBV^2(\mathbf{R}^n)$ s.t. $v_h \to v$ $a.e.x \in \mathbf{R}^n$.
By Theorem 5.13 and (0.5) we get

$$-\infty < -\|c_4\|_{L^1} \leq \mathcal{F}(v,\mathbf{R}^n) \leq \inf \mathcal{F}(v_h,\mathbf{R}^n) < \infty \ .$$

q.e.d.

Proof of Theorem 0.5 – By (0.6) and the properties of w there is a minimizing sequence v_h s.t. $\mathcal{F}(v_h,\mathbf{R}^n) \in \mathbf{R}$.
By Theorem 4.5 we find $v \in SBH(\mathbf{R}^n)$ s.t. $v_h \to v$ $a.e.x \in \mathbf{R}^n$.
By Theorem 5.15 and (0.6) we get $\mathcal{F}(v,\mathbf{R}^n) \leq \inf \mathcal{F}(v_h,\mathbf{R}^n)$.
By (0.6) and Theorem 1.15 we get

$$-\infty \ < \ (\gamma - \lambda C_n(\Omega))\ \|D^2 v\|_{T(\mathbf{R}^n)} \ + \delta_n(\Omega, w) \ \leq \ \mathcal{F}(v,\mathbf{R}^n) \ .$$

q.e.d.

Proof of Theorem 0.6 – By (0.7), (0.8) and the existence of w there is a minimizing sequence v_h s.t. $\mathcal{F}(v_h,\Omega) \in \mathbf{R}$.
By Theorem 4.6 we find $v \in SBH(\Omega)$ s.t. $v_h - pv_h \to v$ $a.e.x \in \Omega$.

By Theorem 5.15 and (0.7),(0.8) we get $\mathcal{F}(v,\Omega) \leq \inf \mathcal{F}(v_h,\Omega)$.
By (0.7), (0.8) and Theorem 1.16 we get

$$-\infty \ < \ (\gamma - \lambda \mathcal{C}_n(\Omega)) \ \|D^2 v\|_{T(\Omega)} \ + \|c_0\|_{L^1} - \frac{\gamma^2}{4} \ \leq \ \mathcal{F}(v,\Omega) \ .$$

q.e.d.

Remark 6.1 - The thesis of Theorem 0.6 holds true when assumption (0.8) is substituted by the following one

$$\begin{aligned}&\forall t \in \mathbf{R}, \ \ r \in \mathbf{R}^n, \ \ \exists \mu \neq 0 \text{ s.t.}\\ &\textstyle\int_\Omega G\big(x, v(x) - \mu(t + r\cdot x)\big) = \int_\Omega G(x, v(x)) \ \forall t \in \mathbf{R}, r \in \mathbf{R}^n, \ v \in \mathit{SBH}(\Omega)\,,\end{aligned}$$

as one can see by inspection of the last proof and referring to the theory of sequential recession functionals (see Def. 2.1 and Th.3.4 in [8])

7. Examples

All along the paper we exploited only weak L^2 lower semicontinuity and positive definiteness in L^2 of the terms $\int |\nabla^2 v|^2$ and $\int |\nabla v|^2$ appearing in the functional (0.1). Hence the following statement for a more general functional holds true as an easy corollary of the previous analysis.

Theorem 7.1. *Consider the functional*

$$\begin{aligned}\mathcal{F}(v,\alpha,\beta,\gamma,\sigma,A) \ &= \int_A \big(Q(\nabla^2 v) + \sigma\, \mathcal{Q}(\nabla v) + G(x,v)\big)\, dx + \\ &+ \alpha\mathcal{H}^{n-1}(S_v \cap A) + \beta\mathcal{H}^{n-1}((S_{\nabla v} \setminus S_v) \cap A) + \gamma \int_{S_{\nabla v} \cap A} |[\nabla v]|\, d\mathcal{H}^{n-1}\end{aligned} \tag{7.1}$$

where both Q and $\mathcal{Q}$ are real valued positive definite quadratic forms. Then the thesis of Theorems 0.3, 0.4, 0.5, 0.6 hold true also when functional (7.1) is substituted to functional (0.1).

