Singular kernels, multiscale decomposition of microstructure, and dislocation models

February 17, 2010

Sergio Conti[1], Adriana Garroni[2], and Stefan Müller[1,3]

[1] *Institut für Angewandte Mathematik, Universität Bonn Endenicher Allee 60, 53115 Bonn, Germany*

[2] *Dipartimento di Matematica, Sapienza, Università di Roma P.le A. Moro 2, 00185 Roma, Italy*

[3] *Hausdorff Center for Mathematics, Universität Bonn Endenicher Allee 60, 53115 Bonn, Germany*

We consider a model for dislocations in crystals introduced by Koslowski, Cuitiño and Ortiz, which includes elastic interactions via a singular kernel behaving as the $H^{1/2}$ norm of the slip. We obtain a sharp-interface limit of the model within the framework of Γ-convergence. From an analytical point of view, our functional is a vector-valued generalization of the one studied by Alberti, Bouchitté and Seppecher to which their rearrangement argument no longer applies. Instead we show that the microstructure must be approximately one-dimensional on most length scales and exploit this property to derive a sharp lower bound.

1 Introduction

1.1 The result

We consider the functional

$$E_\varepsilon^*[u,\Omega] = \int_{\Omega\times\Omega} \sum_{i,j=1}^N \Gamma_{ij}(x-y)\,[u_i(x)-u_i(y)]\,[u_j(x)-u_j(y)]\,dxdy + \frac{1}{\varepsilon}\int_\Omega \operatorname{dist}^2(u(x),\mathbb{Z}^N)dx \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ and Γ is a matrix-valued kernel scaling as $|x-y|^{-3}$, i.e., the first term is bounded from above and below by a multiple of the $H^{1/2}$ norm.

This functional arises in the study of phase field models for dislocations (see Section 1.2 below). Its main feature is that it contains two competing terms:

a nonconvex term which favours integer values of the vector-valued phase field u, and a regularizing term. This is an example of a large class of problems which share this structure, the classical example being the well-known Cahn-Hilliard model from the gradient-theory of fluid-fluid phase transitions, which contains a two-well potential depending on a scalar phase field, and a local regularization given by the Dirichlet integral. The analysis of the asymptotic behavior in terms of Γ-convergence for this functional goes back to Modica and Mortola [21], see also [20, 23]. Generalizations to multiwell problems, to vector-valued problems, and to anisotropic regularizations have been studied by several authors [9, 12, 15, 10]. All these problems give rise in the limit to a sharp-interface model characterized by a line-tension energy density. The local character of the regularization leads to a scaling property that permits to identify the line-tension energy density through a cell-problem formula.

The functional (1.1) is substantially more challenging since the regularization via the Dirichlet integral is replaced by a singular nonlocal term, which behaves as the $H^{1/2}$ norm. The (logarithmic) failure of the embedding of $H^{1/2}$ into continuous functions reflects the fact that all length scales play a role and that the appropriate rescaling is logarithmic. This eliminates the possibility to select one dominant length scale and to focus on a cell problem on that scale. An additional difficulty lies in the fact that (1.1) is a vectorial problem, anisotropic, and that the lower-order term has infinitely many minima.

In the scalar, isotropic case, after the mentioned logarithmic rescaling, the functional (1.1) reduces to

$$\frac{1}{\ln(1/\varepsilon)}\left[\int_{\Omega\times\Omega}\frac{1}{|x-y|^{n+1}}\,|u(x)-u(y)|^2\,dxdy+\frac{1}{\varepsilon}\int_\Omega W(u(x))dx\right], \qquad (1.2)$$

where $W:\mathbb{R}\to[0,\infty)$ is a multiwell potential (and $\Omega\subset\mathbb{R}^n$). A problem of this kind was first studied by Alberti, Bouchitté and Seppecher, for the case of a two-well potential [3, 4]. With this scaling, they proved a compactness result which shows that the domain of the limiting functional is $BV(\Omega;\{W=0\})$. Further, they proved Γ-convergence to a sharp-interface limit. The crucial idea is that even though (1.2) is a nonlocal functional, rearrangement can be used very efficiently. Indeed even though the problem is nonlocal they show by rearrangement that optimal interface profiles are one-dimensional. In particular, the leading-order part of the energy arises from the nonlocal interaction of the area where u is close to one minimum of W with the area where u is close to the other one. The criticality of the singular kernel implies that all distances contribute, and therefore that the overall interaction is logarithmic in the distance of the two sets. For the same reason, the limiting energy does not depend on the precise structure of the profile between the two sets.

The case of infinitely many wells and anisotropic kernel was treated by two of us in [17] (see also [16]). The compactness is more subtle due to the non-coerciveness of the multiwell potential $\mathrm{dist}^2(u, \mathbb{Z})$. The phenomenology is similar, and in particular optimal interface profiles remain one-dimensional, and anisotropy gives rise to an anisotropic line-tension energy of the form

$$\int_{J_u} \gamma(\nu)|u^+ - u^-| \, d\mathcal{H}^1 \,, \qquad u \in BV(\Omega; \mathbb{Z}) \tag{1.3}$$

(in two spatial dimensions). Moreover, the line-tension energy density γ can be completely characterized in terms of the kernel Γ, i.e.

$$\gamma(\nu) = 2 \int_{\{x \cdot \nu = 1\}} \Gamma(x) \, d\mathcal{H}^1(x).$$

In the present case the earlier rearrangement arguments do not apply, since the phase field is vector-valued, and the nonlocal interaction is anisotropic (note, however, that for certain vector-valued problems rearrangement arguments can be used [5]). Nonetheless one can abstractly prove that a Γ-limit exists, and that it has the form

$$\int_{J_u} \gamma(\nu, u^+ - u^-) \, d\mathcal{H}^1 \,, \qquad u \in BV(\Omega; \mathbb{Z}^2) \,, \tag{1.4}$$

but one does not have any further information on the line-tension energy density γ [14, 13]. One can naively try to use the natural generalization of the formula derived in the scalar case (1.3), namely,

$$\gamma_0(\nu, s) = 2 \int_{\{x \cdot \nu = 1\}} s^T \Gamma(x) s \, d\mathcal{H}^1(x) \,. \tag{1.5}$$

However, this does, in general, not produce a lower semicontinuous functional [14, 13], whereas the Γ-limit must be lower semicontinuous. This in particular implies that interfaces are more complicated and produce microstructure. The natural question is whether the Γ-limit is characterized by the BV-relaxation of the 1D interfacial energy (1.5) (see Remark 2.2 below for details).

In this paper we assume that

$$\Gamma(z) = \frac{1}{|z|^3} \hat{\Gamma}\left(\frac{z}{|z|}\right) \,, \tag{1.6}$$

where $\hat{\Gamma} \in L^\infty(S^1; \mathbb{R}^{N \times N}_+)$ obeys, for some $c > 0$,

$$\frac{1}{c}|\xi|^2 \le \xi \cdot \hat{\Gamma}(z)\xi \le c|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^N, \, z \in S^1 \,, \tag{1.7}$$

and $\mathbb{R}^{N \times N}_+$ denote the positive definite, symmetric, $N \times N$ matrices.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and suppose that the kernel Γ satisfies (1.6) and (1.7). Then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \frac{1}{\ln(1/\varepsilon)} E_\varepsilon^* = E_0^{\mathrm{rel}} ,$$

where

$$E_0^{\mathrm{rel}}[u_0, \Omega] = \begin{cases} \displaystyle\int_{J_{u_0} \cap \Omega} \gamma_0^{\mathrm{rel}}(\nu, [u_0]) d\mathcal{H}^1 & \text{if } u_0 \in BV(\Omega; \mathbb{Z}^N) \\ \infty & \text{else.} \end{cases}$$

The surface energy γ_0^{rel} is the BV-relaxation of γ_0, as defined in (1.5) (see Remark 2.2 below for the definition of the BV-relaxation).

The corresponding compactness statement, namely, that sequences u_ε such that $E_\varepsilon^*[u_\varepsilon, \Omega]/\ln(1/\varepsilon)$ is bounded have a subsequence converging to a limit $u \in BV(\Omega; \mathbb{Z}^N)$, can be immediately derived from the scalar results in [16, 17] or from Proposition 4.1 below.

Let us briefly sketch the strategy of our proof. The difficulty is the proof of the lower bound. Due to the logarithmic behaviour, the problem does not have an intrinsic natural scale, and so the lower bound cannot be reduced to the study of an asymptotic cell problem formula. The new idea in dealing with this kind of singular kernels is to perform a dyadic decomposition of the kernel with a sequence of truncated kernels. Each term in this decomposition is then regular, and one could hope to use the ideas of Alberti-Bellettini for non local phase transition models with regular kernels [1, 2]. But there is another obstacle in order to implement this strategy. In principle each regular term in the decomposition could be optimized by very different structures and the choice of one of them could produce a gross underestimation. It is natural to conjecture that this does not happen, but this is not so easy to prove directly, and might depend on finer details.

We thus look for a more robust method which does not require such a detailed analysis of the optimal structures. Roughly speaking we exploit the fact that a BV function cannot have significant microstructure on all scales simultaneously. Since all scales participate roughly equally in the total energy the few potentially bad scales can be ignored in the limit (see Section 2 for a more detailed description of this idea). We focus here on dimension two in view of the physical model which motivated our work. The decomposition strategy, however, is not restricted to two dimensions. A related logarithmic decomposition strategy has also proved useful in the codimension-two context of vortices in Ginzburg-Landau models, see [11, 24, 22] and references therein.

1.2 Connection with a phase field model for dislocations

Functionals of the type under consideration arise in the study of phase field models for dislocations inspired by the Peierls-Nabarro model (see e.g. [18]). Dislocations are line defects in crystals that are responsible for plastic behaviour. They usually arise on special planes (the slip planes) that are determined by the crystalline structure, and can be seen as the discontinuities of a slip on this plane. Depending on the crystalline structure on each slip plane several slip directions (Burgers vectors) are possible, so that the slip can be represented as a vector-valued function whose components represent the slip along a given Burgers vector. The idea of the Peierls-Nabarro model, originally formulated for a one dimensional problem, is to express the free energy in terms of the slip u as follows

$$E_{\text{free}}[u] = E_{\text{elastic}}[u] + E_{\text{interfacial}}[u]\,,$$

where the first term represents the long-range elastic distortion due to the slip and the second term is a nonlinear interfacial potential that remembers the crystal lattice and penalizes slips that are not integer multiples of the Burgers vectors. The main interest of this model is the persistence of discrete features in a continuum setting. The reformulation of this model proposed by Koslowski-Cuitiño-Ortiz [18, 19] for the case of dislocations on a given slip plane, using N different slip systems determined by the Burgers vectors b_1, ..., b_N, considers slips

$$u_1 b_1 + ... + u_N b_N$$

where $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$. The interfacial energy favours values of u close to $\mathbb{Z}^N$. The nonlocal part is the bulk elastic energy, which is given by the integral over the three-dimensional set $\Omega \times \mathbb{R}$ of a quadratic form of the gradient of the displacement $U : \Omega \times \mathbb{R} \to \mathbb{R}^3$ induced by the slip u. Precisely, U minimizes the elastic energy $\int_{\Omega\times\mathbb{R}} \langle C\nabla U, \nabla U\rangle\, dx$ over all vector fields which jump by $\sum u_i b_i$ on $\Omega \times \{x_3 = 0\}$. In the case of isotropic materials this reduces to

$$\int_{\Omega\times\mathbb{R}} \frac{\mu}{2}|e(U)|^2 + \lambda|\text{tr}(e(U)|^2 dx\,,$$

where $e(U) = \frac{\nabla U + \nabla U^t}{2}$ is the linearized strain. Minimizing out U leads to a nonlocal functional of u of the kind of (1.1) (with some differences due to boundary effects, which do not influence the leading-order behavior, see [16, 17]).

The connection with this application on dislocations produces very interesting examples for the functional (1.1). In particular the kernel arising from isotropic elastic interaction can be explicitly computed (up to the boundary terms) for different sets of Burgers vectors. For instance in the case of a pair of orthogonal Burgers vectors (corresponding to square symmetry) the explicit computation shows that the matrix-valued kernel defined in (1.5) takes the form [14]

$$\gamma(\nu, s) = \frac{1}{4\pi(1-\tilde{\nu})} s \cdot \begin{pmatrix} 2 - 2\tilde{\nu}\sin^2\theta & \tilde{\nu}\sin 2\theta \\ \tilde{\nu}\sin 2\theta & 2 - 2\tilde{\nu}\cos^2\theta \end{pmatrix} s \,. \tag{1.8}$$

In this equation $\tilde{\nu} \in [-1, 1/2]$ is the materials' Poisson ratio, and θ characterizes the direction of the normal $\nu = (\cos\theta, \sin\theta)$ to the interface. Notice that the given quadratic form is positive definite but the off-diagonal entries are, for some values of $\tilde{\nu}$ and θ, nonzero.

Consider now for example an interface in direction $\nu = (\cos\theta, \sin\theta)$ between a region where u equals $u_0 = (0, 0)$ and one where u equals $u_1 = (1, 1)$. The energy per unit length is given by

$$\gamma(\nu, u_1 - u_0) = (u_1 - u_0) \cdot \hat{\gamma}\,(u_1 - u_0) = \hat{\gamma}_{11} + 2\hat{\gamma}_{12} + \hat{\gamma}_{22}\,, \tag{1.9}$$

where $\hat{\gamma}$ is the matrix representation of the quadratic form $\gamma(\nu, \cdot)$. If a thin layer where u takes the value $u_2 = (0, 1)$ is inserted in between (see Figure 1, middle panel), then the sum of the two interfaces has the energy

$$\gamma(\nu, u_2 - u_0) = (u_2 - u_0) \cdot \hat{\gamma}\,(u_2 - u_0) + (u_1 - u_2) \cdot \hat{\gamma}\,(u_1 - u_2) = \hat{\gamma}_{11} + \hat{\gamma}_{22}\,. \tag{1.10}$$

If γ takes the form (1.8), then one or the other is more convenient depending on the sign of $\tilde{\nu}\sin 2\theta$. It is therefore clear that the relaxation will choose for each direction the optimal decomposition of the total jump. Cacace and Garroni [14] have shown that for some interfaces a more complex relaxation takes place, and in particular that in some directions a smaller energy is achieved by inserting fine-scale oscillations in the interface (see Figure 1, right panel). The intermediate $(0, 1)$ layer is then inserted only in the part of the interface in which $\tilde{\nu}\sin 2\theta$ is positive. Their construction proves that the BV-relaxation of the surface energy obtained for one-dimensional interfaces is nontrivial. It is possible to prove that this oscillatory construction indeed gives the BV-relaxation for this case.

2 Outline of the proof

We consider, for $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$,

$$\begin{aligned} E_\varepsilon[u, \Omega] = \frac{1}{\ln 1/\varepsilon} \Bigg[& \int_{\Omega\times\Omega} \Gamma_{ij}(x - y)\,[u_i(x) - u_i(y)]\,[u_j(x) - u_j(y)]\,dxdy \\ & + \frac{1}{\varepsilon} \int_\Omega \mathrm{dist}^2(u(x), \mathbb{Z}^N)dx \Bigg] \end{aligned} \tag{2.1}$$

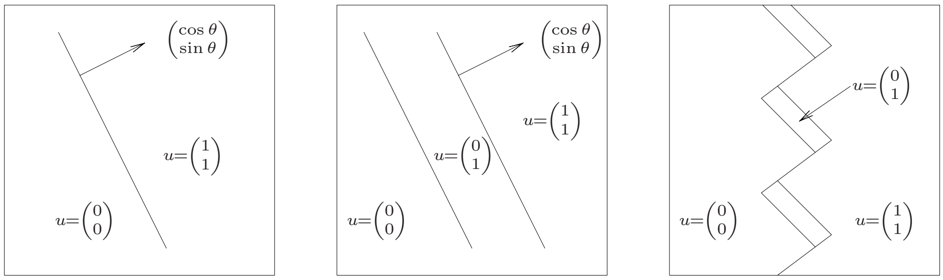

Figure 1: Left: Sketch of the interface as in (1.9). Middle: interface in (1.10). Right: oscillatory interfacial profile corresponding to a macroscopic vertical interfaces which combines the two options.

(sum over i, j from 1 to N is implicit). This differs from (1.1) in that the logarithmic factor is incorporated.

