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PhD Thesis

Intrinsic regular hypersurfaces in Heisenberg groups and weak solutions of non linear first-order PDEs

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To my Family and Antonella

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Introduction

A fundamental problem of geometric analysis, and also of geometric measure theory, is the investigation of the interplay between a surface of a given manifold and its normal. Typically this investigation consists in the study of suitable PDEs once a coordinates' system for the surface has been fixed. In the spirit of this strategy in this thesis we will study the relationships between weak solutions of nonlinear first order PDEs and $\mathbb{H}$-regular intrinsic graphs. $\mathbb{H}$-regular intrinsic graphs are a class of *intrinsic regular* hypersurfaces in the setting of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$, endowed with a left-invariant not euclidean metric d_∞. Here hypersurface simply means a topological codimension 1 surface and by the words "intrinsic" and "regular" we will mean of notions involving respectively the group structure of $\mathbb{H}^n$ and its differential structure as Carnot-Carathéodory manifold in a sense we will define below. In particular we will investigate the problem of the regularity of the parametrization of a hypersurface through well-known results of the theory of weak solutions of conservation laws, a special class of non linear first order PDEs, with which we can describe the normal of the hypersurface.

Given an intrinsic graph $S = G^1_{\mathbb{H},\phi}(\omega) = \Phi(\omega) \subset \mathbb{H}^n$ (see Definition 3.1.12 and (8)) where $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ we will study the relationships between S and ϕ so that S is an $\mathbb{H}$-regular surface (see Definition 3.1.1) and ϕ is a suitable solution of the nonlinear first order PDEs' system

$$\nabla^\phi \phi = w \quad \text{in}\, \omega\,, \tag{1}$$

being ∇^ϕ the family of first order differential operators defined in (9), $w \in C^0(\omega; \mathbb{R}^{2n-1})$ prescribed. In the first Heisenberg group $\mathbb{H}^1$ (1) reads as the classical Burgers' equation whereas in highest Heisenberg groups, i.e. $\mathbb{H}^n$ with $n \geq 2$, it is a real nonlinear system. In [4] $W^\phi \phi$ has been recognized as intrinsic gradient of ϕ in a suitable differential structure projected on $\mathbb{R}^{2n}$ from the CC differential structure of $\mathbb{H}^n$ through the graph parameterization $\Phi : \omega \to S$ as we will define below.

The notion of regular surface in Carnot groups, of which $\mathbb{H}^n$ is the simplest example, and in a more general metric space has been investigated in order to study the classical problem of geometric measure theory of defining regular surfaces, different measures on them and minimal surfaces. This study has been carried out by many authors during the last thirty years and a general account of the many facets and contributions is far beyond the aim of this introduction. Here we limit ourselves to recommend the reader to the general monographs [46, 63, 65, 66, 81, 80, 76, 25], to the articles [83, 47, 48, 68, 24, 64, 60, 26, 1, 55, 82, 56, 57, 73, 44, 85, 78, 29, 30, 58, 77, 3, 32] and references therein.

The notion of intrinsic graph has been introduced in [58] in the setting of a Carnot group and deeply studied in the setting of $\mathbb{H}^n$ in [4], although it was already implicitly used in [55].

Intrinsic graphs in Carnot groups had two main applications so far. The first application has been in the theory of rectifiability in Carnot groups. Indeed in [57] classical De Giorgi's rectifiability and divergence theorems for sets of finite perimeter was fully extended to a Carnot group of step 2. Let us point out that recently an interesting application of this rectifiability result provided in [27] a counterexample in the framework of theoretical computer science. The second one has been in the framework of the Bernstein problem in $\mathbb{H}^n$. Namely in [11] it has been proved that an entire perimeter minimizing regular intrinsic graph in the first Heisenberg group $\mathbb{H}^1$ has to be a an intrinsic plane (see also [43] and [45] for an extension to a wider class of surfaces in $\mathbb{H}^1$). Let us recall that the classical notion of Euclidean graph in $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ does not apply in the two previous topics as proved respectively in [1] for the problem of rectifiability in $\mathbb{H}^n$ (see also [69]) and in [29, 61, 89] for the Bernstein problem in $\mathbb{H}^1$.

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$ is the simplest example of Carnot group, endowed with a left- invariant metric d_∞ equivalent to its Carnot-Carathéodory (CC) metric, but not equivalent to the euclidean metric. We shall denote the points of $\mathbb{H}^n$ by $P = (z,t) = (x+iy,t)$, $z \in \mathbb{C}^n$, $x,y \in \mathbb{R}^n$, $t \in \mathbb{R}$, and also by $P = (x_1,\dots,x_n,y_1,\dots,y_n,t) = (x_1,\dots,x_n,x_{n+1},\dots,x_{2n},t)$. If $P = (z,t)$, $Q = (\zeta,\tau) \in \mathbb{H}^n$ and $r > 0$, following the notations of [92], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := \left(z+\zeta, t+\tau - \frac{1}{2}\Im m(z\cdot\bar{\zeta})\right) \tag{2}$$

and the family of non isotropic dilations

$$\delta_r(P) := (rz, r^2 t), \text{ for } r > 0. \tag{3}$$

We denote as $P^{-1} := (-z,-t)$ the inverse of P and as e the origin of $\mathbb{R}^{2n+1}$.

Moreover $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{4}$$

and the distance d_∞ we shall deal with is defined as

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty. \tag{5}$$

$(\mathbb{H}^n, d_\infty)$ provides the simplest example of a metric space that is not Euclidean, even locally, but is still endowed with a sufficiently rich compatible underlying structure, due to the existence of intrinsic families of left translations and dilations respectively induced from the group law (2.1) and dilations (2.2). Indeed, the geometry of $\mathbb{H}^n$ is noneuclidean at every scale, since it was proved in [91] that there are no bi-Lipschitz maps from $\mathbb{H}^n$ to any Euclidean space. It is well-known that $\mathbb{H}^n$ is a Lie group of topological dimension $2n+1$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $\mathcal{Q} := 2n+2$ (see Proposition 2.1.13).

$\mathbb{H}^n$ is a Carnot group of step 2. Indeed its Lie algebra $\mathfrak{h}_n$ is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t}, \qquad \text{for } j = 1, \dots, n; \qquad T = \frac{\partial}{\partial t}, \tag{6}$$

and the only non-trivial commutator relations are

$$[X_j, Y_j] = T, \qquad \text{for } j = 1, \dots, n.$$

We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [64]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$.

To introduce our results, let us start by recalling some related notions already existing in the literature. The two key points we want to stress now are the notions of intrinsic regular hypersurface and graph in $\mathbb{H}^n$. A general and more complete discussion of these topics in Carnot groups can be found in [58].

Let us recall that in the Euclidean setting $\mathbb{R}^n$, a C^1-hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a notion was easily transposed in [55] to the Heisenberg group, since an intrinsic notion of $C^1_{\mathbb{H}}$-functions has been introduced by Folland and Stein (see [53]): we can state that a continuous real function f on $\mathbb{H}^n$ belongs to $C^1_{\mathbb{H}}(\mathbb{H}^n)$ if its horizontal gradient $\nabla_{\mathbb{H}} f$, defined by $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$ in the sense of distributions, is a continuous vector-valued function. We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$*-regular hypersurface* if it is locally defined as the set of points $P \in \mathbb{H}^n$ such that $f(P) = 0$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on S (see Definition 3.1.1). Since it is not restrictive we will deal in the following with $\mathbb{H}$-regular surfaces S which are locally zero level sets of function $f \in C^1_{\mathbb{H}}$ with $X_1 f \neq 0$. We can also define the *horizontal normal* to S at a point $P \in S$ $\nu_S(P)$, as the unit vector $\nu_S(P) := -\dfrac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|_P} \in \mathrm{H}\mathbb{H}^n_P$.

First of all, we point out that the class of $\mathbb{H}$-regular surfaces is deeply different from the class of Euclidean regular surfaces, in the sense that there are $\mathbb{H}$-regular surfaces in $\mathbb{H}^1 \simeq \mathbb{R}^3$ that are (Euclidean) fractal sets (see [69]), and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$-regular hypersurfaces (see [55], Remark 6.2). We notice that Euclidean continuously differentiable $2n$-manifolds are $\mathbb{H}$-regular surfaces provided they do not contain characteristic points, i.e. points P such that the Euclidean tangent space at P coincides with the horizontal fiber $\mathrm{H}\mathbb{H}^n_P$ at P. According to Frobenius' Theorem, for a general smooth manifold, the set of characteristic points has empty interior; in fact there are few characteristic points ([9], [77]).

The important point supporting the choice of the notion is the fact that this definition yields an Implicit Function Theorem, proved in [55] for the Heisenberg group and in [56] for a general Carnot group (see also [32] for an extension to a CC metric space), so that a $\mathbb{H}$-regular surface locally is a X_1-*graph*, namely (see Definition 3.1.12) there is a continuous parameterization of S

$$\Phi : \omega \subset (\mathbb{V}_1, |\cdot|) \to (S, d_\infty) \tag{7}$$
$$\Phi(A) := A \cdot (\phi(A) e_1) \tag{8}$$

where $\phi : \omega \to \mathbb{R}$ is continuous, $\mathbb{V}_1 := \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\}$, $\omega \subset \mathbb{V}_1$, $\{e_j : \ j = 1, \dots, 2n+1\}$ denotes the standard basis in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ and we consider $|\cdot|$ the Euclidean distance on $\mathbb{V}_1 \simeq \mathbb{R}^{2n}$, (see Theorem 3.1.13). In general, such a parameterization is not continuously differentiable or even Lipschitz continuous. Indeed it has been proved in [69] that generally its best Hölder continuous regularity turns out to be of order $1/2$ with respect to the distances given in (7). A natural question arising is the characterization of the functions $\phi : \omega \to \mathbb{R}$ such that $S = G^1_{\mathbb{H},\phi}(\omega) = \Phi(\omega)$ is $\mathbb{H}$ regular. A characterization has been proposed in [4, 94], see [32] too. Through a natural identification between $\mathbb{V}_1$ and $\mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2} \times \mathbb{R}_\tau$, they consider the parametrization ϕ as a suitable solution of the nonlinear first order PDEs' system (1) $\nabla^\phi \phi = w \quad \text{in}\, \omega$, being ∇^ϕ the family of first order differential operators defined as

$$\nabla_j^\phi := \begin{cases} \dfrac{\partial}{\partial v_j} - \dfrac{v_{j+n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } 2 \leq j \leq n \\ \dfrac{\partial}{\partial \eta} + \phi \dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \dfrac{\partial}{\partial v_j} + \dfrac{v_{j-n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \leq j \leq 2n, \end{cases} \tag{9}$$

when $n \geq 2$ while when $n = 1$ as $\nabla^\phi = \nabla_2^\phi := \dfrac{\partial}{\partial \eta} + \phi \dfrac{\partial}{\partial \tau}$.

In particular let us notice the (nonlinear) differential operator

$$C^1(\omega) \ni \phi \to \mathfrak{B}\phi := \nabla_{n+1}^\phi \phi \tag{10}$$

is a Burgers' type operator which can be also represented in distributional form as

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} + \frac{1}{2}\frac{\partial \phi^2}{\partial \tau}$$

In [4] it has been proved that each $\mathbb{H}$-regular graph $G^1_{\mathbb{H},\phi}(\omega)$ admits an *intrinsic gradient* $\nabla^\phi \phi \in C^0(\omega; \mathbb{R}^{2n})$, in the sense of distributions, which shares a lot of properties with the Euclidean gradient.

Let us recall that the problem of characterizing intrinsic regular graphs was studied also in [32] in the general setting of a CC space. Moreover also a notion of intrinsic Lipschitz graph in $\mathbb{H}^n$ has been introduced in [54] and a study similar to the one in [4] has been recently carried out in [7] in the case of $\mathbb{H}$-regular intrinsic graphs in $\mathbb{H}^n$ with codimension bigger than 1.

In [4] $\nabla^\phi \phi$ has been characterized as intrinsic gradient of ϕ in a suitable differential structure projected on $\mathbb{R}^{2n}$ from the CC differential structure of $\mathbb{H}^n$ through the graph parameterization $\Phi : \omega \to S$. The main results of [4] (Theorems 1.2 and 1.3) prove that if $\phi :\ \omega \to \mathbb{R}$ is a continuous function, then $S = \Phi(\omega) = G^1_{\mathbb{H},\phi}(\omega)$ is an $\mathbb{H}$-regular surface if and only if the distribution $\nabla^\phi \phi$ is represented by a function $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ and there exists a family $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ such that, for any open set $\omega' \Subset \omega$, we have

$$\phi_\epsilon \to \phi \text{ and } \nabla^{\phi_\epsilon}\phi_\epsilon \to w \text{ uniformly in } \omega'. \tag{11}$$

Moreover, for every $P \in S$, the horizontal normal to S ν_S can be represented by

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) (\Phi^{-1}(P)). \tag{12}$$

This characterization motives the methods and techniques used in our work. Indeed they draw mainly from the theory of nonlinear first order PDEs and from the study of Burgers' equation. In particular it is fundamental the study of two classes of weak solutions of PDEs: the distributional solutions of the equation $\phi_\eta + \phi\phi_\tau = w$ (see definition 1.1.2) and the broad* solutions of the system (1), i.e. (see definition 3.3.1) a continuous function $\phi : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}$ such that for every $A \in \omega$, $\forall\, j = 2, ..., 2n$ there exists an exponential map,

$$\gamma_j^B(s) = \exp(s\nabla_j^\phi)(B) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$$

where $0 < \delta_2 < \delta_1$, $s \in [-\delta_2, \delta_2]$ such that $\forall\, B \in I_{\delta_2}(A)$

(E.1) $\gamma_j^B \in C^1([-\delta_2, \delta_2])$

(E.2) $\begin{cases} \dot\gamma_j^B = \nabla_j^\phi \circ \gamma_j^B \\ \gamma_j^B(0) = B \end{cases}$

(E.3) $\phi\left(\gamma_j^B(s)\right) - \phi\left(\gamma_j^B(0)\right) = \int_0^s w_j\left(\gamma_j^B(r)\right)\, dr \qquad \forall\, s \in [-\delta_2, \delta_2]$

In fact, as we will explain below, some of our main results are the following characterizations: if $\phi \in C^0(\omega)$ and $w \in C^0(\omega, \mathbb{R}^{2n-1})$ the following conditions are equivalent:

i $S = G^1_{\mathbb{H},\phi}(\omega)$ is an $\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$ and $\forall\, P \in S$

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1+|w|^2}}, \frac{w}{\sqrt{1+|w|^2}}\right)(\Phi^{-1}(P)).$$

ii ϕ is a broad* solution of the system $\nabla^\phi\phi = w$.

iii ϕ is a distributional solution of the system $\nabla^\phi\phi = w$.

The continuity of the broad* and distributional solutions plays a central rule in the discussion: indeed when the function ϕ is continuous, the concepts of distributional and broad* solution are the same.

The structure of the thesis is the following. In chapter 1 we provide a complete exposition of preliminary and classical results about the theory of conservation laws. As we said before, the fundamental concept of weak solution is investigated. Following [21, 51, 70] we study the broad solution $u : \omega = (0, T) \times (-r_0, r_0) \subseteq \mathbb{R}^2_{t,x} \to \mathbb{R}$ for the quasilinear conservation law

$$u_t + uu_x = g(t, x)$$

and distributional solution for the general equation

$$\begin{cases} u_t + f(u)_x = g(t,x) & \text{in}\,\omega \\ u(0,x) = u_0(x) & \text{on}\,\{0\} \times [-r_0, r_0] \end{cases} \tag{13}$$

In general, a distributional solution of problem (13) could be not unique, see example 1.1.4. In particular distributional solutions are not smooth, but could be discontinuous. Kružhkov define in [70] a special class of distributional solutions, the entropy solutions, which are physically admissible and prove an important uniqueness Theorem for this kind of solutions, see [13].

In section 1.5 we give a short introduction to the study of the Hamilton-Jacobi equation

$$u_t + H(u_x) = G(t,x), \tag{14}$$

see [37, 38, 39, 51, 74]. In particular we study the notion of viscosity solution of (14) and the link between the viscosity solution of (14) and the entropy solution of the conservation law $u_t+H(u)_x = g(t,x)$, where $g(t,x) = \dfrac{\partial}{\partial x}G(t,x)$.

Chapter 2 is devoted to a complete description of the Heisenberg group $\mathbb{H}^n$, that we introduced in the first part of this introduction. After the recalling of the most important and well-know definitions and preliminary results, we give an exhaustive exposition of multilinear algebra in $\mathbb{H}^n$, following [8, 52, 58, 59]. In section 2.4 we revisit the theory of the Rumin complex [90], a complex of intrinsic differential forms that fits the structure of $\mathbb{H}^n$ in the same way as De Rham complex does in Euclidean space. We prove an interesting generalization of some results of [59] in $\mathbb{H}^n$. We define the operator $\text{curl}_{\mathbb{H}}$ through the Rumin theory and we write explicitly its components. Then we establish the explicit compatibility's conditions for the existence of a primitive of $F = (F_1, ..., F_{2n})$ with $F_j \in \mathcal{D}'(\Omega)$, where $\Omega \subset \mathbb{H}^n$ is open and simply connected: similarly to the classical Poincarè Lemma in Euclidean setting, there exists $f \in \mathcal{D}'(\Omega)$ such that $\nabla_{\mathbb{H}} f = F$ if and only if $\text{curl}_{\mathbb{H}} F = 0$, where the equalities have to be understood in distributional sense.

Chapter 3 is dedicated to the study of intrinsic $\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$. As we said, a subset $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurfaces if it is locally defined as zero's level set of a non critical function $f \in C^1_{\mathbb{H}}(\mathbb{H}^n)$, i.e. $\nabla_{\mathbb{H}} f \neq 0$ on S. In this chapter we recall the most important Theorem about $\mathbb{H}$-regular hypersurfaces, following [4, 32, 54, 55, 56, 57, 58, 69, 64, 76, 81, 94]. In particular we study the Implicit Function Theorem and the intrinsic gradient $\nabla^{\phi}\phi$, about which we discussed in the first part of this introduction. The most important original result of this chapter is Theorem 3.3.12, an Hölder continuous regularity result for broad* solutions which extends a previous one given in [4] for C^1 regular solution ϕ of the system $\nabla^\phi\phi = w$ (see [4], Theorem 5.8). This technical Theorem will be the central point of the proofs of the characterization of the parametrization ϕ of $\mathbb{H}$-regular hypersurfaces (Theorem 0.0.2) and of their euclidean regularity results (Theorems 0.0.4 and 0.0.5).

In section 3.4 we consider some regularity problems about the parametrization Φ of S $\mathbb{H}$-regular hypersurface, that are written in collaboration with D.Vittone [20]. Namely we are able to give a negative answer to the question of extending an interesting result of Lipschitz regularity, done by D. R. Cole and S. Pauls in [34] for C^1 surface S in $\mathbb{H}^1$, to general $\mathbb{H}$-regular surfaces. $\Phi : (\mathbb{R}^2, \varrho) \to (\mathbb{H}^1, d_\infty)$ cannot be bi-Lipschitz where $\varrho\big((x,z),(x',z')\big) := |x-x'|+|z-z'|^{1/2}$, see Theorem 3.4.2. The idea for constructing a counterexample lies in the possibility of finding $\mathbb{H}$-regular surfaces which are connected by curves with finite length, and to notice that the parabolic plane $(\mathbb{R}^2, \varrho)$ does not share this property. This first result was obtained by D. Vittone in [94] with the help of G. Citti and Z. Balogh.

An other question risen in [69] was to understand whether the map Φ belongs to some Sobolev class $W^{1,p}_m((\omega,d),(\mathbb{H}^n,d_c))$ of maps between metric spaces. We are able to answer in the negative also to this second question:

Theorem 0.0.1. *The parametrization $\Phi : \omega \to S$ of an $\mathbb{H}$-regular surface S does not belong to $W^{1,p}_m((\omega,d),(\mathbb{H}^n,d_\infty))$ for any $1 \le p \le +\infty$ when d is the Euclidean distance on ω. The same result holds when d is the distance $d_{\infty|\omega}$ on $\omega \subset \mathbb{V}_1 \simeq \mathbb{R}^{2n}$ provided Φ is not the inclusion map $\omega \hookrightarrow \mathbb{H}^n$ (i.e. if $\phi \not\equiv 0$).*

In chapter 4 we explain the original results obtained in collaboration with professor Serra Cassano and exposed in [18, 19]. As we said in our context the notion of broad* solution can be understood as a notion of C^1-differentiability with respect to the vector fields ∇^ϕ. Indeed we prove that the notions of $\mathbb{H}$-regular hypersurface and the one of broad* solution of the system (1) are equivalent.

Theorem 0.0.2. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ and $w = (w_2,...,w_{2n}) : \omega \to \mathbb{R}^{2n-1}$ be continuous functions. Then the following conditions are equivalent:*

i

$$\phi \textit{ is a broad* solution of the system } \nabla^\phi\phi = w \textit{ in } \omega\,; \tag{15}$$

ii *$S = G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular and $\nu^{(1)}_S(P) < 0$ for all $P \in S$, where we denote with $\nu_S(P) = \Big(\nu^{(1)}_S(P),...,\nu^{(2n)}_S(P)\Big)$ the horizontal normal to S at a* *point $P \in S$. Moreover*

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) \left(\Phi^{-1}(P) \right)$$

$\forall\, P \in S$ where $\nabla^\phi \phi$ denotes the intrinsic gradient of ϕ.

Let us explicitly point out that Theorem 0.0.2 extends the characterization of $\mathbb{H}$-regular intrinsic graphs contained in [4] (see Theorems 3.2.12 and 3.3.9). Indeed the results contained in [4] yield the thesis of Theorem 0.0.2 provided the additional assumption that ϕ is little Hölder continuous of order 1/2 (see Lemma 4.1.2) is made. Here the key step to the proof of Theorem 0.0.2 will be to gain 1/2-little Hölder continuity when ϕ is supposed to be only a (continuous) broad* solution of the system (1) (see Theorem 3.3.12).

Theorem 0.0.2 also yields that each Lipschitz continuous solution ϕ of the system (1) with w continuous induces a $\mathbb{H}$-regular graph (see Corollary 4.1.4). Moreover a broad* solution of (1) turns out to be also a distributional solution (see Corollary 4.1.5).

In section 4.2 we prove a second new characterization of the parametrization of $\mathbb{H}$-regular hypersurfaces. By the link between continuous broad* solutions and continuous distributional solutions of Burgers' equation, obtained by a result of [42] (see Theorem 1.4.17), we can show the following:

Theorem 0.0.3. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

i *$S := \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface.*

ii *There exists $w = (w_2, \ldots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of the system* (1).