An important example of functional related to image analysis ([9], [36]) is the following, obtained by choosing $\alpha > 0$, $\ G(x,s) = |g(x) - s|^2$, in the functional (7.1).

$$\mathcal{G}(v,\alpha,\beta,\gamma,\sigma,A) = \int_A \Big(|\nabla^2 v|^2 + \eta|\widetilde{\Delta} v|^2 + \sigma|\nabla v|^2 + |v-g|^2 \Big) dx + \tag{7.2}$$
$$+\alpha\mathcal{H}^{n-1}(S_v\cap A) + \beta\mathcal{H}^{n-1}((S_{\nabla v}\setminus S_v)\cap A) + \gamma\int_{S_{\nabla v}\cap A} |[\nabla v]|\, d\mathcal{H}^{n-1}$$

where $\widetilde{\Delta} v$ stands for the trace of $\nabla^2 v$. Notice that the map $v \to \mathrm{Trace}\nabla^2 v$ is sequentially weakly continuous with respect to the L^2 convergence of $\nabla^2 v$, hence $\int \eta|\widetilde{\Delta} v|^2\, dx$ is sequentially weakly lower semicontinuous with respect to the L^2 convergence of $\nabla^2 v$.

Theorem 7.2. *Let (0.2),(0.4),(0.5), $\eta \geq 0$, $g \in L^2(\Omega)$. Then functional $\mathcal{G}$ in (7.2) achieves a finite minimum over $GSBV^2(\Omega)$.*

Another interesting example arises in continuum mechanics. The following choices in functional (7.1): $n = 2$, $\alpha = 0$, $\sigma = 0$, $G(x,s) = -g(x)s$ and a quadratic form Q dependent on the Lamé coefficients λ and μ, lead to the total energy of the Kirchhoff-Love plate with damage at small scale ([39],[40],[35]):

$$\mathcal{P}(v,0,\beta,\gamma,0,A) = \frac{2}{3}\mu \int_A \Big(|\nabla^2 v|^2 + \frac{\lambda}{\lambda+2\mu}|\Delta^a v|^2 \Big)\, dx - \int_A gv\, dx + \tag{7.3}$$
$$+\beta\mathcal{H}^1(S_{\nabla v}\cap A) + \gamma\int_{S_{\nabla v}\cap A} |[\nabla v]|\, d\mathcal{H}^1 \, .$$

In this context A is the natural state of the unloaded elastic-plastic plate ([40]); the first integral expresses the elastic deformation energy; the second one the potential energy due to the transverse dead load g; the last two terms together express the energy associated to damage along free plastic-yield lines.

Theorem 7.3. *Let $\mu > 0$, $2\mu + 3\lambda > 0$, $g \in \mathcal{M}(\mathbf{R}^2)$, (0.3) and there exists $w \in SBH(\mathbf{R}^2)$ s.t. $\mathcal{P}(w,0,\beta,\gamma,0,\mathbf{R}^2)$ is finite and*

$$|g|_{T(\overline{\Omega})} < 4\gamma \qquad \textbf{(safe load condition for clamped plate)}\, . \tag{7.4}$$

Then there is a minimizer u, with finite energy, of $\mathcal{P}(v,0,\beta,\gamma,0,\mathbf{R}^2)$ among $v \in SBH(\mathbf{R}^2)$ such that $v = w$ in $\mathbf{R}^2 \setminus \overline{\Omega}$.