The upper bound follows directly from the abstract representation result [14, 13] and the analysis of one-dimensional interfaces (see Section 9).

The main point in the proof of Theorem 1.1 is to establish the following lower bound, whose proof will be concluded in Section 8.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and assume $u_0 \in BV(\Omega; \mathbb{Z}^N)$. Then for any sequences $\varepsilon_i \to 0$, $u_i \to u_0$ in $L^1(\Omega; \mathbb{R}^N)$ we have*

$$\liminf_{i\to\infty} E_{\varepsilon_i}[u_i, \Omega] \geq E_0^{\text{rel}}[u_0, \Omega]\,,$$

where

$$E_0^{\text{rel}}[u_0, \Omega] = \int_{J_{u_0}\cap\Omega} \gamma_0^{\text{rel}}(\nu, [u_0])d\mathcal{H}^1\,.$$

Remark 2.2. *The surface energy γ_0^{rel} is the BV-relaxation of γ_0 as defined in (1.5) and is given by*

$$\gamma_0^{\text{rel}}(\nu, s) = \min\left\{\int_{\overline{Q}_\nu\cap J_v} \gamma_0(\nu_v, [v])d\mathcal{H}^1 :\ v \in BV_{\text{loc}}(\mathbb{R}^2; \mathbb{Z}^N) \text{ and } v = u_\nu^s \text{ in } Q_\nu^c\right\}$$

where Q_ν is a unit square with two sides parallel to ν and $u_\nu^s = s\chi_{\{x\cdot\nu>0\}}$. The energy $E_0^{\text{rel}}[u, \Omega]$ is then the lower-semicontinuous envelope of

$$\int_{J_u\cap\Omega} \gamma_0(\nu, [u])d\mathcal{H}^1$$

with respect to the L^1 topology. For more details about the relaxation of functionals defined on partitions we refer to [6, 7, 8].

The main ideas for the proof of Theorem 2.1 are the following.

We first show that the kernel Γ can be rewritten by a dyadic superposition of truncated kernels and that given any function $u \in W^{1,1}$ and any given level of truncation, u can be substituted by a BV function with values in $\mathbb{Z}^N$ whose truncated energy is controlled by the energy of u. Then we show that one dimensional functions with values in $\mathbb{Z}^N$ are good test functions for computing the truncated energy, which in turn can be expressed in term of the line tension γ_0. In general functions with controlled energy do not satisfy the property of being one dimensional, but we can show that this is almost true locally if their total variation does not change much after mollification. We show this last property for a sequence of mollifications on suitably well separated scales of our initial sequence. The key idea is that a sequence with controlled energy can oscillate at many scales, but not at all scales. To illustrate this strategy we first apply it to the one dimensional case in Section 4.1.

In Section 3 we recall some elementary results for nonlocal terms with integrable kernel. In Section 4 we decompose the singular kernel into a sequence of integrable kernels and show that the sequence u_k in the lim inf can be essentially replaced by a sequence v_k with values in $\mathbb{Z}^n$ which is uniformly bounded in BV. In Section 5 we restrict ourselves to one-dimensional functions of the form $w_k(x) = f(x \cdot \nu)$ and show that for those the limit energy can be computed explicitly. Our general philosophy is that on most scales the given function v_k is close to a one-dimensional function. Thus in Section 6 we carefully estimate the energy of almost one-dimensional functions. In Sections 7 and 8 we combine those estimates with the idea that on most length scales a BV function is locally close to a one-dimensional function. To quantify the distance of the BV function from a locally one-dimensional function on a given length scale we use an iterative mollification on the different length scales, starting from the smallest one, and measure the defect in the total variation of the gradient (see Section 8 for the details).

3 Elementary estimates on the nonlocal term

Lemma 3.1. *Given $\Gamma' \in L^1(\mathbb{R}^2; \mathbb{R}_+^{N\times N})$ and $u \in L^2(\Omega; \mathbb{R}^N)$ we define*

$$p_{\Gamma',\Omega}(u) = \sum_{i,j=1}^{N} \int_{\Omega\times\Omega} \Gamma'_{ij}(x-y)\,[u_i(x) - u_i(y)]\,[u_j(x) - u_j(y)]\,dxdy\,. \tag{3.1}$$

Then:

(i) One has

$$0 \le p_{\Gamma',\Omega}(u) \le 4\|\Gamma'\|_{L^1(\mathbb{R}^2;\mathbb{R}^{N\times N})}\|u\|^2_{L^2(\Omega;\mathbb{R}^N)}\,. \tag{3.2}$$

(ii) The function $p^{1/2}_{\Gamma',\Omega}(\cdot)$ *is a seminorm, and in particular*

$$p_{\Gamma',\Omega}(u') \le (1+\eta)p_{\Gamma',\Omega}(u) + (1+\frac{1}{\eta})p_{\Gamma',\Omega}(u-u') \tag{3.3}$$

for all $\eta > 0$, u, $u' \in L^2(\Omega;\mathbb{R}^N)$.

(iii) The function p *is set-superadditive, in the sense that for any pair* A, $B \subset \mathbb{R}^2$, *with* $A \cap B = \emptyset$, *one has*

$$p_{\Gamma',A}(u) + p_{\Gamma',B}(u) \le p_{\Gamma',A\cup B}(u)\,. \tag{3.4}$$

In the following we write $\|\cdot\|_{L^p(\Omega)}$ or simply $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\Omega;\mathbb{R}^N)}$ or $\|\cdot\|_{L^p(\Omega;\mathbb{R}^{N\times N})}$, when no ambiguity arises.

Proof. The upper bound follows from

$$\|(\Gamma' * u)u\|_{L^1} \le \|\Gamma' * u\|_{L^2}\|u\|_{L^2} \le \|\Gamma'\|_{L^1}\|u\|^2_{L^2}\,.$$

The lower bound follows from the fact that $\Gamma'(z)$ is a positive definite matrix.

Since p is a positive semidefinite, continuous quadratic form, its square root is a seminorm.

Finally, observe that $(A\cup B)\times(A\cup B) = (A\times A)\cup(B\times B)\cup(A\times B)\cup(B\times A)$, hence we only have to show that the contributions of the last two terms are nonnegative. Let $\xi^{xy} = u(x) - u(y)$. Then

$$\begin{aligned}\sum_{ij}\int_{A\times B}&\Gamma_{ij}(x-y)\,[u_i(x)-u_i(y)]\,[u_j(x)-u_j(y)]\,dxdy\\ &=\sum_{ij}\int_{A\times B}\Gamma_{ij}(x-y)\xi^{xy}_i\xi^{xy}_j \ge 0\,,\end{aligned}$$

since $\Gamma(x-y) \in \mathbb{R}^{N\times N}_+$ for any x, y. Since we may exchange A and B, this concludes the proof. □

4 Dyadic decomposition and compactness

In this section we show that we can represent the singular kernel with a superposition of truncated kernels for which the regular phase field u can be substituted with a BV function.

Let $\phi(x) = x^{-3}$, $\phi : (0, \infty) \to \mathbb{R}$. We consider the following dyadic decomposition of ϕ. For $k \in \mathbb{N}$ set

$$\phi_k(x) = \begin{cases} 2^{3(k+1)} - 2^{3k} & \text{if } 0 < x \le 2^{-k-1} \\ x^{-3} - 2^{3k} & \text{if } 2^{-k-1} < x \le 2^{-k} \\ 0 & \text{if } x > 2^{-k} \,. \end{cases}$$

Further, we set $\phi_{-1}(x) = 1$ for $x < 1$, and $\phi_{-1}(x) = x^{-3}$ otherwise. A simple check shows that $\phi = \sum_{k=-1}^{\infty} \phi_k$, and each ϕ_k is continuous, nonnegative, and for $k \ge 0$ the function ϕ_k is supported on $\overline{B}_{2^{-k}}$. We denote by $\Gamma_k(z) = \phi_k(|z|)\hat{\Gamma}(z/|z|)$ the "layer" kernel, and by $\Gamma_{0,k} = \sum_{i=0}^{k} \Gamma_i$ the truncated kernel (we shall not need to use the function ϕ_{-1} explicitly). For later reference we remark that

$$\|\Gamma_k\|_{L^1(\mathbb{R}^2)} = c2^k \,, \qquad \text{for all } k \in \mathbb{N} \,. \tag{4.1}$$

We shall replace a function with good energy by a BV function which takes integer values and which has good truncated energy. The truncated energy is defined by

$$E_\varepsilon^k[v, \Omega] = \frac{1}{\ln 1/\varepsilon} p_{\Gamma_{0,k},\Omega}(v) \,.$$

Proposition 4.1. *Assume that $\hat{\Gamma}$ is strictly positive definite, i.e., that there is $c > 0$ such that*

$$\xi \cdot \hat{\Gamma}(z)\xi \ge c|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^N,\ z \in S^1 \,. \tag{4.2}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and $\omega \subset\subset \Omega$, $\delta \in (0, 1/2)$. Then for every sufficiently small $\varepsilon > 0$ (on a scale set by δ and $\operatorname{dist}(\omega, \partial\Omega)$) and every $u \in L^2(\Omega; \mathbb{R}^N)$ there are $k \in \mathbb{N}$ and $v \in BV(\omega; \mathbb{Z}^N)$ such that

$$E_\varepsilon^k[v, \omega] \le E_\varepsilon[u, \Omega] \left(1 + \frac{C}{\delta(\ln 1/\varepsilon)^{1/2}}\right) \,,$$

$$|Dv|(\omega) \le \frac{C}{\delta} E_\varepsilon[u, \Omega] \,, \tag{4.3}$$

and

$$\varepsilon^{1-\delta/2} \le 2^{-k} \le \varepsilon^{1-\delta} \,.$$

Furthermore,

$$\|u - v\|_{L^1(\omega;\mathbb{R}^N)} \le C2^{-k/2}(E_\varepsilon[u, \Omega])^{1/2} \,.$$

Proof. We consider, for $k \in \mathbb{N}$, the quantities

$$p_{\Gamma_k,\Omega}(u)\,.$$

Clearly $E_\varepsilon[u,\Omega] \geq \frac{1}{\ln 1/\varepsilon}\sum_{k=0}^\infty p_{\Gamma_k,\Omega}(u)$. We can assume without loss of generality that ε is sufficiently small that the number of $k \in \mathbb{N}$ such that $\varepsilon^{1-\delta/2} \leq 2^{-k} \leq \varepsilon^{1-\delta}$ is at least $\delta \ln(1/\varepsilon)/(4\ln 2)$. Therefore there is one k such that

$$\varepsilon^{1-\delta/2} \leq 2^{-k} \leq \varepsilon^{1-\delta} \text{ and } p_{\Gamma_k,\Omega}(u) \leq \frac{C}{\delta} E_\varepsilon[u,\Omega]\,. \tag{4.4}$$

Let $\alpha = 2^{-k-4}$, and for $z \in \alpha\mathbb{Z}^2$ consider $q_z = z+(0,\alpha)^2$ and $Q_z = z+(-\alpha, 2\alpha)^2$. Let $Z = \{z \in \alpha\mathbb{Z}^2 : Q_z \subset \Omega\}$. We observe that ω is covered by the disjoint union of the small squares $\{q_z\}_{z\in Z}$ (up to a null set) and that the large squares Q_z have finite overlap and are contained in Ω.

We claim that for any $z \in Z$ there is $v_z \in \mathbb{Z}^N$ such that

$$\int_{Q_z} |u-v_z|^2 dx \leq c \int_{Q_z} \mathrm{dist}^2(u,\mathbb{Z}^N)\, dx + c2^{-k} p_{\Gamma_k,Q_z}(u)\,, \tag{4.5}$$

for some constant c depending only on $\hat\Gamma$ and N.

Since $\mathrm{diam}(Q_z) < 2^{-k-1}$, for any $x, y \in Q_z$ we have $\phi_k(x-y) = 2^{3(k+1)} - 2^{3k}$. Recalling (4.2) we obtain

$$p_{\Gamma_k,Q_z}(u) \geq c2^{3k} \int_{Q_z}\int_{Q_z} |u(x)-u(y)|^2 dxdy \geq c2^k \int_{Q_z} |u(x)-\bar u|^2 dx \tag{4.6}$$

where $\bar u$ is the average of u over Q_z. Fix $w : Q_z \to \mathbb{Z}^N$ measurable and such that $\mathrm{dist}(u,\mathbb{Z}^N) = |u-w|$, and let $\bar w$ be its average. We estimate

$$\int_{Q_z} |w-\bar w|^2 dx \leq 3\int_{Q_z} |w-u|^2 dx + 3\int_{Q_z} |u-\bar u|^2 dx + 3\int_{Q_z} |\bar u - \bar w|^2 dx\,.$$

The last term is controlled by the first term in the right-hand side, which in turn is controlled by the integral of the squared distance of u from $\mathbb{Z}^N$. The second term in the right-hand side is controlled by (4.6). Therefore

$$\int_{Q_z} |w-\bar w|^2 dx \leq 6\int_{Q_z} \mathrm{dist}^2(u,\mathbb{Z}^N)\, dx + c2^{-k} p_{\Gamma_k,Q_z}(u)\,.$$

Recalling that $w \in \mathbb{Z}^N$, we get

$$\mathcal{L}^2(Q_z)\mathrm{dist}^2(\bar w,\mathbb{Z}^N) \leq \int_{Q_z} |w-\bar w|^2 dx \leq 6\int_{Q_z} \mathrm{dist}^2(u,\mathbb{Z}^N) + c2^{-k} p_{\Gamma_k,Q_z}(u)\,.$$

We pick $v_z \in \mathbb{Z}^N$ such that $|\bar{w} - v_z| = \mathrm{dist}(\bar{w}, \mathbb{Z}^N)$, and obtain

$$\int_{Q_z} |u - v_z|^2 dx \leq 3 \int_{Q_z} |u - \bar{u}|^2 dx + 3 \int_{Q_z} |\bar{u} - \bar{w}|^2 dx + 3 \int_{Q_z} |\bar{w} - v_z|^2 dx\,.$$

Collecting the previous estimates proves (4.5).

Repeating the same procedure for all squares we obtain a function $v \in L^\infty(\omega; \mathbb{Z}^N)$, defined by $v = v_z$ on q_z, such that

$$\|u - v\|^2_{L^2(\omega)} \leq \sum_{z \in Z} \|u - v\|^2_{L^2(Q_z)} \leq c2^{-k} p_{\Gamma_k, \Omega}(u) + c \int_\Omega \mathrm{dist}^2(u, \mathbb{Z}^N) dx\,. \quad (4.7)$$

Here we used the superadditivity of $p_{\Gamma_k,\Omega}$ and the fact that the Q_z have finite overlap.

We now turn to the estimate of the measure $|Dv|$. This is obviously concentrated on the union of the boundaries of the squares. Consider two neighbouring squares q_z and $q_{z'}$ (so that they share an edge, i.e., $z \neq z'$ and $\mathcal{H}^1(\partial q_z \cap \partial q_{z'}) > 0$). Then q_z is contained in both Q_z and $Q_{z'}$, and analogously $q_{z'}$. The key idea is that if u is approximately constant on each of the larger cubes, then the jump must be zero (approximately constant on one of the large cubes does not suffice, with the present definition of v_z – consider, for example, $u = 0$ on Q_z and $u = 100$ on $\mathbb{R}^2 \setminus Q_z$). Precisely,

$$\begin{aligned}
\mathcal{L}^2(q_z)|v_z - v_{z'}|^2 \leq& 2 \int_{q_z} |u - v_z|^2 + |u - v_{z'}|^2 dx \\
\leq& 2 \int_{Q_z} |u - v_z|^2 dx + 2 \int_{Q_{z'}} |u - v_{z'}|^2 dx \\
\leq& c \int_{Q_z \cup Q_{z'}} \mathrm{dist}^2(u, \mathbb{Z}^N)\, dx + c2^{-k} p_{\Gamma_k, Q_z \cup Q_{z'}}(u)\,,
\end{aligned}$$

where in the last step we used (4.5).

