Non-local approximation of free-discontinuity functionals with linear growth: the one-dimensional case

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Abstract. We approximate, in the sense of Γ-convergence, one-dimensional free-discontinuity functionals with linear growth in the gradient by means of a sequence of non-local integral functionals depending on the averages of the gradient on small intervals.

Key words. Variational approximation, free discontinuities.

1 Introduction

In the variational approach to many problems in computer vision, such as image segmentation or signal processing, an important rôle has been recently played by the method of "curve evolution". In this framework, a segmentation functional depending on the L^1-norm of the gradient of the unknown function u (which represents the smoothing of the input image) has been proposed: see, e.g. [10], [9] and [1]. In the one-dimensional setting, this gives rise to integral functionals of the type

$$F(u) = \int_a^b \phi(|u'(x)|)dx + \sum_{x\in S_u} f(|u^+(x) - u^-(x)|) + c_0|D^c u|(a,b)\,,$$

where ϕ is a positive increasing convex function with linear growth, f is a positive concave function, and u varies in the space $BV(a,b)$ of the functions of bounded variation. Thus, S_u stands for the set of discontinuity points of u, the term $|u^+(x)-u^-(x)|$ is the jump of u at the point x, and $|D^c u|$ is the total variation of the so-called Cantor part of the distributional derivative of u.

The main difficulty in the study of F is the presence of the term containing the discontinuities of u. So it is natural to try to approximate F by simpler functionals defined on Sobolev spaces. We notice that it is not possible to obtain a variational approximation of F by means of local integrals, leading to the convergence of minimum points; indeed, if such an approximation existed, the functional F would also be the variational limit of the convex functionals obtained by considering the convex envelope of the integrand functions, in contrast with the lack of convexity of F. This difficulty has been overcome in [1] by introducing an auxiliary variable on the line of [3]. We pursue here the same method introduced in [6] for the Mumford-Shah functional, i.e. the use of non-local integrals of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|dy\right)dx$$

on $W^{1,1}(a,b)$. We will prove the existence of a suitable family (f_ε) of convex-concave integrand functions for which the Γ-convergence of (F_ε) to F in the L^1-topology holds as $\varepsilon \to 0$.

Similar results are contained in [8], where the approximation is given by a finite difference scheme.

2 Notation and preliminaries

Functions of bounded variation. For the general theory of functions of bounded variation we refer the reader to [2]. Here we just recall some definitions and results we will need.

Let $I = (a, b)$ be an open interval of $\mathbb{R}$, and $u\colon I \to \mathbb{R}$. We define:

$$pV(u, I) = \left\{ \sum_{i=1}^{n-1} |u(t_{i+1}) - u(t_i)| : \ n \geq 2, \ a < t_1 < \ldots < t_n < b \right\},$$

and

$$eV(u, I) = \inf\{pV(v, I) : \ v = u \text{ a.e. in } I\}. \tag{2.1}$$

Assume $u \in L^1(I)$; we say that u is a *function of bounded variation* if $eV(u, I)$ is finite. This is equivalent to the property that the distributional derivative Du is a finite Radon measure in I; moreover, it turns out that the total variation $|Du|(I)$ coincides with $eV(u, I)$.

It can be proved that for any $u \in L^1(I)$ the infimum in (2.1) is attained. If $u \in BV(I)$ any representative $\overline{u}$ of u with the property that

$$pV(\overline{u}, I) = eV(u, I)$$

will be called a *good representative* of u. Let S_u be the set of atoms of Du, i.e. $t \in S_u$ if and only if $Du(\{t\}) \neq 0$. Then any good representative $\overline{u}$ is uniquely defined and continuous in $I \setminus S_u$ and has a jump discontinuity at any point of S_u. Thus the left and right traces $u^-(x)$ and $u^+(x)$ are well defined at every $x \in (a, b)$.

For any $u \in BV(I)$ the measure Du, as any Radon measure on $\mathbb{R}$, can be split into three parts: an absolutely continuous one $D^a u$ (with respect to the Lebesgue measure), a purely atomic one $D^j u$ (the *jump part*), and a diffuse (i.e. without atoms) singular one $D^c u$ (the *Cantor part*). The density of $D^a u$ with respect to the Lebesgue measure is denoted by u'.

We say that a function $u \in BV(I)$ is a *special function of bounded variation* ($u \in SBV(I)$) if $|D^c u|(I) = 0$.

We say that a function $u \in L^1(I)$ is a *generalized function of bounded variation* ($u \in GBV(I)$) if $u^T = (-T) \vee u \wedge T$ belongs to $BV(I)$ for every $T \geq 0$.

Since $(u^T)' = (u^M)'$ a.e. in $\{|u| \leq T\}$ if $T \leq M$, if we set $u' = (u^T)'$ on $\{|u| \leq T\}$ we obtain a function which is well defined a.e. on I.

Clearly, if $u \in GBV(I)$, for every $x \in I$ the functions of T

$$(u^T)^-(x), \qquad (u^T)^+(x), \qquad \big|(u^T)^+ - (u^T)^-(x)\big|$$

are monotone. The same holds for the set function $T \mapsto S_{u^T}$. Therefore, we have an at most countable set S_u of discontinuity points, given by $S_u = \bigcup_{T \geq 0} S_{u^T}$; and for every $x \in S_u$ we can define the traces:

$$u^-(x) = \lim_{T \to +\infty} (u^T)^-(x), \qquad u^+(x) = \lim_{T \to +\infty} (u^T)^+(x).$$

If $u \in GBV(I)$, the Cantor part of the derivative can be defined as the least upper bound of the measures $|D^c u^T|$. It turns out that for every Borel subset B of I:

$$|D^c u|(B) = \lim_{T \to +\infty} |D^c u^T|(B).$$

Finally, we recall an important compactness theorem in BV.

Theorem 2.1 *Every sequence (u_h) in $BV(I)$ satisfying*

$$\sup_{h \in \mathbb{N}} \left\{ \int_I |u_h(x)| dx + |Du_h|(I) \right\} < +\infty$$

admits a subsequence converging in $L^1(I)$ to a function $u \in BV(I)$.

Relaxation. We recall that the *relaxed* functional $\overline{F}$ of a given functional F is the largest lower semicontinuous functional smaller than F. We will need the following relaxation theorem, which can be easily obtained from the results contained in [4] (see, in particular, the proof of Theorem 3.1):

Theorem 2.2 *Let $\phi, f\colon \mathbb{R} \to [0, +\infty)$ be lower semicontinuous functions with $\phi(0) = f(0) = 0$. Assume that ϕ is convex and f is concave, and*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = \lim_{t\to 0^+} \frac{f(t)}{t} = c_0 > 0\,.$$

Let (a, b) be a bounded open interval, and consider the functional $F\colon BV(a, b) \to [0, +\infty]$ defined by

$$F(u) = \begin{cases} \displaystyle\int_a^b \phi(|u'(x)|)dx + \sum_{x\in S_u} f(|u^+(x) - u^-(x)|) \\ \qquad\qquad\qquad\qquad\qquad \text{if } u \in SBV(a, b) \text{ and } \#S_u < +\infty \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Then the relaxed functional $\overline{F}\colon BV(a, b) \to \mathbb{R}$ of F with respect to the L^1-topology is given by

$$\overline{F}(u) = \int_a^b \phi(|u'(x)|)dx + \sum_{x\in S_u} f(|u^+(x) - u^-(x)|) + c_0|D^c u|(a, b)\,.$$

Γ-*convergence.* For the general theory see [7]. Let (X, d) be a metric space. Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that (F_j) Γ-*converges*, as $j \to +\infty$, to $F\colon X \to \overline{\mathbb{R}}$, if for all $u \in X$ we have:

i) (*lower semicontinuity inequality*) for every sequence (u_j) converging to u

$$F(u) \le \liminf_{j\to+\infty} F_j(u_j);$$

ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that:

$$F(u) \ge \limsup_{j\to+\infty} F_j(u_j);$$

The *lower* and *upper* Γ-*limits* of (F_j) in $u \in X$ are defined as

$$F'(u) = \inf\bigl\{\liminf_{j\to+\infty} F_j(u_j) : \; u_j \to u\bigr\}, \tag{2.2}$$

$$F''(u) = \inf\bigl\{\limsup_{j\to+\infty} F_j(u_j) : \; u_j \to u\bigr\}, \tag{2.3}$$

respectively.

We extend this definition of convergence to families depending on a real parameter. Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-*converges*, as $\varepsilon \to 0$, to $F\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence (ε_j) the sequence (F_{ε_j}) Γ-converges to F.

If we define the *lower* and *upper* Γ-*limits* of (F_ε) as

$$F'(u) = \inf\bigl\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \; u_\varepsilon \to u\bigr\}, \tag{2.4}$$

$$F''(u) = \inf\bigl\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \; u_\varepsilon \to u\bigr\}, \tag{2.5}$$

respectively, then (F_ε) Γ-converges to F in u if and only if $F'(u) = F''(u) = F(u)$. Both F' and F'' are lower semicontinuous on X. In the estimate of F' we shall use the following immediate consequence of the definition:

$$F'(u) = \inf\bigl\{\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) : \; \varepsilon_j \to 0^+, \; u_j \to u\bigr\}. \tag{2.6}$$

It turns out that the infimum is attained.