Theorem 7.4. *Let $\mu > 0,\ 2\mu + 3\lambda > 0$, Ω is $\mathcal{R}$ regular (Def. 1.5), $g \in \mathcal{M}(\Omega)$, (0.3), there is $w \in SBH(\Omega)$ s.t. $\mathcal{P}(w, 0, \beta, \gamma, 0, \Omega)$ is finite and*

$$(7.5)\qquad \int_\Omega g\, v\, dx \;=\; 0 \quad \forall \text{ affine displacement } v$$

(**compatibility condition for free plate**)

$$(7.6)\qquad |g|_{T(\Omega)} \;<\; \frac{4\gamma}{M_2(\Omega) S_2(\Omega)} \;=\; \frac{\gamma}{\Gamma_2(\Omega)}$$

(safe load condition for free plate)

where $M_2,\ S_2,\ \Gamma_2$ are given by Theorems 1.7, 1.11.
Then there is a minimizer with finite energy of $\mathcal{P}(v, 0, \beta, \gamma, 0, \Omega)$ among v in $SBH(\Omega)$.

We consider now the Dirichlet and Neumann problems for an elastic-plastic beam: $\Omega \subset \mathbf{R}$ is an open interval (the undeformed state of the beam), g the transverse load.

Theorem 7.5. *Given $\mu > 0$, $\beta > 0$, $\gamma > 0$, $\Omega \subset \mathbf{R}$ open bounded interval, $g \in \mathcal{M}(\Omega)$, and $w \in SBH(\mathbf{R})$, assume*

$$(7.7)\qquad |g|_{T(\overline{\Omega})} \;<\; \frac{2\gamma}{|\Omega|} \qquad \text{(safe load condition for clamped beam)}$$

Then there is a minimizer u, with finite energy, of

$$\mathcal{B}(v) \;=\; \int_{\mathbf{R}} (\mu |\ddot{v}|^2 - g\, v)\, dx + \beta \#(S_{\dot{v}}) \;+ \gamma \sum_{S_{\dot{v}} \cap A} |[\dot{v}]|$$

among $v \in SBH(\mathbf{R})$ such that $v = w$ in $\mathbf{R} \setminus \Omega$.

Theorem 7.6. *Given $\mu > 0$, $\beta > 0$, $\gamma > 0$, $\Omega \subset \mathbf{R}$ open bounded interval, $g \in \mathcal{M}(\Omega)$, assume*

$$(7.8)\qquad \int_\Omega g\, v\, dx \;=\; 0 \quad \forall \text{ affine displacement } v$$

(**compatibility condition for free beam**)

$$|g|_{T(\Omega)} \; < \; \frac{\gamma}{|\Omega|} \qquad \textbf{(safe load condition for free beam)} \tag{7.9}$$

Then there is a minimizer u, with finite energy, of

$$\mathcal{T}(v) \; = \; \int_{\Omega} (\mu|\ddot{v}|^2 - g\,v)\,dx + \beta\#(S_{\dot{v}}) \, + \gamma \sum_{S_{\dot{v}}\cap A} |[\dot{v}]|$$

among $v \in SBH(\Omega)$.

Remark 7.7 - Notice that in Theorem 7.3 the finiteness of $|\Omega|$ is not required, since the problem has the appropriate homogeneity. In fact, in the proofs of Theorems 7.3, 7.4, 7.5 and 7.6, the estimate of the term $\int_A gv\,dx$ do not require the additional computations performed in Theorems 1.15, 1.16, but it is obtained as a straightforward consequence from Theorems 1.6: (1,6) entails continuity on every $\overline{B_r(0)}$ and by exploiting (1.5) we get $|\int_{\mathbf{R}^2} gv| \leq |g|_T \|v\|_{L^\infty}$.

References

[1] R.Adams: *Sobolev spaces*, Academic Press, New York, 1975.

[2] G. Alberti, Rank one property for derivatives of functions with bounded variation, in:*Proc. Roy. Soc. Edinburgh Sect.A,* **123**(1993), 239–274.

[3] L. Ambrosio: Compactness for a special case of functions of bounded variation, in: *Boll. Un. Mat. Ital.*, **3-B (7)**, (1989) 857–881.

[4] L. Ambrosio, Existence theory for a new class of variational problems, in:*Arch. Rational Mech. Anal.,* **111**(1990), 291–322.

[5] L.Ambrosio, L.Faina & R.March, Variational approximation of a second order free discontinuity problem in computer vision, in: *SIAM J. Math. Anal.,* **32** (2001), 1171-1197.