Recalling that v is integer-valued we obtain, for the same squares,

$$\begin{aligned}
|Dv|(\partial q_z \cap \partial q_{z'}) = 2^{-k}|v_z - v_{z'}| &\leq 2^k \mathcal{L}^2(q_z)|v_z - v_{z'}|^2 \\
&\leq c2^k \int_{Q_z \cup Q_{z'}} \mathrm{dist}^2(u, \mathbb{Z}^N)\, dx + c p_{\Gamma_k, Q_z \cup Q_{z'}}(u)\,.
\end{aligned}$$

Summing over all squares gives

$$|Dv|(\omega) \leq c p_{\Gamma_k, \Omega}(u) + c2^k \int_\Omega \mathrm{dist}^2(u, \mathbb{Z}^N) dx\,.$$

From (4.4) we obtain, for sufficiently small ε,

$$2^k \leq \frac{1}{\varepsilon^{1-(\delta/2)}} \leq \frac{1}{\varepsilon \ln 1/\varepsilon}\,, \tag{4.8}$$

and therefore

$$|Dv|(\omega) \leq \frac{C}{\delta} E_\varepsilon[u]\,. \tag{4.9}$$

This concludes the proof of (4.3).

We compute, recalling (4.7) and Lemma 3.1, for all $\eta \in (0, 1/2)$

$$\begin{aligned} p_{\Gamma_{0,k},\omega}(v) &\leq (1+\eta) p_{\Gamma_{0,k},\omega}(u) + \left(1 + \frac{1}{\eta}\right) p_{\Gamma_{0,k},\omega}(u-v) \\ &\leq (1+\eta) p_{\Gamma_{0,k},\omega}(u) + \frac{2}{\eta} \|\Gamma_{0,k}\|_{L^1(\mathbb{R}^2)} \|u-v\|^2_{L^2(\omega)}\,. \end{aligned}$$

Since $\|\Gamma_{0,k}\|_{L^1(\mathbb{R}^2)} \leq c2^k$, using (4.7) and arguing as done for (4.9) we obtain

$$\begin{aligned} p_{\Gamma_{0,k},\omega}(v) &\leq (1+\eta) p_{\Gamma_{0,k},\omega}(u) + \frac{1}{\eta} c 2^k 2^{-k} p_{\Gamma_k,\Omega}(u) + \frac{1}{\eta} c 2^k \int_\Omega \mathrm{dist}^2(u, \mathbb{Z}^N)\, dx \\ &\leq (1+\eta)(\ln \frac{1}{\varepsilon}) E_\varepsilon[u,\Omega] + \frac{1}{\eta}\frac{c}{\delta} E_\varepsilon[u,\Omega]\,. \end{aligned}$$

Finally,

$$E^k_\varepsilon[v,\omega] = \frac{1}{\ln 1/\varepsilon} p_{\Gamma_{0,k},\omega}(v) \leq (1+\eta) E_\varepsilon[u,\Omega] + \frac{c}{\delta\eta \ln 1/\varepsilon} E_\varepsilon[u,\Omega]\,.$$

Taking $\eta = (\ln 1/\varepsilon)^{-1/2}$ gives

$$E^k_\varepsilon[v,\omega] \leq E_\varepsilon[u,\Omega] + \frac{c}{\delta (\ln 1/\varepsilon)^{1/2}} E_\varepsilon[u,\Omega]\,.$$

Finally, from (4.7) we have

$$\|u-v\|_{L^1(\omega)} \leq c 2^{-k/2} \left[p_{\Gamma_k,\Omega}(u) + c2^k \int_\Omega \mathrm{dist}^2(u,\mathbb{Z}^N) dx \right]^{1/2}\,.$$

Recalling (4.8) we conclude

$$\|u-v\|_{L^1(\omega)} \leq c 2^{-k/2} \left(E_\varepsilon[u,\Omega]\right)^{1/2}\,.$$

□

As consequence any given sequence converging in L^1 to a function in $BV(\Omega;\mathbb{Z}^N)$ can be substituted with a sequence in $BV(\Omega;\mathbb{Z}^N)$ for which we control the energy. More precisely we have the following proposition.

Proposition 4.2. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain, and assume $u_0\in BV(\Omega;\mathbb{Z}^N)$. Then for any $\delta\in(0,1/2)$, any sequences $\varepsilon_i\to 0$, $u_i\to u_0$ in $L^1(\Omega;\mathbb{R}^N)$ and any Lipschitz domain $\omega\subset\subset\Omega$ there is a sequence $v_k\in BV(\omega;\mathbb{Z}^N)$ such that $v_k\to u_0$ in $L^1(\omega;\mathbb{R}^N)$,*

$$\liminf_{k\to\infty}\frac{1}{k}\sum_{h=0}^{k}p_{\Gamma_h,\omega}(v_k)\le(1+2\delta)\ln 2\liminf_{i\to\infty}E_{\varepsilon_i}[u_i,\Omega]\,,$$

and

$$|Dv_k|(\omega)\le C_\delta(\liminf_{i\to\infty}E_{\varepsilon_i}[u_i,\Omega]+1)\,.$$

Proof. If the lim inf equals ∞ there is nothing to prove. By taking a subsequence we can assume that

$$E_{\varepsilon_i}[u_i,\Omega]\le\lim_{i\to\infty}E_{\varepsilon_i}[u_i,\Omega]+1$$

for all i. We apply Proposition 4.1 to each u_i, and obtain $k_{i,\delta}$ and $v_{i,\delta}$. The estimate on the total variation is immediate. From the condition on $k_{i,\delta}$ we obtain

$$-k_{i,\delta}\ln 2\le-(1-\delta)\ln\frac{1}{\varepsilon_i}$$

which implies

$$\frac{1}{k_{i,\delta}}\le(1+2\delta)\frac{\ln 2}{\ln\frac{1}{\varepsilon_i}}\,.$$

Therefore

$$\frac{1}{k_{i,\delta}}p_{\Gamma_{0,k_{i,\delta}},\omega}(v_{i,\delta})\le(1+2\delta)\ln 2\,E_{\varepsilon_i}[u_i,\Omega]\left(1+\frac{C_\delta}{(\ln 1/\varepsilon_i)^{1/2}}\right)\,,$$

which gives

$$\liminf_{i\to\infty}\frac{1}{k_{i,\delta}}p_{\Gamma_{0,k_{i,\delta}},\omega}(v_{i,\delta})\le(1+2\delta)\ln 2\lim_{i\to\infty}E_{\varepsilon_i}[u_i,\Omega]\,.$$

Taking a further subsequence we can assume the map $i\mapsto k_{i,\delta}$ to be nondecreasing. We set for every $K\in\mathbb{N}$

$$w_K=v_{j,\delta}\text{ where }j=\min\{i\in\mathbb{N}:k_{i,\delta}\ge K\}\,.\tag{4.10}$$

Then $w_K \to u$ in L^1, $|Dw_K|(\omega)$ still obeys the desired bound, and from the fact that $\frac{1}{k_{i,\delta}} p_{\Gamma_{0,k_{i,\delta}},\omega}(v_{i,\delta})$ is a subsequence of $\frac{1}{K} p_{\Gamma_0,\omega}(w_K)$ we get

$$\liminf_{K\to\infty} \frac{1}{K} \sum_{h=0}^{K} p_{\Gamma_h,\omega}(w_K) \le \liminf_{i\to\infty} \frac{1}{k_{i,\delta}} p_{\Gamma_{0,k_{i,\delta}},\omega}(v_{i,\delta})$$

and hence the thesis follows. $\square$

4.1 Digression: the one-dimensional case without rearrangement

We pause for a moment to illustrate how Proposition 4.2 can be used to obtain the lower bound without the use of rearrangement in the one-dimensional scalar case, i.e., for the functional (1.2) with $n = 1$, $\Omega = (-L, L)$. For simplicity we only consider the two-well problem, i.e., we take

$$F_\varepsilon[u] = \frac{1}{\ln(1/\varepsilon)} \left[\int_{(-L,L)^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dxdy + \frac{1}{\varepsilon} \int_{-L}^{L} \mathrm{dist}^2(u(x), \{0, 1\})dx \right] .$$

In this case the Γ-limit is $2\#\mathrm{jumps}[u]$ for $u \in BV(\Omega, \{0, 1\})$, and ∞ otherwise. The upper bound is in this situation immediate (it suffices to smooth the jumps on the scale ε).

Assume $\varepsilon_i \to 0$, and u_i to be a sequence such that

$$E^* = \liminf_{i\to\infty} F_{\varepsilon_i}[u_i]$$

is finite. We may assume that the functions u_i take values in $[0, 1]$, since projection of the values to $[0, 1]$ reduces the energy F_ε. Fix $\delta \in (0, 1/2)$. The construction of Proposition 4.2 yields a sequence of characteristic functions v_k such that

$$\#\mathrm{jumps}(v_k) \le C_\delta(E^* + 1) \le C'_\delta \tag{4.11}$$

and

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{h=0}^{k} p_{\Gamma_h,\omega}(v_k) \le (1 + 2\delta)(\ln 2)E^* , \tag{4.12}$$

where $\omega = (-L', L')$, with $0 < L' < L$,

$$\Gamma_k(x) = \begin{cases} 2^{2(k+1)} - 2^{2k} & \text{if } 0 < |x| \le 2^{-k-1} \\ |x|^{-2} - 2^{2k} & \text{if } 2^{-k-1} < |x| \le 2^{-k} \\ 0 & \text{if } |x| > 2^{-k} , \end{cases}$$

and p is defined as in (3.1). In particular (4.11) implies that the limit u_0 is a characteristic function with finitely many jumps, and one sees easily that it suffices to prove the lower bound for the case that u_0 has a single jump, i.e., $u_0 = \chi_{(0,L)}$.

Suppose that v_k has a jump at $\overline{x}$ and no other jump in $I_h = (\overline{x} - 2^{-h}, \overline{x} + 2^{-h}) \subset (-L', L')$, for some $h \in \mathbb{N}$. A change of variables and an explicit integration give

$$\begin{aligned} p_{\Gamma_h,\omega}(v_k) &\geq p_{\Gamma_h,I_h}(v_k) \\ &= 2\int_{I_h^-}\int_{I_h^+} \Gamma_h(x-y)dxdy \\ &= 2\int_{-1}^{0}\int_{0}^{1} \Gamma_0(x-y)dxdy = 2\ln 2\,. \end{aligned} \tag{4.13}$$

Thus, *if* at every scale 2^{-h} the function v_k has a jump which is isolated in the above sense we immediately conclude from (4.12) that $E^* \geq 2/(1+2\delta)$ which gives the desired conclusion, since $\delta > 0$ was arbitrary. Now we cannot expect that v_k has an isolated jump at every scale, but we will see that this is true at most scales, after a small modification of v_k.

To make this precise we use that the jump set $J = J_{v_k}$ contains only finitely many points, with a bound independent of v_k. We now iteratively cluster and remove points in J as follows:

(i) Set $J^{(k+1)} = J$ and $w^{(k+1)} = v_k$.

(ii) Given $J^{(h+1)}$ and $w^{(h+1)}$ define $J^{(h)}$ and $w^{(h)}$ as follows. An ℓ-cluster is a maximal sequence of points $x_1 < x_2 < \ldots < x_\ell$ in $J^{(h+1)}$ with $x_{i+1} - x_i < 2^{-h}$. Now we obtain $J^{(h)}$ by replacing each cluster with odd ℓ by the leftmost point x_1 and each cluster with even ℓ by the empty set. If $J^{(h+1)} = J^{(h)}$, set $w^{(h)} = w^{(h+1)}$. If $J^{(h+1)} \neq J^{(h)}$, let $w^{(h)}$ be the characteristic function which jumps at the points in $J^{(h)}$ and agrees with $w^{(h+1)}$ outside the intervals $[x_1, x_\ell]$ defined by the ℓ-clusters in $J^{(h+1)}$. Thus

$$\|w^{(h)} - w^{(h+1)}\|_{L^1} = \|w^{(h)} - w^{(h+1)}\|_{L^2}^2 \leq (\#J^{h+1})2^{-h} \leq (\#J)2^{-h}\,. \tag{4.14}$$

We say that a level h is *critical* if $J^{(h)} \neq J^{(h+1)}$. Since J is finite we have

$$\#\{h : h \text{ is critical}\} \leq \#J\,.$$

If h is not critical then all jumps of $w^{(h)} = w^{(h+1)}$ are h-isolated, i.e., there is no other jump in a neighbourhood of size 2^{-h}. Thus by (4.13)

$$p_{\Gamma_h,(-L',L')}(w^{(h)}) \geq 2\ln 2\,, \tag{4.15}$$

if h is not critical.

We now would like to exploit that $w^{(h)}$ is very close to v_k if h, $h+1$, ..., $h+m-1$ are not critical. Fix $m \in \mathbb{N}$. We say that h is good if h, $h+1$, ..., $h+m-1$ are not critical. Thus

$$\#\{h \in \{1, \dots, k\} : h \text{ good}\} \geq k - m\#J\,. \tag{4.16}$$

At the same time, if h is good $w^{(h)} = w^{(h+m)}$, and therefore

$$\|w^{(h)} - v_k\|_{L^1} = \|w^{(h+m)} - v_k\|_{L^1} \leq 2(\#J)2^{-(h+m)}$$

(and the same for the squared L^2 norm, since we are dealing with characteristic functions). We compute, using Lemma 3.1,

$$\begin{aligned}
p_{\Gamma_h,\omega}(w^{(h)}) &= p_{\Gamma_h,\omega}(v_k + w^{(h)} - v_k)\\
&\leq (1+\eta)p_{\Gamma_h,\omega}(v_k) + (1+\frac{1}{\eta})p_{\Gamma_h,\omega}(w^{(h)} - v_k)\\
&\leq (1+\eta)p_{\Gamma_h,\omega}(v_k) + (1+\frac{1}{\eta})2^h\|w^{(h)} - v_k\|_{L^2}^2\\
&\leq (1+\eta)p_{\Gamma_h,\omega}(v_k) + 2(1+\frac{1}{\eta})2^{-m}\#J\,. \qquad (4.17)
\end{aligned}$$

Recalling (4.15), (4.16), and (4.17) we obtain

$$\begin{aligned}
\left(1 - \frac{m\#J}{k}\right) 2\ln 2 &\leq \frac{1}{k}\sum_{k \text{ good}} p_{\Gamma_h,\omega}(w^{(h)})\\
&\leq \frac{1+\eta}{k}\sum_{h=1}^{k} p_{\Gamma_h,\omega}(v_k) + 2\left(1+\frac{1}{\eta}\right)2^{-m}\#J\,.
\end{aligned}$$

Taking the limit $k \to \infty$ and recalling (4.11) and (4.12) we get

$$2\ln 2 \leq (1+\eta)(1+2\delta)(\ln 2)E^* + 2\left(1+\frac{1}{\eta}\right)2^{-m}C_\delta(E^*+1)\,. \tag{4.18}$$

Since m, δ, η were arbitrary it follows that $2 \leq E^*$ as desired.

In concluding this digression, we summarize the main points of the argument:

(i) For most levels h the function v_k can be approximated by a function $w^{(h)}$ which is monotone on scale 2^{-h} (near the jump set).

(ii) The function $w^{(h)}$ is close to v_k in L^1 with a bound that scales slightly better than 2^{-h}.

(iii) The (truncated) energy of $w^{(h)}$ is controlled by the energy of v_k.

We will use a similar argument in higher dimension. In that case the approximations $w^{(h)}$ will be one-dimensional and monotone. The good levels g are selected by the condition that the local BV norm does not change under successive mollification on the scales 2^{-h-m}, ..., 2^{-h}.

5 One-dimensional test functions

In this section we show that if a function is one dimensional and takes values in $\mathbb{Z}^N$ it is possible to estimate its nonlocal truncated energy with the right line tension energy. Our efforts then will be devoted to show that these properties are almost satisfied locally by any sequence of finite energy.