We will also use the fact (see [7], Proposition 6.11) that the lower and upper Γ-limits of a sequence of functionals coincide with the lower and upper Γ-limits, respectively, of the sequence of the corresponding relaxed functionals.

An important consequence of the definition of Γ-convergence is the following result about the convergence of minimizers (see, e.g., [7], Corollary 7.20):

Theorem 2.3 *Let $F_j \colon X \to \overline{\mathbb{R}}$ ($j \in \mathbb{N}$) be a sequence of functions which Γ-converges to some $F : X \to \overline{\mathbb{R}}$; assume that $\inf_{v\in X} F_j(v) > -\infty$ for every j. Let (ε_j) be a positive infinitesimal sequence, and for every j let $u_j \in X$ be an ε_j-minimizer of F_j, i.e.*

$$F_j(u_j) \leq \inf_{v\in X} F_j(v) + \varepsilon_j \,.$$

Assume that $u_j \to u$ for some $u \in X$. Then u is a minimum point of F, and $F(u) = \lim_{j\to+\infty} F_j(u_j)$.

Remark 2.4 We also point out the following property, which is a direct consequence of the definition of Γ-convergence: if $F_\varepsilon \stackrel{\Gamma}{\to} F$ then $F_\varepsilon + G \stackrel{\Gamma}{\to} F + G$ whenever G is continuous.

In conclusion we recall the following useful result, which can be found in [5].

Lemma 2.5 (supremum of measures) *Let Ω be an open subset of $\mathbb{R}^n$ and denote the family of its open subsets by $\mathcal{A}(\Omega)$. Let λ be a positive Borel measure on Ω, and $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ a set function which is superadditive on open sets with disjoint compact closures (i.e. if $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$, then $\mu(A \cup B) \geq \mu(A) + \mu(B)$). Let $(\psi_i)_{i\in I}$ be a family of positive Borel functions. Suppose that*

$$\mu(A) \geq \int_A \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I;$$

then

$$\mu(A) \geq \int_A \sup_i \psi_i d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

3 Setting of the problem and main results

Let (a, b) be a bounded open interval of $\mathbb{R}$ and consider the functional $F : L^1(a, b) \to [0, +\infty]$ defined as follows:

$$F(u) = \begin{cases} \displaystyle\int_a^b \phi(|u'(x)|)dx + \sum_{x\in S_u} f\left(|u^+(x) - u^-(x)|\right) + c_0|D^c u|(a, b) & \text{if } u \in GBV(a, b), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\phi, f : [0, +\infty) \to [0, +\infty)$ satisfy the following conditions:

(A0) ϕ is convex and f is concave, with $\phi(0) = f(0) = 0$ and there exists $c_0 \in \mathbb{R}$, with $c_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = \lim_{t\to 0^+} \frac{f(t)}{t} = c_0.$$

By Theorem 5.4 in [2] (see also, e.g., [5] §2.4), F is sequentially lower semicontinuous in the L^1-topology.

We will prove an approximation result for F by means of a family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $L^1(a, b) \to [0, +\infty]$ of the form:

$$F_\varepsilon(u) = \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|dy\right) dx & \text{if } u \in W^{1,1}(a, b)\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

where f_ε is requested to satisfy the conditions *(A1)–(A3)* below.

(A1) For every $\varepsilon > 0$, $f_\varepsilon \colon [0, +\infty) \to [0, +\infty)$ is a non-decreasing continuous function with $f_\varepsilon(0) = 0$; moreover, there exists $a_\varepsilon > 0$ such that $a_\varepsilon \to 0$ as $\varepsilon \to 0$ and f_ε is concave in $(a_\varepsilon, +\infty)$.

REMARK 3.1 It can be easily checked that a non-decreasing and non-negative concave function on $[0, +\infty)$ is subadditive. Hence, f is subadditive, and assumption *(A1)* guarantees that $\tilde{f}_\varepsilon \colon t \mapsto f_\varepsilon(t + a_\varepsilon) \colon [0, +\infty) \to [0, +\infty)$ is subadditive.

Simple heuristic remarks suggest that the volume term in the limit depends on the behaviour of f_ε near zero. Then we require that f_ε behaves as a suitable rescaling of ϕ in a neighborhood of zero; more precisely, we assume that:

(A2)
$$\lim_{(\varepsilon,t)\to(0,0)} \frac{f_\varepsilon(t)}{\varepsilon\phi\left(\frac{t}{\varepsilon}\right)} = 1\,.$$

For reference convenience we point out that, in particular, *(A2)* implies:

$$\lim_{\varepsilon\to 0} \frac{f_\varepsilon(\varepsilon s)}{\varepsilon} = \phi(s) \quad \text{for every} \quad s \geq 0\,; \tag{3.2}$$

moreover, for every $\delta > 0$ there exist $t_\delta > 0$ and $\varepsilon_\delta > 0$ such that:

$$f_\varepsilon(t) \geq (1 - \delta)\varepsilon\phi(t/\varepsilon)\,, \tag{3.3}$$

whenever $0 \leq t \leq t_\delta$ and $0 < \varepsilon < \varepsilon_\delta$.

Analogously, we expect that the jump term in the limit depends on the pointwise behaviour of f_ε. Accordingly, we assume that:

(A3) $f_\varepsilon(t) \to f(t)$ uniformly on the compact subsets of $[0, +\infty)$.

Given f and ϕ as above, a possible choice for f_ε satisfying *(A1)–(A3)* is:

$$f_\varepsilon(t) = \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \leq t \leq t_\varepsilon, \\ f(t - t_\varepsilon) + \varepsilon\phi(\frac{t_\varepsilon}{\varepsilon}) & \text{if } t \geq t_\varepsilon, \end{cases}$$

where $t_\varepsilon \to 0$, and $t_\varepsilon/\varepsilon \to +\infty$. The only non-trivial assumption to verify is *(A2)*: since $(\varepsilon/t)\phi(t/\varepsilon) \to c_0$ as $(\varepsilon, t) \to (0, 0)$, with $t \geq t_\varepsilon$, the check amounts to verify that:

$$\lim_{\substack{(\varepsilon,t)\to(0,0)\\ t\geq t_\varepsilon}} \frac{f(t - t_\varepsilon) + \varepsilon\phi(t_\varepsilon/\varepsilon)}{t} = c_0\,.$$

This follows immediately from $f(t - t_\varepsilon)/(t - t_\varepsilon) \to c_0$ and $(\varepsilon/t_\varepsilon)\phi(t_\varepsilon/\varepsilon) \to c_0$ as $(\varepsilon, t) \to (0, 0)$, with $t > t_\varepsilon$.

We point out two properties we will need in the sequel.

REMARK 3.2 It is easy to see that there exist sequences (c_h) and (d_h) of real numbers with

$$0 \leq c_h \leq c_0\,, \qquad \phi(t) = \sup_{h\in\mathbb{N}}(c_h t + d_h) \qquad \text{for every } t \geq 0. \tag{3.4}$$

REMARK 3.3 There exists $C > 0$ such that

$$f_\varepsilon(t) \leq Ct \qquad \text{for every } t \geq 0 \text{ and } \varepsilon \text{ small enough.}$$

Proof. For every $\delta > 0$ Assumption *(A2)* yields the existence of $t_\delta, \varepsilon_\delta > 0$ such that $f_\varepsilon(t) \leq (1 + \delta)\varepsilon\phi(t/\varepsilon)$ for every $0 \leq t \leq t_\delta$ and $0 < \varepsilon \leq \varepsilon_\delta$; therefore, since $\phi(s) \leq c_0 s$, we have:

$$f_\varepsilon(t) \leq c_0(1 + \delta)t \qquad \text{for every } 0 \leq t \leq t_\delta \text{ and } 0 < \varepsilon \leq \varepsilon_\delta;$$

As to the interval $[t_\delta, +\infty)$, notice that we can suppose f_ε concave in $[t_\delta/2, +\infty)$ for every $0 < \varepsilon \le \varepsilon_\delta$; thus, if $m_\varepsilon t + q_\varepsilon$ is the affine function with coincides with f_ε in $t_\delta/2$ and t_δ, then $f_\varepsilon(t) \le m_\varepsilon t + q_\varepsilon$ if $t \ge t_\delta$. The pointwise convergence of f_ε gives the existence of m_0 and q_0 such that $f_\varepsilon(t) \le m_0 t + q_0$ for every $t \ge t_\delta$ and ε small enough (independently of $t \ge t_\delta$). The existence of a linear function majorizing f_ε on $[0, +\infty)$ uniformly in ε now immediately follows. ■

Let us now state the main results of the paper. We notice that the arguments of the proof require a "localized" version of the functionals F_ε; more precisely, for every open subset A of (a, b) we set

$$F_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|dy\right)dx & \text{if } u \in W^{1,1}(a, b)\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

Clearly, $F_\varepsilon\big(\cdot, (a, b)\big)$ coincides with the functional F_ε defined in (3.1). The lower and upper Γ-limits of $\big(F_\varepsilon(\cdot, A)\big)$ (see § 2) will be denoted by $F'(\cdot, A)$ and $F''(\cdot, A)$, respectively.