The characterization given in Theorem 0.0.3 is the exact counterpart of the distributional one in the Euclidean setting. Namely a function $\phi \in C^1(\omega)$ can be understood as a continuous distributional solution of $\nabla \phi = w$ in ω, provided $w \in C^0(\omega; \mathbb{R}^m)$ and $\omega \subset \mathbb{R}^m$ open set. Let us observe that the strong approximation assumption (11) is not required in the statement **ii** of Theorem 0.0.3. Its equivalence to the statement of Theorems 1.2 and 1.3 of [4] is not immediate. Our strategy will be to prove the equivalence between the statement **ii** of Theorem 0.0.3 and the statement **i** of Theorem 0.0.2.

On the other hand we do not know whether the approximation (11) can be directly obtained by recoursing to technical devices like mollification or approximation by vanishing viscosity of the continuous distributional solutions of the system (1). A very deep study of vanishing viscosity solutions with bounded variation of nonlinear hyperbolic systems has been carried out in [15] (see also the remark in [15], section 1.3). This study does not seem to apply to our context where the solution is supposed to be only continuous.

In the section 4.3 a local uniqueness result for broad* solutions of (1) uniformly bounded in ω is also given provided initial conditions (see Theorem 4.3.1). As far as the existence of broad* solutions for (4.1) is concerned we will prove that there will always be broad* solutions for any assigned initial conditions but for suitable data w (see Theorem 4.3.4). In the case $n \geq 2$ compatibility's conditions among the components of w are needed for the existence of broad* solutions as pointed out in Theorem 4.3.5 and Remark 4.3.6.

In the section 4.4 we will study the Euclidean regularity of an $\mathbb{H}$-regular graph $S = G^1_{\mathbb{H},\phi}(\omega)$ through the regularity of its intrinsic gradient $\nabla^\phi\phi$.

Theorem 0.0.4. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set, let $G^1_{\mathbb{H},\phi}(\omega)$ be $\mathbb{H}$-regular in $\mathbb{H}^n$ and let us assume the component of its intrinsic gradient $\nabla^\phi_{n+1}\phi \in Lip_{loc}(\omega)$. Then $\phi \in Lip_{loc}(\omega)$.*

Let us point out that Theorem 0.0.4 is sharp. Indeed in [11], Example 2.8 it has been proved that if $\phi : \omega := (-1,1) \times \mathbb{R} \to \mathbb{R}$, $\phi(\eta,\tau) := \frac{\tau}{\eta + \frac{\tau}{|\tau|}}$ then $\phi \in Lip_{loc}(\omega) \setminus C^1(\omega)$, $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular in $\mathbb{H}^1$ and its intrinsic gradient $\nabla^\phi\phi \equiv 0$ in ω.

Weakening the assumption $\nabla^\phi_{n+1}\phi \in Lip(\omega)$ with $\nabla^\phi_{n+1}\phi \in C^{0,\alpha}(\omega)$ the thesis of Theorem 0.0.4 can fail. For instance, if $n = 1$ by [4] Corollary 5.11 (see also [94]) we can construct for each $\alpha \in \left(\frac{1}{2}, 1\right)$ a function $\phi \in C^{0,\alpha}(\omega)$ such that $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular and $\nabla^\phi\phi \in C^{0,2\alpha-1}(\omega)$.

Moreover a regularizing effect is stressed when $n \geq 2$ by an higher regularity result which fails if $n = 1$ (see also Theorem 4.4.5, Corollary 4.4.6 and Remark 4.4.7).

Theorem 0.0.5. *Let $n \geq 2$, $\omega \subseteq \mathbb{R}^{2n}$ be an open set and let $\phi \in Lip(\omega)$ and $w = (w_2, \ldots, w_{2n}) \in Lip(\omega; \mathbb{R}^{2n-1})$ such that $\nabla^\phi\phi = w$ a.e. in ω. Then $\phi \in C^1(\omega)$.*

As a consequence of this study we will get a (local) uniqueness result for $\mathbb{H}$-regular graphs of a prescribed horizontal normal (see Corollary 4.3.3).

Eventually let us point out that this regularity technique could help in the approach to the difficult problem of the regularity for the minimizers of sets of finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$ (see Definition 2.2.25). Indeed, by analogy with the Euclidean setting, a key tool was to get regularity for a set of finite perimeter by means of the regularity of its generalized normal (see, for instance, [62], Theorem 4.11). Some problems related to this topic have been studied in [86, 29, 28, 87, 30, 89, 31] for sets whose boundary is an Euclidean graph and in [4, 22, 23] for sets whose boundary is an intrinsic graph, assuming at least Lipschitz regularity. On the other hand Theorem 0.0.4 could be applied to $\mathbb{H}$-perimeter minimizing intrinsic graphs which, *a priori*, could be less regular than Lipschitz.

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Basic notation

$\Subset$	compactly contained
$\#A$	cardinality of a set A
$\oplus$	direct sum of vector spaces
$\circ$	composition of functions
$\mathbb{R}^n$	n-dimensional Euclidean space
$\{e_1, \ldots, e_n\}$	canonical basis of $\mathbb{R}^n$
ω, Ω	open sets in $\mathbb{R}^n$
K	compact set in $\mathbb{R}^n$
$U(x,r)$, $B(x,r)$	open and closed ball with respect to the euclidean metric
$U_\rho(x,r)$, $B_\rho(x,r)$	open and closed ball with respect to the metric ρ
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\mathcal{H}^d_\rho, \mathcal{S}^d_\rho$	d-dim. Hausdorff and spherical measures induced by ρ
ω_s	Lebesgue measure of the unit ball in $\mathbb{R}^s$
∂_i	i-th vector of the standard basis of $\mathbb{R}^n$
$\partial_i f$	partial derivative of the function f along ∂_i
$\frac{\partial f}{\partial x}, \partial_x f, f_x$	partial derivative of f with respect to x
∇f	Euclidean gradient of f
$\operatorname{div} f$	Euclidean divergence of f
χ_E	characteristic function of a measurable set $E \subset \mathbb{R}^n$
supp f	support of f
$Lip(\Omega)$, $C^{0,\alpha}(\Omega)$	Lipschitz or α-Hölder continuous real functions in Ω
$C^k(\Omega)$	continuously k-differentiable real functions in Ω
$C^k_c(\Omega)$	functions in $C^k(\Omega)$ with compact support in Ω
$\dot\gamma$	time derivative of a curve γ
$f * g$	convolution between f and g
$\|P\|_M$	norma of $P \in (M, d_M)$
$W^{1,p}_m(M, \mu, N)$	metrical Sobolev space of $u : M \to N$ with respect to a measure μ

$\mathbb{H}^n$	n-th Heisenberg group
$\{W_j\}_{j=1,\dots,2n+1}$	$=\{X_j,Y_j,T\}_{j=1,\dots,n}$ vector fields on $\mathbb{H}^n$
$\mathfrak{h}$	Heisenberg algebra, i.e. Lie algebra of $\mathbb{H}^n$
$\mathfrak{h}_1$	the subalgebra span$\{X_j,Y_j\}$
$\mathfrak{h}_2$	the subalgebra span$\{T\}$
$[X,Y]$	commutator of vector fields $X,Y\in\mathfrak{h}$
$P\cdot Q$	group product between $P,Q\in\mathbb{H}^n$
d_C	Carnot-Carathodory distance
$\Vert\cdot\Vert_\infty, d_\infty$	infinity norm and associated distance on $\mathbb{H}^n$
τ_P	left translation by an element $P\in\mathbb{H}^n$
δ_λ	homogeneous dilations in $\mathbb{H}^n$
$\mathcal{Q}$	homogeneous dimension of $\mathbb{H}^n$
$T\mathbb{H}^n, T\mathbb{H}^n_P$	tangent bundle to $\mathbb{H}^n$ and tangent space at $P\in\mathbb{H}^n$
$H\mathbb{H}^n, H\mathbb{H}^n_P$	horizontal subbundle to $\mathbb{H}^n$ and horizontal subspace at $P\in\mathbb{H}^n$
$d_{\mathbb{H}}f_P$	Pansu-differential of f at $P\in\mathbb{H}^n$
$\nabla_{\mathbb{H}}f$	horizontal gradient of f with respect to $X_1,\dots,X_n,Y_1,\dots,Y_n$
$\mathrm{div}_{\mathbb{H}}$	horizontal divergence
$\mathrm{curl}_{\mathbb{H}}$	horizontal curl operator
$C^1_{\mathbb{H}}(\Omega)$	continuously $\nabla_{\mathbb{H}}$-differentiable functions in Ω
$C^k_{\mathbb{H}}(\Omega)$	continuous real function f in Ω such that $\nabla_{\mathbb{H}}f\in C^{k-1}(\Omega)$
$Lip_{\mathbb{H}}(\Omega)$	Lipschitz continuous functions with respect to d_∞ in Ω
$C^k_{\mathbb{H}}(\omega;H\mathbb{H}^n)$	C^k-sections of $H\mathbb{H}^n$
$BV_{\mathbb{H}}$	functions with bounded $\mathbb{H}$-variation
$\bigwedge_k\mathfrak{h},\bigwedge^k\mathfrak{h},$	set of k-vector and k-covector in $\mathbb{H}^n$
$\bigwedge_k\mathfrak{h}_1,\bigwedge^k\mathfrak{h}_1,$	set of horizontal k-vector and k-covector in $\mathbb{H}^n$
θ	contact form in $\mathbb{H}^n$
$d\theta$	symplectic form in $\mathbb{H}^n$
${}_H\bigwedge_k, {}_H\bigwedge^k$	simple and integrable k-vector and k-covector in $\mathbb{H}^n$
$\mathcal{D}^{k,m}_{\mathbb{H}}(K)$	Heisenberg k-differential forms of class C^m in K
$\mathcal{D}^m_{\mathbb{H},k}(K)$	Heisenberg k-vector fields of class C^m in K
$\mathbb{V}_j$	$\{P\in\mathbb{H}^n : p_j=0\}$
ι	diffeomorphic identification between $\mathbb{V}_1$ and $\mathbb{R}^{2n}$
ω	open subset of $\mathbb{R}^{2n}$
$I_r(A)=I_r(\eta,v,\tau)$	$=(\eta-r,\eta+r)\times U(v,r)\times(\tau-r,\tau+r)\subset\mathbb{R}_\eta\times\mathbb{R}^{2n-2}_v\times\mathbb{R}_\tau$
$d_{\infty\vert\omega}$	restriction of d_∞ to ω
$G^1_{\mathbb{H},\phi}(\omega)$	X_1-graph induced by $\phi:\omega\to\mathbb{R}$
$S=\Phi(\omega)$	$\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$
$\nu_S(P)$	horizontal normal to S at $P\in S$
$\widetilde{X}_j,\widetilde{Y}_j,\widetilde{T}$	the restriction of X_j,Y_j,T to $\mathbb{V}_1$
W^ϕ	the vector field $\widetilde{Y}_1+\phi\widetilde{T}$
$\mathfrak{B}\phi$	the distribution $\dfrac{\partial\phi}{\partial\eta}+\dfrac{1}{2}\dfrac{\partial(\phi^2)}{\partial\tau}$
$\nabla^\phi\phi$	the intrinsic gradient $\left(\widetilde{X}_2\phi,\dots,\widetilde{X}_n\phi,W^\phi\phi,\widetilde{Y}_2\phi,\dots,\widetilde{Y}_n\phi\right)$
$\hat{w}_{n+1}$	the vector function $(w_2,\dots,w_n,w_{n+2},\dots,w_{2n})$

Chapter 1

Non linear first-order PDEs

A scalar conservation law in one dimensional space is a first-order partial differential equation of the type

$$u_t + f(u)_x = 0 \tag{1.1}$$

with the initial condition

$$u(0,x) = u_0(x) \tag{1.2}$$

where $x \in [-r_0, r_0] \subset \mathbb{R}$, $t \in [0,T]$. This equation is called conservation law because u represents a conserved quantity: the quantity u is neither created or destroyed: the total amount of u contained inside any given interval $[-r_0, r_0]$ can change only due to the flow of u across the two endpoints. In fact, integrating (1.1) over $[-r_0, r_0]$ we obtain

$$\frac{d}{dt}\int_{-r_0}^{r_0} u(t,x)\,dx = \int_{-r_0}^{r_0} u_t(t,x)\,dx = -\int_{-r_0}^{r_0} f(u(t,x))_x\,dx = f(u(t,-r_0)) - f(u(t,r_0)).$$

Example 1.0.6. Let us recall some physical example of conservation laws: the Euler's equations for compressible gas flow in one dimension. Let ρ be the mass, v the velocity, E the energy density per unit mass and p the pressure

$$\begin{array}{ll} \rho_t + (\rho v)_x = 0 & \text{conservation of mass} \\ (\rho v)_t + (\rho v^2 + p)_x = 0 & \text{conservation of momentum} \\ (\rho E)_t + (\rho E v + p v)_x = 0 & \text{conservation of energy} \end{array}$$

By a **classical solution** of (1.1) we main a continuously differentiable function $u = u(t,x)$ which satisfies (1.1) at every point of the domain. In this case the initial condition (1.2) must be regular: $u_0 \in C^1([-r_0, r_0])$. When the initial condition is only locally integrable, we have to give an other interpretation to our "solution", that becomes a weak solution. We will speak about different definitions of solution u, that can be discontinuous too:

distributional solution : $\forall\, \varphi \in C_c^1([0,T] \times [-r_0, r_0])$ we have

$$\int_0^T \int_{-r_0}^{r_0} [u\varphi_t + f(u)\varphi_x]\; dtdx = 0;$$

entropy solution : a distributional solutions of (1.1) which is physically admissible in a suitable sense;

broad solution : the function u depends by particular integral curves, the characteristics.

We will study distributional solutions for the general equation in section 1.1, entropy solutions for the general equation in sections 1.2 and 1.3 and broad solutions for the semilinear and quasilinear equation in section 1.4. In section 1.5 we give a short introduction to the study of Hamilton Jacobi equation $u_t + H(u_x) = G(t,x)$, see [37, 38, 39, 51, 74]. In particular we study the notion of viscosity solution of this equation and its equivalence with the entropy solution of the conservation law $u_t + H(u)_x = g(t,x)$, where $g(t,x) = \dfrac{\partial}{\partial x} G(t,x)$.

1.1 Distributional Solutions of Conservation Laws

In this section we will study the general conservation law $u_t + f(u)_x = g$. Its solution is not smooth in general, but it can be discontinuous, see examples 1.1.5, 1.2.1 and 1.4.14). We will so study the notion of distributional solution, see [6, 14, 21, 35, 41, 50, 51, 70, 72]. In the following let us indicate $I = (-r_0, r_0)$, $T > 0$, $\omega = (0,T) \times (-r_0, r_0)$.

Definition 1.1.1. *A locally measurable function $u : \omega \to \mathbb{R}$ is a distributional solution of the PDE*

$$u_t(t,x) + f(u(t,x))_x = g(t,x) \tag{1.3}$$

where $f \in Lip_{loc}(\mathbb{R})$ and $g : \omega \to \mathbb{R}$ if

$$\int_\omega [u\varphi_t + f(u)\varphi_x + g\varphi]\; dx\, dt = 0 \qquad \forall\, \varphi \in C_c^1(\omega) \tag{1.4}$$

Let us now consider the Cauchy problem

$$\begin{cases} u_t(t,x) + f(u(t,x))_x = g(t,x) & \text{in } \omega \\ u(0,x) = u_0(x) & \text{in } \{0\} \times I \end{cases} \tag{1.5}$$

Definition 1.1.2. *We say that* $u \in C^0([0,T], L^1_{loc}(I))$ *is a distributional solution of the Cauchy problem* (1.5) *with* $u_0 \in L^1_{loc}(I)$ *if* $\forall\, \varphi \in C^1_c(\omega)$

$$\int_\omega [u\varphi_t + f(u)\varphi_x + g\varphi]\; dx\, dt + \int_{-r_0}^{r_0} u_0(x)\varphi(0,x)dx = 0, \tag{1.6}$$

and $\lim\limits_{t\to 0^+} \| u(t,\cdot) - u_0 \|_{L^1_{loc}(I)} = 0$.

As we said distributional solution can be discontinuous. Let us study a condition that a distributional solution u must satisfy on his lines of discontinuity.
Let us consider an open region $V \subseteq \omega$ and a function $u : V \to \mathbb{R}$ with jumps on a finite number of curves: for example let us suppose in the open region V that u is smooth on either side of a smooth curve C, that we can represent parametrically as

$$C = \{(t,x) \in V :\; x = s(t)\} \quad \text{for some smooth function } s : (0,T) \to \mathbb{R}.$$

Let V_- be the part of V on the left of the curve and V_+ the part on the right. Let us denote with u_- the left limit of u to C and u_+ the right limit, i.e. $\forall\, t \in (0,T)$

$$u_-(t) := \lim_{x\to s(t)^-} u(t,x) \qquad u_+(t) := \lim_{x\to s(t)^+} u(t,x).$$

We give a jump condition, the Rankine-Hugoniot condition, that ensures that u is a distributional solution of (1.3), see [21, 35, 51].

Theorem 1.1.3. *Let us assume in* V_- *and* V_+ *that* u *is a distributional solution of* (1.3) *and that its first derivatives are uniformly continuous. If* u *is a distributional solution of* (1.3) *in* V *then*

$$f(u_+) - f(u_-) = \dot{s}(u_+ - u_-). \tag{1.7}$$

Proof. Let us choose in (1.4) $\varphi \in C^1_c(V_-)$. Since φ vanishes near the boundary of V_-, integrating by part we obtain

$$0 = \int_\omega [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt = -\int_\omega [u_t + f(u)_x - g]\, \varphi dx\, dt \tag{1.8}$$

(1.8) holds $\forall \varphi \in C^1_c(V_-)$, and so we have

$$u_t + f(u)_x - g = 0 \qquad \text{in}\, V_- \tag{1.9}$$

In the same way we obtain

$$u_t + f(u)_x - g = 0 \qquad \text{in } V_+ \tag{1.10}$$

Let us select now $\varphi \in C_c^1(V)$, which does not necessarily vanish along C. Again employing (1.4) we deduce

$$0 = \int_\omega [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt =$$

$$= \int_{V_-} [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt + \int_{V_+} [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt. \tag{1.11}$$

Since φ has compact support within V, we obtain by (1.9)

$$\int_{V_-} [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt = -\int_{V_-} [u_t + f(u)_x - g]\, \varphi dx\, dt + \tag{1.12}$$

$$+ \int_C \left(u_-\nu^1 + f(u_-)\nu^2\right) \varphi\, dl = \int_C \left(u_-\nu^1 + f(u_-)\nu^2\right) \varphi\, dl$$

where $\nu = (\nu^1, \nu^2)$ is the unit normal to the curve C, pointing from V_- into V_+. Similarly, we obtain by (1.10)

$$\int_{V_+} [u\varphi_t + f(u)\varphi_x + g\varphi]\, dx\, dt = \int_C \left(u_+\nu^1 + f(u_+)\nu^2\right) \varphi\, dl. \tag{1.13}$$

Adding (1.12) and (1.13) and recalling (1.11), we have (1.12) and (1.13) and recalling (1.11), we have

$$\int_C \left[(u_+ - u_-)\nu^1 + (f(u_+) - f(u_-))\nu^2\right] \varphi\, dl = 0 \qquad \forall\, \varphi \in C_c^1(V)$$

and then

$$(u_+ - u_-)\nu^1 + (f(u_+) - f(u_-))\nu^2 = 0 \qquad \text{along } C \tag{1.14}$$

Let us take $\nu = (\nu^1, \nu^2) = (1 + \dot{s}^2)^{-\frac{1}{2}}(-\dot{s}, 1)$. (1.14) implies

$$f(u_+) - f(u_-) = \dot{s}(u_+ - u_-)$$

in V, along the curve C. We obtain so the thesis. $\square$

The Rankine-Hugoniot relation expresses the fact that the component of the vector field (u, f) in the direction normal to the line of discontinuity is continuous across the line of discontinuity. There are some generalizations of this condition, let us recall the following in the case $g \equiv 0$.

Theorem 1.1.4. *Let $u : \omega \to \mathbb{R}$ be a measurable, bounded and Lipschitz function with a line of discontinuity $C = \{(t, s(t))\}$. Then u is a distributional solution of* (1.3) *if and only if the equation holds at almost every point of ω and across the line of discontinuity the Rankine-Hugoniot condition* (1.7) *holds for almost every $t \in (0, T)$.*

Proof. See [21], chapter 4. □

Example 1.1.5. Let us consider the Cauchy problem

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = 0 & \text{in } (0,T) \times \mathbb{R} \\ u(0,x) = -\dfrac{2}{3}\sqrt{3x} & \text{in } \mathbb{R} \end{cases} \tag{1.15}$$

It is easy to verify that

$$u(t,x) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{if } 3x + t^2 > 0 \\ 0 & \text{if } 3x + t^2 \leq 0 \end{cases} \tag{1.16}$$

is a classical solution in the regions $\{3x + t^2 > 0\}$ and $\{3x + t^2 \leq 0\}$, but it is not a distributional solution of (1.15) on $(0, T) \times \mathbb{R}$. Indeed let us notice that the Rankine-Hugoniot condition (1.7) does not hold: along the line of discontinuity $x = -\dfrac{1}{3}t^2$ we have

$$u_+(t) = -\frac{2}{3}t, \qquad u_-(t) = 0, \qquad \dot{s}(t) = -\frac{2}{3}t$$

$$f\left(u_+(t)\right) = \frac{2t^2}{9}, \qquad f\left(u_-(t)\right) = 0.$$

Then condition (1.7) becomes

$$\frac{2t^2}{9} = \frac{4t^2}{9}$$

that's false for $t > 0$. Let us observe that a distributional solution of (1.15) on $(0, T) \times \mathbb{R}$ is

$$u(t,x) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{if } 4x + t^2 > 0 \\ 0 & \text{if } 4x + t^2 \leq 0 \end{cases} \tag{1.17}$$

Indeed the Rankine-Hugoniot condition (1.7) holds: along the line of discontinuity $x = -\frac{1}{4}t^2$ we have

$$u_+(t) = -t, \qquad u_-(t) = 0, \qquad \dot{s}(t) = -\frac{t}{2}$$

$$f\left(u_{+}(t)\right)=\frac{t^{2}}{2}, \qquad f\left(u_{-}(t)\right)=0.$$

and the condition (1.7) becomes

$$\frac{t^{2}}{2}=\left(-\frac{t}{2}\right)\cdot(-t)\,.$$

□

1.2 Entropy Solutions of Conservation Laws

Definition 1.1.2 of distributional solution to the problem (1.5) is not stringent enough to single out a unique solution.