Given a scalar kernel $\Gamma' \in L^1(\mathbb{R}^2;\mathbb{R})$ and an orientation $\nu \in S^1$ we define the one-dimensional interfacial energy (per unit length) by

$$\begin{aligned}\gamma_{1D}^{\Gamma'}(\nu) =&2\int_{\{x\cdot\nu\le 0\le y\cdot\nu\,,\ x\wedge\nu=0\}} \Gamma'(x-y)d\mathcal{H}^1(x)dy\\ =&2\int_{[0,\infty)^2\times\mathbb{R}} \Gamma'((t_1-t_2)\nu+s\nu^\perp)dt_1dt_2ds\,.\end{aligned}$$

Lemma 5.1. *For $a\in\mathbb{R}^N$, $k\in\mathbb{N}$, and Γ as in Section 4, one has*

$$\gamma_{1D}^{a\cdot\Gamma_k a}(\nu) = 2(\ln 2)\int_{\{x\cdot\nu=1\}} a\cdot\Gamma(x)a\,d\mathcal{H}^1(x) = 2(\ln 2)\int_{-\pi/2}^{\pi/2} a\cdot\hat\Gamma(e_\theta)a\,\cos\theta\,d\theta\,.$$

Proof. We consider polar coordinates centered at $x=-t_1\nu$, and set $y=x+\rho e_\theta$, $e_\theta=\cos\theta\nu+\sin\theta\nu^\perp$. Then $y\cdot\nu=-t_1+\rho\cos\theta$, and

$$\begin{aligned}\gamma_{1D}^{a\cdot\Gamma_k a}(\nu) =&2\int_0^\infty\int_0^\infty \rho\phi_k(\rho)\int_{\{\rho\cos\theta\ge t_1\}} a\cdot\hat\Gamma(e_\theta)a\,d\theta\,d\rho\,dt_1\\ =&2\int_0^\infty \rho^2\phi_k(\rho)\,d\rho\int_{-\pi/2}^{\pi/2} a\cdot\hat\Gamma(e_\theta)a\,\cos\theta\,d\theta\,.\end{aligned}$$

By a direct computation one sees that

$$\int_0^\infty \rho^2\phi_k(\rho)\,d\rho = \ln 2\,.$$

This proves the second expression.

At the same time the set $\{x \cdot \nu = 1\}$ can be parametrized by $x = e_\theta / \cos\theta$, $\theta \in (-\pi/2, \pi/2)$. Therefore

$$\int_{\{x\cdot\nu=1\}} a \cdot \Gamma(x) a \, d\mathcal{H}^1(x) = \int_{-\pi/2}^{\pi/2} (\cos\theta)^3 a \cdot \hat{\Gamma}(e_\theta) a \frac{1}{\cos^2\theta} d\theta \,.$$

Collecting the previous expressions the proof is concluded. □

Lemma 5.2. *Assume $u(x) = u_0 + v_0 \lambda(x \cdot \nu)$, $u \in BV(\mathbb{R}^2; \mathbb{Z}^N)$, with λ monotone and $\nu \in S^1$, $u_0, v_0 \in \mathbb{R}^N$. For $m, k \in \mathbb{N}$ with $k \geq m+1$, set $Q = [0, 2^{-m}]^2$. Then*

$$p_{\Gamma_k,Q}(u) \geq \int_{q \cap J_u} |[u]|^2 \frac{1}{|v_0|^2} \gamma_{1D}^{v_0 \cdot \Gamma_k v_0}(\nu) d\mathcal{H}^1 \,. \tag{5.1}$$

Here $q = [2^{-k}, 2^{-m} - 2^{-k}]^2$.

Proof. We first write out

$$p_{\Gamma_k,Q}(u) = \int_{Q \times Q} (v_0 \cdot \Gamma_k(x-y) v_0)(\lambda(x \cdot \nu) - \lambda(y \cdot \nu))^2 dx dy \,.$$

Notice that by assumption $v_0 \cdot \Gamma_k v_0 \geq 0$ pointwise. By symmetry we can restrict to $x \cdot \nu \leq y \cdot \nu$, and add a factor 2. The last factor can be estimated, using the monotonicity of λ, for $x, y \notin J_u$, by

$$(\lambda(x \cdot \nu) - \lambda(y \cdot \nu))^2 = \left(\sum_{t \in J_\lambda \cap [x\cdot\nu, y\cdot\nu]} [\lambda](t) \right)^2 \geq \sum_{t \in J_\lambda \cap [x\cdot\nu, y\cdot\nu]} [\lambda]^2(t)$$

(here $[a, b]$ is the segment joining a and b). Therefore

$$p_{\Gamma_k,Q}(u) \geq 2 \int_{(x,y) \in Q\times Q, (y-x)\cdot\nu \geq 0} \sum_{t \in J_\lambda \cap [x\cdot\nu, y\cdot\nu]} [\lambda]^2(t) (v_0 \cdot \Gamma_k(x-y) v_0) dx dy \,.$$

Swapping the sum with the integral, we obtain

$$p_{\Gamma_k,Q}(u) \geq 2 \sum_{t \in J_\lambda} [\lambda]^2(t) \int_{(x,y) \in Q\times Q, x\cdot\nu \leq t \leq y\cdot\nu} (v_0 \cdot \Gamma_k(x-y) v_0) dx dy \,.$$

At this point we have separated the different interfaces, and we can deal with a single one. To conclude the proof it suffices to show that for any $t \in J_\lambda$,

$$2 \int_{(x,y) \in Q\times Q, x\cdot\nu \leq t \leq y\cdot\nu} (v_0 \cdot \Gamma_k(x-y) v_0) dx dy \geq \mathcal{H}^1(I_t) \gamma_{1D}^{v_0 \cdot \Gamma_k v_0}(\nu) \,, \tag{5.2}$$

where $I_t = q \cap \{z : z \cdot \nu = t\}$ is the "reduced" interface. Recalling that $\operatorname{supp} \Gamma_k \subset B_{2^{-k}}$ and $B_{2^{-k}}(q) \subset Q$, we obtain, for any $x \in q$,

$$\int_{y\in Q, y\cdot\nu\geq t} (v_0 \cdot \Gamma_k(x-y)v_0)dy = \int_{y\in\mathbb{R}^2, y\cdot\nu\geq t} (v_0 \cdot \Gamma_k(x-y)v_0)dy\,.$$

We restrict in (5.2) the x integration to the set $S_t = I_t + [-2^{-k}, 0]\nu = \{z + w : z \in I_t, w \in [-2^{-k}, 0]\nu\} \subset Q$, and decompose the integral into the component parallel and orthogonal to ν. We obtain

$$\begin{aligned}&2\int_{(x,y)\in S_t\times\mathbb{R}^2, y\cdot\nu\geq t} (v_0 \cdot \Gamma_k(x-y)v_0)dxdy \\ =&2\mathcal{H}^1(I_t)\int_{(x,y)\in([t-2^{-k},t]\nu\times\mathbb{R}^2), y\cdot\nu\geq t} (v_0 \cdot \Gamma_k(x-y)v_0)d\mathcal{H}^1(x)dy\,.\end{aligned}$$

Since $\operatorname{supp} \Gamma_k \in B_{2^{-k}}$ the integral in x can be extended to $(-\infty, t)\nu$. This concludes the proof. □

The next lemma deals with the reduction of one-dimensional functions to integer-valued one-dimensional functions.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable, $M > 0$. Let $u : \mathbb{R}^n \to \mathbb{R}^N$ be of the form*

$$u(x) = a\lambda(x \cdot \nu) + b\,,$$

for some $a, b \in \mathbb{R}^N$, $\lambda \in L^\infty(\mathbb{R}; \mathbb{R})$, $\nu \in S^{n-1}$. If

$$\|a\lambda\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq M$$

then there are $a^*, b^* \in \mathbb{Z}^N$, $\lambda^* \in L^\infty(\mathbb{R}; \mathbb{Z})$ such that the function $u^*(x) = a^*\lambda^*(x \cdot \nu) + b^*$ obeys

$$\|u - u^*\|_{L^1(\Omega;\mathbb{R}^N)} \leq C\|\operatorname{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)}\,.$$

Here C depends only on N and M.

Notice that the L^∞ bound is needed, as the following example on $\Omega = (0, 3)$ shows:

$$b = 0\,, \qquad a = \begin{pmatrix} 1 \\ 1/k \end{pmatrix}\,, \qquad \lambda(x) = \begin{cases} 0 & \text{if } x \in (0, 1] \\ 1 & \text{if } x \in (1, 2] \\ k & \text{if } x \in (2, 3)\,. \end{cases}$$

Here $\|\operatorname{dist}(u_k, \mathbb{Z}^2)\|_{L^1} = 1/k$, but for any u_k^* as stated one has $\|u_k - u_k^*\|_{L^1} \geq 1/2$. Indeed, since the three values $(0, 0)$, $(1, 0)$, $(k, 1)$ do not lie on a straight line, u_k^* cannot take all three of them; hence at least one entry must be off by at least $1/2$.

Proof of Lemma 5.3. Let

$$\eta = \|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)} \,.$$

We can assume without loss of generality that $|a| = 1$ and $|b| \le N$ (otherwise we prove the lemma for the function $v(x) = u(x) - [b]$, where $[b]$ denotes a vector whose components are the integer parts of those of b). We define $z : \mathbb{R} \to \mathbb{Z}^N$ measurable and such that $\mathrm{dist}(a\lambda(t) + b, \mathbb{Z}^N) = |a\lambda(t) + b - z(t)|$, for all $t \in \mathbb{R}$. Clearly $\|z\|_\infty \le M + 2N$.

For $w \in \mathbb{Z}^N \cap B_{M+2N}$, define

$$\Omega(w) = \{x \in \Omega : z(x \cdot \nu) = w\} \,,$$

so that

$$\|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1} = \sum_w \|a\lambda(x \cdot \nu) + b - w\|_{L^1(\Omega(w))} = \eta \,. \tag{5.3}$$

Choose $w_1 \neq w_2$ such that

$$\mathcal{L}^n(\Omega(w_1)) \ge \mathcal{L}^n(\Omega(w_2)) \ge \mathcal{L}^n(\Omega(w)) \text{ for all } w \neq w_1 \,.$$

Since $\mathbb{Z}^N \cap B_{M+2N}$ contains a finite number of points we also have that

$$\mathcal{L}^n(\Omega) \le c\mathcal{L}^n(\Omega(w_1)) \,, \qquad \mathcal{L}^n(\Omega \setminus \Omega(w_1)) \le c\mathcal{L}^n(\Omega(w_2)) \,,$$

with c depending only on M and N. Let λ_1 and λ_2 be the average of $\lambda(x \cdot \nu)$ over $\Omega(w_1)$ and $\Omega(w_2)$ respectively. Then

$$\mathcal{L}^n(\Omega(w_1))|a\lambda_1 + b - w_1| \le \|a\lambda(x \cdot \nu) + b - w\|_{L^1(\Omega(w_1))} \le \eta \,,$$

and since $\mathcal{L}^n(\Omega) \le c\mathcal{L}^n(\Omega(w_1))$, we obtain

$$\mathcal{L}^n(\Omega)|a\lambda_1 + b - w_1| \le c\eta \,. \tag{5.4}$$

We set $b^* = w_1$. Argueing as above we obtain

$$\mathcal{L}^n(\Omega \setminus \Omega(w_1))|a\lambda_2 + b - w_2| \le c\eta \,,$$

which implies

$$\mathcal{L}^n(\Omega \setminus \Omega(w_1))|a(\lambda_2 - \lambda_1) - (w_2 - w_1)| \le c^*\eta \,. \tag{5.5}$$

If $\mathcal{L}^n(\Omega \setminus \Omega(w_1)) \le 2c^*\eta$ then setting $a^* = \lambda^* = 0$ will do. Otherwise, since $|w_2 - w_1| \ge 1$, by (5.5) we obtain that $|a(\lambda_2 - \lambda_1)| = |\lambda_2 - \lambda_1| \ge 1/2$. Let

$$\xi = \min\{t > 0 : t(w_2 - w_1) \in \mathbb{Z}^N \setminus \{0\}\} \,,$$

clearly $\xi \in [1/(2M+4N), 1]$. We set

$$a^* = \xi(w_2 - w_1) \in \mathbb{Z}^N\,, \qquad \tilde\lambda = \frac{\lambda - \lambda_1}{(\lambda_2 - \lambda_1)\xi}\,.$$

Then

$$|(a^*\tilde\lambda + w_1) - (a\lambda + b)| \le \left|(w_2 - w_1)\frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} - a(\lambda - \lambda_1)\right| + |w_1 - (a\lambda_1 + b)|\,.$$

The second term can be controlled by (5.4). The first one is bounded by $2|\lambda - \lambda_1|\,|(w_2 - w_1) - (\lambda_2 - \lambda_1)a|$. Integrating separately over $\Omega(w_1)$ and over $\Omega\setminus\Omega(w_1)$, using the estimate $\|\lambda - \lambda_1\|_{L^1(\Omega(w_1))} \le c\eta$ and (5.5), we obtain

$$\|(a^*\tilde\lambda + w_1) - (a\lambda + b)\|_{L^1(\Omega)} \le c\eta\,,$$

and recalling the definition of w

$$\sum_w \|a^*\tilde\lambda + w_1 - w\|_{L^1(\Omega(w))} \le c\eta\,. \tag{5.6}$$

It remains to replace $\tilde\lambda$ by an integer-valued function λ^*. To do this, consider

$$\zeta = \inf\{\mathrm{dist}(\mathbb{R}z, \mathbb{Z}^N \cap B_{2M+4N} \setminus \mathbb{R}z) : z \in \mathbb{Z}^N \cap B_{2M+4N}\}\,.$$

We remark that $\zeta > 0$. Indeed, if this was not the case there would be sequences $z_i, w_i \in \mathbb{Z}^N \cap B_{2M+4N}$ and $t_i \in \mathbb{R}$ such that $|t_i z_i - w_i| \to 0$ and $w_i \notin \mathbb{R}z_i$. By compactness the sequences z_i and w_i have a constant subsequence, hence we obtain $|t_i z - w| \to 0$. Since $\mathbb{R}z$ is closed this implies $w \in \mathbb{R}z$, a contradiction.

Fix one $w \in \mathbb{Z}^N \cap B_{M+2N}$. If there is $\lambda_w \in \mathbb{R}$ such that $\lambda_w a^* = w - w_1$, then from the definition of ξ we obtain $\lambda_w \in \mathbb{Z}$, and we can set $\lambda^* = \lambda_w$ in $\Omega(w)$. Otherwise, $w - w_1 \notin \mathbb{R}a^*$, hence $|ta^* - w + w_1| \ge \zeta$ for all $t \in \mathbb{R}$. In this case we set $\lambda^* = 0$ in $\Omega(w)$, and estimate

$$|w - w_1| \le 2M + 4N \le \frac{2M+4N}{\zeta}|a^*\tilde\lambda + w_1 - w|$$

pointwise in $\Omega(w)$, which gives

$$\|a^*\lambda^* + w_1 - w\|_{L^1(\Omega(w))} \le \frac{2M+4N}{\zeta}\|a^*\tilde\lambda + w_1 - w\|_{L^1(\Omega(w))}\,.$$

Recalling (5.3) and (5.6) the proof is concluded. $\square$

Remark 5.4. *The function u^* constructed in the previous Lemma always satisfies*

$$\|u - u^*\|_{L^\infty} \leq C(M + N)\,.$$

We conclude this section with the following rigidity Lemma for affine functions, which states that if an affine function on a square is close to the set of integers, then it is close to a single integer.

Lemma 5.5. *There is a constant $\delta > 0$ such that the following holds: For every $Q = (-\ell, \ell)^2$, with $\ell > 0$, every $A \in \mathbb{R}^{N\times 2}$, $b \in \mathbb{R}^N$, if*

$$\frac{1}{\ell^2}\|\mathrm{dist}(Ax + b, \mathbb{Z}^N)\|_{L^1(Q)} \leq \delta \tag{5.7}$$

then there is $z \in \mathbb{Z}^N$ such that

$$\|Ax + b - z\|_{L^1(Q;\mathbb{R}^N)} = \|\mathrm{dist}(Ax + b, \mathbb{Z}^N)\|_{L^1(Q)}\,.$$

Proof. Let $w : Q \to \mathbb{Z}^N$ be such that $\mathrm{dist}(Ax+b, \mathbb{Z}^N) = |Ax-b-w|$ pointwise. We claim that for an appropriate δ the condition (5.7) implies that w is constant. To prove this, it suffices to show that any component is constant. Since $\mathrm{dist}((Ax + b)_i, \mathbb{Z}) \leq \mathrm{dist}(Ax + b, \mathbb{Z}^N)$, it suffices to consider the case $N = 1$.