Theorem 3.4 *Let $(F_\varepsilon)_{\varepsilon>0}$ be as in (3.1), with f_ε satisfying conditions (A0), (A1), (A2) and (A3). Then (F_ε) Γ-converges, in the L^1-topology, as $\varepsilon \to 0$, to $\mathcal{F}\colon L^1(a, b) \to [0, +\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_a^b \phi(|u'(x)|)dx + 2\sum_{x\in S_u} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) + c_0|D^c u|(a, b) & \text{if } u \in GBV(a, b)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 3.5 It is worth noticing that actually we will prove (see Theorem 6.1) that for every $u \in GBV(a, b)$ the estimate

$$F'(u, A) \ge \int_A \phi(|u'(x)|)\,dx + 2\sum_{x\in S_u\cap A} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) + c_0|D^c u|(A)$$

holds for every A open subset of (a, b).

Theorem 3.6 (Compactness) *Let (ε_j) be a positive infinitesimal sequence, and let (u_j) be a sequence in $L^1(a, b)$ such that*

$$\|u_j\|_\infty \le M, \quad F_{\varepsilon_j}(u_j) \le M$$

for a suitable constant M independent of j. Then there exists a subsequence (u_{j_k}) converging in $L^1(a, b)$ to a function $u \in BV(a, b)$.

As an example of application of these results we consider the following corollary.

Corollary 3.7 *Let (ε_j) be a positive infinitesimal sequence and $g \in L^\infty(a, b)$. For every $u \in L^1(a, b)$ and $j \in \mathbb{N}$, define:*

$$G_j(u) = F_{\varepsilon_j}(u) + \int_a^b |u(x) - g(x)|\,dx\,,$$

and

$$\mathcal{G}(u) = \mathcal{F}(u) + \int_a^b |u(x) - g(x)|\,dx\,.$$

For every j let u_j be a σ_j-minimizer of G_j in $L^1(a, b)$ (with $\sigma_j \searrow 0$), i.e. $G_j(u_j) \le \inf_{L^1(a,b)} G_j + \sigma_j$. Then (u_j) converges, up to a subsequence, to a minimizer of $\mathcal{G}$ in $L^1(a, b)$.

Proof. Since $g \in L^\infty(a,b)$ and F_{ε_j} decreases by truncation, we can assume that (u_j) is equibounded. By Theorem 3.6 there exists $u \in BV(a,b)$ such that (up to a subsequence) $u_j \to u$ in $L^1(a,b)$. By Theorem 2.3, since (G_j) Γ-converge to $\mathcal{G}$ (recall Remark 2.4), u is a minimum point of $\mathcal{G}$ on $L^1(a,b)$. ∎

Theorem 3.6 will be proved in the next section (Corollary 4.3) while Theorem 3.4 will be completed in § 7.

We end this section with two remarks related to the convergence of (F_ε).

REMARK 3.8 The Γ-convergence of (F_ε) easily implies the convergence of $(\tilde{F}_\varepsilon)$, where

$$\tilde{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\Big(\frac{1}{2}\int_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|\,dy\Big)\,dx$$

if $u \in W^{1,1}(a,b)$, and $\tilde{F}_\varepsilon(u) = +\infty$ otherwise in $L^1(a,b)$. Since $\tilde{F}_\varepsilon \le F_\varepsilon$, we have only to check the inequality

$$\liminf_{j\to+\infty} \tilde{F}_{\varepsilon_j}(u_j) \ge \mathcal{F}(u)$$

whenever $\varepsilon_j \to 0^+$ and $u_j \to u$ in $L^1(a,b)$. Assume that the left-hand side is finite. Notice that $|(x-\varepsilon,x+\varepsilon)\cap(a,b)| \ge \varepsilon/2$ for every $x \in (a,b)$ and $\varepsilon < (b-a)/2$; thus, if we set $\overline{f}_\varepsilon(t) = f_\varepsilon(t/4)$, it turns out that:

$$\tilde{F}_{\varepsilon_j}(u_j) \ge \frac{1}{\varepsilon}\int_a^b \overline{f}_{\varepsilon_j}\left(\varepsilon_j \fint_{(x-\varepsilon_j,x+\varepsilon_j)\cap(a,b)} |u_j'(y)|dy\right)dx\,.$$

By Theorem 3.4 (applied with $\overline{f}_\varepsilon$ in place of f_ε) the boundedness of $\big(\tilde{F}_{\varepsilon_j}(u_j)\big)$ implies that $u \in GBV(a,b)$.

Fix now $\sigma > 0$; for j sufficiently large it turns out that:

$$\tilde{F}_{\varepsilon_j}(u_j) \ge \frac{1}{\varepsilon_j}\int_{a+\sigma}^{b-\sigma} f_{\varepsilon_j}\Big(\frac{1}{2}\int_{(x-\varepsilon_j,x+\varepsilon_j)\cap(a,b)} |u_j'(y)|\,dy\Big)\,dx = F_{\varepsilon_j}\big(u_j,(a+\sigma,b-\sigma)\big)\,;$$

since $u \in GBV(a,b)$, from Remark 3.5 we deduce that $\liminf_{j\to+\infty} \tilde{F}_{\varepsilon_j}(u_j) \ge \mathcal{F}\big(u,(a+\sigma,b-\sigma)\big)$: let now σ tend to zero.

REMARK 3.9 The Γ-convergence of (F_ε) implies the Γ-convergence of the relaxed functionals $(\overline{F}_\varepsilon)$ with respect to the L^1-topology. If $u \in BV(a,b)$ it turns out that the relaxed functional $\overline{F}_\varepsilon$ is given by

$$\overline{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(\frac{\varepsilon}{|I_\varepsilon(x)\cap(a,b)|}|Du|\big(I_\varepsilon(x)\cap(a,b)\big)\right)dx\,, \tag{3.6}$$

where $I_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$.

Proof. Denote by H_ε the functional on the right-hand side of (3.6), defined with value $+\infty$ on $L^1 \setminus BV$. Let $c_\varepsilon(x) = \varepsilon/|I_\varepsilon(x)\cap(a,b)|$. It is easy to prove the L^1-lower semicontinuity of H_ε at every $u \in BV$. Indeed: let (u_h) be a sequence in $L^1(a,b)$ converging, in $L^1(a,b)$, to a function $u \in BV(a,b)$; if $\liminf_{h\to+\infty} H_\varepsilon(u_h) < +\infty$ we can suppose that this lower limit is a limit, and that each u_h is in BV; then by Fatou's lemma and the lower semicontinuity of the total variation,

$$\begin{aligned}
\liminf_{h\to+\infty} H_\varepsilon(u_h) &\ge \frac{1}{\varepsilon}\int_a^b \liminf_{h\to+\infty} f_\varepsilon\left(c_\varepsilon(x)|Du_h|\big(I_\varepsilon(x)\cap(a,b)\big)\right)dx \\
&\ge \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon(x)\liminf_{h\to+\infty}|Du_h|\big(I_\varepsilon(x)\cap(a,b)\big)\right)dx \\
&\ge \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon(x)|Du|\big(I_\varepsilon(x)\cap(a,b)\big)\right)dx = H_\varepsilon(u).
\end{aligned}$$

Since $H_\varepsilon(u) \le F_\varepsilon(u)$ for all $u \in L^1(a,b)$, the relaxed functional $\overline{F}_\varepsilon$ is estimated from below by H_ε on $BV(a,b)$. Consider now the opposite inequality. Given $u \in BV(a,b)$, if (v_h) denotes the sequence obtained from u (extended to a neighborhood of (a,b)) by standard mollification, then $v_h \to u$ in $L^1(a,b)$ and

$$|Dv_h|\big(I_\varepsilon(x) \cap (a,b)\big) \to |Du|\big(I_\varepsilon(x) \cap (a,b)\big)$$

for a.e. $x \in (a,b)$ (see, e.g., [2] Proposition 3.7 and Remark 3.8). Then by the dominated convergence theorem (recall Remark 3.3 and that $c_\varepsilon(x) \le 2$):

$$\lim_{h\to+\infty} F_\varepsilon(v_h) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon(x)|Du|\big(I_\varepsilon(x)\cap(a,b)\big)\right)dx = H_\varepsilon(u).$$

This shows that H_ε coincides with the relaxed functional $\overline{F}_\varepsilon$ on $BV(a,b)$. ■

4 Estimate from below of the volume and Cantor terms. Compactness

The main result of this section is Proposition 4.8. Moreover, the technical Lemma 4.2 will immediately imply the compactness result of Corollary 4.3.

Let us state a preliminary result (from [6], Lemma 4.2) whose proof follows by applying the mean value theorem for integrals to the function $\sum_{\alpha\in\mathbb{Z}} \psi(x+\alpha\eta)$.

Lemma 4.1 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. For every $\eta > 0$ there exists $x_0 \in (-\eta/2, \eta/2)$ such that*

$$\int_{\mathbb{R}} \psi(x)dx = \sum_{\alpha\in\mathbb{Z}} \eta\psi(x_0 + \alpha\eta).$$

This allows to estimate $F_\varepsilon(u)$ by a useful integral sum (see (4.1) below).