[6] L.Ambrosio, N.Fusco & D.Pallara, *Functions of bounded variations and free discontinuity problems,* Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[7] P. Aviles & Y. Giga, Singularities and rank one properties of Hessian measure, in: *Duke Math. J.,* **58** (1989), 441–467.

[8] C.Baiocchi, G.Buttazzo, F.Gastaldi & F.Tomarelli, General existence theorems for unilateral problems in continuum mechanics, in: *Arch. Rat. Mech. Anal.,* **100** (1988), 149-189.

[9] A.Blake & A.Zisserman, *Visual Reconstruction,* The MIT Press, Cambridge, 1987.

[10] G.Buttazzo & F.Tomarelli, Compatibility conditions for nonlinear Neumann problems, in: Advances in Math., **89** (1991), 127-143.

[11] M.Carriero, A.Leaci & F.Tomarelli, Plastic free discontinuities and special bounded hessian, in: *C. R. Acad. Sci. Paris,* ser.I **314** (1992), 595-600.

[12] M.Carriero, A.Leaci & F.Tomarelli, Special Bounded Hessian and elastic-plastic plate, in: *Rend. Accad. Naz. delle Scienze (dei XL), (109)* **XV** (1992), 223-258.

[13] M.Carriero, A.Leaci & F.Tomarelli, Strong solution for an elastic-plastic plate, in: *Calc. Var.,* **2** (1994), 219-240.

[14] M.Carriero, A.Leaci & F.Tomarelli, Free gradient discontinuities, in: Buttazzo, Bouchitté, Suquet (Eds.), *Calculus of Variations, Homogeneization and Continuum Mechanics,* Luminy 1993, World Scientific, Singapore, 1994, pp.131-147.

[15] M.Carriero, A.Leaci & F.Tomarelli, A second order model in image segmentation: Blake & Zisserman functional, in: R.Serapioni, F.Tomarelli (Eds.), *Variational Methods for Discontinuous Structures,* Villa Olmo 1994, PNLDE 25, Birkäuser, (1996) pp.57-72.

[16] M.Carriero, A.Leaci & F.Tomarelli, Strong minimizers of Blake & Zisserman functional, in: *Ann. Scuola Normale Sup. Pisa, s.IV,* **25** (1997), 257-285.

[17] M.Carriero, A.Leaci & F.Tomarelli, Density estimates and further properties of Blake & Zisserman functional, in R.Gilbert & P.Pardalos (Eds.), *From Convexity to Nonconvexity,* Nonconvex Optimization and Its Applications 55, Kluwer, Amsterdam, 2001, pp.381-392.

[18] M.Carriero, A.Leaci & F.Tomarelli, Necessary conditions for extremals of Blake & Zisserman functional, in: *C. R. Acad. Sci. Paris,* ser.I **334** (2002), 343-348.

[19] M.Carriero, A.Leaci & F.Tomarelli, Local minimizers for a free gradient discontinuity problem in image segmentation, in: G.Dal Maso & F.Tomarelli Eds., Proc. *Variational Methods for Discontinuous Structures,* PNLDE 51, 67-80, Birkhäuser, Basel, (2002).

[20] M.Carriero, A.Leaci & F.Tomarelli, Calculus of Variations and image segmentation, in: *J. of Physiology, Paris,* vol. 97, 2-3, (2003), pp. 343-353.

[21] M.Carriero, A.Leaci & F.Tomarelli, Euler equations for Blake & Zisserman functional, to appear.

[22] F.Colombo & F.Tomarelli, Boundary value problems and obstacle problem for elastic bodies with free cracks, in: *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi,* D. Pallara (ed.), Quaderni di Matematica, Series edited by Dipartimento di Matematica, Seconda Universita' di Napoli, (2004).

[23] A. Coscia, Existence result for a new variational problem in one dimensional segmentation theory, in: *Ann. Univ. Ferrara,* XXXVII (1991), 185–203.