Assume that w is not constant. Then there is $\bar{x} \in Q$ such that $|A\bar{x}+b-w| = 1/2$. Since $x \mapsto \mathrm{dist}(Ax + b, \mathbb{Z})$ is $|A|$-Lipschitz, we have

$$\mathrm{dist}(Ay + b, \mathbb{Z}) \geq \frac{1}{2} - |A|\,|\bar{x} - y|\,.$$

Let $r = 1/(4|A|\sqrt{2})$, and assume $\bar{x} \in (0, \ell/2)^2$ (otherwise a few signs have to be changed). On $\bar{x} + (0, r)^2$ we have $\mathrm{dist}(Ax + b, \mathbb{Z}) \geq 1/4$. Now if $r \geq \ell/4\sqrt{2}$ we have

$$\int_Q \mathrm{dist}(Ax + b, \mathbb{Z})\,dx \geq \int_{\bar{x}+(0,\ell/4\sqrt{2})^2} \mathrm{dist}(Ax + b, \mathbb{Z})\,dx \geq \frac{\ell^2}{32}\frac{1}{4}$$

and the proof is concluded (with $\delta < 1/128$).

Otherwise, set $R = 1/|A| = 4\sqrt{2}r < \ell$. Choose at least $\frac{1}{4}\ell^2/R^2$ disjoint squares of side $2R$ contained in Q. Let $q = y + (-R, R)^2$ be one of them. Since $|A|R = 1$, there is $\bar{x} \in y + (-R, 0)^2$ such that $\mathrm{dist}(A\bar{x} + b, \mathbb{Z}) = 1/2$. Since $r = \frac{R}{4\sqrt{2}}$, argueing as above, we obtain

$$\int_q \mathrm{dist}(Ax + b, \mathbb{Z})\,dx \geq \int_{\bar{x}+(0,R/4\sqrt{2})^2} \mathrm{dist}(Ax + b, \mathbb{Z})\,dx \geq \frac{R^2}{32}\frac{1}{4}\,.$$

Summing over all squares the thesis follows with δ reduced by $1/4$. ☐

6 Local approximation by one-dimensional functions

Our next goal is to approximate functions with well-controlled energy by one-dimensional functions. We first state the result and then explain the meaning of the different quantities involved in the statement. Here and below we use the euclidean norm and scalar product for matrices, i.e., $A\cdot B = \mathrm{Tr} A^T B$ and $|A|^2 = \mathrm{Tr} A^T A$. For notational simplicity we focus on $W^{1,1}$ (resp. L^1) functions, the arguments in this section also hold in the case of BV functions (resp. measures).

Theorem 6.1. *Let $\ell > 0$, $Q = (-\ell, \ell)^2$, $A \in \mathbb{R}^{N\times 2}$ with $|A| = 1$, $u \in W^{1,1}(Q;\mathbb{R}^N)$, and define*

$$\begin{aligned}
\eta_1 &= \frac{1}{\ell^2}\|\mathrm{dist}(u,\mathbb{Z}^N)\|_{L^1(Q)}\,,\\
\eta_2 &= \frac{1}{\ell}\|Du\|_{L^1(Q;\mathbb{R}^{N\times 2})}\,,\\
\eta_3 &= \frac{1}{\ell}\|Du - A(A\cdot Du)_+\|_{L^1(Q;\mathbb{R}^{N\times 2})}\,.
\end{aligned}$$

Assume $\eta_1 \le \delta/2$, δ being as in Lemma 5.5. Then there are $a, b \in \mathbb{R}^N$, $\nu \in S^1$, $\lambda \in W^{1,1}(\mathbb{R};\mathbb{R})$, nondecreasing, such that the function

$$\tilde u(x) = a\lambda(x\cdot\nu) + b$$

obeys

$$\frac{1}{\ell}\|u-\tilde u\|_{L^2(q;\mathbb{R}^N)} \le c\eta_2^{2/3}\eta_3^{1/3} + c\eta_2\eta_3^{1/2} + c\eta_1$$

and

$$\|a\lambda\|_{L^\infty(\mathbb{R};\mathbb{R}^N)} \le c\eta_2\,.$$

Here $q = (-\ell/4, \ell/4)^2$, and the constant c depends only on the dimension N.

Here and below, $a_\pm = \max\{\pm a, 0\}$.

Note that the quantities η_1 and η_2 can be controlled by the energy. In contrast the quantity η_3 is small whenever the L^1 norm of a suitable mollification (on scale l) of Du almost agrees with the L^1 norm of Du (see Lemma 6.2 below). We will see in Section 8 that this property holds for many scales.

Before presenting the proof we discuss how this fundamental ingredient of our construction can be made quantitative. Since the L^1 norm is not strictly convex, the norm of a function $f \in L^1(\mathbb{R};[0,\infty))$ is the same as the norm of any mollification, $\|f\|_{L^1(\mathbb{R})} = \|f * \varphi\|_{L^1(\mathbb{R})}$. The same, however, does not hold for functions without a sign, or for vectorial functions. The next lemma makes this quantitative, in a localized way. We assert that if mollification does not decrease the L^1 norm of a function substantially, then the function $f:\mathbb{R}^n\to\mathbb{R}^p$ is approximately scalar, in the sense that there is a vector $\nu\in S^{p-1}$ such that f is close to the line $\nu[0,\infty)$.

We shall apply this Lemma to the gradient of u, i.e., with $f=Du$, $n=2$, $p=2N$, and the direction ν shall be an $N\times 2$ matrix.

Lemma 6.2. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^p)$, $\psi\in C_c(B_1,[0,\infty))$ be such that $\psi\geq 1$ on $B_{1/2}(0)$ and $\int_{B_1}\psi dx=1$, and let $\psi_r(x)=r^{-n}\psi(x/r)$. Set $Q=(-r/2^{2+n/2},r/2^{2+n/2})^2$ and*

$$\eta=\int_{\mathbb{R}^n}|f|(\chi_Q*\psi_r)dx-\int_Q|f*\psi_r|dx\,.$$

Then the function f is approximately scalar, in the sense that there is $\nu\in S^{p-1}$ such that

$$\int_Q(|f|-f\cdot\nu)dx\leq c\eta$$

and

$$\int_Q|f-\nu(f\cdot\nu)_+|dx\leq c\|f\|^{1/2}_{L^1(Q;\mathbb{R}^p)}\eta^{1/2}\,.$$

Proof. By scaling we can assume $r=1$. For $x\in Q$, let $\nu(x)\in S^{p-1}$ be a unit vector parallel to $(f*\psi)(x)$, so that

$$|f*\psi|(x)=\int_{\mathbb{R}^n}f(y)\cdot\nu(x)\psi(x-y)\,dy\,. \tag{6.1}$$

We define $\tilde\eta:Q\to[0,\infty)$ by

$$\begin{aligned}\tilde\eta(x)&=\int_{\mathbb{R}^n}|f|(y)\psi(x-y)dy-|f*\psi|(x)\\&=\int_{\mathbb{R}^n}(|f|(y)-f(y)\cdot\nu(x))\,\psi(x-y)dy\,.\end{aligned}$$

The integrand is obviously nonnegative. Since for $x,y\in Q$ we have $|x-y|\leq 1/2$, it follows that

$$\tilde\eta(x)\geq\int_Q(|f|(y)-f(y)\cdot\nu(x))\;dy\geq\int_Q(|f|(y)-(f(y)\cdot\nu(x))_+)\;dy\,.$$

But by the definition of $\tilde\eta$ we obtain

$$\int_Q\tilde\eta(x)\,dx=\eta\,.$$

Therefore there is at least a point $x \in Q$ such that $\tilde{\eta}(x) \leq 2^{n+2}\eta$. Setting $\nu = \nu(x)$ concludes the proof of the first part.

To prove the second part we observe that

$$|f - \nu(f \cdot \nu)_+| = (|f|^2 - (f \cdot \nu)_+^2)^{1/2} \leq 2|f|^{1/2}(|f| - (f \cdot \nu)_+)^{1/2} .$$

Using Hölder's inequality we obtain the thesis. $\square$

We now prove that if the gradient of a function is one-dimensional, in the sense that it can be well approximated by a scalar multiple of a fixed matrix, then either the matrix is almost rank-one or the function is almost affine. As usual in this kind of inequalities, when working in $W^{1,p}$ with $1 < p < \infty$ we can obtain full control in the same space, whereas in the for us most relevant case $p = 1$ one can only estimate the function u in the corresponding space $L^{1^*} = L^2$.

Proposition 6.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $1 < p < \infty$. For any $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, $A \in \mathbb{R}^{N\times 2}$ with $\operatorname{rank} A = 2$, $\xi \in L^p(\Omega; \mathbb{R})$ there is $\bar{\xi} \in \mathbb{R}$ such that*

$$\|Du - \bar{\xi}A\|_{L^p(\Omega;\mathbb{R}^{N\times 2})} \leq C\frac{a_1}{a_2}\|Du - \xi A\|_{L^p(\Omega;\mathbb{R}^{N\times 2})} .$$

Further, for any $u \in W^{1,1}(\Omega; \mathbb{R}^N)$, $A \in \mathbb{R}^{N\times 2}$, $\xi \in L^1(\Omega; \mathbb{R})$ there are $\bar{\xi} \in \mathbb{R}$ and $b \in \mathbb{R}^N$ such that

$$\|u(x) - \bar{\xi}Ax - b\|_{L^2(\Omega;\mathbb{R}^{N\times 2})} \leq C\frac{a_1}{a_2}\|Du - \xi A\|_{L^1(\Omega;\mathbb{R}^{N\times 2})} .$$

Here $a_1 \geq a_2 > 0$ are the singular values of A, i.e., the eigenvalues of $(A^TA)^{1/2}$. The constant depends on p, Ω and N.

Proof. Set $\eta = \|Du - \xi A\|_{L^p(\Omega)}$ ($p = 1$ in the second case). By replacing u with $\tilde{u}(x) = Qu(Rx)$, and A with $\tilde{A} = QAR$, with suitable $Q \in O(N)$, $R \in O(2)$, we can assume A to be diagonal, in the sense that $A = a_1e_1 \otimes e_1 + a_2e_2 \otimes e_2$, with $a_1 \geq a_2 > 0$. For all $i = 3, \dots N$ one has

$$\|Du_i\|_{L^p} \leq \eta$$

which implies the thesis for those components, hence it suffices to treat the case $N = 2$. Define $v \in W^{1,p}(\Omega; \mathbb{R}^2)$ by

$$v_1(x) = a_1u_2(x) , \qquad v_2(x) = -a_2u_1(x) .$$

Then $(a_1e_1 \otimes e_2 - a_2e_2 \otimes e_1)(Du - \xi A) = Dv - a_1a_2\xi(e_1 \otimes e_2 - e_2 \otimes e_1)$, which implies

$$\left|\frac{Dv + Dv^T}{2}\right| \leq |Dv - a_1a_2\xi(e_1 \otimes e_2 - e_2 \otimes e_1)| \leq a_1 \, |Du - A\xi| \tag{6.2}$$

pointwise. Therefore Korn's inequality shows that there is $\bar\xi \in \mathbb{R}$ such that, for any $p > 1$,

$$\|Dv - a_1 a_2 \bar\xi (e_1 \otimes e_2 - e_2 \otimes e_1)\|_{L^p} \le C a_1 \left\|Du - A\xi\right\|_{L^p} ,$$

which in turn implies, using (6.2),

$$a_1 a_2 \|\xi - \bar\xi\|_{L^p} \le C a_1 \left\|Du - A\xi\right\|_{L^p} .$$

Thus $\|\xi A - \bar\xi A\|_{L^p} \le C a_1 \|\xi - \bar\xi\|_{L^p} \le C\frac{a_1}{a_2}\|Du - A\xi\|_{L^p}$, and the proof of the first part is concluded.

For $p = 1$ the same estimates hold in weak-L^1, which does not embed in L^2. However, from the Korn-Poincaré inequality (or the embedding of BD into L^2), one still has the existence of $\bar\xi \in \mathbb{R}$ and $b \in \mathbb{R}^n$ such that

$$\|v(x) - \bar\xi a_1 a_2 (e_1 \otimes e_2 - e_2 \otimes e_1)x - b\|_{L^2} \le C a_1 \left\|Du - A\xi\right\|_{L^1} ,$$

which in turn implies

$$\|u(x) - \bar\xi A x - b\|_{L^2} \le C\frac{a_1}{a_2} \left\|Du - A\xi\right\|_{L^1} .$$

□

Next we prove a Poincaré-type inequality where we only have half-sided control on one component of the gradient. We show that u is close to an increasing function of one scalar variable alone.

Lemma 6.4. *Let $\ell > 0$, $\nu \in S^1$, $Q^*_\nu = \{x : |x \cdot \nu| \le \ell, |x \cdot \nu^\perp| \le \ell/2\}$, $u \in W^{1,1}(Q^*_\nu; \mathbb{R})$. Set*

$$\eta = \int_{Q^*_\nu} (|\partial_{\nu^\perp} u| + |(\partial_\nu u)_-|)\, dx$$

and $Q_\nu = \{x : |x \cdot \nu| \le \ell/2, |x \cdot \nu^\perp| \le \ell/2\}$ (see Figure 2). Then there is a nondecreasing function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{Q_\nu} |u(x) - h(x\cdot\nu)|^2\, dx \le C\eta |Du|(Q^*_\nu)$$

and

$$\|h(x \cdot \nu) - \bar h\|_{L^\infty(Q_\nu)} \le \frac{1}{\ell}|Du|(Q^*_\nu)$$

for some $\bar h \in \mathbb{R}$.

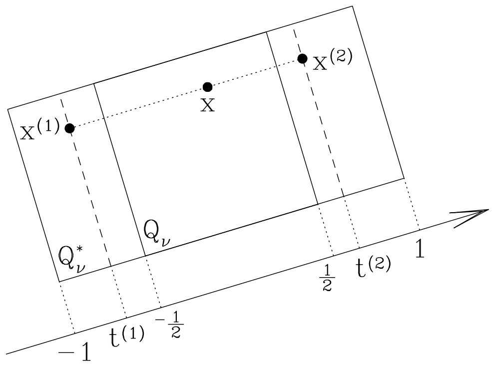

Figure 2: Geometry in Lemma 6.4.

Proof. By scaling we can assume $\ell = 1$. Define $g : (-1, 1) \to \mathbb{R}$ by

$$g(t) = \int_{I(t)} u \, d\mathcal{H}^1$$

where $I(t) = Q_\nu^* \cap \{x : x \cdot \nu = t\}$, notice that $\mathcal{H}^1(I(t)) = 1$ for all $t \in (-1, 1)$. For almost every $x \in Q_\nu^*$ we have

$$|u(x) - g(x \cdot \nu)| \le \int_{I(x\cdot\nu)} |\partial_{\nu^\perp} u| \, d\mathcal{H}^1 \,. \tag{6.3}$$

Choose $t^{(1)} \in (-1, -1/2)$, $t^{(2)} \in (1/2, 1)$ such that

$$\int_{I(t^{(1)}) \cup I(t^{(2)})} |\partial_{\nu^\perp} u| d\mathcal{H}^1 \le 2\eta \,. \tag{6.4}$$

We observe that $g \in W^{1,1}((-1, 1))$, with

$$g'(t) = \int_{I(t)} \partial_\nu u \, d\mathcal{H}^1 \,,$$

which implies

$$|g'_-|(t) \le \int_{I(t)} |(\partial_\nu u)_-| d\mathcal{H}^1$$

for almost all $t \in (-1,1)$, and

$$\int_{-1}^{1} |g'|\, dt \leq |Du|(Q_\nu^*)\,.$$

Therefore

$$\int_{(-1,1)} |g'_-|(t) dt \leq \eta\,.$$

For any $x \in Q_\nu$ we set $x^{(2)} = t^{(2)}\nu + \nu^\perp(x \cdot \nu^\perp) = x + (t^{(2)} - x\cdot\nu)\nu$, and estimate

$$\begin{aligned} u(x) &= u(x^{(2)}) - \int_{[x,x^{(2)}]} \partial_\nu u \, d\mathcal{H}^1 \\ &\leq g(t^{(2)}) + |u(x^{(2)}) - g(t^{(2)})| + \int_{[x,x^{(2)}]} |(\partial_\nu u)_-| \, d\mathcal{H}^1\,. \end{aligned}$$

As above, $[a,b]$ is the segment with endpoints a and b. From

$$|g(x\cdot\nu) - g(t^{(2)})| \leq \int_{[t_1,t_2]} |g'| dt \leq |Du|(Q_\nu^*)\,, \tag{6.5}$$

(6.3), and (6.4) we obtain

$$|u(x^{(2)}) - g(t^{(2)})| \leq 2\eta \leq 2|Du|(Q_\nu^*)\,,$$

and therefore

$$u(x) \leq g(x\cdot\nu) + 3|Du|(Q_\nu^*) + \int_{[x,x^{(2)}]} |(\partial_\nu u)_-| \, d\mathcal{H}^1\,.$$

Analogously

$$u(x) \geq g(t^{(1)}) - |u(x^{(1)}) - g(t^{(1)})| - \int_{[x^{(1)},x]} |(\partial_\nu u)_-| d\mathcal{H}^1$$

gives

$$u(x) \geq g(x\cdot\nu) - 3|Du|(Q_\nu^*) - \int_{[x^{(1)},x]} |(\partial_\nu u)_-| \, d\mathcal{H}^1\,.$$

We conclude that

$$|u(x) - g(x\cdot\nu)| \leq 3|Du|(Q_\nu^*) + \int_{[x^{(1)},x^{(2)}]} |(\partial_\nu u)_-| d\mathcal{H}^1\,.$$

We multiply by (6.3) and integrate over Q_ν, to obtain

$$\int_{Q_\nu} |u(x) - g(x\cdot\nu)|^2\,dx \le \left(\int_{Q_\nu^*} |\partial_{\nu^\perp} u|dx\right)\left(3|Du|(Q_\nu^*) + \int_{Q_\nu^*} |(\partial_\nu u)_-|dx\right).$$

The second factor in the right-hand side can be controlled by $4|Du|(Q_\nu^*)$.