Let us fix $u \in W^{1,1}(a,b)$, $\varepsilon > 0$ and φ_ε a cut-off function in $C_c^\infty(a,b)$ such that $0 \le \varphi_\varepsilon \le 1$ in (a,b) and $\varphi_\varepsilon = 1$ in $(a+\varepsilon, b-\varepsilon)$. Let ψ_ε be the continuous function defined by

$$\psi_\varepsilon(x) = \varphi_\varepsilon(x) f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(t)|dt\right)$$

for $x \in (a,b)$, and 0 for $x \in \mathbb{R}\setminus(a,b)$. Then, taking Lemma 4.1 into account, with $\eta = 2\varepsilon$, we have

$$F_\varepsilon(u) \ge \frac{1}{\varepsilon}\int_a^b \psi_\varepsilon(x)dx = \frac{1}{\varepsilon}\int_{\mathbb{R}} \psi_\varepsilon(x)dx = \sum_{\alpha\in\mathbb{Z}} 2\psi_\varepsilon(x_\varepsilon + 2\varepsilon\alpha) \tag{4.1}$$

for a suitable $x_\varepsilon \in \mathbb{R}$. Therefore, if we set $x_\alpha = x_\alpha^\varepsilon = x_\varepsilon + 2\alpha\varepsilon$, $(\alpha \in \mathbb{Z})$, and

$$J_\varepsilon := \{\alpha \in \mathbb{Z} :\ x_\alpha \in (a+\varepsilon, b-\varepsilon)\}$$

we get ($\varphi_\varepsilon(x_\alpha) = 1$ if $\alpha \in J_\varepsilon$):

$$F_\varepsilon(u) \ge 2\sum_{\alpha\in J_\varepsilon} f_\varepsilon\left(\varepsilon \fint_{x_\alpha-\varepsilon}^{x_\alpha+\varepsilon} |u'(t)|dt\right). \tag{4.2}$$

Lemma 4.2 *Let $u \in W^{1,1}(a,b)$ and $\delta > 0$; let t_δ and ε_δ be as in (3.3). Fix $\varepsilon < \varepsilon_\delta$. With the notation above let*

$$\mathcal{P}_\varepsilon = \{(x_\alpha - \varepsilon, x_\alpha + \varepsilon) : \alpha \in J_\varepsilon\}.$$

Then we can select a subfamily $\mathcal{P}'_\varepsilon$ of $\mathcal{P}_\varepsilon$ such that

$$(i) \quad \#(\mathcal{P}_\varepsilon \setminus \mathcal{P}'_\varepsilon) \le \frac{1}{2} F_\varepsilon(u)/f_\varepsilon(t_\delta);$$

$$(ii) \quad F_\varepsilon(u) \ge (1-\delta) \int_{\bigcup \mathcal{P}'_\varepsilon} l(|u'(t)|)dt$$

whenever l is an affine function satisfying $l \le \phi$.

Since $f_\varepsilon(t_\delta) \to f(t_\delta) > 0$, the inequality (i) states that for any given $\delta > 0$ we can estimate $\#(\mathcal{P}_\varepsilon \setminus \mathcal{P}'_\varepsilon)$ by $F_\varepsilon(u)/f(t_\delta)$ for ε sufficiently small.

Proof. From (3.3) and (4.2), we get

$$F_\varepsilon(u) \ge 2(1-\delta) \sum_{\alpha \in H_\varepsilon} \varepsilon \phi \left(\fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)|dt \right),$$

where $H_\varepsilon = \left\{ \alpha \in J_\varepsilon : \ \varepsilon \fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)|dt < t_\delta \right\}$. Let us now fix an affine function $l : \mathbb{R} \to \mathbb{R}$ with $l \le \phi$, and define $\mathcal{P}'_\varepsilon = \{(x_\alpha - \varepsilon, x_\alpha + \varepsilon) : \ \alpha \in H_\varepsilon\}$. Then

$$F_\varepsilon(u) \ge 2\varepsilon(1-\delta) \sum_{\alpha \in H_\varepsilon} l \left(\fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)|dt \right)$$

$$= (1-\delta) \sum_{\alpha \in H_\varepsilon} \int_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} l(|u'(t)|)dt = (1-\delta) \int_{\bigcup \mathcal{P}'_\varepsilon} l(|u'(t)|)dt.$$

Finally, from (4.2), and the monotonicity of f_ε, it follows that

$$F_\varepsilon(u) \ge 2 \sum_{\alpha \in J_\varepsilon \setminus H_\varepsilon} f_\varepsilon(t_\delta) = 2\#(J_\varepsilon \setminus H_\varepsilon) f_\varepsilon(t_\delta).$$

■

Corollary 4.3 (Compactness) *Let (ε_j) be a positive infinitesimal sequence, and let (u_j) be a sequence in $L^1(a,b)$ such that $\|u_j\|_\infty \le M$, and $F_{\varepsilon_j}(u_j) \le M$ for a suitable constant M independent of j. Then there exists a subsequence (u_{j_k}) converging in $L^1(a,b)$ to a function $u \in BV(a,b)$.*

Proof. Clearly, $u_j \in W^{1,1}(a,b)$ for every j. Since $\lim_{t \to +\infty} \phi(t)/t = c_0$ there exists $c_1 \in \mathbb{R}$ such that $\phi(t) \ge c_0 t + c_1$. Fix $\delta > 0$ and apply Lemma 4.2 with $u = u_j$ and $l(t) = c_0 t + c_1$; we determine a family $\mathcal{P}_j$ of intervals of width $2\varepsilon_j$ covering a.e. $(a + \varepsilon_j, b - \varepsilon_j)$, and a subfamily $\mathcal{P}'_j$ such that $\#(\mathcal{P}_j \setminus \mathcal{P}'_j)$ is bounded independently of j, and

$$F_{\varepsilon_j}(u_j) \ge (1-\delta) \int_{\bigcup \mathcal{P}'_j} l(|u'_j(x)|)dx.$$

Then, if we define

$$v_j(x) = \begin{cases} u_j(x) & x \in \bigcup \mathcal{P}'_j \\ 0 & \text{otherwise} \end{cases}$$

it turns out that $v_j \in SBV(a,b)$, $\|v_j\|_\infty \le M$, $\#S_{v_j}$ bounded independently of j, and

$$M \ge (1-\delta) \int_a^b l(|v'_j(x)|)dx = c'_0 \int_a^b |v'_j(x)|dx + c'_1$$

with $c_0' > 0$. Therefore (v_j) is bounded in $BV(a,b)$ and we can extract a subsequence (v_{j_k}) converging in $L^1(a,b)$ to a function $u \in BV(a,b)$. Since

$$\|v_j - u_j\|_1 \le 2\varepsilon_j M(\#(\mathcal{P}_j \setminus \mathcal{P}_j') + 2)$$

we conclude that u_{j_k} converges to u in $L^1(a,b)$. ■

Corollary 4.4 *Let (ε_j) be a positive infinitesimal sequence and let (u_j) be a converging sequence in $L^1(a,b)$. If $\big(F_{\varepsilon_j}(u_j)\big)$ is bounded, then the limit of (u_j) belongs to $GBV(a,b)$. In particular, if $F'(u) < +\infty$ then $u \in GBV(a,b)$.*

Proof. Let u be the L^1-limit of (u_j). For each $T > 0$ apply the previous Corollary to $u_j^T = (-T) \vee u_j \wedge T$: we get $(-T) \vee u \wedge T \in BV(a,b)$; hence $u \in GBV(a,b)$. ■

Lemma 4.5 *Let $u \in BV(a,b)$ and let (u_j) be a sequence in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and a.e. in (a,b). Suppose that there exists $\sigma \ge 0$ such that for every $x \in S_u$:*

$$|u^+(x) - u^-(x)| \le \sigma.$$

Then, for every $j \in \mathbb{N}$, there exists $\tilde{u}_j \in W^{1,1}(a,b)$ such that

$$F_\varepsilon(\tilde{u}_j) \le F_\varepsilon(u_j) \qquad \text{for every } \varepsilon > 0,$$

$\tilde{u}_j \to u$ in $L^1(a,b)$, and

$$\limsup_{j\to+\infty} \|\tilde{u}_j - u\|_\infty \le \sigma.$$

Proof. Let us denote by u a good representative. It is not difficult to see that for every $\eta > 0$ there exists $\delta > 0$ such that whenever $x, y \in (a,b)$:

$$|x - y| < \delta \Rightarrow |u(x) - u(y)| < \sigma + \eta.$$

Indeed, suppose by contradiction that there exists $\eta_0 > 0$ such that for every $n \in \mathbb{N}$ we can find $x_n, y_n \in (a,b)$ satisfying $|x_n - y_n| < 1/n$ and

$$|u(x_n) - u(y_n)| \ge \sigma + \eta_0. \tag{4.3}$$

Up to a subsequence we can assume that (x_n) and (y_n) converge to a point $x_0 \in [a,b]$ and, moreover, that

$$x_n \to x_0^+ \quad \text{or} \quad x_n \to x_0^- \quad \text{and} \quad y_n \to x_0^+ \quad \text{or} \quad y_n \to x_0^-.$$

Then by taking the limit as $n \to +\infty$ in (4.3), we have a contradiction both if x_0 is a continuity or a jump point of u, and if x_0 is an end point of (a,b).