Example 1.2.1. Let us consider the Burgers' equation

$$u_t+\left(\frac{u^2}{2}\right)_x=0$$

with initial data

$$u_0(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}$$

For every $\alpha\in(0,1)$ let us define the piecewise constant function $u_\alpha:[0,T]\times\mathbb{R}\to\mathbb{R}$ as

$$u_\alpha(t,x):=\begin{cases}0 & \text{if } x<\frac{\alpha t}{2}\\ \alpha & \text{if } \frac{\alpha t}{2}\leq x\leq\frac{(\alpha+1)t}{2}\\ 1 & \text{if } x\geq\frac{(\alpha+1)t}{2}\end{cases}$$

Then each u_α is a solution to the Cauchy problem, because it satisfies the equation a.e. and the Rankine-Hugoniot condition (1.7) holds along the two lines of discontinuity $\gamma_1(t)=\frac{\alpha t}{2}$ and $\gamma_2(t)=\frac{(\alpha+1)t}{2}$. □

In order to achieve the uniqueness and continuous dependence of the initial data, the notion of distributional solution must be supplemented with further *admissibility conditions*, possibly motivated by physical considerations, see [21]. This conditions are usually called entropy conditions because they are motivated by the second law of thermodynamics for gas dynamics.

Definition 1.2.2. *Let $f\in Lip_{loc}(\mathbb{R})$. Two smooth functions $e,d:\mathbb{R}\to\mathbb{R}$ comprise an entropy/entropy-flux for the conservation law $u_t+f(u)_x=g(t,x)$ provided*

i *e is convex*

ii $e' \cdot f' = d'$

Remark 1.2.3. Any convex function $e : \mathbb{R} \to \mathbb{R}$ provides an entropy. Indeed the condition **ii** of definition 1.2.2 reduces to the ODE $d'(u) = e'(u)f'(u)$. As entropy flux one can take

$$d(u) = \int_c^u e'(v)f'(v)\,dv.$$

The lower limit of the integral is an arbitrary constant.

Definition 1.2.4. *Let $f \in Lip_{loc}(\mathbb{R})$, $g \in L^1(\omega)$, $u_0 \in L^\infty(I)$. We call $u \in C^0([0,T]; L^1(I)) \cap L^\infty(\omega)$ an entropy solution of*

$$\begin{cases} u_t + f(u)_x = g(t,x) & \text{in } \omega \\ u = u_0 & \text{on } \{0\} \times I \end{cases} \tag{1.18}$$

provided that u satisfies

i $\forall \varphi \in C_c^\infty(\omega)$ *with* $\varphi \geq 0$

$$\int_\omega [e(u)\varphi_t + d(u)\varphi_x + e'(u)g\varphi]\ dt\,dx \geq 0,$$

for each smooth entropy/entropy flux pairs $e, d : \mathbb{R} \to \mathbb{R}$ of the conservation law $u_t + f(u)_x = g(t,x)$

ii $\lim_{t \to 0^+} \| u(t,\cdot) - u_0 \|_{L^1(I)} = 0.$

Remark 1.2.5. Suppose that u is a smooth solution of $u_t + f(u)_x - g = 0$. Then, multiplying the equation by $e'(u)$

$$0 = u_t + f(u)_x - g = e'(u)u_t + e'(u)f'(u)u_x - e'(u)g = e(u)_t + d(u)_x - e'(u)g \tag{1.19}$$

In general distributional solution of (1.1) will not be smooth enough, owing to shocks and other irregularities, to justify the foregoing computation. The idea is instead to replace (1.19) with an inequality

$$e(u)_t + d(u)_x - e'(u)g \leq 0 \qquad \text{in } \omega. \tag{1.20}$$

In the case $g \equiv 0$, equation (1.19) says that the quantity $e(u)$ satisfies a scalar conservation law. In applications $e(u)$ will sometimes be the negative of physical entropy and $d(u)$ the entropy flux. The inequality (1.20) therefore asserts entropy evolves according to its flux, but may also undergo sharp increases, for instance along shocks.

Remark 1.2.6. Condition **i** of definition 1.2.4 imposes a restriction only on lines of discontinuity. Indeed, let u be a Lipschitz function with a line of discontinuity C parametrizated by $s(t)$. Then u is an entropy solution if and only if

$$\dot{s}\,[e(u_+) - e(u_-)] \geq d(u_+) - d(u_-). \tag{1.21}$$

The statement is right for a function $u \in BV(\omega)$, for the proof in case $g \equiv 0$ see [21] chapter 4, the generalization to case $g \neq 0$ is given in [41], Theorem 1.8.2 and section 4.3.

Volpert [95] introduce an equivalent definition of entropy solution for scalar conservation laws, as we say in the following lemma:

Lemma 1.2.7. *Let $f \in Lip_{loc}(\mathbb{R})$, $g \in L^1(\omega)$, $u_0 \in L^\infty(I)$. $u \in C^0([0,T]; L^1(I)) \cap L^\infty(\omega)$ is an entropy solution of* (1.18) *if and only if*

$$|u-k|_t + \frac{\partial}{\partial x}[\mathrm{sign}(u-k)(f(u)+f(k))] \leq g\,\mathrm{sign}(u-k) \qquad \forall\, k \in \mathbb{R} \qquad \text{in } \mathcal{D}'(\omega) \tag{1.22}$$

and $\lim_{t\to 0^+} \| u(t,\cdot) - u_0 \|_{L^1(I)} = 0.$

Remark 1.2.8. Notice that for each $k \in \mathbb{R}$

$$e(u) = |u-k|, \qquad\qquad d(u) = \mathrm{sgn}(u-k)(f(u) - f(k))$$

are Lipschitz entropy/entropy flux pairs for the conservation law (1.18).

We will see in Theorem 1.5.9 that a continuous distributional solution of (1.18) is an entropy solution too. At the moment let us recall that a well-known method to construct an entropy solution u of the problem (1.18) is the approximation of u by suitable regular solutions, (see for instance [21] section 4.4 and [51] section 11.4.2, Theorem 2). For our purpose the following result will be crucial.

Proposition 1.2.9. *Let $(u^\epsilon)_\epsilon \subset Lip([0,T] \times [-r_0, r_0])$, $(g^\epsilon)_\epsilon \subset L^1([0,T] \times [-r_0, r_0])$, $f \in Lip_{loc}(\mathbb{R})$ such that*

$$u^\epsilon_t + f'(u^\epsilon) u^\epsilon_x = g^\epsilon \qquad \mathcal{L}^2 - \text{a.e. in}\,(0,T) \times (-r_0, r_0) \tag{1.23}$$

Let us assume that

$$u^\epsilon \to u \qquad \text{uniformly in}\,[0,T] \times [-r_0, r_0] \tag{1.24}$$

$$g^\epsilon \to g \qquad \text{in}\, L^1([0,T] \times [-r_0, r_0]) \tag{1.25}$$

Then u is an entropy solution of (1.18) *with $u_0(x) = u(0,x)$.*

Proof. Let $e, d : \mathbb{R} \to \mathbb{R}$ be two smooth function comprising an entropy/entropy flux for the conservation law $u_t + f(u)_x = g(t,x)$. Then by (1.23)

$$\frac{\partial}{\partial t}(e(u^\epsilon)) + \frac{\partial}{\partial x}(d(u^\epsilon)) = e'(u^\epsilon)g^\epsilon \qquad \mathcal{L}^2 - \text{a.e. in}\, \omega. \tag{1.26}$$

Therefore multiplying both sides of (1.26) for a given $v \in C^1_c((0,T) \times (-r_0, r_0))$, integrating by parts and taking the limit as $\epsilon \to 0^+$, by (1.24) and (1.25) we get Definition 1.2.4 **i** (actually with an equality, so with no entropy production). On the other hand by (1.24) $u \in C^0([0,T] \times [-r_0, r_0]) \subseteq C^0([0,T]; L^1(-r_0, r_0)) \cap L^\infty(\omega)$, then also Definition 1.2.4 **ii** follows. □

Remark 1.2.10. By Remark 1.4.10 and Proposition 1.2.9 it follows that each Lipschitz continuous broad solution of (1.18) is an entropy solution too.

In the spirit of proposition 1.2.9, let us study that an other physical admissibility condition. We will approximate the problem (1.5) with a small viscosity effect, through the so called "vanishing viscosity method".

Definition 1.2.11. (Vanishing viscosity) *A distributional solution u of* (1.18) *is admissible with vanishing viscosity if there exists a sequence of smooth functions $\{u^\epsilon\}_\epsilon \subset C^\infty(\omega)$ such that*

$$\begin{cases} u^\epsilon_t + f(u^\epsilon)_x - \epsilon u^\epsilon_{xx} = g & in\, \omega \\ u^\epsilon(0,x) = u_0(x) & in\ \{0\} \times I \end{cases} \tag{1.27}$$

and $u^\epsilon \to u$ in $L^1_{loc}([0,T] \times [-r_0, r_0])$ as $\epsilon \to 0^+$.

Remark 1.2.12. Physically we regard the term ϵu^ϵ_{xx} as imposing an artificial viscosity effect, which we are now sending to 0. We expect that this vanishing viscosity technique should allow us to recover the correct entropy solution, which may have discontinuities across shock waves, as the limit of the solutions u^ϵ, which are smooth. Mathematically the term ϵu^ϵ_{xx} makes equation (1.27) similar to the heat equation, in fact the solution u^ϵ is smooth, in spite of the nonlinearity. On the other hand an obvious guess is that the solution u^ϵ should converge as $\epsilon \to 0$ to a solution of the conservation laws.

Remark 1.2.13. (Hopf-Cole transform) Let us give a generalization of the Hopf-Cole transform [33, 51, 67], which linearizes the viscous Burgers' equation

$$u^\epsilon_t + u^\epsilon u^\epsilon_x = g + \epsilon u^\epsilon_{xx} \tag{1.28}$$

where $g \in C^0(\omega)$. Let us apply the substitution $u^\epsilon = v_x$ in (1.28), we obtain

$$v_{xt} + v_x v_{xx} = g + \epsilon v_{xxx}. \tag{1.29}$$

Let us integrate (1.29) with respect to x, finding

$$v_t + \frac{1}{2}(v_x)^2 = G + \epsilon v_{xx} \tag{1.30}$$

where $G(x,t) = \int_0^x g(x,t)\,dx$. If we apply a second substitution $v = -3\epsilon \log \varphi$ in (1.30) we obtain

$$-2\epsilon\frac{\varphi_t}{\varphi} + \frac{1}{2}\left(-2\epsilon\frac{\varphi_x}{\varphi}\right)^2 = G - 2\epsilon^2\frac{\varphi_{xx}\varphi - (\varphi_x)^2}{\varphi^2}, \tag{1.31}$$

therefore

$$-2\epsilon\frac{\varphi_t}{\varphi} + 2\epsilon^2\frac{\varphi_x^2}{\varphi^2} = G - 2\epsilon^2\frac{\varphi_{xx}}{\varphi} + 2\epsilon^2\frac{\varphi_x^2}{\varphi^2} \tag{1.32}$$

and finally

$$\varphi_t = \epsilon\varphi_{xx} - \frac{G}{2\epsilon}\varphi \tag{1.33}$$

[51, 67] show that, in the case $g \equiv 0$, the solution u^ϵ of equation (1.28) obtained using the Hopf-Cole transform is such that $u^\epsilon(t,x) \to u(t,x)\ \forall\ (t,x) \in \omega$, where u is the entropy solution of the equation $u_t + uu_x = 0$. □

Let us now study the relationship between solutions admissible with vanishing viscosity and entropy solutions.

Theorem 1.2.14. *Let us assume that $\{u^\epsilon\}_\epsilon$ is uniformly bounded in $L^\infty([0,T]\times[-r_0,r_0])$, u is a distributional solution of (1.18) admissible with vanishing viscosity and that $u^\epsilon \to u\ \mathcal{L}^2$ -a.e. $(t,x) \in [0,T]\times[-r_0,r_0]$ as $\epsilon \to 0$. Then u is an entropy solution of (1.18).*

Proof. Let us choose any smooth entropy/entropy flux pair (e,d). Left multiplying (1.1) by $e'(u^\epsilon)$ and recalling **ii** of Definition 1.2.2, we compute

$$e(u^\epsilon)_t + d(u^\epsilon)_x = \epsilon e'(u^\epsilon)u^\epsilon_{xx} + ge'(u^\epsilon) = \epsilon e(u^\epsilon)_{xx} - \epsilon\left(e''(u^\epsilon)(u^\epsilon_{xx})^2\right) + ge'(u^\epsilon). \tag{1.34}$$

As e is convex

$$e''(u^\epsilon)(u^\epsilon_{xx})^2 \geq 0 \tag{1.35}$$

Let us multiply (1.34) by $\varphi \in C_c^\infty(\omega)$, $\varphi \geq 0$ and let us integrate by parts. We have

$$\int_\omega [e(u^\epsilon)\varphi_t + d(u^\epsilon)\varphi_x + ge'(u^\epsilon)\varphi]\,dx\,dt =$$

$$= \int_\omega \left[\epsilon\left(e''(u^\epsilon)(u^\epsilon_{xx})^2\right)\varphi - \epsilon e(u^\epsilon)\varphi_{xx}\right]dx\,dt \geq -\int_\omega \epsilon e(u^\epsilon)\varphi_{xx}dx\,dt,$$

the last inequality holding in view of (1.35) and the non negativity of φ. Now let $\epsilon \to 0$. Since $u^\epsilon \to u$ $\mathcal{L}^2$-a.e. $(t,x) \in [0,T] \times [-r_0, r_0]$ and by the dominated convergence Theorem, we obtain

$$\int_\omega e(u)\varphi_t + d(u)\varphi_x + ge'\varphi \geq 0.$$

Thus u verifies the condition **i** of Definition 1.2.4. If e and d are not smooth, we obtain the same conclusion after an approximation.
Finally let us consider (1.27), multiplying for $\varphi \in C_c^\infty(\omega)$ and integrating by parts we obtain

$$\int_\omega [u^\epsilon \varphi_t + f(u^\epsilon)\varphi_x + \epsilon u^\epsilon \varphi_{xx}]dx\,dt + \int_{-r_0}^{r_0} u_0 \varphi_{|t=0} dx = -\int_\omega g\varphi\,dx\,dt.$$

We send $\epsilon \to 0$ and we deduce that u is a distributional solution of (1.18). $\square$

Remark 1.2.15. By the proof of Theorem 5 in [6] we see that an entropy solution is a distributional solution admissible with vanishing viscosity.

1.3 Existence and Uniqueness Theorems

In this section we will expone existence and uniqueness results and a maximum principle for the entropy solution of equation (1.18), see [6, 12, 13, 14, 50, 51, 70].

Theorem 1.3.1. *For any $u_0 \in L^\infty(-r_0, r_0)$, $g \in L^\infty(\omega)$, there exists an entropy solution of* (1.18)*.*

Remark 1.3.2. For the proof see [13]. The statement of Theorem is true for $u_0 \in L^1(-r_0, r_0)$, $g \in L^1(\omega)$ too, see [12].

We will now prove a refinement of a well-known uniqueness result due to Kružhkov in order to get a local uniqueness result for entropy solutions of (1.18), see [13] too.

Theorem 1.3.3. *Let $g \in L^1(\omega)$ and let $u, \tilde{u} \in C^0([0,T]; L^1(I)) \cap L^\infty(\omega)$ be two entropy solutions of the problem* (1.18)*. Let M, L be constant such that*

$$|u(t,x)| \leq M, \quad |\tilde{u}(t,x)| \leq M \qquad (t,x) \in \omega \tag{1.36}$$

$$|f(u_1) - f(u_2)| \leq L|u_1 - u_2| \qquad \forall\, u_1, u_2 \in [-M, M] \tag{1.37}$$

Then $\forall\, r \in (0, r_0)$, *if* $r + LT < r_0$ *then* $\forall\, 0 \le \tau_0 \le \tau \le T$ *we get*

$$\int_{|x|\le r} |u(\tau,x) - \tilde{u}(\tau,x)|\; dx \le \int_{|x|\le r+L(\tau-\tau_0)} |u(\tau,x) - \tilde{u}(\tau,x)|\; dx \qquad (1.38)$$

In particular when $\tau_0 = 0$ *and* $u(0,\cdot) = \tilde{u}(0,\cdot)$ *a.e. in* I *then*

$$u(t,x) = \tilde{u}(t,x) \qquad \text{a.e. } (t,x) \in (0,T) \times (-r,r).$$

The classical proof of Theorem 3.1 is contained in Kružhkov [70], section 3 Theorem 1, when $r_0 = +\infty$, $f \in C^1(\mathbb{R}^2)$ $g \in C^1(\mathbb{R}^2)$. A simpler proof with $g \equiv 0$ can be found in Evans [51], section 11.4.3. A statement similar to ours is proved in Bressan [21], Theorem 6.2, when $r_0 = +\infty$, $f \in Lip_{loc}(\mathbb{R})$, $g \equiv 0$. We will adapt the techniques contained in these two last references in order to get the proof.

Proof. We are going to divide the proof in 4 steps.
1. step: Let u be an entropy solution of (1.18). Then by Definition 1.2.4 **i**

$$\int_0^T \int_{-r_0}^{r_0} [e(u)v_t + d(u)v_x + e'(u)gv]\; dtdx \ge 0 \qquad (1.39)$$

$\forall\, v \in C_c^\infty(\omega)$, $v \ge 0$ where e is smooth, convex and

$$d(z) = \int_{z_0}^{z} e'(w)f'(w)\, dw$$

for any $z_0 \in \mathbb{R}$. Fix $\alpha \in \mathbb{R}$ and take $e_k(z) = \beta_k\,(z-\alpha)$ where $z \in \mathbb{R}$ and for each $k = 1, \dots$ $\beta_k : \mathbb{R} \to \mathbb{R}$ is smooth, convex and

$$\begin{cases} \beta_k(z) \to |z| & \text{uniformly} \\ \beta_k'(z) \to \operatorname{sgn}(z) & \text{boundedly a.e} \end{cases}$$

Consequently $\forall\, z$

$$d_k(z) \to \int_0^z \operatorname{sgn}(w-\alpha)f'(w)dw = \operatorname{sgn}(z-\alpha)(f(z) - f(\alpha)).$$

Putting e_k, d_k in (1.39) and sending $k \to \infty$ we deduce

$$\int_0^T \int_{-r_0}^{r_0} [|u-\alpha|v_t + \operatorname{sgn}(u-\alpha)((f(u) - f(\alpha))v_x + gv)]\; dt\, dx \ge 0 \qquad (1.40)$$

$\forall \alpha \in \mathbb{R}$, and $v \in C_c^\infty(\omega)$, $v \ge 0$.

2. step: Let $\tilde{u}$ be an another entropy solution. Then by (1.40)

$$\int_0^T \int_{-r_0}^{r_0} [|\tilde{u}-\tilde{\alpha}|\tilde{v}_s + \mathrm{sgn}(\tilde{u}-\tilde{\alpha})((f(\tilde{u})-f(\tilde{\alpha}))\tilde{v}_y + g\tilde{v})]\, ds\, dy \geq 0 \quad (1.41)$$

$\forall\, \tilde{\alpha} \in \mathbb{R}$, $\tilde{v} \in C_c^\infty(\omega)$, $\tilde{v} \geq 0$. Now let $w \in C_c^\infty((-r_0,r_0)\times(-r_0,r_0)\times(0,T)\times(0,T))$, $w \geq 0$, $w = w(x,y,t,s)$. Fixing $(s,y) \in (0,T)\times(-r_0,r_0)$ we take $\alpha = \tilde{u}(s,y)$, $v(t,x) = w(x,y,t,s)$ in (1.40). Integrating with respect to y, s we produce the inequality

$$\int_0^T \int_0^T \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} \{|u(t,x)-\tilde{u}(s,y)|w_t + \mathrm{sgn}\,(u(t,x)-\tilde{u}(s,y))\cdot$$

$$\cdot[(f(u(t,x)) - f(\tilde{u}(s,y))w_x + g(t,x)w)]\}dx\,dy\,dt\,ds \geq 0 \quad (1.42)$$

Likewise for each fixed $(t,x) \in (0,T)$ we take $\tilde{\alpha} = u(t,x)$, $\tilde{v}(s,y) = w(x,y,t,s)$ in (1.41). Integrating with respect to (t,x) gives

$$\int_0^T \int_0^T \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} \{|\tilde{u}(s,y)-u(t,x)|w_s + \mathrm{sgn}\,(\tilde{u}(s,y)-u(t,x))\cdot$$

$$\cdot[(f(\tilde{u}(s,y) - f(u(t,x)))w_y + g(s,y)w)]\}dx\,dy\,dt\,ds \geq 0 \quad (1.43)$$

Add (1.42) and (1.43)

$$\int_0^T \int_0^T \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} \{|u(t,x)-\tilde{u}(s,y)|(w_t+w_s)+$$

$$+\mathrm{sgn}\,(u(t,x)-\tilde{u}(s,y))\,(f(u(t,x)) - f(\tilde{u}(s,y))(w_x+w_y)+$$

$$+\mathrm{sgn}\,(u(t,x)-\tilde{u}(s,y))\,(g(t,x)-g(s,y))w)\}dx\,dy\,dt\,ds \geq 0 \quad (1.44)$$

$\forall\, w \in C_c^\infty((-r_0,r_0)\times(-r_0,r_0)\times(0,T)\times(0,T))$, $w \geq 0$.