Finally, we define $h : \mathbb{R} \to \mathbb{R}$ by $h(0) = g(0)$, $h' = g'_+$, and observe that

$$\|h - g\|_{L^\infty(-1/2,1/2)} \le \int_{[-1,1]} |g'_-|dt \le \eta\,,$$

which concludes the proof of the first inequality. The uniform bound follows from the definition of h and (6). ☐

Proof of Theorem 6.1. Recall that

$$\eta_1 = \frac{1}{\ell^2}\|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(Q)}\,,$$
$$\eta_2 = \frac{1}{\ell}\|Du\|_{L^1(Q)}\,,$$
$$\eta_3 = \frac{1}{\ell}\|Du - A(A\cdot Du)_+\|_{L^1(Q)}\,,$$

with $\eta_1 \le \delta/2$, δ being as in Lemma 5.5, and that we have to show that there exists a function $\tilde{u}(x) = a\lambda(x\cdot\nu) + b$ (with $a, b \in \mathbb{R}^N$, $\nu \in S^1$, $\lambda \in W^{1,1}(\mathbb{R};\mathbb{R})$), such that

$$\frac{1}{\ell}\|u - \tilde{u}\|_{L^2(q)} \le c\eta_2^{2/3}\eta_3^{1/3} + c\eta_2\eta_3^{1/2} + c\eta_1$$

and

$$\|a\lambda\|_{L^\infty(\mathbb{R})} \le c\eta_2\,.$$

By scaling we can assume $\ell = 1$; from $|A| = 1$ one obtains $\eta_3 \le \eta_2$. Let $a_1 \ge a_2 \ge 0$ be the singular values of A.

The argument is based on obtaining two different estimates, and then choosing one or the other depending on the value of a_2 relative to the $\eta_{1,2,3}$.

Step 1. Assume first $a_2 > 0$, i.e., $\operatorname{rank} A = 2$. Setting $\xi = (A\cdot Du)_+$, from Proposition 6.3 and $|A| = 1$ we have

$$\|u(x) - \bar{\xi}Ax - b\|_{L^2(Q)} \le C\frac{a_1}{a_2}\eta_3 \le C\frac{1}{a_2}\eta_3\,,$$

for some $\bar{\xi} \in \mathbb{R}$, $b \in \mathbb{R}^N$. This implies

$$\|\mathrm{dist}(\bar{\xi}Ax + b, \mathbb{Z}^N)\|_{L^1(Q)} \le c_*\frac{1}{a_2}\eta_3 + \eta_1\,.$$

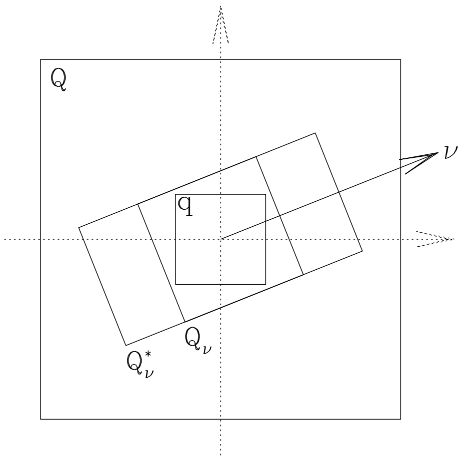

Figure 3: Geometry in Step 2 of the proof of Proposition 6.1.

We distinguish two cases. If $c_*\eta_3 \le \delta a_2/2$, then the right-hand side is less then δ, and by Lemma 5.5 there is $z \in \mathbb{Z}^N$ such that

$$\|\bar\xi Ax - b - z\|_{L^1(Q)} \le c_* \frac{1}{a_2}\eta_3 + \eta_1 \, .$$

This immediately implies

$$|\bar\xi A| \le C(\frac{1}{a_2}\eta_3 + \eta_1) \, .$$

We conclude that at least one of the two inequalities

$$\|u - b\|_{L^2(Q)} \le C\frac{1}{a_2}\eta_3 + C\eta_1 \tag{6.6}$$

or

$$a_2 \le C'\eta_3 \tag{6.7}$$

holds. Here both constants may only depend on N. It is clear that the same conclusion holds also in the remaining case $a_2 = 0$.

Step 2. Choose $\alpha, \alpha' \in S^{N-1}$, $\nu, \nu' \in S^1$ orthogonal and so that $A = a_1\alpha\otimes\nu + a_2\alpha'\otimes\nu'$, then

$$|A - \alpha\otimes\nu| \le |1 - a_1| + |a_2| \le 2a_2 \, .$$

Writing $B = \alpha \otimes \nu$, we obtain

$$
\begin{aligned}
|A(A\cdot Du)_+ - B(B\cdot Du)_+| &\le |A-B|\,|(B\cdot Du)_+| + |A|\,|(A\cdot Du)_+ - (B\cdot Du)_+| \\
&\le 2|A-B|\,|Du|\,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|Du - &\alpha\otimes\nu(\partial_\nu u\cdot\alpha)_+\|_{L^1(Q)} \\
&\le \|Du - A(A\cdot Du)_+\|_{L^1(Q)} + \|A(A\cdot Du)_+ - B(B\cdot Du)_+\|_{L^1(Q)} \\
&\le \eta_3 + 4a_2\eta_2\,. \qquad (6.8)
\end{aligned}
$$

Let $P_\alpha^\perp = \mathrm{Id}_N - \alpha\otimes\alpha$ be the projection on the space orthogonal to α. Since $P_\alpha^\perp(\alpha\otimes\nu) = 0$ we deduce

$$
\|D(P_\alpha^\perp u)\|_{L^1(Q)} \le \eta_3 + 4a_2\eta_2\,.
$$

Therefore there is $b\in\mathbb{R}^N$, $b\cdot\alpha = 0$, such that

$$
\|P_\alpha^\perp u - b\|_{L^2(Q)} \le c\|DP_\alpha^\perp u\|_{L^1(Q)} \le c\eta_3 + ca_2\eta_2\,. \qquad (6.9)
$$

The component $u\cdot\alpha$ is treated using Lemma 6.4. Indeed, with the notation in that statement (using the present ν, $\ell = 1/\sqrt{2}$) we have $q\subset Q_\nu\subset Q_\nu^*\subset Q$ (see Figure 3). We conclude together with (6.8) that there is a monotone function h such that

$$
\|(u\cdot\alpha)(x) - h(x\cdot\nu)\|_{L^2(q)} \le C(\eta_3 + a_2\eta_2)^{1/2}\eta_2^{1/2}\,.
$$

Combining this with (6.9) and dropping irrelevant terms we obtain

$$
\|u(x) - \alpha h(x\cdot\nu) - b\|_{L^2(q)} \le ca_2^{1/2}\eta_2 + c\eta_2^{1/2}\eta_3^{1/2}\,. \qquad (6.10)
$$

We finally come back to the two cases we distinguished at the end of Step 1. If (6.7) holds, then (6.10) becomes

$$
\|u(x) - \alpha h(x\cdot\nu) - b\|_{L^2(q)} \le c\eta_2\eta_3^{1/2} + c\eta_2^{1/2}\eta_3^{1/2}\,.
$$

In this case the proof is concluded.

Assume now that (6.7) does not hold. If $\frac{1}{a_2}\eta_3 + \eta_1 > a_2^{1/2}\eta_2 + \eta_2^{1/2}\eta_3^{1/2}$ we set $\tilde u(x) = \alpha h(x\cdot\nu) + b$, otherwise $\tilde u = b$. From (6.6) and (6.10) we then obtain

$$
\begin{aligned}
\|u - \tilde u\|_{L^2(q)} &\le c\min\left\{\frac{1}{a_2}\eta_3 + \eta_1,\, a_2^{1/2}\eta_2 + \eta_2^{1/2}\eta_3^{1/2}\right\} \\
&\le c\min\left\{\frac{\eta_3}{a_2},\, a_2^{1/2}\eta_2\right\} + c\eta_1 + c\eta_2^{1/2}\eta_3^{1/2}\,.
\end{aligned}
$$

But $\min\{\eta_3/a_2, a_2^{1/2}\eta_2\} \le \eta_3^{1/3}\eta_2^{2/3}$, and we conclude

$$
\|u - \tilde u\|_{L^2(q)} \le c\eta_2^{2/3}\eta_3^{1/3} + c\eta_1\,.
$$

□

7 Control of the line energy with the truncated energy

In the previous section we saw that functions with low energy (and small differences $\|Du\|_{L^1}-\|\varphi*Du\|_{L^1}$) are well approximated by one-dimensional functions. In Section 5 we saw that for one-dimensional functions the truncated energy is well approximated by the line energy. Now we combine these results to obtain a global approximation: given a function $u \in BV(\Omega;\mathbb{R}^N)$ and $\omega \subset\subset \Omega$ we construct a new function $w \in BV(\omega;\mathbb{Z}^N)$ such that the relaxed line energy of w

$$E_0^{\mathrm{rel}}[w,\omega] = \int_{J_w\cap\omega} \gamma_0^{\mathrm{rel}}(\nu,[w])\, d\mathcal{H}^1 \tag{7.1}$$

is essentially controlled by the truncated energy of u (we switch from γ_0 to the smaller γ_0^{rel}, which has linear growth, since boundary terms are only controlled in L^1). We fix a mollifier $\varphi \in C_c(B_1;[0,\infty))$, with $\int_{\mathbb{R}^2}\varphi\, dx = 1$ and $\varphi \ge 1$ on $B_{1/2}(0)$. Let $\varphi_h(x) = 2^{2h}\varphi(2^h x)$.

Proposition 7.1. *Let $\omega \subset\subset \Omega$ be two Lipschitz sets, $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, $M > 1$, $h, t \in \mathbb{N}$ with $t \ge 3$, $\eta \in (0,1)$. Assume $\mathrm{dist}(\omega,\partial\Omega) \ge 2^{-h+1}$.*

Then there is $w = w_{M,h,t,\eta} \in BV(\omega;\mathbb{Z}^N)$ such that

$$\begin{aligned}(\ln 2)\int_{J_w\cap\omega} \gamma_0^{\mathrm{rel}}(\nu,[w])\, d\mathcal{H}^1 \le& (1+\eta+c2^{-t})p_{\Gamma_{h+t},\Omega}(u) + \frac{C_M}{\eta}2^{h+t}\|\mathrm{dist}(u,\mathbb{Z}^N)\|_{L^1(\Omega)}\\ &+\frac{C_M}{\eta}2^t A^{5/6}\left(|Du|(\Omega) - |D(u*\varphi_h)|(\omega)\right)^{1/6}\\ &+\frac{c}{M^{1/2}}2^{t/2}A\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\|u-w\|_{L^1(\omega)} \le& \frac{c}{M^{1/2}}2^{-h+t/2}A + C_M\|\mathrm{dist}(u,\mathbb{Z}^N)\|_{L^1(\Omega)}\\ &+ C_M 2^{-h}A^{2/3}\left(|Du|(\Omega)-|D(u*\varphi_h)|(\omega)\right)^{1/3}.\end{aligned} \tag{7.3}$$

Here $A = \max\{|Du|(\Omega), p_{h+t,\Omega}(u)\}$.

Proof. **Step 1. Domain subdivision.** For $z \in \mathbb{Z}^2$, define $Q_z^* = (z + [-1,1]^2)2^{-h-5}$ and $Q_z^{**} = (z+[-4,4]^2)2^{-h-5}$. We shall consider those z for which the larger square touches ω, i.e., those in

$$Z = \{z \in \mathbb{Z}^2 : Q_z^{**} \cap \omega \ne \emptyset\}.$$

The larger squares have finite overlap, are contained in Ω, and give a uniform cover of ω, in the sense that

$$64\chi_\omega \le \sum_{z\in Z} \chi_{Q_z^{**}} \le 64\chi_\Omega \qquad \text{a.e.}\,. \tag{7.4}$$

The set ω is covered by the smaller squares, in the sense that

$$4\chi_\omega \le \sum_{z\in Z} \chi_{Q_z^*}\,. \tag{7.5}$$

We assert that we can find for each $z \in Z$ a function $u_z^* \in BV(Q_z^*, \mathbb{Z}^N)$ such that

$$\begin{aligned}\sum_{z\in Z} \|u - u_z^*\|_{L^1(Q_z^*)} \le & \frac{c}{M^{1/2}} 2^{-h+t/2} A + C_M \|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)} \\ & + C_M 2^{-h} A^{2/3} \left(|Du|(\Omega) - |D(u * \varphi_h)|(\omega)\right)^{1/3}\,. \end{aligned} \tag{7.6}$$

Further, for some $B \subset Z$, u_z^* is constant if $z \in B$, and

$$\begin{aligned}\sum_{z\in Z\setminus B} \|u - u_z^*\|^2_{L^2(Q_z^*)} \le & \; C_M \|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)} \\ & + C_M 2^{-h} A^{5/6} \left(|Du|(\Omega) - |D(u * \varphi_h)|(\omega)\right)^{1/6}\,. \end{aligned} \tag{7.7}$$

We treat in Step 2 the squares in B, in Step 3 those in $Z \setminus B$. We start by defining the set of "bad" squares by

$$B = \{z \in Z : |Du|(Q_z^{**}) \ge M 2^{-h}\}\,, \tag{7.8}$$

and at the same time the set of "good" squares

$$G = Z \setminus B = \{z \in Z : |Du|(Q_z^{**}) < M 2^{-h}\}\,. \tag{7.9}$$

Step 2. The "bad" squares. This argument is similar to the one used in proving Proposition 4.1. Fix one $z \in B$. We shall subdivide Q_z^* into smaller squares of side $\alpha = 2^{-(h+t+2)}$, and show that u does not change much from small square to small square. This will allow us to replace u by a constant in the entire square Q_z^*.