By means of this uniform control on the oscillation of u, we truncate now u_j. For any given $n \in \mathbb{N}$, let $\delta_n > 0$ be such that

$$|x - y| < \delta_n \Rightarrow |u(x) - u(y)| < \sigma + \frac{1}{n}$$

whenever $x, y \in (a,b)$. Consider a finite partition P_n of (a,b):

$$a = x_0 < x_1 < x_2 < \cdots < x_{k+1} = b$$

(we drop the dependence on n) with mesh size less than δ_n and such that each x_i (for every $i = 1, \dots, k$) is a point of convergence for the sequence (u_j). Let $j_n \in \mathbb{N}$ be such that if $j \ge j_n$

$$|u_j(x_i) - u(x_i)| \le \frac{1}{n} \quad \text{for every } i = 1, \dots, k.$$

We can choose (j_n) strictly increasing. Fix $j \in \mathbb{N}$ and let $j_n \le j \le j_{n+1}$. On the first and last interval of the partition P_n we define $\tilde{u}_j$ with constant value $u_j(x_1)$ and $u_j(x_k)$ respectively. Let now $[\xi, \eta]$ be any of the subintervals of $[x_i, x_{i+1}]$ for $i = 1, \dots, k-1$. Without loss of generality we can assume $u_j(\xi) \le u_j(\eta)$. Now define

$$\tilde{u}_j(x) = [u_j(x) \vee u_j(\xi)] \wedge u_j(\eta)$$

for every $x \in (\xi, \eta)$. Clearly, $\tilde{u}_j$ equals u_j at the endpoints ξ and η, hence $\tilde{u}_j \in W^{1,1}(a,b)$. Moreover, for every $x \in [\xi, \eta]$

$$u(\xi) - 1/n \le u_j(\xi) \le \tilde{u}_j(x) \le \tilde{u}_j(\eta) \le u(\eta) + 1/n.$$

Therefore:

$$|\tilde{u}_j(x) - u(x)| \le \operatorname{osc}_{[\xi,\eta]} u + 2/n \le \sigma + 3/n.$$

These inequalities hold in the first and last intervals of the decomposition, too. Hence we get the stated estimate about the upper limit of $\|\tilde{u}_j - u\|_\infty$. Since F_ε decreases by truncation, we have $F_\varepsilon(\tilde{u}_j) \le F_\varepsilon(u_j)$. As to the L^1 convergence of the sequence $(\tilde{u}_j)$, consider, as above, the generic subinterval $[\xi, \eta]$ of the partition P_n; let $\tilde{u}'_j$ be the pointwise projection of u_j onto the interval $[u_j(\xi) - 1/n, u_j(\eta) + 1/n]$; since u takes values in this interval, we have:

$$|\tilde{u}_j - u| \le |\tilde{u}_j - \tilde{u}'_j| + |\tilde{u}'_j - u| \le 1/n + |u_j - u|, \quad \text{in } [\xi, \eta].$$

This implies the pointwise convergence of $\tilde{u}_j$ to u a.e. on the compact subsets of (a,b), hence on (a,b); the equiboundedness of the sequence gives the L^1 convergence. ■

Remark 4.6 Given $u \in L^1(a,b)$, the set functions $A \mapsto F_\varepsilon(u, A)$ are increasing and superadditive. Consequently also the set function $A \mapsto F'(u, A)$ is increasing and superadditive, i.e.

(i) $F'(u, A_1) \le F'(u, A_2)$, whenever $A_1 \subseteq A_2 \subseteq (a,b)$;

(ii) $F'(u, A_1 \cup A_2) \ge F'(u, A_1) + F'(u, A_2)$, whenever $A_1 \cap A_2 = \emptyset$.

These properties imply, in particular, the following result. Let $u \in BV(a,b)$ and let λ be a positive Radon measure on (a,b); suppose

$$F'(u, I) \ge \int_I g(x)\, d\lambda$$

for every interval $I \subseteq (a,b)$, with g a non-negative Borel function. Then

$$F'(u, A) \ge \int_A g(x)\, d\lambda \tag{4.4}$$

for every A open subset of (a,b). Indeed, every open subset A of (a,b) is a countable union of pairwise disjoint intervals I_h; by the monotonicity and superadditivity of F', for every $N \in \mathbb{N}$ we have

$$F'(u, A) \ge F'\left(u, \bigcup_{h=1}^N I_h\right) \ge \sum_{h=1}^N F'(u, I_h) \ge \int_{\bigcup_{h=1}^N I_h} g(x)\, d\lambda.$$

Passing to the limit as $N \to +\infty$ we obtain (4.4).

Before addressing the main result of this section (Proposition 4.8) we need to relate (see Lemma 4.7 below) $F'(u, I)$, where I is an open subinterval of (a,b) and $u_{|_I} \in BV(I)$, with $F'\big(v, (a,b)\big)$, where v is the extension of $u_{|_I}$ to the whole (a,b) with the inner traces on ∂I.

Lemma 4.7 *Let $I = (\alpha, \beta)$ be a subinterval of (a,b) and $u \in BV(a,b)$; let $v \in BV(a,b)$ defined by*

$$v(x) = \begin{cases} u^+(\alpha) & x \in (a, \alpha) \\ u^-(\beta) & x \in (\beta, b) \\ u(x) & \textit{otherwise} \end{cases}$$

Then $F'(u, I) \ge F'(v)$.

Proof. There exist a positive infinitesimal sequence (ε_j) and a sequence (u_j) in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and satisfying: $F'(u,I) = \lim_{j\to+\infty} F_{\varepsilon_j}(u_j, I)$. We can also suppose that $u_j \to u$ a.e. in (a,b). Let $0 < \sigma < (\beta-\alpha)/2$, and let $x^\sigma \in (\alpha, \alpha+\sigma)$ and $y^\sigma \in (\beta-\sigma, \beta)$ be such that

$$u_j(x^\sigma) \to u(x^\sigma), \quad u_j(y^\sigma) \to u(y^\sigma)$$

as $j \to +\infty$. Let

$$u^\sigma = \begin{cases} u(x^\sigma) & x \in (a, x^\sigma) \\ u(y^\sigma) & x \in (y^\sigma, b) \\ u(x) & \text{otherwise} \end{cases} \quad \text{and} \quad u_j^\sigma = \begin{cases} u_j(x^\sigma) & x \in (a, x^\sigma) \\ u_j(y^\sigma) & x \in (y^\sigma, b) \\ u_j(x) & \text{otherwise} \end{cases}$$

Then $u_j^\sigma \in W^{1,1}(a,b)$ and $u_j^\sigma \to u^\sigma$ in $L^1(a,b)$ as $j \to +\infty$; hence

$$\begin{aligned} F'(u^\sigma) &\le \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j^\sigma, (a,b)) = \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j^\sigma, I) \\ &\le \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, I) = F'(u,I)\,. \end{aligned}$$

Finally, since $u^\sigma \to v$ in $L^1(a,b)$ as $\sigma \to 0$, by the lower semicontinuity of F' we have $F'(v) \le \liminf_{\sigma\to 0} F'(u^\sigma) \le F'(u,I)\,.$ ■

Proposition 4.8 *For every $u \in BV(a,b)$ and A open subset of (a,b)*

$$F'(u,A) \ge \int_A \phi(|u'(x)|)dx, \quad \textit{and} \quad F'(u,A) \ge c_0|D^c u|(A).$$

Proof. *Step 1.* We claim that if $u \in BV(a,b)$ then: for every $\delta > 0$ and $l_h \le \phi$ affine function, with $l_h(t) = c_h t + d_h$ as in (3.4), the following inequality holds:

$$F'(u)\left(1 + \frac{6\sigma c_0}{f(t_\delta)}\right) \ge (1-\delta)\left(\int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)\right),$$

where t_δ is as in the assumption (3.3), and $\sigma = \sup_{x\in S_u} |u^+(x) - u^-(x)|$.

Without loss of generality we can assume that there exist a positive infinitesimal sequence (ε_j), and a sequence (u_j) in $W^{1,1}(a,b)$ such that $u_j \to u$ in $L^1(a,b)$ and a.e., and $F_{\varepsilon_j}(u_j) \to F'(u) < +\infty$. Let $\sigma = \sup_{x\in S_u} |u^+(x) - u^-(x)|$. Lemma 4.5 gives a sequence $(\tilde u_j)$ in $W^{1,1}(a,b)$, converging to u in $L^1(a,b)$, such that

$$F_{\varepsilon_j}(\tilde u_j) \le F_{\varepsilon_j}(u_j) \quad \text{and} \quad \limsup_{j\to+\infty} \|\tilde u_j - u\|_\infty \le \sigma.$$

In particular $F_{\varepsilon_j}(\tilde u_j) \to F'(u)$. Let now $\eta > 0$ be fixed; we can suppose that $\|\tilde u_j - u\|_\infty \le \sigma + \eta$ for every $j \in \mathbb{N}$. As shown in the first part of the proof of Lemma 4.5, there exists $\gamma > 0$ such that, if $J \subseteq (a,b)$, then $osc_J u < \sigma + \eta$ whenever $\mathrm{diam} J < \gamma$; therefore

$$\mathrm{diam} J < \gamma \Rightarrow \mathrm{osc}_J \tilde u_j < 3(\sigma + \eta)\,. \tag{4.5}$$