3. step: Let $\delta : \mathbb{R} \to [0,1]$ be a continuous function such that

$$\int_{-\infty}^{+\infty} \delta(z)\,dz = 1, \qquad \delta(z) = 0 \quad \text{if} \quad |z| > 1$$

and define

$$\delta_h(z) := h\delta(hz), \qquad \alpha_h(z) := \int_{-\infty}^{z} \delta_h(s)\,ds.$$

Let $\phi = \phi(\bar{t},\bar{x}) \in C_c^\infty((0,T)\times(-r_0,r_0))$, $\phi \geq 0$ and let us define

$$w(x,y,t,s) := \phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\cdot\delta_h\left(\frac{t-s}{2}\right)\cdot\delta_h\left(\frac{x-y}{2}\right) \qquad (x,y,t,s) \in \mathbb{R}^4.$$

Since we can assume that

$$\mathrm{spt}(\phi) \subseteq [\rho, T-\rho] \times [-r_0+\rho, r_0-\rho] \tag{1.45}$$

for a suitable $0 < \rho < \min\{T/2, r_0\}$, if h is large enough we have that $w \in C_c^\infty((-r_0,r_0)\times(-r_0,r_0)\times(0,T)\times(0,T))$. Moreover a direct computation yields

$$(w_x + w_y)(x,y,t,s) = \phi_{\overline{x}}\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\cdot \delta_h\left(\frac{t-s}{2}\right)\cdot \delta_h\left(\frac{x-y}{2}\right)$$

$$(w_t + w_s)(x,y,t,s) = \phi_{\overline{t}}\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\cdot \delta_h\left(\frac{t-s}{2}\right)\cdot \delta_h\left(\frac{x-y}{2}\right).$$

We insert this choice of w in (1.44) and we get

$$\int_0^T\int_0^T\int_{-r_0}^{r_0}\int_{-r_0}^{r_0}\left(I_h^{(1)} + I_h^{(2)}\right)dx\,dy\,dt\,ds \geq 0 \qquad \forall\, h\in\mathbb{N} \tag{1.46}$$

where

$$I_h^{(1)} := \left[|u(t,x)-\tilde{u}(s,y)|\phi_{\overline{t}}\left(\frac{t+s}{2}, \frac{x+y}{2}\right)+\right.$$

$$\left.+\mathrm{sgn}(u(t,x)-\tilde{u}(s,y))(f(u(t,x))-f(\tilde{u}(s,y)))\phi_{\overline{x}}\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\right]\delta_h\left(\frac{t-s}{2}\right)\delta_h\left(\frac{x-y}{2}\right),$$

$$I_h^{(2)} := \mathrm{sgn}(u(t,x)-\tilde{u}(s,y))\cdot(g(t,x)-g(s,y))\cdot\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{t-s}{2}\right)\delta_h\left(\frac{x-y}{2}\right).$$

Now let us perform the change of variables in $\mathbb{R}^4$

$$\overline{x} = \frac{x+y}{2} \qquad \overline{t} = \frac{t+s}{2}$$

$$\overline{y} = \frac{x-y}{2} \qquad \overline{s} = \frac{t-s}{2}$$

and we get that, for h large enough,

$$\int_0^T\int_0^T\int_{-r_0}^{r_0}\int_{-r_0}^{r_0} I_h^{(1)}\,dx\,dy\,dt\,ds = \int_{-\rho}^{\rho}\int_{-\rho}^{\rho} G_1(\overline{s},\overline{y})\delta_h(\overline{s})\delta_h(\overline{y})\,d\overline{s}\,d\overline{y} \tag{1.47}$$

where

$$G_1(\overline{s},\overline{y}) := \int_\rho^{T-\rho}\int_{-r_0+\rho}^{r_0-\rho}\left\{|u(\overline{t}+\overline{s},\overline{x}+\overline{y}) - \tilde{u}(\overline{t}-\overline{s},\overline{x}-\overline{y})|\phi_{\overline{t}}(\overline{t},\overline{x})+\right.$$

$$+\mathrm{sgn}(u(\overline{t}+\overline{s},\overline{x}+\overline{y})-\tilde{u}(\overline{t}-\overline{s},\overline{x}-\overline{y}))(f(u(\overline{t}+\overline{s},\overline{x}+\overline{y}))-f(\tilde{u}(\overline{t}-\overline{s},\overline{x}-\overline{y})))\phi_{\overline{x}}(\overline{t},\overline{x})\Big\}\,d\overline{t}\,d\overline{x}$$

if $-\rho \le \overline{s} \le \rho$, $-\rho \le \overline{y} \le \rho$. On the other hand

$$\int_{-\rho}^{\rho}\int_{-\rho}^{\rho} G_1(\overline{s},\overline{y})\delta_h(\overline{s})\delta_h(\overline{y})\,d\overline{s}\,d\overline{y} = \int_{-h\rho}^{h\rho}\int_{-h\rho}^{h\rho} G_1\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right)\cdot\delta(\tilde{s})\delta(\tilde{y})\,d\tilde{s}\,d\tilde{y} \tag{1.48}$$

and since the mappings $[-M,M]^2 \ni (a,b) \mapsto |a-b|$, $\mathrm{sgn}(a-b)(f(a)-f(b))$ are Lipschitz continuous, then

$$G_1\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right) \longrightarrow \int_0^T\int_{-R_0}^{R_0}\Big\{|u(\overline{t},\overline{x})-\tilde{u}(\overline{t},\overline{x})|\phi_{\overline{t}}(\overline{t},\overline{x})+$$

$$+\mathrm{sgn}(u(\overline{t},\overline{x})-\tilde{u}(\overline{t},\overline{x}))\cdot(f(u(\overline{t},\overline{x}))-f(\tilde{u}(\overline{t},\overline{x})))\cdot\phi_{\overline{x}}(\overline{t},\overline{x})\Big\}\,d\overline{t}\,d\overline{x} =: L_1(\phi)$$

By (1.47) and (1.48) we get

$$\lim_{h\to\infty}\int_0^T\int_0^T\int_{-r_0}^{r_0}\int_{-r_0}^{r_0} I_h^{(1)}\,dx\,dy\,dt\,ds = L_1(\phi). \tag{1.49}$$

Analogously by performing the same change of variables in the second term of (1.46) we get

$$\int_0^T\int_0^T\int_{-r_0}^{r_0}\int_{-r_0}^{r_0} I_h^{(2)}\,dx\,dy\,dt\,ds = \int_{-\rho}^{\rho}\int_{-\rho}^{\rho} G_2(\overline{s},\overline{y})\delta_h(\overline{s})\delta_h(\overline{y})\,d\overline{s}\,\overline{y} \tag{1.50}$$

where

$$G_2(\overline{s},\overline{y}) = \int_{\rho}^{T-\rho}\int_{-r_0+\rho}^{r_0-\rho}\Big\{\mathrm{sgn}(u(\overline{t}+\overline{s},\overline{x}+\overline{y})-\tilde{u}(\overline{t}-\overline{s},\overline{x}-\overline{y}))\cdot$$

$$\cdot(g(\overline{t}+\overline{s},\overline{x}+\overline{y})-g(\overline{t}-\overline{s},\overline{x}-\overline{y}))\cdot\phi(\overline{t},\overline{x})\Big\}\,d\overline{t}\,d\overline{x}$$

if $-\rho \le \overline{s} \le \rho$, $-\rho \le \overline{y} \le \rho$. But

$$\int_{-\rho}^{\rho}\int_{-\rho}^{\rho} G_2(\overline{s},\overline{y})\delta_h(\overline{s})\delta_h(\overline{y})\,d\overline{s}\,d\overline{y} = \int_{-h\rho}^{h\rho}\int_{-h\rho}^{h\rho} G_2\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right)\delta(\tilde{s})\delta(\tilde{y})\,d\tilde{s}\,d\tilde{y} \tag{1.51}$$

Let us prove now that

$$\lim_{h\to\infty} G_2\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right) = 0 \qquad \forall\,(\tilde{s},\tilde{y}) \in \mathbb{R}^2. \tag{1.52}$$

Indeed fixed $(\tilde{s},\tilde{y}) \in \mathbb{R}^2$ and let h large enough then

$$\left|G_2\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right)\right| \le \sup_{\omega}|\phi|\int_{\rho}^{T-\rho}\int_{-r_0+\rho}^{r_0-\rho}\left|g\left(\overline{t}+\frac{\tilde{s}}{h},\overline{x}+\frac{\tilde{y}}{h}\right)-g\left(\overline{t}-\frac{\tilde{s}}{h},\overline{x}-\frac{\tilde{y}}{h}\right)\right|\,d\overline{t}\,d\overline{x}. \tag{1.53}$$

On the other hand $\forall\epsilon > 0$ there exists $g^* \in C_c^0(\omega)$ such that

$$\| g - g^* \|_{L^1(\omega)} = \int_0^T \int_{-r_0}^{r_0} |g(\bar{t},\bar{x}) - g^*(\bar{t},\bar{x})| \, d\bar{t}\, d\bar{x} < \epsilon \tag{1.54}$$

By (1.53) and (1.54) we get

$$\left| G_2\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right)\right| \le C\left(2\epsilon + \int_\rho^{T-\rho}\int_{-r_0+\rho}^{r_0-\rho}\left| g^*\left(\bar{t}+\frac{\tilde{s}}{h},\bar{x}+\frac{\tilde{y}}{h}\right) - g^*\left(\bar{t}-\frac{\tilde{s}}{h},\bar{x}-\frac{\tilde{y}}{h}\right)\right| d\bar{t}\, d\bar{x}\right)$$

Taking the limit as $h \to \infty$ in the previous inequality we get, since g^* is continuous

$$\limsup_{h\to\infty}\left| G_2\left(\frac{\tilde{s}}{h},\frac{\tilde{y}}{h}\right)\right| \le 2\epsilon \qquad \forall\,\epsilon > 0.$$

Then (1.52) follows. By (1.51) and (1.52) we get

$$\lim_{h\to\infty}\int_0^T\int_0^T\int_{-r_0}^{r_0}\int_{-r_0}^{r_0} I_h^{(2)}\, dx\, dy\, dt\, ds = 0. \tag{1.55}$$

Combining (1.46), (1.49) and (1.55) we get the inequality

$$\int_\omega \{|u(t,x)-\tilde{u}(t,x)|\phi_t + \mathrm{sgn}(u(t,x)-\tilde{u}(t,x))[f(u(t,x))-f(\tilde{u}(t,x))]\phi_x\}\, dt\, dx \ge 0 \tag{1.56}$$

for every $\phi \in C_c^\infty(\omega)$, $\phi \ge 0$.

4. step: Now let $0 < \tau_0 < \tau < T$ and $0 < r < r_0$ be given such that $r + LT < r_0$. For $(t,x) \in \omega = (0,T)\times(-r_0,r_0)$, let

$$\phi(t,x) := [\alpha_h(t-\tau_0) - \alpha_h(t-\tau)]\cdot\left[1-\alpha_k\left(|x|-r-L(\tau-t)+\frac{1}{k}\right)\right] =$$

$$= \alpha_h(t-\tau_0)\alpha_h(t-\tau)\cdot\chi_k(t,x)$$

where

$$\alpha_h(z) := \int_{-\infty}^z \delta_h(s)\, ds.$$

It is easy to see that for h,k large $\phi \in Lip_0(\omega)$ and $\phi \ge 0$. Using (1.56) with this particular test function ϕ_h we obtain

$$\int_\omega |u(t,x)-\tilde{u}(t,x)|(\delta_h(t-\tau_0)-\delta_h(t-\tau))\left[1-\alpha_k\left(|x|-r-L(\tau-t)+\frac{1}{k}\right)\right] dt\, dx \ge$$

$$\ge \int_\omega\left\{\frac{x}{|x|}[f(u(t,x))-f(\tilde{u}(t,x))]\mathrm{sgn}(u(t,x)-\tilde{u}(t,x)) + L|u(t,x)-\tilde{u}(t,x)|\right\}\cdot$$

$$\cdot[\alpha_h(t-\tau_0)-\alpha_h(t-\tau)]\cdot\delta_h\left(|x|-r-L(\tau-t)+\frac{1}{k}\right)\,dt\,dx.$$

By (1.36) and (1.37) we have

$$|f(u(t,x))-f(\tilde u(t,x))|\le L|u(t,x)-\tilde u(t,x)|.$$

Hence $\forall\, h,k$

$$\int_\omega |u(t,x)-\tilde u(t,x)|(\delta_h(t-\tau_0)-\delta_h(t-\tau))\chi_k(t,x)\,dt\,dx\ge 0 \tag{1.57}$$

Let us denote

$$C:=\{(t,x)\in\omega:\ |x|\le r+L(\tau-t)\}$$
$$S(t):=\{x\in(-r_0,r_0):\ (t,x)\in C\}.$$

Let us observe that $\chi_k\to\chi_C$ a.e. $(t,x)\in\omega$. Thus passing to the limit as $k\to\infty$ in (1.57) we get

$$\int_\omega |u(t,x)-\tilde u(t,x)|(\delta_h(t-\tau_0)-\delta_h(t-\tau))\chi_C(t,x)\,dt\,dx\ge 0 \tag{1.58}$$

for every h. Let

$$\psi(t):=\int_{S(t)}|u(t,x)-\tilde u(t,x)|\,dx=\int_\omega \chi_C(t,x)|u(t,x)-\tilde u(t,x)|\,dx$$

because $u,\tilde u\in C^0([0,T];L^1(I))$ $\psi\in C^0([0,T])$. Thus by (1.58)

$$\int_0^T \delta_h(t-\tau)\psi(t)\,dt\le\int_0^T\delta_h(t-\tau_0)\psi(t)\,dt$$

and taking the limit as $h\to+\infty$ we get $\psi(\tau)\le\psi(\tau_0)$. Thus we obtain (1.38) when $0<\tau_0<\tau\le T$. By continuity (1.38) still holds when $\tau_0=\tau$ or if $\tau_0=0$. $\square$

By Theorem 1.3.3 we get the following local uniqueness result for Burgers' entropy solutions that will be crucial in the next chapters.

Corollary 1.3.4. *Let $g\in L^1((0,T)\times(-r_0,r_0))$, $u_0\in L^\infty(-r_0,r_0)$, $M>0$. Let us denote by $\mathcal{E}_M(T,r_0)$ the class of functions $u\in C^0([0,T];L^1(-r_0,r_0))$ such that*

$$|u(t,x)|\le M\quad \mathcal{L}^2-a.e.(t,x)\in(0,T)\times(r_0,r_0)\,.$$

Let u, $\tilde u\in\mathcal{E}_M(T,r_0)$ be entropy solutions of the initial value problem

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = g & \text{in}\,(0,T)\times(-r_0,r_0) \\ u(0,x) = u_0(x) & \forall\, x\in(-r_0,r_0) \end{cases}$$

Then, if $r + MT < r_0$*,*

$$u(t,x) = \tilde{u}(t,x) \qquad \mathcal{L}^2 - \text{a.e. } (t,x)\in(0,T)\times(-r,r).$$

Proof. Applying Theorem 1.3.3 with $f(u) = \dfrac{u^2}{2}$, (1.37) holds with $L = M$ and the thesis follows. □

Finally, let us recall the following maximum result obtained by Theorem 1 in [6] and corollary 2.1 of [14]

Theorem 1.3.5. *Let* $g\in C^0(\bar{\omega})$*,* $u_0\in C^0([0,T])$ *and* $\omega = (0,T)\times(-r_0,r_0)$ *and let* u *entropy solution of the problem*

$$\begin{cases} u_t + uu_x = g(t,x) & \text{in}\,\omega \\ u(0,x) = u_0(x) & \forall\, x\in(-r_0,r_0) \end{cases} \tag{1.59}$$

then

$$\inf_{x\in[-r_0,r_0]} u_0(x) + \int_0^t \inf_{x\in[-r_0,r_0]} g(y,x)dy \le u(t,x) \le \sup_{x\in[-r_0,r_0]} u_0(x) + \int_0^t \sup_{x\in[-r_0,r_0]} g(y,x)dy$$

Corollary 1.3.6. *Let* $g\in C^0(\bar{\omega})$*,* $u_0\in C^0([0,T])$ *and* $\omega = (0,T)\times(-r_0,r_0)$ *and let* u *entropy solution of the problem* (1.59)*. Then*

$$|u(t,x)| \le \|u_0\|_\infty + \|g\|_\infty r_0. \tag{1.60}$$

Proof. By Theorem 1.3.5

$$\inf_{x\in[-r_0,r_0]} u_0(x) + \int_0^t \inf_{x\in[-r_0,r_0]} g(y,x)dy \le u(t,x) \le \sup_{x\in[-r_0,r_0]} u_0(x) + \int_0^t \sup_{x\in[-r_0,r_0]} g(y,x)dy \tag{1.61}$$

Therefore

$$|u(t,x)| \le \sup_{x\in[-r_0,r_0]} |u_0(x)| + \int_0^t \sup_{x\in[-r_0,r_0]} |g(y,x)|dy \tag{1.62}$$

Let us notice that, since $g\in C^0(\overline{\omega})$,

$$\int_0^t \sup_{x\in[-r_0,r_0]} |g(y,x)|\,dy \le \int_0^t \sup_{(y,x)\in\overline{\omega}} |g(y,x)|\,dy \le \|g\|_\infty t \le \|g\|_\infty r_0. \tag{1.63}$$

Therefore, by (1.62) and (1.63) we obtain

$$|u(t,x)| \le \|u_0\|_\infty + \|g\|_\infty r_0.$$

□

1.4 Broad Solutions of Conservation Laws

Scalar semilinear equation

Let us consider the scalar semilinear equation

$$u_t + a(t,x)u_x = g(t,x,u) \tag{1.64}$$

with initial condition (1.2). Let us assume that

$$a : \mathbb{R}^2 \to \mathbb{R} \text{ is locally Lipschitz continuous,} \tag{1.65}$$

$$g : \mathbb{R}^3 \to \mathbb{R} \text{ is locally Lipschitz continuous,} \tag{1.66}$$

$$u_0 : [-r_0, r_0] \to \mathbb{R} \text{ continuous.} \tag{1.67}$$

Observe that (1.64) describes the directional derivative of $u : \mathbb{R}^2 \to \mathbb{R}$ at each point (t,x) in the direction $V(t,x) := (1, a(t,x))$. Let us recall that the integral curve of V are called characteristic curves, i.e. the function $x = x(t)$ where $\dot{x} = \dfrac{dx}{dt} = a(t,x)$, see [51] chapter 3.2 and [21], chapter 3.

For any point $(\tau, y) \in \mathbb{R}^2$ let us denote by $I_{(\tau,y)} \ni t \longmapsto x(t;\tau,y)$ the maximal solution of the Cauchy problem

$$\dot{x}(t) = a(t, x(t)), \qquad x(\tau) = y. \tag{1.68}$$

If $x = x(t)$ is a solution of (1.68) then

$$\frac{d}{dt}u(t,x(t)) = u_t + u_x\dot{x} = g(t,x(t),u(t,x(t))).$$

Therefore, provided that

$$0 \in I_{(\tau,y)} \tag{1.69}$$

the value of a C^1 solution at a point (τ, y) coincides the value at time τ of the solution to the Cauchy problem for the ODE

$$\frac{d}{dt}u = g(t, x(t;\tau,y), u) \qquad u(0,x) = u_0(x(0;\tau,y)).$$

Remark 1.4.1. Notice that the value of u_0 at a point x_0 determines the value of the solution u along the entire characteristic line $t \mapsto x(t; 0, x_0)$. The "information" contained in the initial data is transported along the characteristic lines. In the semilinear case the characteristics are entirely determined by the equation (1.68) and don't depend on the initial condition u_0.

Remark 1.4.2. Notice that (1.69) can be not true. Indeed, for instance, let $a(t,x) = x^2$, $h \equiv 0$, $u_0 \equiv 0$, then $\forall\,(\tau, y) \in [0,\infty) \times \mathbb{R}$

$$x(t;\tau,y) = \frac{y}{1 - y(t-\tau)}$$

$$I_{\tau,y} = \begin{cases} 0 & \text{if } y = 0 \\ \left(-\infty, \tau + \frac{1}{y}\right) & \text{if } y \neq 0 \end{cases}$$

thus if $\tau + \frac{1}{y} \leq 0$ then $0 \notin I_{\tau,y}$.

Definition 1.4.3. (Domain of determinacy): *A closed region* $D \subseteq [0,\infty)\times \mathbb{R}$ *is called a domain of determinacy for the initial problem* (1.64,1.2) *if* $\forall\,(\tau, y) \in D$ (1.69) *holds, the characteristic line*

$$\{(t, x(t;\tau,y)) : 0 \leq t \leq \tau\} \subseteq D$$

and $x(0;\tau,y) \in [-r_0, r_0]$.

The largest domain of determinacy is $D_{max} = \{(\tau, y) : \tau \geq 0,\ x(0;\tau,y) \in [-r_0, r_0]\}$.

Remark 1.4.4. We have the representation $D_{max} = \{(t,x) : t \geq 0,\ x \in I(t, -r_0, r_0)\}$ where

$$I(t, -r_0, r_0) := \begin{cases} [x(t;0,-r_0),\ x(t;0,r_0)] & \text{if } x(t;0,-r_0) \leq x(t;0,r_0) \\ [x(t;0,r_0),\ x(t;0,-r_0)] & \text{if } x(t;0,-r_0) > x(t;0,r_0) \end{cases}$$

Using the characteristic's method, let us introduce a notion of weak solution: the broad solution.

Definition 1.4.5. (Broad solution): *Let* $D \subseteq [0,\infty) \times \mathbb{R}$ *a domain of determinacy of initial value problem* (1.64,1.2). *Then a locally integrable function* $u : D \longrightarrow \mathbb{R}$ *is called broad solution of initial value problem* (1.64,1.2) *if* $\mathcal{L}^2$ *a.e.* $(\tau, y) \in D$

i $I_{\tau,y} \ni t \longmapsto u(t, x(t;\tau,y))$ *is of class* C^1,

ii $\dfrac{d}{dt}u(t, x(t;\tau,y)) = g(t, x(t;\tau,y), u(t, x(t;\tau,y))) \quad \forall\, t \in I_{\tau,y}$.

The following local existence and continuous dependence on the data result hold for scalar semilinear case, see [21], Theorem 3.3.

Theorem 1.4.6. (local existence) *Let us assume* (1.65) *holds. For any* $u_0 : [-r_0, r_0] \to \mathbb{R}$ *bounded and measurable there exist* $C, \varepsilon > 0$ *such that the semilinear initial value problem* (1.64), (1.2) *has a unique broad solution* u *on the domain*

$$D = D_{C,\varepsilon} := \{(t,x) : t \in [0,\varepsilon], \ -r_0 + Ct \leq x \leq r_0 - Ct\} \tag{1.70}$$

If the functions a, g, u_0 *are continuously differentiable then* u *actually is a classical solution.*

Proof: [21], section 3.4, using a fixed point's method.

Theorem 1.4.7. (continuous dependence) *Consider a sequence* $(a_\epsilon, g_\epsilon, u_{0,\epsilon})_{\epsilon \geq 0}$, *with* $a_\epsilon : \mathbb{R}^2 \to \mathbb{R}$, $g_\epsilon : \mathbb{R}^2 \to \mathbb{R}$, $u_{0,\epsilon} : [-r_0, r_0] \to \mathbb{R}$ *that satisfy the condition* (1.65,1.66,1.67). *Let* $D \subset [0, +\infty) \times \mathbb{R}$ *be a common domain of determinacy for the problem* (1.64,1.2) *with data* $a_\epsilon, g_\epsilon, u_{0,\epsilon}$ *and let* u_ϵ *the solution of this problem. If* $a_{ep} \to \bar{a}$ *uniformly on* D, $u_{0,\epsilon} \to \bar{u}_0$ *uniformly on* $[-r_0, r_0]$ *and* $g_\epsilon \to \bar{g}$ *uniformly on every compact set* $K \subseteq \mathbb{R}^2$, *then* $u_\epsilon \to \bar{u}$ *uniformly on* D, *where* $\bar{u}$ *is the solutions of the problem* (1.64,1.2) *with data* $bara, \bar{g}, \bar{u}_{0,\epsilon}$.

Proof: [21], section 3.5.