Precisely, for $\zeta \in \mathbb{Z}^2$ we set $q_\zeta = \alpha\zeta + (0, \alpha)^2$, $Q_\zeta = \alpha\zeta + (-\alpha, 2\alpha)^2$, and $W_z = \{\zeta \in \mathbb{Z}^2 : Q_\zeta \subset Q_z^{**}\}$. We define $\hat u_z : W_z \to \mathbb{R}^N$ by setting $\hat u_z(\zeta)$ equal to the average of u over the square Q_ζ. Reasoning as in (4.6) of Proposition 4.1 we obtain

$$\int_{Q_\zeta} |u - \hat u_z(\zeta)|^2\, dx \le c 2^{-h-t} p_{h+t, Q_\zeta}(u)\,.$$

For $|\zeta-\zeta'|=1$ we have $\mathcal{L}^2(Q_\zeta\cap Q_{\zeta'})\geq\alpha^2$, and therefore

$$\sum_{\zeta,\zeta'\in W_z:|\zeta-\zeta'|=1}\alpha^2|\hat u_z(\zeta)-\hat u_z(\zeta')|^2\leq c2^{-h-t}p_{h+t,Q_z^{**}}(u)\,.$$

Since W_z is a discrete square, the discrete Poincaré inequality yields a $v_z\in\mathbb{R}^N$ such that

$$\sum_{\zeta\in W_z}\alpha^4|\hat u_z(\zeta)-v_z|^2\leq c2^{-2h}2^{-h-t}p_{h+t,Q_z^{**}}(u)\,.$$

Choose $u_z^*\in\mathbb{Z}^N$ such that $|u_z^*-v_z|\leq N$. Then

$$\begin{aligned}\|u-u_z^*\|^2_{L^2(Q_z^{**})}\,dx&\leq 3\sum_{\zeta\in W_z}\left[\int_{q_\zeta}|u-\hat u_z(\zeta)|^2\,dx+\alpha^2|\hat u_z(\zeta)-v_z|^2+\alpha^2N^2\right]\\&\leq c2^{-h-t}p_{h+t,Q_z^{**}}(u)+c\alpha^{-2}2^{-2h}2^{-h-t}p_{h+t,Q_z^{**}}(u)+c2^{-2h}\\&\leq c2^{-h+t}p_{h+t,Q_z^{**}}(u)+c2^{-2h}\,,\end{aligned}$$

and since $z\in B$,

$$\begin{aligned}\|u-u_z^*\|_{L^1(Q_z^*)}&\leq c2^{-h}\|u-u_z^*\|_{L^2(Q_z^*)}\\&\leq c2^{-h+t/2}2^{-h/2}\left(p_{h+t,Q_z^{**}}(u)\right)^{1/2}+c2^{-2h}\\&\leq\frac{c}{M^{1/2}}2^{-h+t/2}\left(|Du|(Q_z^{**})\right)^{1/2}\left(p_{h+t,Q_z^{**}}(u)\right)^{1/2}+\frac{c}{M}2^{-h}|Du|(Q_z^{**})\,.\end{aligned}$$

We conclude that

$$\begin{aligned}\sum_{z\in B}\|u-u_z^*\|_{L^1(Q_z^*)}&\leq\frac{c}{M^{1/2}}2^{-h+t/2}\left(|Du|(\Omega)\right)^{1/2}\left(p_{h+t,\Omega}(u)\right)^{1/2}+\frac{c}{M}2^{-h}|Du|(\Omega)\\&\leq\frac{c}{M^{1/2}}2^{-h+t/2}\max\left\{|Du|(\Omega),p_{h+t,\Omega}(u)\right\}\,.\end{aligned}\tag{7.10}$$

This proves (7.6) for the "bad" squares.

Step 3. The "good" squares. Let $z\in Z$. We apply Lemma 6.2 to $f=Du$ on the square Q_z^{**}, with $r=2^{-h}$, and the mollifier φ_h (it is here important that the side of Q_z^{**} is 2^{-h-2}). For each of them we obtain a matrix $A_z\in\mathbb{R}^{N\times 2}$ such that the quantity

$$\eta_3^z=2^h\|Du-A_z(A_z\cdot Du)_+\|_{L^1(Q_z^{**})}$$

obeys

$$\eta_3^z\leq c2^h\left(\int_{\mathbb{R}^2}|Du|(\varphi_h*\chi_{Q_z^{**}})-\int_{\mathbb{R}^2}|D(u*\varphi_h)|\chi_{Q_z^{**}}\right)^{1/2}\left(|Du|(Q_z^{**})\right)^{1/2}\,.\tag{7.11}$$

We intend to apply Theorem 6.1 to the pair of squares $q = Q_z^* \subset Q = Q_z^{**}$, with $\ell = 2^{-h-3}$ and $z \in G$. Therefore we define, analogously to Proposition 6.1 (but, for notational convenience, without the factors 2^{-3}),

$$\eta_1^z = 2^{2h} \|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(Q_z^{**})}$$

and

$$\eta_2^z = 2^h |Du|(Q_z^{**})\,.$$

For the values of z such that $\eta_1^z \leq \delta/2^7$, i.e.,

$$\|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(Q_z^{**})} \leq \delta 2^{-2h-7}\,, \tag{7.12}$$

we can apply Theorem 6.1 to the square Q_z^{**}, and obtain $\nu_z \in S^1$, $a_z, b_z \in \mathbb{R}^N$, and a monotone function λ_z, such that the function $\tilde{u}_z(x) = a_z \lambda_z(x \cdot \nu_z) + b_z$ obeys

$$\begin{aligned}\|u - \tilde{u}_z\|_{L^2(Q_z^*)} &= \|u(x) - a_z\lambda_z(x \cdot \nu_z) - b_z\|_{L^2(Q_z^*)} \\ &\leq c2^{-h}((\eta_2^z)^{2/3}(\eta_3^z)^{1/3} + \eta_2^z(\eta_3^z)^{1/2} + \eta_1^z)\,,\end{aligned}$$

with

$$\|a_z\lambda_z\|_{L^\infty(\mathbb{R})} \leq c\eta_2^z\,.$$

Since $z \in G$ we have $\eta_3^z \leq \eta_2^z \leq M$, and the above conditions imply

$$\|u - \tilde{u}_z\|_{L^2(Q_z^*)} \leq C_M 2^{-h}((\eta_2^z)^{2/3}(\eta_3^z)^{1/3} + \eta_1^z)\,, \tag{7.13}$$

and

$$\|a_z\lambda_z\|_{L^\infty(\mathbb{R})} \leq cM\,.$$

Therefore we can apply Lemma 5.3 to the function $\tilde{u}_z$, and obtain a_z^*, $b_z^* \in \mathbb{Z}^N$ and $\lambda_z^* \in L^1(\mathbb{R}; \mathbb{Z})$ such that $u_z^*(x) = a_z^*\lambda_z^*(x \cdot \nu_z^*) + b_z^*$ obeys

$$\|\tilde{u}_z - u_z^*\|_{L^1(Q_z^*)} \leq C_M \|\mathrm{dist}(\tilde{u}_z, \mathbb{Z}^N)\|_{L^1(Q_z^*)}\,. \tag{7.14}$$

Here and below the dependence of the constant on M is indicated explicitly, whereas we do not indicate the dependence on N. In turn, using Remark 5.4, (7.14) gives

$$\begin{aligned}\|\tilde{u}_z - u_z^*\|^2_{L^2(Q_z^*)} &\leq C(M+N)\|\tilde{u}_z - u_z^*\|_{L^1(Q_z^*)} \\ &\leq C_M \|\mathrm{dist}(\tilde{u}_z, \mathbb{Z}^N)\|_{L^1(Q_z^*)} \\ &\leq C_M \left(\|\mathrm{dist}(u, \mathbb{Z}^N)\|_{L^1(Q_z^*)} + \|u - \tilde{u}_z\|_{L^1(Q_z^*)}\right)\,.\end{aligned}$$

Recalling the definition of η_1^z and (7.13), we obtain

$$\begin{aligned}\|\tilde u_z - u_z^*\|^2_{L^2(Q_z^*)} \leq & C_M\left(2^{-2h}\eta_1^z + 2^{-h}\|u-\tilde u_z\|_{L^2(Q_z^*)}\right)\\ \leq & C_M 2^{-2h}\left((\eta_2^z)^{2/3}(\eta_3^z)^{1/3}+\eta_1^z\right)\,.\end{aligned}$$

Since we assumed $\eta_3^z \leq \eta_2^z \leq M$ and $\eta_1^z \leq \delta/2^7$, estimate (7.13) implies

$$\|u-\tilde u_z\|^2_{L^2(Q_z^*)} \leq C_M 2^{-2h}((\eta_2^z)^{2/3}(\eta_3^z)^{1/3}+\eta_1^z)\,.$$

Therefore for those $z \in G$ for which (7.12) holds we have

$$\begin{aligned}\|u-u_z^*\|^2_{L^2(Q_z^*)} \leq & 2\left(\|u-\tilde u_z\|^2_{L^2(Q_z^*)} + \|\tilde u_z - u_z^*\|^2_{L^2(Q_z^*)}\right)\\ \leq & C_M 2^{-2h}\left((\eta_2^z)^{2/3}(\eta_3^z)^{1/3}+\eta_1^z\right)\,. \end{aligned}\tag{7.15}$$

If instead $z \in G$ is such that (7.12) does not hold, then we take u_z^* constant, equal to the integer closest to the average of u. Then by the Sobolev-Poincarè inequality

$$\|u-u_z^*\|^2_{L^2(Q_z^*)} \leq c2^{-2h} + c\left(|Du|(Q_z^*)\right)^2 \leq c(1+M^2)2^{-2h} \leq C_M \eta_1^z 2^{-2h}\,.$$

Therefore the estimate (7.15) holds for all $z \in G$.

We conclude that

$$\begin{aligned}\sum_{z\in G}\|u-u_z^*\|^2_{L^2(Q_z^*)} \leq & C_M 2^{-2h}\left(\sum_{z\in G}(\eta_2^z)^{2/3}(\eta_3^z)^{1/3} + \sum_{z\in G}\eta_1^z\right)\\ \leq & C_M 2^{-h}\left(\sum_{z\in G}2^{-h}\eta_2^z\right)^{2/3}\left(\sum_{z\in G}2^{-h}\eta_3^z\right)^{1/3} + C_M\sum_{z\in G}2^{-2h}\eta_1^z\,.\end{aligned}$$

Since $\sum \chi_{Q_z^{**}} \leq C\chi_\Omega$, we have

$$\begin{aligned}&\sum_{z\in G}\|u-u_z^*\|^2_{L^2(Q_z^*)}\\ &\leq C_M 2^{-h}\left(|Du|(\Omega)\right)^{2/3}\left(\sum_{z\in G}2^{-h}\eta_3^z\right)^{1/3} + C_M\|\mathrm{dist}(u,\mathbb{Z}^N)\|_{L^1(\Omega)}\,.\end{aligned}$$

The term containing η_3^z is estimated using (7.11),

$$\begin{aligned}&\sum_{z\in G}2^{-h}\eta_3^z \leq \sum_{z\in Z}2^{-h}\eta_3^z\\ \leq & c\left(\int_{\mathbb{R}^2}|Du|\left[\varphi_h * \sum_{z\in Z}\chi_{Q_z^{**}}\right] - \int_{\mathbb{R}^2}|D(u*\varphi_h)|\sum_{z\in Z}\chi_{Q_z^{**}}\right)^{1/2}\left(|Du|(\Omega)\right)^{1/2}\,.\end{aligned}$$

Recalling (7.4), and the fact that $\mathrm{dist}(Q_z^{**},\partial\Omega)\ge \mathrm{dist}(\omega,\partial\Omega)-\mathrm{diam}(Q_z^{**})\ge 2^{-h}$ for all $z\in Z$, we obtain

$$\varphi_h * \sum_{z\in Z}\chi_{Q_z^{**}} \le 64\varphi_h * \chi_{\cup \chi_{Q_z^{**}}} \le 64\chi_\Omega$$

and $\sum_{z\in Z}\chi_{Q_z^{**}}\ge 64\chi_\omega$. Therefore

$$\sum_{z\in G} 2^{-h}\eta_3^z \le c\left(|Du|(\Omega)-|D(u*\varphi_h)|(\omega)\right)^{1/2}\left(|Du|(\Omega)\right)^{1/2} .$$

We conclude

$$\begin{aligned}&\sum_{z\in G}\|u-u_z^*\|^2_{L^2(Q_z^*)}\\&\le C_M 2^{-h}\left(|Du|(\Omega)\right)^{5/6}\left(|Du|(\Omega)-|D(u*\varphi_h)|(\omega)\right)^{1/6}+C_M\|\mathrm{dist}(u,\mathbb{Z}^N)\|_{L^1(\Omega)} .\end{aligned}$$

This concludes the proof of (7.7).

Finally, from (7.13) and (7.14) we have

$$\|u-u_z^*\|_{L^1(Q_z^*)}\le C_M\|\mathrm{dist}(\tilde u_z,\mathbb{Z}^N)\|_{L^1(Q_z^*)}+C_M 2^{-2h}((\eta_2^z)^{2/3}(\eta_3^z)^{1/3}+\eta_1^z) .$$

Estimating the sum over all squares as above,

$$\begin{aligned}\sum_{z\in G}\|u_z^*-u\|_{L^1(Q_z^*)}&\le C_M\|\mathrm{dist}(\tilde u_z,\mathbb{Z}^N)\|_{L^1(\Omega)}\\&\quad + C_M 2^{-h}\left(|Du|(\Omega)-|D(u*\varphi_h)|(\omega)\right)^{1/3}\left(|Du|(\Omega)\right)^{2/3} .\end{aligned}$$

This, together with (7.10), concludes the proof of (7.6).

Step 4. Global construction. Based on the functions constructed above on each square, which obey the estimates (7.6) and (7.7), we shall now construct the global function w. The first idea is to set $w=u_z^*$ in $Q_z=(z+[0,1]^2)2^{-h-5}$. Since (7.6) gives only control of $u-u_z^*$ in L^1 the function w could have large jumps on ∂Q_z, and may not be in $BV(\omega)$. The standard device to avoid this is to set $w=u_z^*$ on the shifted squares $Q_{a,z}=(a+z+[-1/2,1/2]^2)2^{-h-5}$. Then one can use Fubini's theorem to show that there exists an $a\in[-1/4,1/4]^2$ such that w has good BV bound, see (7.18) and (7.19) below. Since Lemma 5.2 gives a control of the line energy in terms of a slightly enlarged square we also introduce the squares $\hat Q_{a,z}$.

Precisely, for any $a\in[-1/4,1/4]^2$ and $z\in\mathbb{Z}^2$ we define $Q_{a,z}=(a+z+[-1/2,1/2]^2)2^{-h-5}$ and $\hat Q_{a,z}=(a+z+[-1/2-2^{-t},1/2+2^{-t}]^2)2^{-h-5}$. We observe that $Q_{a,z}\subset\hat Q_{a,z}\subset Q_z^*$ for all admissible a, t, z. Further,

$$\chi_\omega\le\sum_{z\in Z}\chi_{Q_{a,z}}\le\chi_\Omega \qquad \text{a.e.}$$

for all admissbile choices of a.