For every $h \in \mathbb{N}$ let $l_h(t) = c_h t + d_h$ as in (3.4). Let $\delta > 0$ be fixed; apply Lemma 4.2 with $u = \tilde u_j$ and $l = l_h$. Thus, for j sufficiently large, we determine a uniform mesh $(x_\alpha)_{\alpha\in\mathbb{Z}}$ of $\mathbb{R}$ with size $2\varepsilon_j$ and a subfamily $\mathcal{P}_j'$ of

$$\mathcal{P}_j = \{(x_\alpha - \varepsilon_j, x_\alpha + \varepsilon_j) : \ x_\alpha \in (a+\varepsilon_j, b-\varepsilon_j)\}$$

with the following property

$$\#(\mathcal{P}_j \setminus \mathcal{P}_j') \le \frac{1}{2 f_{\varepsilon_j}(t_\delta)} F_{\varepsilon_j}(\tilde u_j), \qquad F_{\varepsilon_j}(\tilde u_j) \ge (1-\delta) \int_{\bigcup \mathcal{P}_j'} l_h(|\tilde u_j'(x)|)dx. \tag{4.6}$$

Let $a_0 = \inf(\bigcup \mathcal{P}_j)$ and $b_0 = \sup(\bigcup \mathcal{P}_j)$; define (a.e.) v_j in (a_0, b_0) as follows

$$v_j(x) = \begin{cases} \tilde{u}_j(x) & \text{if } x \in \bigcup \mathcal{P}'_j \\ \displaystyle -\!\!\!\!\!\!\int_J \tilde{u}_j(z)dz & \text{if } x \in J \in \mathcal{P}_j \setminus \mathcal{P}'_j. \end{cases}$$

We consider v_j extended by continuity on (a,b) with values $v_j^+(a_0)$ and $v_j^-(b_0)$. Then $v_j \in SBV(a,b)$ and, by (4.5):

$$|v_j^+(x) - v_j^-(x)| \leq 3(\sigma + \eta)$$

for j sufficiently large (such that $2\varepsilon_j < \gamma$), for every $x \in S_{v_j}$. Moreover, by (4.6), $v_j \to u$ in $L^1(a,b)$ and

$$\#S_{v_j} \leq \frac{2}{f(t_\delta)} F_{\varepsilon_j}(\tilde{u}_j).$$

Therefore,

$$F_{\varepsilon_j}(\tilde{u}_j)\left(1 + \frac{6c_0}{f(t_\delta)}(\sigma+\eta)\right) \geq$$

$$\geq (1-\delta)\left(\int_a^b l_h(|v'_j(x)|)dx + c_0|D^s v_j|(a,b)\right) - (1-\delta)d_h 2\varepsilon_j\big(\#(\mathcal{P}_j \setminus \mathcal{P}'_j) + 2\big)$$

$$\geq (1-\delta)\big(c_h|Dv_j|(a,b) + d_h(b-a)\big) - (1-\delta)d_h 2\varepsilon_j\big(\#(\mathcal{P}_j \setminus \mathcal{P}'_j) + 2\big),$$

and, as $j \to +\infty$,

$$F'(u)\left(1 + \frac{6c_0}{f(t_\delta)}(\sigma+\eta)\right) \geq (1-\delta)\big(c_h|Du|(a,b) + d_h(b-a)\big)$$

$$\geq (1-\delta)\left(\int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)\right).$$

By the arbitrariness of η we conclude.

Step 2. Let $u \in BV(a,b)$; let us fix $\sigma > 0$ and consider the finite set of points $\{x_1, \ldots, x_{n-1}\} \subseteq S_u$ such that $|u^+(x_i) - u^-(x_i)| > \sigma$ for $i = 1, \ldots, n-1$. Let $a = x_0$ and $b = x_n$; then for every $i = 0, \ldots, n-1$,

$$\sup_{x \in S_u \cap (x_i, x_{i+1})} |u^+(x) - u^-(x)| \leq \sigma.$$

Let

$$v(x) = \begin{cases} u^+(x_i) & x \in (a, x_i] \\ u^-(x_{i+1}) & x \in [x_{i+1}, b) \\ u(x) & \text{otherwise.} \end{cases}$$

By *Step 1* and by Lemma 4.7

$$F'\big(u,(x_i,x_{i+1})\big)\left(1 + \frac{6c_0}{f(t_\delta)}\sigma\right) \geq F'(v)\left(1 + \frac{6c_0}{f(t_\delta)}\sigma\right)$$

$$\geq (1-\delta)\left(\int_{x_i}^{x_{i+1}} l_h(|u'(x)|)dx + c_h|D^c u|(x_i,x_{i+1})\right)$$

for every $i = 0, \ldots, n-1$. Then, by Remark 4.6

$$F'(u)\left(1 + \frac{6c_0}{f(t_\delta)}\sigma\right) \geq (1-\delta)\left(\int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)\right).$$

By the arbitrariness of $\sigma > 0$ and $\delta > 0$, we obtain

$$F'(u) \geq \int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)$$

Again, by applying Lemma 4.7 it is easy to see that

$$F'(u, I) \geq \int_I l_h(|u'(x)|)dx + c_h|D^c u|(I)$$

for every I open subinterval of (a, b). Thus, by Remark 4.6

$$F'(u, A) \geq \int_A l_h(|u'(x)|)dx + c_h|D^c u|(A)$$

for every A open subset of (a, b); we conclude that

$$F'(u, A) \geq \int_A l_h(|u'(x)|)dx \quad \text{and} \quad F'(u, A) \geq c_h|D^c u|(A).$$

Step 3. Let $d_0 = \lim_{t\to+\infty}\big(\phi(t) - c_0 t\big) < 0$; without loss of generality we can suppose $d_h > d_0$ and $c_h \geq 0$. Fix $u \in BV(a, b)$; by Step 2:

$$F'(u, A) - d_0|A| \geq \int_A (l_h(|u'(x)|) - d_0)dx \quad \text{and} \quad F'(u, A) \geq c_h|D^c u|(A) \tag{4.7}$$

for every h and for every A open subset of (a, b). The first inequality and Lemma 2.5 yield:

$$F'(u, A) - d_0|A| \geq \int_A (\phi(|u'(x)|) - d_0)dx$$

and then $F'(u, A) \geq \int_A \phi(|u'(x)|)dx$. By taking the supremum on h in the second inequality of (4.7) we have $F'(u, A) \geq c_0|D^c u|(A)$. ■

5 Estimate from below of the jump part

Lemma 5.1 *Let $u \in BV(a, b)$ and let (u_j) be a sequence in $W^{1,1}(a, b)$ converging to u in $L^1(a, b)$. Let $x \in S_u$; then there exist $x_j^+, x_j^- \in (a, b)$ such that $x_j^\pm \to x^\pm$ and such that*

$$|u_j(x_j^+) - u^+(x)| \to 0, \qquad |u_j(x_j^-) - u^-(x)| \to 0.$$

Proof. For all $\sigma > 0$ there exists $\delta = \delta_\sigma > 0$ such that

$$\left| u^+(x) - \fint_x^{x+\delta} u(y)dy \right| < \frac{\sigma}{2};$$

clearly, we can assume that $\delta < \sigma$. By the L^1-convergence of (u_j) there exists j_σ such that

$$\left| \fint_x^{x+\delta} u_j(y)dy - \fint_x^{x+\delta} u(y)dy \right| < \frac{\sigma}{2}$$

for all $j \geq j_\sigma$; by the mean value theorem for integrals, for every $j \geq j_\sigma$ we can find $x_j^+ \in (x, x + \delta)$ such that $|u_j(x_j^+) - u^+(x)| < \sigma$. For every $k \in \mathbb{N}$ let j_k be the integer j_σ corresponding to $\sigma = 1/k$. We can assume that j_k is strictly increasing. If we select the points x_j^+, defined above, with $j_k \leq j < j_{k+1}$, we get a sequence satisfying the required conditions. An analogous argument yields x_j^-. ■

Proposition 5.2 *For every $u \in BV(a,b)$ and A open subset of (a,b) we have*

$$F'(u,A) \geq 2 \sum_{x \in S_u \cap A} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right).$$

Proof. *Step 1.* We claim that for any $\overline{x} \in S_u \cap A$

$$F'(u,A) \geq 2f\left(\frac{1}{2}|u^+(\bar{x}) - u^-(\bar{x})|\right).$$

Let (ε_j) be a positive infinitesimal sequence, and let (u_j) be a sequence in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and satisfying: $F'(u,A) = \lim_{j \to +\infty} F_{\varepsilon_j}(u_j, A)$.