Scalar quasilinear equation

Let us consider the equation

$$u_t + a(t,x,u)u_x = g(t,x,u) \tag{1.71}$$

with initial condition (1.2). Let us recall that a classical solution of (1.71) is a function $u : \omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuously differentiable (i.e. $u \in C^1(\omega)$) satisfying (1.71) where ω is an open set. To find classical solutions, let us apply the characteristics' method to the problem (1.71,1.2), following [21]. At the moment let us assume $\omega = \mathbb{R}^2$, $a, g \in C^1(\mathbb{R}^3)$, $u_0 \in C^1(\mathbb{R}^3)$ and consider the vector field $V = (1, a(t,x), g(t,x,u))$. His integral curves are the characteristic curves of the equation, obtained by solving for $y \in \mathbb{R}$ the system of ODEs

$$\begin{cases} \dfrac{dx(t;y)}{dt} = a(t, x(t;y), u(t, x(t;y))) \\ x(0) = y \end{cases} \qquad \begin{cases} \dfrac{du(t,x(t;y))}{dt} = g(t, x(t;y), u(t, x(t;y))) \\ u(0) = u_0(x(0;y)) \end{cases} \tag{1.72}$$

where $I_y \ni t \mapsto x(t;y)$ denote the maximal solution of the first problem in (1.72) and $I_y \ni t \mapsto (x(t;y), u(t, x(t;y)))$ the maximal solution of the

system (1.72). As y varies, the graph of all these solutions generates a two-dimensional surface $S \subset \mathbb{R}^3$, parametrized by (t, y). We are going to see that S is the graph of a function $u = u(t, x)$, which provides a classical solution to (1.71).
Indeed, by classical Theorems on ODEs, the map $(t, y) \mapsto (t, x(t; y), u(t, x(t; y)))$ defining S in parametric form is continuously differentiable. Let $\bar{y} \in \mathbb{R}$ be fixed, then we have

$$\frac{dt}{dt} = 1, \qquad \frac{dt}{dy} = 0, \qquad \frac{dx}{dt} = a(0, \bar{y}, u_0(\bar{y})), \qquad \frac{dx}{dy} = 1.$$

By the Implicit Function Theorem, the map $(t, y) \mapsto (t, x(t; y))$ is locally invertible in a neighbourhood I of $(0, \bar{y})$. Therefore S is locally the graph of a C^1 function $u = u(t, x)$. By (1.72) the initial data clearly holds. Now let (t, x) be any point in I, say with $x = x(t; y)$ for some y. Then

$$u_t + a(t, x, u)u_x = \frac{d}{dt}u(t, x(t; y)) = g(t, x(t; y), u(t, x(t; y))),$$

proving that u is a solution of (1.71).

Let us notice, in the following example, that the invertibility of the map $(t, y) \mapsto (t, x(t; y))$ is necessary to ensure the existence of the classical solution of the problem (1.71). Where this map is not invertible, the characteristic lines can intersect and the classical solution doesn't exist.

Example 1.4.8. Consider the problem

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = 0 \\ u(0, x) = u_0(x) = \dfrac{1}{1 + x^2} \end{cases} \tag{1.73}$$

Let us apply the characteristics' method. The system (1.72) is, $\forall y \in \mathbb{R}$,

$$\begin{cases} \dfrac{dx}{dt} = u \\ x(0) = y \end{cases} \qquad\qquad \begin{cases} \dfrac{du}{dt} = 0 \\ u(0) = \dfrac{1}{1 + y^2} \end{cases}$$

Since $\dfrac{du}{dt} = 0$, u must be constant along the characteristic lines

$$t \mapsto (t, y + tu_0(y)) = \left(t, y + \frac{t}{1 + y^2}\right)$$

and the surface S is parametrized by

$$(t,y) \mapsto \left(t, y + \frac{t}{1+y^2}, \frac{1}{1+y^2}\right).$$

The function $y \mapsto x(t,y) = y + \dfrac{t}{1+y^2}$ has a smooth inverse, say $y = y(t,x)$, for $t < \dfrac{8}{\sqrt{27}}$. Indeed the solution of (1.73) can be written as

$$u(t,x) = \frac{1}{1+y^2(t,x)} \qquad \text{for } t < \frac{8}{\sqrt{27}}.$$

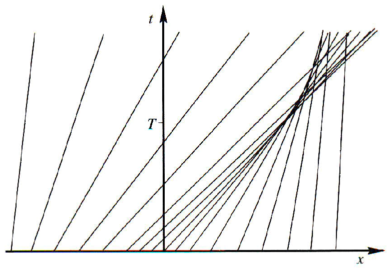


Figure 1.1: The intersection of the characteristic lines

For $t > \dfrac{8}{\sqrt{27}}$ the function $y \mapsto x(t,y) = y + \dfrac{t}{1+y^2}$ is not one-to-one, the characteristic lines start to intersect and no classical solution exists, see figure 1.1, by [21]. $\square$

Similarly to the semilinear case, we can introduce the concept of broad solution for the equation (1.71,1.2). Let us assume that (1.66) and

$$a : \mathbb{R}^3 \longrightarrow \mathbb{R} \quad \text{is locally Lipschitz continuous.} \tag{1.74}$$

Definition 1.4.9. *Let $D \subseteq \mathbb{R}^2$ be a closed region, then $u : D \longrightarrow \mathbb{R}$ is called a broad solution of the quasilinear problem* (1.71) *with initial datum* (1.2) *provided that the following conditions hold:*

i *Let $\hat{a}(t,x) := a(t,x,u(t,x))$ then $\hat{a} : D \longrightarrow \mathbb{R}$ is locally Lipschitz continuous.*

ii *D is a domain of determinacy for the initial value problem*

$$\begin{cases} u_t + \hat{a}(t,x)u_x = g(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \tag{1.75}$$

iii *u is a broad solution of the semilinear problem* (1.75)*.*

Proposition 1.4.10. *Let us suppose that $u : D \longrightarrow \mathbb{R}$ is a broad solution of the quasilinear problem* (1.71) *with initial condition* (1.2) *and that it is locally Lipschitz continuous. Then*

$$u_t(t,x) + a(t,x,u(t,x))u_x(t,x) = g(t,x,u(t,x)) \qquad \mathcal{L}^2 - a.e.\,(t,x) \in D \tag{1.76}$$

Proof. By definition $\mathcal{L}^2 - a.e.\,(\tau, y) \in D$, if $x(\cdot;\tau,y)$ denotes the characteristic curve of the semilinear equation

$$u_t + \hat{a}(t,x)u_x = g(t,x,u),$$

i.e.

$$\dot{x}(t;\tau,y) = \hat{a}(t,x(t;\tau,y)) = a(t,x(t;\tau,y),u(t,x(t;\tau,y))$$

and $x(\tau;\tau,y) = y$, then

i $I_{\tau,y} \ni t \longmapsto u(t,x(t;\tau,y))$ is of class C^1

ii $\dfrac{d}{dt}u(t,x(t;\tau,y)) = g(t,x(t;\tau,y),u(t,x(t;\tau,y)))\ \forall\, t \in I_{\tau,y}$.

Since u is locally Lipschitz continuous, by the chain rule for Lipschitz functions

iii $\exists\, \dfrac{d}{dt}u(t,x(t;\tau,y)) = u_t(t,x(t;\tau,y)) + u_x(t,x(t;\tau,y))\dot{x}(t;\tau,y) =$

$$= (u_t + a(t,x,u)u_x)(t;x(t;\tau,y)) \qquad a.e.\, t \in I_{\tau,y}.$$

Let us prove now that (1.76) holds. Indeed let

$$D_1 := \{(t,x) \in D : \quad \text{u is differentiable at } (t,x)\}$$

and denote by D_2 the points $(\tau,y) \in D$ for which there exists the characteristic curve $x(\cdot;\tau,y)$.
Then $\mathcal{L}^2(D \setminus (D_1 \cap D_2)) = 0$ and $\forall\,(\tau,y) \in D_1 \cap D_2$

$$g(\tau,y,u(\tau,y)) = \frac{du}{dt}(\tau,y) = u_t(\tau,y) + a(\tau,y,u(\tau,y))u_x(\tau,y).$$

□

The following local existence result hold for scalar quasilinear case.

Theorem 1.4.11. *Let us assume* (1.74) *and* (1.66) *hold and* $u_0 \in Lip([-r_0, r_0])$. *Then there exist* $C, \varepsilon > 0$ *and a Lipschitz continuous function* $u : D \to \mathbb{R}$ *which is the unique broad solution of the quasilinear initial value problem* (1.71,1.2) *on the domain*

$$D = D_{C,\varepsilon} := \{(t, x) : t \in [0, \varepsilon], -r_0 + Ct \leq x \leq r_0 - Ct\} \tag{1.77}$$

If the functions a, g, u_0 *are continuously differentiable then* u *actually is a classical solution.*

Proof: see [21], Theorem 3.8.

Remark 1.4.12. We remark that, in contrast with semilinear case, the characteristic lines and hence the domains of determinacy depend on the solution u. In the semilinear case the broad solution were defined within the set of locally integrable functions, in quasilinear case the study of broad solutions is restricted within the class of Lipschitz-continuous functions. It depends to the fact that a depends on u and that the lipschitz continuity of a is essential in order to define uniquely the characteristic lines.

Looking at the proof of Theorem 1.4.11 we can extract the following approximation result:

Corollary 1.4.13. *Let* u *be the function in Theorem* 1.4.11 *and let* D *be the set in* (1.77). *Then there exists a sequence of* C^1*-functions*

$$u^{(k)} : D \longrightarrow \mathbb{R}$$

such that

i $u^{(k)} \to u$ *uniformly in* D

ii $u^{(k)}$ *is the solution of the semilinear problem*

$$\begin{cases} u_t^{(k)} + a(t, x, u^{(k-1)}(t, x))u_x^{(k)} = g(t, x, u^{(k)}) & \text{in } D \\ u^{(k)}(0, x) = u_0(x) & x \in [-r_0, r_0] \end{cases}$$

If (1.66) doesn't hold or $u_0 \notin Lip([-r_0, r_0])$ we can find a discontinuous solution through the characteristics' method, this solution is not a broad solution. Indeed in this case the existence of broad solution is not guaranteed by Theorem 1.4.11.

Example 1.4.14. Let us consider the problem

$$\begin{cases} u_t + uu_x = 0 & \text{in } \omega = (0,T)\times(-r_0,r_0) \\ u(0,x) = u_0(x) & \text{in } (-r_0,r_0) \end{cases} \tag{1.78}$$

with $u_0(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1-\sqrt{x} & \text{if } x > 0 \end{cases}$. Let us solve the problem

$$\begin{cases} \dot{x}(t) = u(t,x(t)) \\ x(0) = y \end{cases} \tag{1.79}$$

Since $\dfrac{d}{dt}u(t,x(t)) = u_t + u_x\dot{x} = u_t + uu_x = 0$ we have

$$u(t,(x(t)) = u(0,y) = u_0(y). \tag{1.80}$$

Then (1.79) becomes $\dot{x}(t) = u(0,y)$ and so we obtain

$$x(t) = u_0(y)t + y.$$

In our case, by the definition of u_0 in (1.78)

$$\begin{cases} x = t+y & \text{if } y \leq 0 \\ x = (1-\sqrt{y})t + y & \text{if } y > 0 \end{cases} \tag{1.81}$$

By (1.81) we obtain

$$\begin{cases} y = x-t & \text{if } x-t \leq 0 \\ \sqrt{y} = \frac{t+\sqrt{t^2+4(x-t)}}{2} & \text{if } x-t > 0 \end{cases}$$

Then, by (1.80), the solution of (1.78) is

$$u(t,x) = \begin{cases} 1 & \text{if } x-t \leq 0 \\ 1-\frac{t+\sqrt{t^2+4(x-t)}}{2} & \text{if } x-t > 0 \end{cases} \tag{1.82}$$

This solution is discontinuous on the line $\{x-t=0\}$, indeed for $t>0$ fixed

$$\lim_{x\to t^-} u(t,x) = 1$$

$$\lim_{x\to t^+} u(t,x) = 1-t$$

It is easy to verify that u is a classical solution on the two regions $\{(t,x)\in \omega : x-t\leq 0\}$ and $\{(t,x)\in\omega : x-t>0\}$, but it is not a broad solution on the entire region ω. It is not a distributional solution too, indeed let us show that the solution found in example 1.4.14 does not satisfy the Rankine-Hugoniot condition (1.7), therefore it is not a distributional solution on $(0,T)\times(-r_0,r_0)$. Indeed along the discontinuity's line $x=t$ we have

$$u_+(t)=1-t, \qquad u_-(t)=1, \qquad \dot{s}(t)=1$$

$$f\left(u_+(t)\right)=\frac{(1-t)^2}{2}, \qquad f\left(u_-(t)\right)=\frac{1}{2}.$$

Then condition (1.7) becomes

$$1(1-t-1)=\frac{(1-t)^2}{2}-\frac{1}{2}$$

that's false, because $0\neq\dfrac{t^2}{2}$ for $t>0$. A distributional solution of the problem of example 1.4.14 is of the type

$$u(t,x)=\begin{cases} 1 & \text{if } x\leq s(t)\\ 1-\frac{t+\sqrt{t^2+4(x-t)}}{2} & \text{if } x>s(t)\end{cases}$$

where $s(t)$, which parametrizes the line of discontinuity, is such that the Rankine-Hugoniot condition (1.7) hold and $s(0)=0$. $\square$

In examples 1.4.14 we have seen that if $u_0\notin Lip([-r_0,r_0])$ we can not find a broad solution of the conservation laws but a discontinuous distributional solution. Otherwise if u is a continuous distributional solution of the equation (1.3), then u has good regularity properties along the characteristic lines. Indeed let us indicate the characteristic line associated with the continuous distributional solution u of the equation (1.3)

$$u_t+\left(f(u)\right)_x=g(t,x),$$

as the solution $x=\xi(t)$ of the ODE $\dfrac{dx}{dt}=f'(u(t,x))$, defined on $(0,T)$. Then the following regularity results hold.

Theorem 1.4.15. *Let $g:\omega\to\mathbb{R}$ be bounded and measurable such that $g(t,\cdot)$ is continuous on $(-r_0,r_0)$ for any $t\in[0,T)$. Let $x=\xi(\cdot)$ be a characteristic associated with a continuous distributional solution u of (1.3) where f is strictly convex. Set $\nu(t)=u(t,\xi(t))$ for $t\in[0,T)$. Then $(\xi(t),\nu(t))$ satisfies the system of ODEs*

$$\begin{cases}\dot{\xi}(t)=f'(\nu(t))\\ \dot{\nu}(t)=g(t,\xi(t))\end{cases}$$

on $[0,T)$. In particular $\nu(t)$ and $\dot{\xi}(t)$ are Lipschitz on $[0,T)$.

Proof. (Cfr. [42] Theorem 3, [40] Theorem 3.3.)
Equation $\dot{\xi}(t) = f'(\nu(t))$ holds directly by definition of $\xi(t)$. Let us show $\dot{\nu}(t) = g(t, \xi(t))$. Let σ, τ be fixed such that $0 \leq \sigma < \tau < T$ and let $\epsilon > 0$. Let us consider the functions $\psi(t,x)$ and $h(t)$ defined for small $\delta > 0$ by

$$\psi(t,x) = \begin{cases} 0 & 0 < t < T,\ -r_0 < x \leq \xi(t) - \epsilon - \delta \\ \frac{1}{\delta}(x - \xi(t) + \epsilon + \delta) & 0 < t < T,\ \xi(t) - \epsilon - \delta < x \leq \xi(t) - \epsilon \\ 1 & 0 < t < T,\ \xi(t) - \epsilon < x \leq \xi(t) \\ \frac{1}{\delta}(-x + \xi(t) + \delta) & 0 < t < T,\ \xi(t) < x \leq \xi(t) + \delta \\ 0 & 0 < t < T,\ \xi(t) + \delta < x < r_0 \end{cases} \tag{1.83}$$

$$h(t) = \begin{cases} 0 & 0 < t \leq \sigma - \delta \\ \frac{1}{\delta}(t - \sigma + \delta) & \sigma - \delta < t \leq \sigma \\ 1 & \sigma < t \leq \tau \\ \frac{1}{\delta}(-t + \tau + \delta) & \tau < t \leq \tau + \delta \\ 0 & \tau + \delta < t < T \end{cases} \tag{1.84}$$

and write the distributional equation (1.4)

$$\int_0^T \int_{-r_0}^{r_0} [u\varphi_t + f(u)\varphi_x + g\varphi] \ dx\, dt = 0 \tag{1.85}$$

for the test function $\varphi(t,x) = \psi(t,x)h(t)$ and compute the limit $\delta \to 0^+$:

$$\lim_{\delta \to 0^+} \left(\int_0^T \int_{-r_0}^{r_0} u(t,x)\psi_t(t,x)h(t)\, dx\, dt \ + \int_0^T \int_{-r_0}^{r_0} u(t,x)\psi(t,x)h_t(t)\, dx\, dt \ + \right.$$
$$\left. + \int_{-r_0}^{r_0} f(u(t,x))\psi_x(t,x)h(t) \ + \int_0^T \int_{-r_0}^{r_0} g(t,x)\psi(t,x)h(t) \ dx\, dt \right) = \lim_{\delta \to 0^+} (I_1 + I_2 + I_3 + I_4) \tag{1.86}$$

Using the Integral Average Theorem we obtain that

$$\lim_{\delta \to 0^+} I_1 = \lim_{\delta \to 0^+} \left[\int_{\sigma-\delta}^{\sigma} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} u \frac{-\dot{\xi}(t)}{\delta}(t - \sigma + \delta) dx + \right.\right.$$
$$\left. + \int_{\xi(t)}^{\xi(t)+\delta} u \frac{\dot{\xi}(t)}{\delta}(t - \sigma + \delta) dx \right) + \int_{\sigma}^{\tau} dt \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} u \frac{-\dot{\xi}(t)}{\delta} dx + \int_{\xi(t)}^{\xi(t)+\delta} u \frac{\dot{\xi}(t)}{\delta} dt \right) +$$
$$\left. + \int_{\tau}^{\tau+\delta} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} u \frac{-\dot{\xi}(t)}{\delta}(-t + \tau + \delta) dx + \int_{\xi(t)}^{\xi(t)+\delta} u \frac{\dot{\xi}(t)}{\delta}(-t + \tau + \delta) dx \right) \right] =$$

$$= 0 + \int_{\sigma}^{\tau} \left[-\dot{\xi}(t) u(t, \xi(t) - \epsilon) + \dot{\xi}(t) u(t, \xi(t)) \right] dt + 0 \tag{1.87}$$

$$\lim_{\delta \to 0^+} I_2 = \lim_{\delta \to 0^+} \left[\int_{\sigma-\delta}^{\sigma} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} u \frac{x-\xi(t)+\epsilon+\delta}{\delta} dx + \int_{\xi(t)-\epsilon}^{\xi(t)} u \, dx + \int_{\xi(t)}^{\xi(t)+\delta} u \frac{-x+\xi(t)+\delta}{\delta} dx \right) + \right.$$

$$\left. - \int_{\tau}^{\tau+\delta} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} u \frac{x-\xi(t)+\epsilon+\delta}{\delta} dx + \int_{\xi(t)-\epsilon}^{\xi(t)} u \, dx + \int_{\xi(t)}^{\xi(t)+\delta} u \frac{-x+\xi(t)+\delta}{\delta} dx \right) \right] =$$

$$= \int_{\xi(\sigma)-\epsilon}^{\xi(\sigma)} u(x, \sigma) dx - \int_{\xi(\tau)-\epsilon}^{\xi(\tau)} u(x, \tau) dx \tag{1.88}$$

$$\lim_{\delta \to 0^+} I_3 = \lim_{\delta \to 0^+} \left[\int_{\sigma-\delta}^{\sigma} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} \frac{f(u)}{\delta} dx - \int_{\xi(t)}^{\xi(t)+\delta} \frac{f(u)}{\delta} dx \right) + \right.$$

$$\left. + \int_{\sigma}^{\tau} dt \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} \frac{f(u)}{\delta} dx - \int_{\xi(t)}^{\xi(t)+\delta} \frac{f(u)}{\delta} dx \right) + \int_{\tau}^{\tau+\delta} \frac{dt}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} \frac{f(u)}{\delta} dx - \int_{\xi(t)}^{\xi(t)+\delta} \frac{f(u)}{\delta} dx \right) \right] =$$

$$= 0 + \int_{\sigma}^{\tau} \left[f(u(t, \xi(t) - \epsilon)) - f(u(t, \xi(t))) \right] dt + 0 \tag{1.89}$$

$$\lim_{\delta \to 0^+} I_4 = \lim_{\delta \to 0^+} \left[\int_{\sigma-\delta}^{\sigma} \frac{(t-\sigma+\delta)}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx + \int_{\xi(t)-\epsilon}^{\xi(t)} g \, dx + \int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx + \right) dt + \right.$$

$$+ \int_{\sigma}^{\tau} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx + \int_{\xi(t)-\epsilon}^{\xi(t)} g \, dx + \int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx \right) dt +$$

$$\left. + \int_{\sigma-\delta}^{\sigma} \frac{(-t+\tau+\delta)}{\delta} \left(\int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx + \int_{\xi(t)-\epsilon}^{\xi(t)} g \, dx + \int_{\xi(t)-\epsilon-\delta}^{\xi(t)-\epsilon} g \frac{x-\xi(t)+\epsilon+\delta}{\delta} \, dx \right) dt \right] =$$

$$= 0 + \int_{\sigma}^{\tau} \int_{\xi(t)-\epsilon}^{\xi(t)} g(t, x) \; dx \, dt + 0 \tag{1.90}$$

Replacing (1.87), (1.88), (1.89) and (1.90) in (1.85) we obtain

$$\int_{\xi(\tau)-\epsilon}^{\xi(\tau)} u(x, \tau) dx - \int_{\xi(\sigma)-\epsilon}^{\xi(\sigma)} u(x, \sigma) dx - \int_{\sigma}^{\tau} \int_{\xi(t)-\epsilon}^{\xi(t)} g(t, x) \; dx \, dt = \tag{1.91}$$

$$= \int_\sigma^\tau \Big\{ f(u(t,\xi(t)-\epsilon)) - f(u(t,\xi(t))) - \dot{\xi}(t) \left[u(t,\xi(t)-\epsilon) - u(t,\xi(t)) \right] \Big\} \, dt \geq 0$$

In the same way we obtain

$$\int_{\xi(\tau)}^{\xi(\tau+\epsilon)} u(x,\tau)dx - \int_{\xi(\sigma)}^{\xi(\sigma)+\epsilon} u(x,\sigma)dx - \int_\sigma^\tau \int_{\xi(t)}^{\xi(t)+\epsilon} g(t,x) \; dx \, dt = \tag{1.92}$$

$$= \int_\sigma^\tau \Big\{ f(u(t,\xi(t)+\epsilon)) - f(u(t,\xi(t))) - \dot{\xi}(t) \left[u(t,\xi(t)+\epsilon) - u(t,\xi(t)) \right] \Big\} \, dt \leq 0$$

Notice that since $\dot{\xi}(t) = f'(u(t,\xi(t)))$ and f is convex, the right-hand side of (1.91) is nonnegative and the right-hand side of (1.92) is nonpositive.
Upon dividing (1.91) and (1.92) by ϵ and then letting $\epsilon \to 0$, we obtain by the Integral Mean Theorem that

$$\nu(\tau) - \nu(\sigma) - \int_\sigma^\tau g(t,\xi(t)) \, dt = 0,$$

and so we have the thesis. □

Remark 1.4.16. let $u \in C^1(\omega)$ be a classical solution of (1.4), let $\sigma, \tau \in (0,T)$ and $\epsilon > 0$. If we integrate (1.4) over the two domains $D_1 := \{(t,x) \in \omega : \sigma < t < \tau, \, \xi(t) - \epsilon < x < \xi(t)\}$ and $D_2 := \{(t,x) \in \omega : \sigma < t < \tau, \, \xi(t) < x < \xi(t) + \epsilon\}$, we obtain respectively (1.91) and (1.92) by an easy application of the Green's formula. Let us show (1.91). By (1.4) we have

$$\int_\sigma^\tau \int_{\xi(t)-\epsilon}^{\xi(t)} \left[u_t(t,x) + f(u(t,x))_x - g(t,x) \right] \; dx \, dt = 0$$

By Green's formula it is

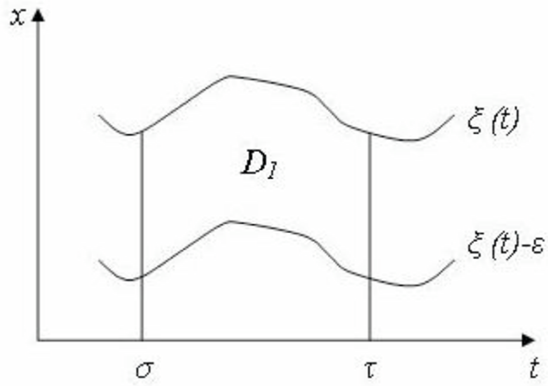
$$\oint_C \left[u(t,x)dx \; - f(u(t,x))dt \right] \; - \int_\sigma^\tau \int_{\xi(t)-\epsilon}^{\xi(t)} g(t,x) \; dx \, dt = 0 \tag{1.93}$$

where C is the boundary of the domain D_1.