We define

$$f(a)=\sum_{z,z'\in Z}\int_{\partial Q_{a,z}\cap\partial Q_{a,z'}}|u_z^*-u_{z'}^*|(x)d\mathcal{H}^1(x)$$

and observe that, by Fubini's theorem,

$$\int_{(-1/4,1/4)^2} f(a)da\le c2^h\sum_{z,z'\in Z}\|u_z^*-u_{z'}^*\|_{L^1(Q_z^*\cap Q_{z'}^*)}\le c2^h\sum_{z\in Z}\|u-u_z^*\|_{L^1(Q_z^*)}\,. \tag{7.16}$$

In order to control the error done by enlarging the squares we define analogously

$$g(a)=\sum_{z\in Z}\left[p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u)-p_{\Gamma_{h+t},Q_{a,z}}(u)\right]\,.$$

and claim that

$$\int_{(-1/4,1/4)^2} g(a)da\le c2^{-t}p_{h+t,\Omega}(u)\,. \tag{7.17}$$

To see this, we write

$$\begin{aligned}p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u)-p_{\Gamma_{h+t},Q_{a,z}}(u)=&\int_{\mathbb{R}^2\times\mathbb{R}^2}(u(x)-u(y))\cdot\Gamma_{h+t}(x-y)(u(x)-u(y))\\&\times\left[\chi_{\hat{Q}_{a,z}}(x)\chi_{\hat{Q}_{a,z}}(y)-\chi_{Q_{a,z}}(x)\chi_{Q_{a,z}}(y)\right]dxdy\end{aligned}$$

and observe that

$$\begin{aligned}&\chi_{\hat{Q}_{a,z}}(x)\chi_{\hat{Q}_{a,z}}(y)-\chi_{Q_{a,z}}(x)\chi_{Q_{a,z}}(y)=\\&\qquad\chi_{\hat{Q}_{a,z}}(x)\left[\chi_{\hat{Q}_{a,z}}(y)-\chi_{Q_{a,z}}(y)\right]+\left[\chi_{\hat{Q}_{a,z}}(x)-\chi_{Q_{a,z}}(x)\right]\chi_{Q_{a,z}}(y)\,.\end{aligned}$$

Focussing on the second term we note that $\chi_{Q_{a,z}}\le\chi_{Q_{0,z}^*}$ and $\chi_{Q_{a,z}}(x)=\chi_{Q_{0,z}}(x-2^{-h-5}a)$. Therefore

$$\begin{aligned}&\int_{(-1/4,1/4)^2}\left[\chi_{\hat{Q}_{a,z}}(x)-\chi_{Q_{a,z}}(x)\right]\chi_{Q_{a,z}}(y)da\\&\le\chi_{Q_z^*}(y)\chi_{Q_z^*}(x)\int_{\mathbb{R}^2}\chi_{\hat{Q}_{0,z}\setminus Q_{0,z}}(x-2^{-h-5}a)\,da\\&\le 2^{2h+10}\mathcal{L}^2(\hat{Q}_{0,z}\setminus Q_{0,z})\chi_{Q_z^*}(x)\chi_{Q_z^*}(y)\le c2^{-t}\chi_{Q_z^*}(x)\chi_{Q_z^*}(y)\,.\end{aligned}$$

An analogous estimate holds for the other term. We conclude that

$$\begin{aligned}\sum_{z\in Z}\int_{(-1/4,1/4)^2}&\Big[\chi_{\hat{Q}_{a,z}}(x)\chi_{\hat{Q}_{a,z}}(y)-\chi_{Q_{a,z}}(x)\chi_{Q_{a,z}}(y)\Big]\,da\\&\le c2^{-t}\sum_{z\in Z}\chi_{Q^*_z}(x)\chi_{Q^*_z}(y)\\&\le c2^{-t}\chi_\Omega(x)\chi_\Omega(y)\end{aligned}$$

and

$$\begin{aligned}\int_{(-1/4,1/4)^2}g(a)da\le&\int_{\mathbb{R}^2\times\mathbb{R}^2}(u(x)-u(y))\cdot\Gamma_{h+t}(x-y)(u(x)-u(y))\\&\times c2^{-t}\chi_\Omega(x)\chi_\Omega(y)\\=&c2^{-t}p_{h+t,\Omega}(u)\,.\end{aligned}$$

This concludes the proof of (7.17).

By (7.16) and (7.17) there exists $a\in(-1/4,1/4)^2$ such that

$$f(a)\le c2^h\sum_{z\in Z}\|u-u^*_z\|_{L^1(Q^*_z)} \tag{7.18}$$

and

$$g(a)\le c2^{-t}p_{h+t,\Omega}(u)\,.$$

We define

$$w=\sum_{z\in Z}u^*_z\chi_{Q_{a,z}}\,.$$

Clearly $w\in BV(\Omega;\mathbb{Z}^N)$, and (7.3) follows from (7.6). In order to prove (7.2) we first observe that

$$|Dw|\left(\bigcup_{z\in Z}\partial Q_{a,z}\right)\le cf(a)\,. \tag{7.19}$$

The fact that γ_0^{rel} is convex in the first argument and subadditive in the second easily implies $|\gamma_0^{\mathrm{rel}}(\nu,s)|\le C|s|$ (to see this, consider that $\gamma_0^{\mathrm{rel}}(\nu,s)\le\sum_{i=1}^N\sum_{j=1}^2|s_i||\nu_j|\gamma_0^{\mathrm{rel}}(e_j,e_i)$). Therefore

$$\int_{\omega\cap\bigcup_{z\in Z}\partial Q_{a,z}}\gamma_0^{\mathrm{rel}}(\nu,[w])d\mathcal{H}^1\le cf(a)\,.$$

By Lemma 5.2 we obtain

$$(\ln 2)E_0[w,Q_{a,z}]\le p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u^*_z)\qquad\forall\ z\in G\,.$$

Since $E_0^{\rm rel} \le E_0$ and $E_0^{\rm rel}[w, Q_{a,z}] = 0$ whenever $z \in B$,

$$
\begin{aligned}
(\ln 2) E_0^{\rm rel}[w,\omega] &\le \sum_{z\in G} (\ln 2) E_0[w, Q_{a,z}] + cf(a) \\
&\le \sum_{z\in G} p_{\Gamma_{h+t}, \hat{Q}_{a,z}}(u_z^*) + cf(a)\,.
\end{aligned}
$$

The first term can be estimated by

$$
\begin{aligned}
\sum_{z\in G} p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u_z^*) &\le (1+\eta) \sum_{z\in G} p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u) + \left(1+\frac{1}{\eta}\right) \sum_{z\in G} p_{\Gamma_{h+t},\hat{Q}_{a,z}}(u_z^* - u) \\
&\le (1+\eta) \sum_{z\in G} p_{\Gamma_{h+t},Q_{a,z}}(u) + (1+\eta) g(a) + \left(1+\frac{1}{\eta}\right) c2^{h+t} \sum_{z\in G} \|u_z^* - u\|^2_{L^2(Q_z^*)} \\
&\le (1+\eta+c2^{-t}) \sum_{z\in G} p_{\Gamma_{h+t},Q_{a,z}}(u) + \left(1+\frac{1}{\eta}\right) c2^{h+t} \sum_{z\in G} \|u_z^* - u\|^2_{L^2(Q_z^*)}
\end{aligned}
$$

Recalling (7.6), (7.7) and (7.18) we obtain (7.2). This finishes the proof of Proposition 7.1. □

8 Iterative mollification and conclusion of the proof

We now prove the following key result.

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and assume $u_0 \in BV(\Omega;\mathbb{Z}^N)$. Then for any sequences $\varepsilon_i \to 0$, $u_i \to u_0$ in $L^1(\Omega;\mathbb{R}^N)$ and any Lipschitz domain $\omega \subset\subset \Omega$ there is a sequence $w_j \in BV(\omega;\mathbb{Z}^N)$ such that $w_j \to u_0$ in $L^1(\omega;\mathbb{R}^N)$ and*

$$
\liminf_{j\to\infty} E_0^{\rm rel}[w_j,\omega] \le \liminf_{i\to\infty} E_{\varepsilon_i}[u_i,\Omega]\,.
$$

This result directly implies Theorem 2.1.

Proof of Theorem 2.1. Since $E_0^{\rm rel}$ is lower semicontinuous,

$$
E_0^{\rm rel}[u_0,\omega] \le \liminf_{j\to\infty} E_0^{\rm rel}[w_j,\omega] \le \liminf_{i\to\infty} E_{\varepsilon_i}[u_i,\Omega]\,.
$$

The conclusion follows by considering an increasing sequence ω_k with $\bigcup \omega_k = \Omega$. □

As already mentioned in Section 2, Theorem 2.1 yields the lower bound in the proof of Theorem 1.1. The upper bound is instead obtained by a more standard argument which we recall in the next section.

Proof of Proposition 8.1. We choose a Lipschitz set Ω' such that $\omega \subset\subset \Omega' \subset\subset \Omega$. For any $\delta > 0$, Proposition 4.2 applied to the pair of sets $\Omega' \subset\subset \Omega$ gives functions $v_{k,\delta} \in BV(\Omega'; \mathbb{Z}^N)$ such that

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{h=0}^{k} p_{\Gamma_h,\Omega'}(v_{k,\delta}) \le (\ln 2)(1+\delta) \liminf_{i\to\infty} E_{\varepsilon_i}[u_i, \Omega]\,, \tag{8.1}$$

with $\lim_{k\to\infty} \|v_{k,\delta} - u_0\|_{L^1(\Omega')} = 0$ and we can also assume that

$$\frac{1}{k} \sum_{h=0}^{k} p_{\Gamma_h,\Omega'}(v_{k,\delta}) + |Dv_{k,\delta}|(\Omega') \le A_\delta \quad \forall k$$

(since such a bound holds for the subsequence in k that realizes the liminf in (8.1)). The quantity A_δ may depend both on δ and on the original sequence u_i, but not on the parameters which will be chosen below.

We define, for $h \in \mathbb{N}$,

$$\Omega_h = \{x \in \mathbb{R}^2 : B_{2^{-h}}(x) \subset \Omega'\}\,.$$

For a fixed $m \ge 3$ we define iteratively, for all $h \in \mathbb{N} \cap [0, k+m]$, the functions $u_{k,\delta,m,h} \in BV(\Omega_h; \mathbb{R}^N)$ by

$$u_{k,\delta,m,h} = \begin{cases} v_{k,\delta} & \text{if } h \ge k\,, \\ u_{k,\delta,m,h+m} * \varphi_h & \text{else.} \end{cases}$$

The mollifier φ_h was defined at the beginning of Section 7. From the definition of $u_{k,\delta,m,h}$ we obtain, dropping the first three indices to simplify the notation,

$$\begin{aligned} \|u_h - u_{h+m}\|_{L^1(\Omega_h)} = \|u_{h+m} - u_{h+m} * \varphi_h\|_{L^1(\Omega_h)} &\le C2^{-h}|Du_{h+m}|(\Omega_{h+m}) \\ &\le C2^{-h}|Dv_{k,\delta}|(\Omega') \le C2^{-h}A_\delta\,, \end{aligned}$$

which, summing the geometric iteration, gives

$$\|u_{k,\delta,m,h} - v_{k,\delta}\|_{L^1(\Omega_h)} \le C2^{-h}|Dv_{k,\delta}|(\Omega) \le C2^{-h}A_\delta\,.$$

Recalling that $v_{k,\delta}$ has value in $\mathbb{Z}^N$ a.e. we also obtain

$$\|\mathrm{dist}(u_{k,\delta,m,h}, \mathbb{Z}^N)\|_{L^1(\Omega_h)} \le C2^{-h}|Dv_{k,\delta}|(\Omega) \le C2^{-h}A_\delta\,. \tag{8.2}$$

We further observe that by summing the telescoping series we get

$$\sum_{h=0}^{k} [|Du_{h+m}|(\Omega_{h+m}) - |Du_h|(\Omega_h)] = \sum_{h=k+1}^{k+m} |Du_h|(\Omega_h) - \sum_{h=0}^{m-1} |Du_h|(\Omega_h)$$
$$\leq CmA_\delta\,. \tag{8.3}$$

Pick $\zeta \in (0, 1/4)$ and $t \in \mathbb{N}$, with $m \geq t \geq 2$ and suppose that $\zeta k \geq m$ (we shall focus on large k). We claim that there exists $h \in (\zeta k, k - \zeta k) \cap \mathbb{N}$ such that

$$p_{\Gamma_{h+t},\Omega'}(v_{k,\delta}) \leq (1 + 5\zeta)\frac{1}{k}\sum_{j=0}^{k} p_{\Gamma_j,\Omega'}(v_{k,\delta})\,. \tag{8.4}$$

By (8.3) we can choose h such that (8.4) holds and additionally

$$|Du_{k,\delta,m,h+m}|(\Omega_{h+m}) - |Du_{k,\delta,m,h}|(\Omega_h) \leq c\frac{m}{k\zeta}A_\delta\,.$$

We apply Proposition 7.1 to the (smooth) function $u_{k,\delta,m,h+m}$, with the chosen value of h and the pair of domains $\omega \subset\subset \Omega_h$, with parameters M and η still to be chosen. Since h was chosen in dependence on the other parameters, we denote the result by $w_{k,\delta,m,t,M,\eta}$. Since $h \geq \zeta k$, for k large enough (on a scale depending on ζ) the assumption on the domains is fulfilled. By the convexity of $p_{\Gamma_{h+t}}(u)$ and the translation invariance of the kernel, denoting $u_z(x) = u(x-z)$, we have

$$\begin{aligned} p_{\Gamma_{h+t},\Omega_h}(u * \varphi_{h+m}) &\leq \int_{\mathbb{R}^2} \varphi_{h+m}(z) p_{\Gamma_{h+t},\Omega_h}(u_z)\,dz \\ &\leq \int_{\mathbb{R}^2} \varphi_{h+m}(z) p_{\Gamma_{h+t},\Omega_{h+m}}(u)\,dz = p_{\Gamma_{h+t},\Omega_{h+m}}(u) \end{aligned}$$

Since by definition $u_{k,\delta,m,h+m} = u_{k,\delta,m,h+2m} * \varphi_{h+m}$, iterating the above inequality we get

$$p_{\Gamma_{h+t},\Omega_h}(u_{k,\delta,m,h+m}) \leq p_{\Gamma_{h+t},\Omega'}(v_{k,\delta})$$

We then obtain

$$\begin{aligned} (\ln 2)\, E_0^{\mathrm{rel}}[w_{k,\delta,m,t,M,\eta}, \omega] \leq &\frac{1}{k}\sum_{h=0}^{k} p_{\Gamma_h,\Omega'}(v_{k,\delta}) + (4\zeta + \eta + c2^{-t})A_\delta + \frac{C_M}{\eta}2^{h+t}2^{-h-m}A_\delta \\ &+ \frac{C_M}{\eta}2^t A_\delta^{5/6}\left(\frac{m}{k\zeta}A_\delta\right)^{1/6} + \frac{c}{M^{1/2}}2^{t/2}A_\delta \end{aligned}$$

(for all k large enough). Therefore, setting $\eta = \zeta$ and recalling (8.1), we get

$$\begin{aligned}
&\liminf_{t\to\infty}\liminf_{M\to\infty}\liminf_{\zeta\to 0}\liminf_{m\to\infty}\liminf_{k\to\infty} E_0^{\mathrm{rel}}[w_{k,\delta,m,t,M,\eta},\omega] \\
&\qquad \le \liminf_{k\to\infty}\frac{1}{\ln 2}\frac{1}{k}\sum_{h=0}^{k} p_{\Gamma_h,\Omega'}(v_{k,\delta}) \\
&\qquad \le (1+\delta)\liminf_{i\to\infty} E_{\varepsilon_i}[u_i,\Omega]\,,
\end{aligned}$$

and therefore

$$\liminf_{\delta\to 0}\liminf_{t\to\infty}\liminf_{M\to\infty}\liminf_{\zeta\to 0}\liminf_{m\to\infty}\liminf_{k\to\infty} E_0^{\mathrm{rel}}[w_{k,\delta,m,t,M,\eta},\omega] \le \liminf_{i\to\infty} E_{\varepsilon_i}[u_i,\Omega]\,.$$

Analogously

$$\limsup_{\delta\to 0}\limsup_{t\to\infty}\limsup_{M\to\infty}\limsup_{\zeta\to 0}\limsup_{m\to\infty}\limsup_{k\to\infty} \|w_{k,\delta,m,t,M,\eta}-u_0\|_{L^1(\omega)} = 0\,.$$

Taking a diagonal subsequence we conclude the proof. □

9 Upper bound

As regards to the upper bound required for the proof of Theorem 1.1 one can use the abstract result of [14]. Indeed one can show that the abstract Γ-limit E exists and takes the form, for $u \in BV(\Omega;\mathbb{Z}^N)$,

$$(\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon)[u,\Omega] = \int_{\Omega\cap J_u} \varphi([u],\nu_u)\,\mathcal{H}^1\,,$$

for some φ to be determined. Now take for any $\nu \in S^1$ and $s \in \mathbb{Z}^N$ a one-dimensional function with a single interface, i.e.,

$$u(x) = \begin{cases} 0 & \text{if } x\cdot\nu < 0 \\ s & \text{if } x\cdot\nu \ge 0\,. \end{cases} \tag{9.1}$$

Let u_ε be a mollification of u at scale ε. By an explicit computation one can show that

$$\lim_{\varepsilon\to 0} E_\varepsilon[u_\varepsilon, B_1(0)] = 2\gamma_0(\nu,s)\,. \tag{9.2}$$

Therefore $\varphi \le \gamma_0$. By the lower semicontinuity of the $\Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon$ and the abstract relaxation results of [6, 7, 8] the integrand φ is BV-elliptic, and therefore $\varphi \le \gamma_0^{\mathrm{rel}}$. Equivalently, $E_0^{\mathrm{rel}} \le \Gamma\text{-}\lim_{\varepsilon\to 0} E_\varepsilon$. This yields the upper bound and finishes the proof of Theorem 1.1.

Acknowledgements

This work was partially supported by the Deutsche Forschungsgemeinschaft through the Forschergruppe 797 *"Analysis and computation of microstructure in finite plasticity"*, projects Co304/4-1 and Mu1067/9-1.

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