[24] G.Dal Maso, R.Toader, A model for the quasi-static growth of brittle fractures: existence and approximation results in: *Arch.Ration. Mech. Anal.,* **162** (2002), 101–135.

[25] G.Dal Maso & F.Tomarelli (Eds.), *Variational Methods for Discontinuous Structures,* PNLDE 51, Birkhäuser, Basel, 2002.

[26] E. De Giorgi, Free discontinuity problems in Calculus of Variations, in: R.Dautray (Ed.), *Frontiers in Pure & Applied Mathematics,* North-Holland, Amsterdam, 1991, 55-61.

[27] E.De Giorgi & L.Ambrosio, Un nuovo tipo di funzionale del Calcolo delle Variazioni, in: *Atti Accad. Naz. Lincei, s.8* **82** (1988), 199-210.

[28] F.Demengel, Fonctions a Hessian Borné, in: *Ann. Inst. Fourier,* **34** (1984), 155–190.

[29] F.Demengel, Compactness theorem for spaces of functions with bounded derivatives and applications to limit analysis problems in plasticity, in *Arch. Rational Mech. Anal.,* **105** (1989), 123–161.

[30] H.Federer, *Geometric Measure Theory,* Springer, Berlin, 1969.

[31] I.Fonseca & G.Francfort, 3D-2D asymptotic analysis of optimal design problem for thin films,in: *J.Reine angew.Math.,***505** (1998), 173-202

[32] I.Fonseca, G.Leoni & R.Paroni, On hessian matrices in the space BH, Preprint 2001.

[33] G.Francfort & J.J.Marigo, Revisiting brittle fracture as an energy minimization problem, J.Mech.Phys. Solids, **46** (1998), 1319-1342.

[34] B.M. ter Haar Romeny (Ed.), *Geometry-Driven Diffusione in Computer Vision,* Kluwer, Dordrecht, 1994.

[35] G.Kirchhoff, Über das Gleichgewichtund die Bewegung einer elastischen Scheibe, J.Reine Angew. Math., **40** (1850), 51-88.

[36] J.M.Morel & S.Solimini, *Variational Models in Image Segmentation,* Birkhäuser, Basel, 1994.

[37] D.Mumford, J.M.Morel & C. von der Malsburg, The Mathematical, Computational and Biological Study of Vision, Mathematisches Forschunginstitut Oberwolfach, Rep. No. 49/2991, (2001), http://www.mfo.de/Meetings/Documents/2001/45/Report_49_01.ps

[38] D.Mumford & J.Shah, Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems, in: *Comm. Pure Appl. Math.,* **42** (1989), 577-685.

[39] D.Percivale & F.Tomarelli, Scaled Korn-Poincaré inequality in BD and a model of elestic plastic cantilever , in: *Asymptotic Analysis,* **23** (2000), 291-311.

[40] D.Percivale & F.Tomarelli, From SBD to SBH: the elastic plastic plate, in: *Interfaces and Free Boundaries,* **4** (2002), 1-29.

[41] D.Percivale & F.Tomarelli, From special bounded deformation to special bounded hessian: the elastic-plastic beam, Quaderni Dip. Matematica Politecnico Milano **551/P** 2003, 1-41.

[42] D.Percivale & F.Tomarelli, Regular extremals of free discontinuity problems, to appear .

[43] G.Savaré, On the regularity of the positive part of functions, in: *Nonlinear Anal.*, **27** (1996), 1055–1074.

[44] G.Savaré & F.Tomarelli, Superposition and chain rule for bounded Hessian functions, in: *Adv. Math.*, **140** (1998), 237–281.

[45] R.Temam, *Problèmes Mathématiques en Plasticité*, Dunod, Paris, 1983.

[46] F.Tomarelli, Special Bounded Hessian and partial regularity of equilibrium for a plastic plate, in: G.Buttazzo, G.P.Galdi & L.Zanghirati Eds., *Developments in PDE and applications to Mathematical Physics*, 235–240, Plenum Press, N.Y., 1992.

[47] W.P.Ziemer, *Weakly Differentiable Functions*, Springer–Verlag, N.Y., 1988.