Let us calculate

$$\oint_C u(t,x)dx = \int_\sigma^\tau u(t,\xi(t)-\epsilon)\dot{\xi}(t) \, dt \; + \int_{\xi(\tau)-\epsilon}^{\xi(\tau)} u(\tau,x)dx \; + \tag{1.94}$$

$$+ \int_\tau^\sigma u(t,\xi(t))\dot{\xi}(t) \, dt \; + \int_{\xi(\sigma)}^{\xi(\sigma)-\epsilon} u(\sigma,x)dx \qquad \text{and}$$



$$\oint_C f(u(t,x))dt = \int_\sigma^\tau f(u(t,\xi(t)-\epsilon))\,dt\ +0+\ \int_\tau^\sigma f(u(t,\xi(t)))\,dt\ +0 \quad (1.95)$$

By (1.93), (1.94) and (1.95) we obtain

$$\int_{\xi(\tau)-\epsilon}^{\xi(\tau)} u(\tau,x)dx - \int_{\xi(\sigma)-\epsilon}^{\xi(\sigma)} u(\sigma,x)dx\ - \int_\sigma^\tau \int_{\xi(t)-\epsilon}^{\xi(t)} g(t,x)\ dx\,dt +$$

$$+\int_\sigma^\tau \Big[u(t,\xi(t)-\epsilon)\dot{\xi}(t) - u(t,\xi(t))\dot{\xi}(t) - f(u(t,\xi(t)-\epsilon)) + f(u(t,\xi(t)))\Big]\ dt = 0$$

and finally (1.91). □

In the case of Theorem 1.4.15, if $g \equiv 0$ the solution u is constant along the characteristic lines.

Corollary 1.4.17. *Let $x = \xi(t)$ be any characteristic line associated with the continuous distributional solution of $u_t + f(u)_x = 0$. Then $u(t,\xi(t)) \equiv \overline{u}$ constant, $\forall t \in (0,T)$. In particular, a unique characteristic emanates from each point $(\bar{t},\bar{x}) \in \omega$.*

Proof. Directly by Theorem 1.4.15, see [41], Theorem 1. □

1.5 Hamilton Jacobi equations and applications to Conservation Laws

Let us consider the inizial value problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(t,x,u,u_x) = 0 & \text{in } \omega = (0,T)\times(-r_0,r_0) \\ u(0,x) = u_0(x) & \forall\, x \in (-r_0,r_0) \end{cases} \quad (1.96)$$

where $H : (0,T) \times (-r_0, r_0) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, called the Hamiltonian. It is not possible to consider distributional solutions of (1.96) defined as usual by integration by parts. Lions and Crandall introduced in [37], see [39, 38, 74, 51] too, a concept of generalized solutions, the viscosity solutions, a class of generalized solutions that play the role of weak solutions for Hamilton-Jacobi equation (1.96).

Definition 1.5.1. *A function $u \in BUC(\omega)$ is called a viscosity solution of* (1.96) *if*

i *$\forall \varphi \in C^1_c(\omega)$, $\forall k \in \mathbb{R}$, if $\varphi \cdot (u-k)$ has a local positive maximum at $(t_0, x_0) \in \omega$ then*

$$-\frac{\varphi_t(t_0,x_0)}{\varphi(t_0,x_0)}(u(t_0,x_0)-k)+H\left(t_0,x_0,u(t_0,x_0),-\frac{\varphi_x(t_0,x_0)}{\varphi(t_0,x_0)}(u(t_0,x_0)-k)\right) \leq 0,$$

and if $\varphi \cdot (u-k)$ has a local negative minimum at $(t_0, x_0) \in \omega$ then

$$-\frac{\varphi_t(t_0,x_0)}{\varphi(t_0,x_0)}(u(t_0,x_0)-k)+H\left(t_0,x_0,u(t_0,x_0),-\frac{\varphi_x(t_0,x_0)}{\varphi(t_0,x_0)}(u(t_0,x_0)-k)\right) \geq 0,$$

ii *$u(0,x) = u_0(x)$ $\forall x \in (-r_0, r_0)$.*

Remark 1.5.2. The notion of viscosity solution is a notion of "weak" solution, since u is assumed to be only continuous and ∇u could not exist. Definition 1.5.1 show that, in some sense, at a point of maximum of $\varphi \cdot (u-k)$ a good candidate for the definition of ∇u is $-\frac{\nabla \varphi}{\varphi} \cdot (u-k)$. Indeed there exists some analogy between this notion of solutions and the standard distribution theory: integration by parts is replaced here by differentiation by parts and "is done inside the nonlinearity". There is some parallel between this notion and the notion of entropy solution for scalar conservation laws, see [74].

The notion of viscosity solution is consistent with the classical concept of solution, indeed

Proposition 1.5.3. **i** *Let $u \in C^1(\omega)$ a classical solution of* (1.96)*. Then u is a viscosty solution too.*

ii *Let $u \in C^0(\omega)$ a viscosity solution of* (1.96) *and suppose u is differentiable at $(t_0, x_0) \in \omega$. Then*

$$u_t(t_0,x_0) + H(t_0,x_0,u(t_0,x_0),u_x(t_0,x_0)) = 0.$$

Proof. cfr. [39] Theorem 1.2 □

Remark 1.5.4. The name "*viscosity solutions*" refers to the "vanishing viscosity" method, i.e. the viscosity solution u as the limit of the sequence $\{u^\epsilon\}_\epsilon$ of solutions of the parabolic problem

$$\begin{cases} u_t^\epsilon + H(t,x,u^\epsilon,u_x^\epsilon) = \epsilon u_{xx}^\epsilon & \text{in } \omega = (0,T)\times(-r_0,r_0) \\ u^\epsilon(0,x) = u_0(x) & \forall\, x \in (-r_0,r_0) \end{cases} \tag{1.97}$$

as we will see in the next Theorem.

Theorem 1.5.5. *Let $u^\epsilon \in C^2(\omega)$ be a solution of* (1.97) *and assume that there exist $u \in C^0(\omega)$ and a subsequence $\{u^{\epsilon_n}\}_n \subset \{u^\epsilon\}_\epsilon$ such that $u^{\epsilon_n} \to u$ in $C^0(\omega)$. Then u is a viscosity solution of* (1.96).

Proof. cfr. [37] Proposition IV.1 □

By Theorem 1.5.5 and using the vanishing viscosity method, we infers the following existence and uniqueness theorems.

Theorem 1.5.6. (Existence) *Assume $H(t,x,z,p) \in C^0(\omega\times\mathbb{R}\times\mathbb{R})$ satisfies*

i *for each $R > 0$ there exists a nondecreasing $\sigma_R : [0,2R] \to [0,+\infty)$ such that $\lim_{s\to 0^+} \sigma_R(s) = 0$ and*

$$|H(t,x,z,p)|-|H(t,x,z,q)| \le \sigma_R(|p-q|) \qquad \forall\,(t,x)\in\omega,\ z\in\mathbb{R},\ p,q\in B(0,R)$$

ii *there exists $\gamma \in \mathbb{R}$ for which*

$$|H(t,x,z,p)| - |H(t,x,z',p)| \ge \gamma|z-z'| \qquad \forall\,(t,x)\in\omega,\ z,z',p\in\mathbb{R}$$

iii *there exists a nondecreasing $\theta : [0,+\infty) \to [0,+\infty)$ such that $\lim_{s\to 0^+}\theta(s) = 0$ and $\forall\, t\in[0,T]$, $x,y\in(-r_0,r_0)$, $z,p\in\mathbb{R}$*

$$|H(t,x,z,p)| - |H(t,y,z,p)| \le \theta(|x-y|(1+|p|)).$$

Then there exists $u \in C^0(\omega)$ unique viscosity solution of (1.96).

Idea of Proof: see [71], Theorem 4.2: We prove that (1.97) is solvable, then we conclude by precompactness of $\{u^\epsilon\}_\epsilon$ using estimates on u^ϵ and u_x^ϵ in $L^\infty(\omega)$ and Theorem 1.5.5. □

Theorem 1.5.7. (Local uniqueness) *Let $u, v \in C^0(\omega)$ be viscosity solution of (1.96) such that $m = \max(\|u\|_\infty, \|v\|_\infty)$, $C = \max(\|u_x\|_\infty, \|v_x\|_\infty)$. Let $H(t,x,z,p)$ be nondecreasing in r for $(t,x,p) \in \omega \times \mathbb{R}$ and such that*

$$|H(t,x,z,p) - H(t,x,z,q)| \le L|p-q| \quad \text{for } |p|, |q| \le C,\ |z| \le m, |x| \le R - Lt.$$

Then $u = v$ on $|x| \le R - Lt$.

Proof. see [37] Theorem V.3 □

Let us now notice a very interesting link between Hamilton-Jacobi equation and Conservation laws:

Theorem 1.5.8. *Let $f \in C^1(\mathbb{R})$, $w_0 \in W^{1,\infty}((-r_0, r_0))$ and $G(t,x) = \int_0^x g(t,y)dy$, $g \in C^0(\omega)$. If $w \in W^{1,\infty}((-r_0, r_0))$ is the unique viscosity solution of*

$$\begin{cases} w_t + f(w_x) = G(t,x) & \text{in } \omega \\ w(0,x) = w_0(x) & \forall\, x \in [-r_0, r_0] \end{cases} \tag{1.98}$$

then $u := w_x$ is the unique entropy solution of (1.18).

Proof. Following the proof of Theorem 2.2 in [36], by the proof of Theorem 1.5.6, see [71] Theorem 4.2, w is the limit as $\epsilon \to 0^+$ in $C^0(\omega)$ of the regular solution w^ϵ of

$$\begin{cases} w^\epsilon_t + f(w^\epsilon_x) = G + \epsilon w^\epsilon_{xx} & \text{in } \omega \\ w^\epsilon(0,x) = w_0(x) & \forall\, x \in [-r_0, r_0] \end{cases} \tag{1.99}$$

Let us notice that for any $\varphi \in C^1_c(\omega)$

$$\lim_{\epsilon \to 0^+} \int_\omega w^\epsilon_x \varphi \, dx\, dt = -\lim_{\epsilon \to 0^+} \int_\omega w^\epsilon \varphi_x \, dx\, dt = -\int_\omega w \varphi_x \, dx\, dt = \int_\omega w_x \varphi \, dx\, dt. \tag{1.100}$$

On the other hand, by Theorems 1.3.1 and 1.3.3 there exists u unique entropy solution of (1.18) and let us observe that $u^\epsilon = w^\epsilon_x$ solves the problem

$$\begin{cases} u^\epsilon_t + f(u^\epsilon)_x = g + \epsilon u^\epsilon_{xx} & \text{in } \omega \\ w^\epsilon(0,x) = w_0(x) & \forall\, x \in [-r_0, r_0] \end{cases} \tag{1.101}$$

According to the proof of Theorem 5 in [6] $u^\epsilon \to u$ in $L^\infty(\omega)$ as $\epsilon \to 0^+$. Then

$$\lim_{\epsilon \to 0^+} \int_\omega u^\epsilon \varphi \, dx\, dt = \int_\omega u \varphi \, dx\, dt \tag{1.102}$$

Consequently, by (1.100) and by the uniqueness of the weak limit $w_x = u$ a.e. in ω. □

Let us show finally an interesting regularity result of continuous distributional solution of a conservation laws. By this Theorem we conclude that the discontinuities of distributional solutions are the reasons of the non uniqueness of these solutions. This result was obtained with the help of S. Bianchini [16].

Theorem 1.5.9. *Let $u, g \in C^0(\overline{\omega})$ such that u is a distributional solution of* (1.18). *Then u is the entropy solution of* (1.18).

Proof. Equation (1.18) is equivalent to the problem

$$\operatorname{curl}\begin{pmatrix} u \\ f(u) - G \end{pmatrix} = 0 \tag{1.103}$$

where $G(t,x) = \displaystyle\int_0^x g(t,y)dy$. Then there exists $w \in C^1(\omega)$ such that

$$w_x = u \qquad w_t = -f(u) + G \tag{1.104}$$

By (1.104) we obtain the Hamilton-Jacobi equation associated with (1.18)

$$w_t + f(w_x) = G \tag{1.105}$$

By Theorems 1.3.1, 1.3.3 there exists $\bar{u}$ unique entropy solution of (1.18) and by Theorem 1.5.8 $w_x = \bar{u}$ a.e. in ω, therefore we conclude by (1.104). $\square$

Chapter 2

The Heisenberg Group

In this chapter we introduce and describe the Heisenberg Group $\mathbb{H}^n$. The Heisenberg Group is the simplest example of subriemannian Carnot-Caratheodory space, endowed with a not euclidean metric. The Heisenberg Group has a rich differential structure and is in particular a Carnot Group, i.e. his Lie Algebra is simply connected and nilpotent.
In sections 2.1 and 2.2 we recall the most important definitions and preliminary results about $\mathbb{H}^n$, see [64, 83, 1, 2, 55, 82, 56, 57, 60, 81, 80, 76, 44, 25, 94, 32, 22, 23] too. In section 2.3 we speak about multilinear algebra in $\mathbb{H}^n$, in particular we recall some definitions and results by [8, 52, 58, 59, 90]. In the last section 2.4 we discuss about the Rumin complex, a complex of intrinsic differential forms that fits the structure if $\mathbb{H}^n$ in the same way as De Rham complex does in Euclidean spaces. In this section we establish the explicit compatibility's conditions of $F \in \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{2n})$ for the existence of $\phi \in \mathcal{D}'(\mathbb{H}^n)$ such that $\nabla_{\mathbb{H}}\phi = F$ (see Theorem 2.4.16), giving a generalization of proposition 2.6 in [59].

2.1 Definition and preliminary results

We indicate by $\mathbb{H}^n$ the n-dimensional Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$. We shall denote the points of $\mathbb{H}^n$ by $P = (z,t) = (x+iy,t)$, $z \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$, let us write $(x_1, ...x_n, y_1, ..., y_n, t) = (x_1, ..., x_{2n}, t)$ too.

Definition 2.1.1. *If $P = (z,t)$, $Q = (\zeta, \tau) \in \mathbb{H}^n$ let us define the group operation*

$$P \cdot Q := \left(z + \zeta, t + \tau + \frac{1}{2} \Im m(\zeta \cdot \bar{z}) \right). \tag{2.1}$$

We denote as $P^{-1} := (-z, -t)$ the inverse of P and as e the origin of $\mathbb{H}^n$.

Definition 2.1.2. *Let us introduce the group of left- translations defined by* $\tau_P : \mathbb{H}^n \to \mathbb{H}^n$ *as*

$$Q \mapsto \tau_P(Q) := P \cdot Q$$

for any fixed $P \in \mathbb{H}^n$ *and the family of non isotropic dilations*

$$\delta_r(P) := (rz, r^2 t), \qquad \text{for } r > 0. \tag{2.2}$$

$\mathbb{H}^n$ is the simplest example of subriemannian Carnot-Carathèodory space.

Definition 2.1.3. *We define on* $\mathbb{H}^n$ *the family of left invariant vector fields, for* $j = 1, \ldots, n$

$$\begin{aligned} X_j &:= \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t} = W_j, \\ Y_j &:= \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t} = W_{j+n}, \\ T &:= \frac{\partial}{\partial t} = W_{2n+1} \end{aligned} \tag{2.3}$$

and their commutator

$$[W_j, W_i] = W_j W_i - W_i W_j.$$

Remark 2.1.4. Let us notice that for $i, j = 1, ..., n$

$$[X_i, X_j] = [Y_i, Y_j] = 0$$

and

$$[X_i, Y_j] = \delta_{i,j} T \qquad \text{for } i, j = 1, \ldots, n. \tag{2.4}$$

Remark 2.1.5. The family of left invariant vector fields X_j, Y_j, T introduced in Definition 2.1.3 generates a Lie algebra, that we indicate with $\mathfrak{h}$. $\mathfrak{h}$ is nilpotent of step 2, i.e. there exists two subalgebra of $\mathfrak{h}$

$$\mathfrak{h}_1 := \operatorname{span}\{X_j, Y_j\}, \qquad \mathfrak{h}_2 := \operatorname{span}\{T\}$$

such that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and

$$[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2, \qquad [\mathfrak{h}_1, \mathfrak{h}_2] = 0, \qquad [\mathfrak{h}_2, \mathfrak{h}_2] = 0.$$

Remark 2.1.6. The family of vector fields $X_1, ..., Y_n$ satisfies the Hörmander's condition: the rank of the Lie algebra $\mathfrak{h}_1$ is maximal, i.e.

$$\operatorname{rank} \mathfrak{h}_1 = 2n. \tag{2.5}$$

Let us now define the Carnot-Caratheodory metric associated with the vector fields $X_1, ..., Y_n$.

Definition 2.1.7. *We say that an absolutely continuous curve* $\gamma : [0,T] \to \mathbb{H}^n$ *is a subunit curve with respect to* $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ *if there exist real measurable functions* $a_1(s), \ldots, a_{2n}(s)$, $s \in [0,T]$ *such that* $\sum_j a_j^2 \leq 1$ *and*

$$\dot{\gamma}(s) = \sum_{j=1}^{n} a_j(s) X_j(\gamma(s)) + \sum_{j=1}^{n} a_{j+n}(s) Y_j(\gamma(s)), \qquad \text{for a.e. } s \in [0,T].$$

If $P_1, P_2 \in \mathbb{H}^n$, *their Carnot–Carathéodory distance* $d_C(P_1, P_2)$ *is*

$$d_C(P_1, P_2) = \inf\left\{T > 0 : \exists \gamma : [0,T] \to \mathbb{H}^n \text{ subunit}, \, \gamma(0) = P_1, \, \gamma(T) = P_2\right\}.$$

Remark 2.1.8. Notice that the set of curves joining P_1 and P_2 is not empty for Chow's Theorem, since the Hörmander's condition (2.5) is satisfied and hence d_C is a distance on $\mathbb{H}^n$.

Definition 2.1.9. *Let us denote the open balls for* d_C *by* $U_C(P_0, r) := \{P \in \mathbb{H}^n : d_C(P, P_0) < r\}$ *and the closed balls for* d_C *by* $B_C(P_0, r) := \{P \in \mathbb{H}^n : d_C(P, P_0) \leq r\}$.

Let us define an other distance d_∞, equivalent to d_C, which is more easy to compute.

Definition 2.1.10. *On* $\mathbb{H}^n$ *we can define the homogeneous norm*

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \qquad \text{for } P = (z,t) \in \mathbb{H}^n \tag{2.6}$$

and the distance (see proposition 2.1.11*)*

$$d_\infty(P, Q) := \|P^{-1} \cdot Q\|_\infty. \tag{2.7}$$

We shall denote by $U_\infty(P_0, r)$ and $B_\infty(P_0, r)$ respectly the open and closed balls with centre $P_0 \in \mathbb{H}^n$ and radius $r > 0$ with respect to the distance d_∞ in $\mathbb{H}^n$.

Proposition 2.1.11. *The function* d_∞ *defined by* (2.7) *is a distance in* $\mathbb{H}^n$.

Theorem 2.1.12. *The Carnot-Carathéodory distance* d_C *is (globally) equivalent to the distance* d_∞ *defined in* (2.7).

Proof: see [93].

Proposition 2.1.13. *The following properties of d_∞ hold: $\forall\, P, Q, Q' \in \mathbb{H}^n$ and $\forall r > 0$*

$$d_C(\tau_P Q, \tau_P Q') = d_C(Q, Q') \qquad d_C(\delta_r Q, \delta_r Q') = r\, d_C(Q, Q'). \tag{2.8}$$

The metrics d_C and d_∞ are not equivalent to the euclidean:

Theorem 2.1.14. *For any bounded subset $\Omega \in \mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that*

$$c_1(\Omega)|P - Q|_{\mathbb{R}^{2n+1}} \le d_C(P, Q) \le c_2(\Omega)|P - Q|^{1/2}_{\mathbb{R}^{2n+1}} \tag{2.9}$$

for $P, Q \in \Omega$.

Remark 2.1.15. The topologies defined by d_C and by the Euclidean distance coincide on $\mathbb{H}^n$, therefore the topological dimension of $\mathbb{H}^n$ is $2n + 1$. On the contrary the Hausdorff dimension of $(\mathbb{H}^n, d_C)$ is $\mathcal{Q} = 2n + 2$, see Theorem 2.1.19. Indeed it was proved in [91] that there are no bi-Lipschitz maps from $\mathbb{H}^n$ to any Euclidean space.

Remark 2.1.16. $U_\infty(P, r)$ is an Euclidean Lipschitz domain in $\mathbb{R}^{2n+1}$.

There is a natural measure dh on $\mathbb{H}^n$ which is given by the Lebesgue measure $d\mathcal{L}^{2n+1} = dz\, dt$ on $\mathbb{C}^n \times \mathbb{R}$. The measure dh is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^n$ then $|E|$ denotes its Lebesgue measure.