Fix $\bar{x} \in S_u \cap A$ and let $(x_j^{\pm})$ be the sequences provided by Lemma 5.1 applied with $x = \bar{x}$. Assume $u^-(\bar{x}) < u^+(\bar{x})$; thus we can suppose that $u_j(x_j^-) < u_j(x_j^+)$ for every j. Let $\tilde{u}_j$ be the continuous extension of u_j from (x_j^-, x_j^+) to (a,b) with the constant values $u_j(x_j^-)$ and $u_j(x_j^+)$. Clearly, for j sufficiently large

$$F_{\varepsilon_j}(u_j, A) \geq F_{\varepsilon_j}(\tilde{u}_j, A) = F_{\varepsilon_j}\big(\tilde{u}_j, (a,b)\big) = F_{\varepsilon_j}(\tilde{u}_j). \tag{5.1}$$

We can also assume that $\tilde{u}_j$ is non-decreasing, otherwise we replace $\tilde{u}_j$ by

$$\left(\tilde{u}_j(a) + \int_a^x (\tilde{u}_j'(t))^+ dt\right) \wedge \tilde{u}_j(b),$$

(this lowers the value of F_{ε_j}). Apply now estimate (4.2) with $\varepsilon = \varepsilon_j$ and $u = \tilde{u}_j$; then

$$F_{\varepsilon_j}(\tilde{u}_j) \geq 2 \sum_{\alpha \in I_j} f_{\varepsilon_j}\left(\varepsilon_j ⨍_{x_\alpha - \varepsilon_j}^{x_\alpha + \varepsilon_j} \tilde{u}_j'(t)dt\right),$$

where

$$I_j = \{\alpha \in \mathbb{Z} : \ (x_\alpha - \varepsilon_j, x_\alpha + \varepsilon_j) \cap (x_j^-, x_j^+) \neq \emptyset\}.$$

The convergence $x_j^{\pm} \to \bar{x}$ yields that $(\# I_j)\varepsilon_j \to 0$ as $j \to +\infty$. For each $\alpha \in I_j$ let

$$\delta_j^\alpha = \varepsilon_j ⨍_{x_\alpha - \varepsilon_j}^{x_\alpha + \varepsilon_j} \tilde{u}_j'(t)dt = \frac{1}{2}[\tilde{u}_j(x_\alpha + \varepsilon_j) - \tilde{u}_j(x_\alpha - \varepsilon_j)] \geq 0.$$

It turns out that

$$2 \sum_{\alpha \in I_j} \delta_j^\alpha = u_j(x_j^+) - u_j(x_j^-) \to u^+(\bar{x}) - u^-(\bar{x}). \tag{5.2}$$

Given $\delta > 0$, we can find $t_\delta > 0$ and j_δ such that (see (3.3))

$$f_{\varepsilon_j}(t) \geq (1-\delta)\varepsilon_j \phi(t/\varepsilon_j) \tag{5.3}$$

whenever $0 \leq t \leq t_\delta$ and $j \geq j_\delta$. Define

$$I_j' = \{\alpha \in I_j : \ \delta_j^\alpha \geq t_\delta\}, \qquad I_j'' = \{\alpha \in I_j : \ \delta_j^\alpha < t_\delta\}.$$

Then

$$F_{\varepsilon_j}(\tilde{u}_j) \geq 2\left(\sum_{\alpha \in I_j'} f_{\varepsilon_j}(\delta_j^\alpha) + \sum_{\alpha \in I_j''} f_{\varepsilon_j}(\delta_j^\alpha)\right).$$

As we noted in Remark 3.1, the function $\tilde{f}_\varepsilon : t \mapsto f_\varepsilon(t + a_\varepsilon) : [0, +\infty) \to [0, +\infty)$ is subadditive; hence

$$\begin{aligned}\sum_{\alpha \in I_j'} f_{\varepsilon_j}(\delta_j^\alpha) &= \sum_{\alpha \in I_j'} \tilde{f}_{\varepsilon_j}(\delta_j^\alpha - a_{\varepsilon_j}) \\ &\geq \tilde{f}_{\varepsilon_j}\Big(\sum_{\alpha \in I_j'} (\delta_j^\alpha - a_{\varepsilon_j})\Big) = f_{\varepsilon_j}\Big(\sum_{\alpha \in I_j'} \delta_j^\alpha - a_{\varepsilon_j}(\# I_j' - 1)\Big).\end{aligned}$$

By (5.2) the lower bound $\delta_j^\alpha \geq t_\delta$ for every $\alpha \in I'_j$ implies that $\# I'_j$ is bounded independently of j. Therefore there exists $C_\delta > 0$ such that for every $j \in \mathbb{N}$

$$\sum_{\alpha \in I'_j} f_{\varepsilon_j}(\delta_j^\alpha) \geq f_{\varepsilon_j}\left(\sum_{\alpha \in I'_j} \delta_j^\alpha - C_\delta a_{\varepsilon_j}\right).$$

Let $\# I''_j = N_j$; then, by the convexity of ϕ, and (5.3)

$$2\sum_{\alpha \in I''_j} f_{\varepsilon_j}(\delta_j^\alpha) \geq 2(1-\delta)\varepsilon_j N_j \sum_{\alpha \in I''_j} \frac{1}{N_j}\phi\left(\frac{\delta_j^\alpha}{\varepsilon_j}\right) \geq (1-\delta)\phi\left(\frac{\sum_{\alpha \in I''_j}\delta_j^\alpha}{N_j \varepsilon_j}\right) 2N_j\varepsilon_j.$$

By (5.2) we can suppose that, up to a subsequence:

$$\sum_{\alpha \in I'_j} \delta_j^\alpha \to s_1 \quad \text{and} \quad \sum_{\alpha \in I''_j} \delta_j^\alpha \to s_2,$$

with $s_1 + s_2 = \frac{1}{2}\big(u^+(\bar{x}) - u^-(\bar{x})\big)$. Then, by the uniform convergence of f_{ε_j} to a concave function f, (see assumption *(A3)*) and taking into account the linear growth of ϕ (notice that $N_j\varepsilon_j \to 0$) we have:

$$\liminf_{j\to+\infty} f_{\varepsilon_j}\left(\sum_{\alpha \in I'_j} \delta_j^\alpha - C_\delta a_{\varepsilon_j}\right) \geq f(s_1),$$

$$\liminf_{j\to+\infty}(1-\delta)\phi\left(\frac{\sum_{\alpha \in I''_j}\delta_j^\alpha}{N_j\varepsilon_j}\right) N_j\varepsilon_j \geq (1-\delta)c_0 s_2.$$

Therefore:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(\tilde{u}_j) \geq 2(f(s_1) + (1-\delta)c_0 s_2) \geq 2[f(s_1) + (1-\delta)f(s_2)] \geq$$

$$\geq 2(1-\delta)f\left(\frac{|u^+(\bar{x}) - u^-(\bar{x})|}{2}\right),$$

where, in the last inequality, we have used the subadditivity of f. Finally, letting $\delta \to 0$, by (5.1), we have:

$$F'(u, A) \geq \liminf_{j\to+\infty} F_{\varepsilon_j}(\tilde{u}_j) \geq 2f\left(\frac{|u^+(\bar{x}) - u^-(\bar{x})|}{2}\right).$$

Step 2. For any $N \in \mathbb{N}$ consider a finite subset $\{x_1, \dots, x_N\}$ of $S_u \cap A$ and let $I_1, \dots, I_N$ be pairwise disjoint intervals contained in A an such that $x_i \in I_i \subseteq (a, b)$ for every $i = 1, \dots, N$. By *Step 1* and the superadditivity of $F'(u, \cdot)$ (see Remark 4.6) it turns out that:

$$F'(u, A) \geq F'\left(u, \bigcup_{i=1}^N I_i\right) \geq \sum_{i=1}^N F'(u, I_i) \geq 2\sum_{i=1}^N f\left(\frac{1}{2}|u^+(x_i) - u^-(x_i)|\right).$$

Since N is arbitrary, we conclude. ∎

6 Estimate from below of the lower Γ-limit

We are now in a position to collect the previous partial results and prove that the Γ-lower limit of (F_ε) is estimated from below by the functional $\mathcal{F}$ defined in Theorem 3.4. Taking Corollary 4.4 into account, it is sufficient to prove the folling result, which includes what stated in Remark 3.5.

Theorem 6.1 *For every $u \in GBV(a, b)$ and A open subset of (a, b) the inequality $F'(u, A) \geq \mathcal{F}(u, A)$ holds.*

Proof. *Step 1.* Let us first consider the case $u \in BV(a,b)$. By Propositions 4.8 and 5.2 we get

$$F'(u,A) \geq \int_A \phi(|u'(x)|)dx, \qquad F'(u,A) \geq c_0|D^c u|(A)$$

and

$$F'(u,A) \geq 2 \sum_{x \in S_u \cap A} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right)$$

for every A open subset of (a,b). Let λ be the Borel measure defined by

$$\lambda(B) = \mathcal{L}^1(B) + \#(S_u \cap B) + c_0|D^c u|(B)$$

for every Borel subset B of (a,b). Let E be a Borel subset of $(a,b) \setminus S_u$ with $|E| = 0$ and on which $|D^c u|$ is concentrated, i.e. $|D^c u|\big((a,b) \setminus E\big) = 0$. Then

$$\mu(A) := F'(u,A) \geq \int_A \psi_i(x) d\lambda$$

for $i = 1,2,3$, and for every A open subset of (a,b), where

$$\psi_1(x) = \begin{cases} \phi(|u'(x)|) & x \in (a,b) \setminus (S_u \cup E) \\ 0 & x \in S_u \\ 0 & x \in E \end{cases}$$

$$\psi_2(x) = \begin{cases} 0 & x \in (a,b) \setminus (S_u \cup E) \\ 2f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) & x \in S_u \\ 0 & x \in E \end{cases}$$

$$\psi_3(x) = \begin{cases} 0 & x \in (a,b) \setminus (S_u \cup E) \\ 0 & x \in S_u \\ c_0 & x \in E\ . \end{cases}$$

Obviously

$$\psi(x) := \sup_i \psi_i(x) = \begin{cases} \phi(|u'(x)|) & x \in (a,b) \setminus (S_u \cup E) \\ 2f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) & x \in S_u \\ c_0 & x \in E \end{cases}$$

and then, from Lemma 2.5 and Remark 4.6 (superadditivity of $F'(u,\cdot)$):

$$\mu(A) \geq \int_A \sup_i \psi_i(x) d\lambda = \int_A \psi(x) d\lambda = \mathcal{F}(u,A)$$

for every open subset A of (a,b).