Notation 2.1.17. (see [52]) *We shall denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, and by $\mathcal{H}^m_\infty$ the m-dimensional Hausdorff measure obtained from the distance d_∞ in $\mathbb{H}^n$. Analogously, $\mathcal{S}^m$ and $\mathcal{S}^m_\infty$ will denote the corresponding spherical measures.*

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (2.8), more precisely we have

Proposition 2.1.18. *Let $\Omega \subseteq \mathbb{H}^n$, $P \in \mathbb{H}^n$ and m,$r \in [0, \infty)$. Then*

$$\mathcal{H}^m_\infty(\tau_P \Omega) = \mathcal{H}^m_\infty(\Omega) \quad \text{and} \qquad \mathcal{H}^m_\infty(\delta_r(\Omega)) = r^m \mathcal{H}^m_\infty(\Omega). \tag{2.10}$$

As we said in remark 2.1.15, the Hausdorff dimension of $\mathbb{H}^n$ as a metric space is $\mathcal{Q} = 2n + 2$ (see [79, 84]). $\mathcal{Q}$ is called homogeneous dimension too. For this purpose, let us recall that the Eulero's Γ function is defined as

$$\Gamma(t) := \int_0^{+\infty} r^{t-1} e^{-r} dr$$

and let us define for $s \geq 0$

$$\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(1 + \frac{s}{2}\right)}. \tag{2.11}$$

Theorem 2.1.19. *Let ω_s defined as in* (2.11)*. Then*

$$h = \mathcal{L}^{2n+1} = \frac{2\omega_{2n}}{\omega_{2n+2}} \mathcal{S}_\infty^{2n+2} = \frac{2\omega_{2n}}{\mathcal{H}_\infty^{2n+2}(B_\infty(0,1))} \mathcal{H}_\infty^{\mathcal{Q}}.$$

In the following we shall identify the vector fields and the associated first-order differential operators.

Definition 2.1.20. *Let $W \in \mathfrak{h}$, $K \Subset \mathbb{H}^n$ a compact set and $p \in K$. Let us consider the Cauchy problem*

$$\begin{cases} \dot{\gamma}_p(s) = W\left(\gamma_p(s)\right) \\ \gamma_p(0) = p \end{cases} \tag{2.12}$$

Let us denote as the exponential map $\exp(sW)(p) := \gamma_p(s)$ *the solution of the problem* (2.12)*.*

Theorem 2.1.21. (Campbell-Hausdorff formula)
Let $V, W \in \mathfrak{h}$, then $\exp(V)\exp(W) = \exp(P(V,W))$ *where*

$$P(V,W) = V + W + \frac{1}{2}[V,W].$$

Let us now introduce the concept of tangent bundle and horizontal bundle in $\mathbb{H}^n$.

Definition 2.1.22. *Let us indicate by $T\mathbb{H}^n$ the tangent vector bundle of $\mathbb{H}^n$, generated by the vector fields $X_1, ..., Y_n, T$.*

Definition 2.1.23. *Let us indicate by $H\mathbb{H}^n$ the horizontal vector bundle of $\mathbb{H}^n$, generated by the vector fields $X_1, ..., Y_n$.*

Remark 2.1.24. $H\mathbb{H}^n$ can be canonically identified with a vector subbundle of $T\mathbb{H}^n$. Let us indicate as $H\mathbb{H}^n_p$ the horizontal fiber at each point $p \in \mathbb{H}^n$. Each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_p$ and the norm $|\cdot|_p$ that make the vector fields $X_1, ..., Y_n$ orthonormal.
Hence we shall identify a section of $H\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame.

Remark 2.1.25. Since remark 2.1.24 we will identified each section F of $H\mathbb{H}^n$ with a function $F = (F_1, ..., F_{2n}) : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$.

Finally let us define the projection of a point $[z,t] \in \mathbb{H}^n$ on a horizontal fiber $H\mathbb{H}^n_{p_0}$.

Definition 2.1.26. *Let $(z,t), p_0 \in \mathbb{H}^n$ be given. We set*

$$\pi_{p_0}((z,t)) := \sum_{j=1}^{n} x_j X_j(p_0) + \sum_{j=1}^{n} y_j Y_j(p_0).$$

The map $p_0 \mapsto \pi_{p_0}((z,t))$ is a smooth section of $H\mathbb{H}^n$.

2.2 Some recalls of Functional Analysis

Following [55] let us give now some definitions and results concerning intrinsic differentiability in $\mathbb{H}^n$, see [83, 84] too.

Notation 2.2.1. If Ω is an open subset of $\mathbb{H}^n$ and $k \geq 0$ is a non negative integer, let us indicate by $C^k(\Omega)$, $C^\infty(\Omega)$ the usual (Euclidean) spaces of real valued continuously differentiable functions. We will denote by $Lip(\Omega)$ and $Lip_{loc}(\Omega)$ respectively the set of Lipschitz and locally Lipschitz continuous in Ω. Let us denote by $C^k(\Omega; \mathrm{H}\mathbb{H}^n)$ the set of all C^k-sections of $\mathrm{H}\mathbb{H}^n$ where the C^k regularity is understood as regularity between smooth manifolds. The notions of $C^k_c(\Omega; \mathrm{H}\mathbb{H}^n)$, $C^\infty(\Omega; \mathrm{H}\mathbb{H}^n)$ and $C^\infty_c(\Omega; \mathrm{H}\mathbb{H}^n)$ are defined analogously.

Definition 2.2.2. *A map $L : \mathbb{H}^n \to \mathbb{R}$ is $\mathbb{H}$-linear if and only if it is a homomorphism and if $\forall\ p \in \mathbb{H}^n$ and $\forall \lambda > 0 \quad L(\delta_\lambda(p)) = \lambda L(p)$.*

Proposition 2.2.3. *A map $L : \mathbb{H}^n \to \mathbb{R}$ is $\mathbb{H}$-linear if and only if there exists $(a,b) \in \mathbb{R}^{2n}$ such that $L(p) = \langle (a,b), (x,y) \rangle_{\mathbb{R}^{2n}}$ for $p = (x+iy, t)$.*

Definition 2.2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $f : \Omega \longrightarrow \mathbb{R}$. We say that f is P-differentiable at $p_0 \in \Omega$ if there is a unique $\mathbb{H}$-linear map $L : \mathbb{H}^n \to \mathbb{R}$ such that*

$$\lim_{p \to p_0} \frac{f(p) - f(p_0) - L\left(p_0^{-1} \cdot p\right)}{d_\infty(p, p_0)} = 0 \tag{2.13}$$

or equivalently there exists a homomorphism $L : \mathbb{H}^n \to \mathbb{R}$ such that

$$\lim_{\lambda \to 0^+} \frac{f\left(\tau_{p_0}(\delta_\lambda v)\right) - f(p)}{\lambda} = L(v) \tag{2.14}$$

uniformly with respect to v belonging to a compact set in $\mathbb{H}^n$. We shall write $L = d_{\mathbb{H}} f_{p_0}$.

Definition 2.2.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $f : \Omega \longrightarrow \mathbb{R}$. We say that f is differentiable along X_j or Y_j at $p_0 \in \Omega$ if the map $\lambda \mapsto f(\tau_{p_0}(\delta_\lambda e_j))$ or respectively $\lambda \mapsto f(\tau_{p_0}(\delta_\lambda e_{j+n}))$ is differentiable at $\lambda = 0$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^{2n+1}$.*

Remark 2.2.6. If $f \in C^1(\Omega)$, then f is differentiable along X_j and Y_j at all points of Ω.

We can introduce now the notions of gradient for functions $\mathbb{H}^n \to \mathbb{R}$ and divergence for sections of $H\mathbb{H}^n$.

Definition 2.2.7. *Let $\Omega \subset \mathbb{H}^n$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}$ be differentiable along X_j and Y_j at $p_0 \in \mathbb{H}^n$ for $j = 1, ..., n$. We define*

$$\nabla_{\mathbb{H}} f := \sum_{j=1}^{n} (X_j f) X_j + (Y_j f) Y_j$$

Remark 2.2.8. $\nabla_{\mathbb{H}} f$ is a section of $\mathrm{H}\mathbb{H}^n$, whose canonical coordinates are $(X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$. Therefore $\nabla_{\mathbb{H}} f$ can be defined alternatively in the following way: if $f \in C^1(\Omega)$

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \tag{2.15}$$

Remark 2.2.9. Let us notice that $\nabla_{\mathbb{H}} = C\nabla$, where ∇ is the Euclidean gradient and C is the $2n \times (2n+1)$ matrix whose rows are the components of the vectors $X_1, ..., Y_n$:

$$C(p) := \begin{bmatrix} X_1(p) \\ \vdots \\ X_n(p) \\ Y_1(p) \\ \vdots \\ Y_n(p) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{2} y_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 & -\frac{1}{2} y_n \\ 0 & \cdots & 0 & 1 & \cdots & 0 & \frac{1}{2} x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & \frac{1}{2} x_n \end{bmatrix}$$

Definition 2.2.10. *If $F = (F_1, \dots, F_{2n}) \in C^1(\Omega; \mathrm{H}\mathbb{H}^n)$,*

$$div_{\mathbb{H}} F := \sum_{j=1}^{n} X_j F_j + Y_j F_{j+n}. \tag{2.16}$$

The definitions above can be understood in distributional sense too.

Proposition 2.2.11. *Let $\Omega \subseteq \mathbb{H}^n$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}$ be P-differentiable at $p_0 \in \Omega$, then f is differentiable along X_j and Y_j at p_0 for $j = 1, ..., n$ and*

$$d_{\mathbb{H}} f(p_0)(v) = \langle \nabla_{\mathbb{H}} f, \pi_{p_0}(v) \rangle_{p_0}.$$

Lemma 2.2.12. *Let $\Omega \subseteq \mathbb{H}^n$ be a connected open set and let $f \in L^1_{loc}(\Omega)$ such that*

$$\nabla_{\mathbb{H}} f = 0.$$

in distributional sense. Then $f \equiv cost$ in Ω.

Definition 2.2.13. *If $\Omega \in \mathbb{H}^n$ let us denote by $C^1_{\mathbb{H}}(\Omega)$ the set of continuous real function in Ω such that $\nabla_{\mathbb{H}} f$ in distributional sense is continuous in Ω. Moreover let us denote by $C^1_{\mathbb{H}}(\Omega, H\mathbb{H}^n)$ the set of all sections F of $H\mathbb{H}^n$ whose canonical coordinates $F_j \in C^1_{\mathbb{H}}(\Omega)$ for $j \in 1, ..., 2n$.*

Example 2.2.14. Let us notice that $C^1(\Omega) \subset C^1_{\mathbb{H}}(\Omega)$ and the inclusion is strict; consider the following example (see [55], Remark 5.9.): let $n = 1$ and $f([z,t]) := x - g(y, 2xy + t)$, where

$$g(p,q) := \frac{|p|^\alpha q}{p^4 + q^2} \quad \text{if } (p,q) \neq (0,0) \quad \text{and } g(0,0) = 0.$$

Then $f \in C^1_{\mathbb{H}}(\Omega)$ if $3 < \alpha < 4$, but f is not locally Lipschitz continuous with respect to the euclidean metric of $\mathbb{R}^3$. □

Let us recall also the following characterizations of the functions in $C^1_{\mathbb{H}}(\Omega)$ (see [55], section 5)

Proposition 2.2.15. *Let $\Omega \subset \mathbb{H}^n$ be an open set and let $f \in C^0(\Omega)$. Then the following conditions are equivalent:*

i *$f \in C^1_{\mathbb{H}}(\Omega)$;*

ii *there exist $g_j \in C^0(\Omega)$ $(j = 1, \ldots, 2n)$ such that f is differentiable along X_j in Ω with derivative g_j. Namely for each $p \in \Omega$ there exists $\delta_p > 0$ such that $(-\delta_p, \delta_p) \to \exp(s\,X_j)(p) = p \cdot s\,e_j \in \Omega$, $(-\delta_p, \delta_p) \to f(\exp(s\,X_j)(p))$ is C^1 and*

$$\frac{d}{ds} f(\exp(s\,X_j)(p)) = g_j(\exp(s\,X_j)(p)) \quad \forall s \in (-\delta_p, \delta_p)\,.$$

Proposition 2.2.16. *Let $\Omega \in \mathbb{H}^n$ an open set and $f : \Omega \longrightarrow \mathbb{R}$ a continuous function. $f \in C^1_{\mathbb{H}}(\Omega)$ if and only if its distributional derivatives $X_j f, Y_j f$ are continuous in Ω for $j = 1, ..., n$.*

Theorem 2.2.17. *Let $\Omega \subset \mathbb{H}^n$ be an open set. If $f \in C^1_{\mathbb{H}}(\Omega)$ then f is P-differentiable at any point $p_0 \in \Omega$.*

Similarly to Definition 2.2.13, let us give the following

Definition 2.2.18. *We shall denote by $C^k_{\mathbb{H}}(\Omega)$ the set of continuous real functions f in Ω such that $\nabla_{\mathbb{H}} f$ is of class C^{k-1} in Ω. Moreover, we shall denote by $C^k_{\mathbb{H}}(\Omega; \mathrm{H}\mathbb{H}^n)$ the set of all sections φ of $\mathrm{H}\mathbb{H}^n$ whose canonical coordinates φ_j belong to $C^k_{\mathbb{H}}(\Omega)$ for $j = 1, \dots, 2n$.*

Definition 2.2.19. *Let $\Omega \subset \mathbb{H}^n$ we will denote by $Lip_{\mathbb{H}}(\Omega)$ the set of functions $f : \Omega \to \mathbb{R}$ such that there exists $L > 0$ for which*

$$|f(P) - f(Q)| \leq L\, d_\infty(P, Q) \quad \forall P, Q \in \Omega\,. \tag{2.17}$$

Remark 2.2.20. By (2.9) $Lip_{\mathbb{H}}(\Omega) \subset C^0(\Omega)$.

The following characterization of $Lip_{\mathbb{H}}(\Omega)$ holds (see, for instance, [81], Theorem 2.21).

Theorem 2.2.21. *Let $\Omega \subset \mathbb{H}^n$ be a connected open set then the following are equivalent*

i *$f \in Lip_{\mathbb{H}}(\Omega)$;*

ii *$f \in L^\infty_{loc}(\Omega)$ and there exists $\nabla_{\mathbb{H}} f \in (L^\infty(\Omega))^{2n}$ in distributional sense.*

Moreover the constant L in (2.17) can be chosen as $L = \|\nabla_{\mathbb{H}} f\|_{(L^\infty(\Omega))^{2n}}$.

Remark 2.2.22. Let $\Omega \subset \mathbb{H}^n$ be an open set. Then $C^1_{\mathbb{H}}(\Omega) \subset \mathrm{Lip}_{\mathbb{H},loc}(\Omega)$.

Let us report the classic technique of intrinsic convolution in $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, see [53]. Let $\rho \in C^\infty(\mathbb{H}^n)$ be such that ρ is a standard mollifier, i.e. $0 \leq \rho \leq 1$, $\displaystyle\int_{\mathbb{H}^n} \rho d\mathcal{L}^{2n+1} = 1$, $\mathrm{spt}\rho \subset B_\infty(0,1)$ and $\rho(p^{-1}) = \rho(p)$ for all $p \in \mathbb{H}^n$. Let us denote for $f : \mathbb{H}^n \to \mathbb{R}$ measurable and $p \in \mathbb{H}^n$

$$\rho_\epsilon(p) := \epsilon^{-2n-2} \rho\left(\delta_{\frac{1}{\epsilon}}(p)\right)$$

$$f_\epsilon(p) = (\rho_\epsilon * f)(p) := \int_{\mathbb{H}^n} \rho_\epsilon(q) f(q^{-1} \cdot p)\, d\mathcal{L}^{2n+1}(q) = \int_{\mathbb{H}^n} \rho_\epsilon(p \cdot q^{-1}) f(q)\, d\mathcal{L}^{2n+1}(q).$$

The following results hold:

Lemma 2.2.23. **i** *$\mathrm{spt} f_\epsilon \subset B_\infty(0,\epsilon) \cdot \mathrm{spt} f$;*

ii *If $f \in L^p(\mathbb{R}^{2n+1})$, $1 \leq p < \infty$, then $f_\epsilon \to f$ in $L^p(\mathbb{R}^{2n+1})$ as $\epsilon \to 0$;*

iii *$W_j(\rho_\epsilon * \varphi) = \rho_\epsilon * W_j \varphi$ for any $\varphi \in C^\infty_c(\mathbb{H}^n)$ and each $j = 1, ..., 2n$;*

iv *$\displaystyle\int_{\mathbb{H}^n} f_\epsilon g\, d\mathcal{L}^{2n+1} = \int_{\mathbb{H}^n} g_\epsilon f\, d\mathcal{L}^{2n+1}$ for every $f \in L^\infty(\mathbb{H}^n)$, $g \in L^1(\mathbb{H}^n)$*

v *If* $f \in L^\infty(\mathbb{H}^n) \cap C^0(\Omega)$ *for a suitable open set* $\Omega \subset \mathbb{H}^n$ *then* $f_\epsilon \to f$ *uniformly on compact subsets of* Ω *as* $\epsilon \to 0$.

In $\mathbb{H}^n$ there is a natural definition of bounded variation functions and of finite perimeter sets (see [60]).

Definition 2.2.24. *We say that* $f : \Omega \to \mathbb{R}$ *is of bounded* $\mathbb{H}$*-variation in an open set* $\Omega \subset \mathbb{H}^n$*,* $(f \in BV_{\mathbb{H}}(\Omega))$*, if* $f \in L^1(\Omega)$ *and if*

$$\int_\Omega d|\nabla_{\mathbb{H}} f| := \sup\left\{ \int_\Omega f\, div_{\mathbb{H}}\varphi\, dh \ : \varphi \in C_c^1(\Omega; \mathrm{H}\mathbb{H}^n),\ |\varphi(P)|_P \le 1 \right\} < +\infty. \tag{2.18}$$

Analogously the space $BV_{\mathbb{H},\mathrm{loc}}(\Omega)$ *is defined in the usual way.*

Definition 2.2.25. *We say that* $E \subset \mathbb{H}^n$ *is a locally finite* $\mathbb{H}$*-perimeter set (or a* $\mathbb{H}$*-Caccioppoli set) if* $\mathbf{1}_E \in BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$*, where we indicate as* $\mathbf{1}_E$ *the characteristic function of the set* E*. In this case, the measure* $|\nabla_{\mathbb{H}} \mathbf{1}_E|$ *will be called* $\mathbb{H}$*-perimeter of* E *and will be denoted by* $|\partial E|_{\mathbb{H}}$.

Theorem 2.2.26. *There exists a* $|\partial E|_{\mathbb{H}}$*-measurable section* ν_E *of* $\mathrm{H}\mathbb{H}^n$ *such that*

$$-\int_E div_{\mathbb{H}}\varphi\, dh = \int_{\mathbb{H}^n} \langle \nu_E, \varphi\rangle\, d|\partial E|_{\mathbb{H}} \qquad \forall \varphi \in C_c^\infty(\Omega; \mathrm{H}\mathbb{H}^n);$$

$$|\nu_E(P)|_P = 1 \qquad for\, |\partial E|_{\mathbb{H}} - a.e.\, P \in \mathbb{H}^n.$$

The measurability of ν_E *is meant in the sense that its coordinates* $\nu_1, \dots, \nu_{2n}$ *are* $|\partial E|_{\mathbb{H}}$*-measurable functions.*

The function ν_E can be interpreted $|\partial E|_{\mathbb{H}}$-almost everywhere as a generalized inward "horizontal" normal to the set E.

Finally let s recall the following definition, see [8].

Definition 2.2.27. *Let* $\Omega \subset \mathbb{H}^n$ *be a bounded open set,* $1 \le p \le \infty$ *and* $m \in \mathbb{N}$*. Let us denote by* $W^{m,p}_{\mathbb{H}}(\Omega)$ *the space of all* $u \in L^p(\Omega)$ *such that* $W^I u \in L^p(\Omega)$ *for any mlti-index* I *with* $d(I) \le m$*, endowed with the natural norm* $\|u\|_{W^{m,p}_{\mathbb{H}}(\Omega)}$*. Let us denote by* $W^{m,p}_{\mathbb{H},0}(\Omega)$ *the completion of* $\mathcal{D}(\Omega)$ *in* $W^{m,p}_{\mathbb{H}}(\Omega)$.

2.3 Multilinear algebra in $\mathbb{H}^n$

Following [8, 52, 58, 59], let us study some definitions and results of multilinear algebra in the Heisenberg group $\mathbb{H}^n$.

Definition 2.3.1. *Let us denote by* $\bigwedge^1 \mathfrak{h}$ *the dual space of* $\mathfrak{h} := \mathrm{span}\{X_1, ..., Y_n, T\}$.

The basis of $\bigwedge^1 \mathfrak{h}$ is the family of covectors $\{dx_1, ..., dx_n, dy_1, ..., dy_n, \theta\}$, where

$$\theta := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is the *contact form* in $\mathbb{H}^n$. Let us indicate as $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, ..., dx_n, dy_1, ..., dy_n, \theta$ an orthonormal basis.

Definition 2.3.2. *If we set for* $i = 1, ..., n$

$$W_i := X_i, \qquad W_{i+n} := Y_i, \qquad W_{2n+1} := T,$$

$$\theta_i := dx_i, \qquad \theta_{i+n} := dy_i, \qquad \theta_{2n+1} := \theta,$$

we put $\bigwedge_0 \mathfrak{h}; = \bigwedge^0 \mathfrak{h} = \mathbb{R}$ *and for* $1 \le k \le 2n+1$

$$\bigwedge\nolimits_k \mathfrak{h} := \mathrm{span}\left\{W_{i_1} \wedge ... \wedge W_{i_k} : 1 \le i_1 < ... < i_k \le 2n+1\right\},$$

$$\bigwedge\nolimits^k \mathfrak{h} := \mathrm{span}\left\{\theta_{i_1} \wedge ... \wedge \theta_{i_k} : 1 \le i_1 < ... < i_k \le 2n+1\right\}.$$

The elements of $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$ are called k-*vectors* and k-*covectors*. By Definition 2.3.2 we obtain the graded algebras

$$\bigwedge\nolimits_* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge\nolimits_k \mathfrak{h} \qquad \text{and} \qquad \bigwedge\nolimits^* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge\nolimits^k \mathfrak{h}.$$

Remark 2.3.3. The dual space $\bigwedge^1(\bigwedge_k \mathfrak{h})$ of $\bigwedge_k \mathfrak{h}$ can be naturally identified with $\bigwedge^k \mathfrak{h}$. The action of a k-covector φ on a k-vector v is denoted by $\langle \varphi | v \rangle$. The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making the bases $\{W_{i_1} \wedge ... \wedge W_{i_k}\}$ and $\{\theta_{i_1} \wedge ... \wedge \theta_{i_k}\}$ orthonormal. Let us finally notice that

$$\dim \bigwedge\nolimits_k \mathfrak{h} = \dim \bigwedge\nolimits^k \mathfrak{h} = \begin{pmatrix} 2n+1 \\ k \end{pmatrix}$$

Definition 2.3.4. *An element* $v \in \bigwedge_k \mathfrak{h}$ *is called simple (or decomposable) if and only if it equals the exterior product of* k *elements of* $\mathfrak{h}$*, i.e. there exists* $v_1, ..., v_k \in \mathfrak{h}$ *such that*

$$v = v_1 \wedge ... \wedge v_k.$$

An element $\varphi \in \bigwedge^k \mathfrak{h}$ *is called simple (or decomposable) if and only if it equals the alternating product of* k *elements of* $\bigwedge^1 \mathfrak{h}$*, i.e. there exists* $\varphi_1, ..., \varphi_k \in \mathfrak{h}$ *such that*

$$\varphi = \varphi_1 \wedge ... \wedge \varphi_k.$$

The same algebraic construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal k-vectors* and *horizontal k-covectors*

$$\bigwedge\nolimits_k \mathfrak{h}_1 := \text{span}\,\{W_{i_1} \wedge ... \wedge W_{i_k} : 1 \le i_1 < ... < i_k \le 2n\},$$

$$\bigwedge\nolimits^k \mathfrak{h}_1 := \text{span}\,\{\theta_{i_1} \wedge ... \wedge \theta_{i_k} : 1 \le i_1 < ... < i_k \le 2n\}$$

and the graded algebra

$$\bigwedge\nolimits_* \mathfrak{h}_1 = \bigoplus_{k=0}^{2n} \bigwedge\nolimits_k \mathfrak{h}_1 \qquad \text{and} \qquad \bigwedge\nolimits^* \mathfrak{h}_1 = \bigoplus_{k=0}^{2n} \bigwedge\nolimits^k \mathfrak{h}_1$$

Let us observe that $H\mathbb{H}^n = \bigwedge_1 \mathfrak{h}_1$, where $H\mathbb{H}^n$ is the bundle generated by $X_1, ..., X_n, Y_1, ..., Y_n$.