Step 2. Let $u \in GBV(a,b)$; we can find a positive infinitesimal sequence (ε_j) and a sequence (u_j) in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and such that $F'(u,A) = \liminf\limits_{j \to +\infty} F_{\varepsilon_j}(u_j, A)$. Define

$$u_j^T = (-T) \vee u_j \wedge T, \qquad u^T = (-T) \vee u \wedge T.$$

Since $u_j^T \to u^T$ in $L^1(a,b)$, by *Step 1* we have

$$F'(u,A) = \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j, A) \geq \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j^T, A) \geq$$

$$\geq \int_A \phi(|(u^T)'(x)|dx + 2 \sum_{x \in S_{u^T} \cap A} f\left(\frac{1}{2}|(u^T)^+(x) - (u^T)^-(x)|\right) + c_0|D^c u^T|(A).$$

We conclude by taking the limit as $j \to +\infty$ and recalling that (see the definition of the space GBV in § 2)

$$(u^T)' = \begin{cases} u' \text{ a.e. on } & \{|u| \leq T\} \\ 0 \text{ a.e. on } & \{|u| > T\} \end{cases}, \qquad |D^c(u^T)|(A) \to |D^c u|(A)$$

$$(u^T)^{\pm}(x) \to u^{\pm}(x) \quad \text{for every } x \in S_u = \bigcup_{T \geq 0} S_{u^T}\,.$$

■

7 Estimate from above of the upper Γ-limit

In this section we conclude the proof of Theorem 3.4, proving the estimate from above of the upper Γ-limit of (F_ε) (Theorem 7.3).

Lemma 7.1 *For every $u \in W^{1,1}(a,b)$ we have*

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(u) = \int_a^b \phi(|u'(x)|)dx\,.$$

Proof. Extend u as a function in $W^{1,1}_{\rm loc}(\mathbb{R})$. For every open subset A of $\mathbb{R}$ let:

$$\tilde F(u,A) = \frac{1}{\varepsilon}\int_A f_\varepsilon\left(\frac{1}{2}\int_{x-\varepsilon}^{x+\varepsilon}|u'(y)|dy\right)dx\,.$$

Fix $\sigma > 0$; then, whenever $\varepsilon < \sigma$,

$$F_\varepsilon(u) = \tilde F\big(u,(a+\sigma,b-\sigma)\big) + R_\varepsilon(\sigma)\,, \tag{7.1}$$

where:

$$R_\varepsilon(\sigma) = \frac{1}{\varepsilon}\int_{A_\sigma} f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)}|u'(y)|\,dy\right)dx\,,$$

with $A_\sigma = (a,a+\sigma)\cup(b-\sigma,b)$. Since $|(x-\varepsilon,x+\varepsilon)\cap(a,b)| \geq \varepsilon$ if $x \in (a,b)$, by Remark 3.3 we have:

$$R_\varepsilon(\sigma) \leq \frac{C}{\varepsilon}\int_{A_\sigma}\int_{(x-\varepsilon,x+\varepsilon)\cap(a,b)}|u'(y)|\,dy\,dx \leq \frac{C}{\varepsilon}\int_{A_\sigma}\int_{x-\varepsilon}^{x+\varepsilon}|u'(y)|\,dy\,dx\,.$$

Consider the functions

$$g_\varepsilon(x) = \fint_{(x-\varepsilon,x+\varepsilon)}|u'(y)|dy = \int_{\mathbb{R}}\varphi_\varepsilon(x-y)|u'(y)|dy\,,$$

where $\varphi_\varepsilon = \frac{1}{2\varepsilon}\chi_{(-\varepsilon,\varepsilon)}$. Then

$$\begin{aligned} &\tilde F_\varepsilon\big(u,(a+\sigma,b-\sigma)\big) = \frac{1}{\varepsilon}\int_{a+\sigma}^{b-\sigma} f_\varepsilon\big(\varepsilon g_\varepsilon(x)\big)\,dx\,, \\ &R_\varepsilon(\sigma) \leq 2C\int_{A_\sigma} g_\varepsilon(x)\,dx\,. \end{aligned} \tag{7.2}$$

Since $|u'| \in L^1(a,b)$, the sequence (g_ε) converges to $|u'|$ in $L^1(a,b)$ and a.e. in (a,b); by (3.2), and since f_ε is non-decreasing, it turns out that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}f_\varepsilon(\varepsilon g_\varepsilon(x)) = \phi(|u'(x)|)$$

for a.e. $x \in (a,b)$. Then, by Remark 3.3, we can pass to the limit under the integrals in (7.2); thus

$$\begin{aligned} &\lim_{\varepsilon\to 0}\tilde F_\varepsilon\big(u,(a+\sigma,b-\sigma)\big) = \int_{a+\sigma}^{b-\sigma}\phi(|u'(x)|)dx\,, \\ &\limsup_{\varepsilon\to 0} R_\varepsilon(\sigma) \leq 2C\int_{A_\sigma}|u'(y)|\,dy\,. \end{aligned} \tag{7.3}$$

Thanks to (7.1), by the arbitrariness of σ we conclude. ■

Proposition 7.2 *For every $u \in SBV(a,b)$ we have $F''(u) \le \mathcal{F}(u)$.*

Proof. Every $u \in SBV(a,b)$ is the L^1-limit of a sequence (u_h) such that

$$\# S_{u_h} < +\infty, \qquad \mathcal{F}(u_h) \to \mathcal{F}(u).$$

Indeed, if $S_u = \{x_i : \ i \in \mathbb{N}\}$, define, e.g.

$$u_h(x) = \int_a^x u'(t)\,dt + \sum_{x_i < x;\ i=1,\dots,h} (u^+(x_i) - u^-(x_i)).$$

Therefore, by the semicontinuity of F'', we can prove the stated inequality only in the case that $\# S_u$ is finite. We shall even suppose that S_u consists of a single point x_0, since the argument applied will be easily generalized to a finite number of jump points.

Let (ε_j) be a positive infinitesimal sequence. For any x, let $I_j(x) = (x - \varepsilon_j, x + \varepsilon_j)$. By Remark 3.9 we have (ε_j sufficiently small):

$$\overline{F}_{\varepsilon_j}(u) \le F_{\varepsilon_j}\big(u, (a, x_0 - \varepsilon_j)\big) + F_{\varepsilon_j}\big(u, (x_0 + \varepsilon_j, b)\big) + R_j, \tag{7.4}$$

where

$$R_j = \frac{1}{\varepsilon_j} \int_{I_j(x_0)} f_{\varepsilon_j}\Big(\frac{1}{2}|Du|(I_j(x))\Big)\,dx.$$

Let u_- be the extension of $u_{|(a,x_0)}$ to the whole (a,b) with the constant value $u^-(x_0)$; an analogous definition for u_+ with the value $u^+(x_0)$. Then:

$$F_{\varepsilon_j}\big(u, (a, x_0 - \varepsilon_j)\big) \le F_{\varepsilon_j}(u_-)\,, \quad F_{\varepsilon_j}\big(u, (x_0 + \varepsilon_j, b)\big) \le F_{\varepsilon_j}(u_+)\,.$$

Apply the previous lemma to u_- and u_+; then:

$$\limsup_{j \to +\infty} \big[F_{\varepsilon_j}\big(u, (a, x_0 - \varepsilon_j)\big) + F_{\varepsilon_j}\big(u, (x_0 + \varepsilon_j, b)\big)\big] \le \int_a^b \phi(|u'(x)|)\,dx\,. \tag{7.5}$$

As to the term R_j, notice that for $x \in I_j(x_0)$

$$\begin{aligned} |Du|(I_j(x)) &= \int_{I_j(x)} |u'(t)|\,dt + |D^s u|(I_j(x)) \\ &\le \int_{x_0 - 2\varepsilon_j}^{x_0 + 2\varepsilon_j} |u'(t)|\,dt + |u^+(x_0) - u^-(x_0)|; \end{aligned}$$

then for any $\sigma > 0$ there exists $j_\sigma \in \mathbb{N}$ such that for every $j \ge j_\sigma$ and $x \in I_j(x_0)$

$$|Du|\big(I_j(x)\big) \le |u^+(x_0) - u^-(x_0)| + \sigma.$$

By *(A3)* and the arbitrariness of σ we immediately conclude that

$$\limsup_{j \to +\infty} R_j \le 2f\Big(\frac{1}{2}|u^+(x_0) - u^-(x_0)|\Big).$$

This, together with (7.4) and (7.5), yields (the upper Γ-limit of a sequence coincides, as recalled in § 2, with the upper Γ-limit of the sequence of the corresponding relaxed functionals):

$$F''(u) \le \limsup_{j \to +\infty} \overline{F}_{\varepsilon_j}(u) \le \int_a^b \phi(|u'(x)|)\,dx + 2f\Big(\frac{1}{2}|u^+(x_0) - u^-(x_0)|\Big).$$

■

Theorem 7.3 *For every $u \in GBV(a,b)$ we have $F''(u) \leq \mathcal{F}(u)$.*

Proof. By the lower semicontinuity of F'' and by the relaxation Theorem 2.2 we have $F''(u) \leq \mathcal{F}(u)$ for $u \in BV(a,b)$. Now, it sufficies to pass to the limit as $T \to +\infty$ in $F''(u^T) \leq \mathcal{F}(u^T)$. ■

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