Definition 2.3.5. *Let us define the horizontal differential $d_{\mathbb{H}} : \bigwedge^k \mathfrak{h}_1 \to \bigwedge^{k+1} \mathfrak{h}_1$ by linearity by*

$$d_{\mathbb{H}}(f\theta_{i_1} \wedge ... \wedge \theta_{i_k}) := \sum_{j=1}^{2n} (W_j f)\theta_j \wedge \theta_{i_1} \wedge ... \wedge \theta_{i_k}$$

for $f : \mathbb{H}^n \to \mathbb{R}$, $1 \le i_1, ..., i_k \le 2n$.

Remark 2.3.6. The *symplectic 2-form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is

$$d\theta := d_{\mathbb{H}}\theta = -\sum_{j=1}^{n} dx_j \wedge dy_j.$$

Let us indicate $I = \{i_1, ..., i_k\}$ with $1 \le i_1 < ... < i_k \le 2n+1$, $W_I = W_{i_1} \wedge ... \wedge W_{i_k}$, $\theta_I = \theta_{i_1} \wedge ... \wedge \theta_{i_k}$, $I^* = \{i_1^* < ... < i_{2n+1-k}^*\} = \{1, ..., 2n+1\} \setminus I$ and $\sigma(I)$ the number of couples (i_h, i_l^*) with $i_h > i_l^*$. Following [52, 58] let us define the Hodge operator

Definition 2.3.7. *For $1 \le k \le 2n+1$ we define the linear isomorphisms*

$$* : \bigwedge\nolimits_k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits_{2n+1-k} \mathfrak{h} \qquad \text{and} \qquad * : \bigwedge\nolimits^k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h}$$

putting for $v = \sum_I v_I W_I$ and $\varphi = \sum_I \varphi_I \theta_I$

$$*v := \sum_I v_I (*W_I) \qquad \text{and} \qquad *\varphi = \sum_I \varphi_I (*\theta_I)$$

where

$$*W_I := (-1)^{\sigma(I)} W_{I^*} \qquad \text{and} \qquad *\theta_I := (-1)^{\sigma(I)} \theta_{I^*}.$$

Remark 2.3.8. Notice that, if $v = v_1 \wedge ... \wedge v_k$ is a simple k-vector, the $*v$ is a simple $(2n+1-k)$-vector. Moreover, notice that if $v \in \bigwedge_k \mathfrak{h}_1$, then $*v = \xi \wedge T$ with $\xi \in \bigwedge_{2n-k} \mathfrak{h}_1$.

Proposition 2.3.9. *The following properties of the $*$-operator hold $\forall\, v, w \in \bigwedge_k \mathfrak{h}$ and $\forall\, \varphi, \psi \in \bigwedge^k \mathfrak{h}$*

$$**\,v = (-1)^{k(2n+1-k)}v = v, \qquad **\,\varphi = (-1)^{k(2n+1-k)}\varphi = \varphi,$$

$$v \wedge *w = \langle v, w\rangle W_1 \wedge ... \wedge W_{2n+1}, \qquad \varphi \wedge *\psi = \langle \varphi, \psi\rangle \theta_1 \wedge ... \wedge \theta_{2n+1}$$

$$\langle *\varphi | *v\rangle = \langle \varphi | v\rangle.$$

Definition 2.3.10. *If $v \in \bigwedge_k \mathfrak{h}$ let us define $v^\natural \in \bigwedge^k \mathfrak{h}$ by the identity*

$$\langle v^\natural | w\rangle := \langle v, w\rangle \qquad \forall\, w \in \bigwedge_k \mathfrak{h}$$

Analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$ by the identity

$$\langle \varphi^\natural | \psi\rangle := \langle \varphi, \psi\rangle \qquad \forall\, \psi \in \bigwedge^k \mathfrak{h}$$

Remark 2.3.11. A simple non-zero k-vector $v = v_1 \wedge ... \wedge v_k \in \bigwedge_k \mathfrak{h}$ is naturally associated with a left invariant distribution of k-dimensional planes in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$. v is said to be *integrable* if the distribution of k-planes $\text{span}\{v_1, ..., v_k\}$ is integrable. In general, if $k > 1$, this distribution is not integrable because not necessarily $[v_i, v_j] \in \text{span}\{v_1, ..., v_k\}$ (by Frobenius Theorem), for example the 2-vector $X_1 \wedge Y_1 \in \bigwedge_2 \mathfrak{h}_1$.

Let us define the vector spaces of integrable k-vectors and k-covectors as follows.

Definition 2.3.12. *We set ${}_H\bigwedge_0 = \mathbb{R}$ and for $1 \le k \le n$*

$${}_H\bigwedge_k := \text{span}\left\{v \in \bigwedge_k \mathfrak{h}_1 : v \ \textit{simple and integrable}\right\}$$

$${}_H\bigwedge_{2n+1-k} := *\left({}_H\bigwedge_k\right)$$

Integrable covectors are defined for $0 \le k \le 2n+1$ by

$${}_H\bigwedge^k := \left\{\varphi \in \bigwedge^k \mathfrak{h} : \varphi^\natural \in_H \bigwedge_k\right\}$$

and ${}_H\bigwedge^k$ turn to be isomorphic to $\bigwedge^1 \left({}_H\bigwedge_k\right)$.

Remark 2.3.13. Notice that ${}_H\bigwedge_1 = \bigwedge_1 \mathfrak{h}_1 = \mathfrak{h}_1$. On the contrary, for $1 < k \le n$, $0 \ne {}_H\bigwedge_k \subsetneq \bigwedge_k \mathfrak{h}_1$. Finally we have for $1 \le k \le n$

$$ {}_H\bigwedge_k = * \left({}_H\bigwedge_{2n+1-k} \right). $$

Proposition 2.3.14. *If $1 \le k \le 2n+1$, then the following diagram is commutative*

$$\begin{tikzcd} {}_H\bigwedge^k \arrow[r, "*"] & {}_H\bigwedge_{2n+1-k} \\ {}_H\bigwedge^k \arrow[r, "*"] \arrow[u, "\natural"] & {}_H\bigwedge_{2n+1-k} \arrow[u, "\natural"'] \end{tikzcd}$$

Proof. (see [8]) By proposition 2.3.9, if $\varphi \in_H \bigwedge^k$ is given and $\alpha \in \bigwedge^{2n+1-k}\mathfrak{h}$ is arbitrarily taken, then

$$ \left\langle \alpha | * \left(\varphi^\natural\right)\right\rangle = \left\langle *\alpha | \varphi^\natural \right\rangle = \langle \alpha | * \varphi \rangle = \left\langle \alpha | \left(*\varphi^\natural \right) \right\rangle. $$

□

Proposition 2.3.15. *We have for $i = 1, ..., 2n$ $W_i^\natural = \theta_i$ and $T^\natural = \theta$. In particular if $\alpha, \beta \in_H \bigwedge^k$ with $1 \le k \le 2n+1$, then $\langle \alpha, \beta \rangle = \langle \alpha^\natural, \beta^\natural \rangle$.*

The following characterization holds, see [58].

Theorem 2.3.16. *Assume $2 \le k \le n$ and $v = v_1 \wedge ... \wedge v_k \in \bigwedge_k \mathfrak{h}_1$, $v \ne 0$. The following three statements are equivalent:*

i $v \in_H \bigwedge_k$

ii $[v_i, v_j] = 0$ *for* $1 \le i, j \le k$

iii $\langle \gamma \wedge d\theta | v \rangle = 0 \ \forall \gamma \in \bigwedge^{k-2} \mathfrak{h}$

Proof. **i** $\Rightarrow$ **ii**. Because $[v_i, v_j]$ is always a multiple of T and $v_i, v_j \in \mathfrak{h}_1$, the necessity of **ii** for the integrability of the distribution associated with v is just Frobenius Theorem.
ii $\Rightarrow$ **i**. follows from Frobenius Theorem.
ii $\Leftrightarrow$ **iii**. A direct computation yields $[v_i, v_j] = \langle d\theta | v_i \wedge v_j \rangle = -\langle v_i, v_j \rangle_{\mathbb{R}^{2n}}$. If $v = v_1, ..., v_k \in \mathfrak{h}_1$ and if $\gamma \in \bigwedge^{k-2}\mathfrak{h}_1$ then

$$ \langle \gamma \wedge d\theta | v \rangle = \sum_\pi \sigma(\pi) \langle \gamma | v_{\pi(1)} \wedge ... \wedge v_{\pi(k-2)} \rangle \langle d\theta | v_{\pi(k-1)} \wedge v_{\pi(k)} \rangle $$

where the sum is extended to all the permutations π of $\{1, ..., k\}$ and $\sigma(\pi)$ is ± 1 accordingly with the parity of the permutations π. Hence, $\forall \gamma \in \bigwedge^{k-2}\mathfrak{h}_1$, $\langle \gamma \wedge d\theta | v_1 \wedge ... \wedge v_k \rangle = 0$ is equivalent with $[v_i, v_j] = \langle d\theta | v_i \wedge v_j \rangle = 0$ for $1 \le i \le j \le k$. □

Remark 2.3.17. In general let us set $N_k := \dim {}_H \bigwedge_k = \dim {}_H \bigwedge^k$. Let us show now that

$$\dim {}_H{\bigwedge}_2 = \binom{2n}{2} - 1$$

and that a vectorial basis of ${}_H \bigwedge_2$ is

$$\mathcal{B} := \{W_i \wedge W_j\}_{\substack{1\le i<j\le 2n \\ j\neq i+n}} \cup \left\{ \frac{1}{\sqrt{2}}(W_h \wedge W_{h+n} - W_{h+1} \wedge W_{h+1+n}) \right\}_{h=1,\dots,n-1} \tag{2.19}$$

We know by Remark 2.3.13 that ${}_H \bigwedge_2 \subsetneq \bigwedge_2 \mathfrak{h}_1$ $\forall n \in \mathbb{N}$ and that $\{W_i \wedge W_j\}_{i,j=1,\dots,2n}$ is an orthonormal basis of $\bigwedge_2 \mathfrak{h}_1$. Let us notice that ${}_H \bigwedge_2$ is a vectorial subspace of $\bigwedge_2 \mathfrak{h}_1$, since $0 \in_H \bigwedge_2$ and ${}_H \bigwedge_2$ is closed respect the sum and the scalar product. Therefore a vectorial basis of ${}_H \bigwedge_2$ can be composed by vectors of the basis of $\bigwedge_2 \mathfrak{h}_1$ or by their linear combinations and $\dim {}_H \bigwedge_2 \le \binom{2n}{2} - 1$. Let us observe that

$$\{W_i \wedge W_j\}_{\substack{1\le i<j\le 2n \\ j\neq i+n}} \subset {}_H{\bigwedge}_2 \tag{2.20}$$

By Theorem 2.3.16 we know that if $v \in {}_H \bigwedge_2$

$$\langle \gamma \wedge d\theta | v \rangle = 0 \qquad \forall \gamma \in {\bigwedge}^{k-2} \mathfrak{h} \tag{2.21}$$

Since $k = 2$, γ is a constant. To verify (2.20), let us notice that $\langle d\theta | v \rangle = 0$ for $v = W_i \wedge W_j$ with $j \neq i + n$. Indeed

$$\left\langle \sum_{k=1}^{n} \theta_k \wedge \theta_{k+n} \middle| W_i \wedge W_j \right\rangle = 0 \qquad \text{if } \ j \neq i + n.$$

Let us considerate $\{W_i \wedge W_{i+n}\}_{i=1,\dots,n}$ and notice that these vectors do not belong to ${}_H \bigwedge_2$. Indeed let us verify (2.21) does not hold for $v = W_i \wedge W_{i+n}$. Indedd $\forall\, i = 1, \dots, n$

$$\left\langle \sum_{k=1}^{n} \theta_k \wedge \theta_{k+n} \middle| W_i \wedge W_{i+n} \right\rangle = \langle \theta_i \wedge \theta_{i+n} | W_i \wedge W_{i+n} \rangle = |W_i \wedge W_{i+n}| = 1.$$

But there are linear combinations of $\{W_i \wedge W_{i+n}\}_{i=1,\dots,n}$ that belong to ${}_H \bigwedge_2$. Indeed let us consider the $n - 1$ vectors

$$v_h = \{W_h \wedge W_{h+n} - W_{h+1} \wedge W_{h+1+n}\}_{h=1,\dots,n-1} \tag{2.22}$$

They belong to ${}_H\bigwedge_2$, indeed

$$\langle d\theta|v_h\rangle = \left\langle \sum_{k=1}^{n} \theta_k \wedge \theta_{k+n} \middle| W_h \wedge W_{h+n} - W_{h+1} \wedge W_{h+1+n} \right\rangle =$$

$$= \langle \theta_h \wedge \theta_{h+n} | W_h \wedge W_{h+n}\rangle - \langle \theta_{h+1} \wedge \theta_{h+1+n} | W_{h+1} \wedge W_{h+1+n}\rangle =$$

$$= |W_h \wedge W_{h+n}| - |W_{h+1} \wedge W_{h+1+n}| = 0$$

It is easy to verify that they are linearly independent and by an easy combinatory calculation we have that the cardinality of $\mathcal{B}$ is $\begin{pmatrix} 2n \\ 2 \end{pmatrix} - 1 = d$

Therefore $\mathcal{B}$ is a vectorial basis of ${}_H\bigwedge_2$ and $\dim_H \bigwedge_2 = d$. Finally we normalize some vectors of the basis with the coefficient $\frac{1}{\sqrt{2}}$, because

$$|W_h \wedge W_{h+n} - W_{h+1} \wedge W_{h+1+n}| = \sqrt{2}.$$

$\square$

Our previous algebraic construction yields, by left translation, several bundles over $\mathbb{H}^n$. These are the bundles of k-vector and k-covector, that thanks to the left invariance of the structure we can still indicate as $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$. Analogously we have the bundles $\bigwedge_k \mathfrak{h}_1$ and $\bigwedge^k \mathfrak{h}_1$ of the horizontal k-vectors and k-covectors and the bundles ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of the simple and integrable k-vectors and k-covectors. The fiber of $\bigwedge_k \mathfrak{h}$ over $p \in \mathbb{H}^n$ is denoted by $\bigwedge_{k,p} \mathfrak{h}$, analogously the other ones.
More formally, $\mathfrak{h}$ can be identified with $T\mathbb{H}^n_e$, the tangent space to $\mathbb{H}^n$ at the origin. Thus ${}_H\bigwedge_k \simeq_H \bigwedge_{k,e}$ is a subspace of $\bigwedge_k T\mathbb{H}^n_e$.

Definition 2.3.18. *Let us introduce for $q, q' \in \mathbb{H}^n$ and for any linear map $f : T\mathbb{H}^n_q \to T\mathbb{H}^n_{q'}$ the linear map*

$$\Lambda_k f : \bigwedge\nolimits_k T\mathbb{H}^n_q \to \bigwedge\nolimits_k T\mathbb{H}^n_{q'}$$

defined by

$$(\Lambda_k f)(v_1 \wedge ... \wedge v_k) = f(v_1) \wedge ... \wedge f(v_k).$$

Let us define, for any $p \in \mathbb{H}^n$,

$${}_H\bigwedge\nolimits_{k,p} := (\Lambda_k d\tau_p)\left({}_H\bigwedge\nolimits_{k,e}\right)$$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge_k \mathfrak{h}$ induces an inner product on each fiber ${}_H \bigwedge_{k,p}$ by the identity

$$\langle \Lambda_k d\tau_p(v), \Lambda_k d\tau_p(w) \rangle_p := \langle v, w \rangle.$$

Definition 2.3.19. *Let us introduce for $q, q' \in \mathbb{H}^n$ and for any linear map $f : T\mathbb{H}^n_q \to T\mathbb{H}^n_{q'}$ the linear map*

$$\Lambda^k f : \bigwedge^k T\mathbb{H}^n_{q'} \to \bigwedge^k T\mathbb{H}^n_q$$

defined by

$$\left\langle (\Lambda^k f)(\alpha) | v_1 \wedge ... \wedge v_k \right\rangle = \langle \alpha | (\Lambda_k f)(v_1 \wedge ... \wedge v_k) \rangle$$

for any $\alpha \in_H \bigwedge^k_{q'}$ and any simple k-vector $v_1 \wedge ... \wedge v_k \in_H \bigwedge_{k,q}$. Let us define, for any $p \in \mathbb{H}^n$,

$$_H\bigwedge^k_p := (\Lambda^k d\tau_{p^{-1}}) \left(_H\bigwedge^k_e \right)$$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^k \mathfrak{h}$ induces an inner product on each fiber ${}_H \bigwedge^k_p$ by the identity

$$\left\langle \Lambda^k d\tau_{p^{-1}}(v), \Lambda^k d\tau_{p^{-1}}(w) \right\rangle_p := \langle v, w \rangle.$$

Lemma 2.3.20. *If $p, q \in \mathbb{H}^n$, then*

$$\Lambda_k d\tau_q :_H \bigwedge_{k,p} \to_H \bigwedge_{k,qp} \qquad \textit{and} \qquad \Lambda^k d\tau_{q^{-1}} :_H \bigwedge^k_p \to_H \bigwedge^k_{qp}$$

are isometries.

Proof. see [8] By the identity $\tau_p \cdot \tau_q = \tau_{pq}$ we obtain

$$\Lambda_k d\tau_p \cdot \Lambda_k d\tau_q = \Lambda_k d\tau_{pq} \quad \text{and} \quad \Lambda^k d\tau_{p^{-1}} \cdot \Lambda^k d\tau_{q^{-1}} = \Lambda^k d\tau_{(pq)^{-1}}$$

and so the thesis. □

We give now some definitions of spaces of k-differential forms and k-vector fields in $\mathbb{H}^n$.

Definition 2.3.21. *Let $K \subset \mathbb{H}^n$ be a compact set. If $0 \leq m \leq \infty$ and $1 \leq k \leq 2n+1$, let us denote by $\mathcal{D}^{k,m}_{\mathbb{H}}(K)$ the space of all C^m-sections of ${}_H \bigwedge^k$ over K, i.e. the Heisenberg k-differential forms of class C^m in K. Let us denote by*

$$\mathcal{D}^{*,m}_{\mathbb{H}}(K) = \mathcal{D}^{0,m}_{\mathbb{H}}(K) \oplus ... \oplus \mathcal{D}^{2n+1,m}_{\mathbb{H}}(K)$$

the graded algebra of all Heisenberg differential forms in K of class C^m, where $\mathcal{D}^{0,m}_{\mathbb{H}}(K) = C^m(K)$.

Definition 2.3.22. *Analogously to definition* 2.3.21 *, let us denote by* $\mathcal{D}^m_{\mathbb{H},k}(K)$ *the space of all* C^m*-sections of* ${}_H\bigwedge_k$ *over* K*, i.e. the Heisenberg* k*-vector fields of class* C^m *in* K *and by*

$$\mathcal{D}^m_{\mathbb{H},*}(K) = \mathcal{D}^m_{\mathbb{H},0}(K) \oplus ... \oplus \mathcal{D}^m_{\mathbb{H},2n+1}(K)$$

their graded algebra.

Let us set also $\mathcal{D}^k_{\mathbb{H}}(K) := \mathcal{D}^{k,\infty}_{\mathbb{H}}(K)$ and $\mathcal{D}_{\mathbb{H},k}(K) := \mathcal{D}^\infty_{\mathbb{H},k}(K)$.
Let $U \subset \mathbb{H}^n$ be an open set, we define $\mathcal{D}^{k,m}_{\mathbb{H}}(U)$ and $\mathcal{D}^m_{\mathbb{H},k}(U)$ as the spaces of all C^m-sections of ${}_H\bigwedge^k$ and ${}_H\bigwedge^k$ over the compact sets $K \Subset U$. Finally we give the definition of $\mathcal{E}^{k,m}_{\mathbb{H}}(U)$ and $\mathcal{E}^m_{\mathbb{H},k}(U)$ in the same way, as the spaces of all C^m-sections of ${}_H\bigwedge^k$ and ${}_H\bigwedge_k$ over U.
Repeating the procedure when replacing ${}_H\bigwedge^k$ with $\bigwedge^k \mathfrak{h}_1$ and ${}_H\bigwedge_k$ with $\bigwedge_k \mathfrak{h}_1$, we obtain the definitions of the horizontal k-differential forms $\mathcal{D}^{k,m}_{H\mathbb{H}}(U)$, $\mathcal{E}^{k,m}_{H\mathbb{H}}(U)$ and k-vector fields $\mathcal{D}^m_{H\mathbb{H},k}(U)$, $\mathcal{E}^m_{H\mathbb{H},k}(U)$.

Remark 2.3.23. Let $\{\xi_1, ..., \xi_{N_k}\}$ be an orthonormal basis of ${}_H\bigwedge^k_e$. Then we can define N_k smooth sections of ${}_H\bigwedge^k$, that we still denote by $\xi_1, ..., \xi_{N_k}$ by taking $\xi_{j,p} := \Lambda^k d\tau_{p^{-1}}(\xi_j)$, for $p \in \mathbb{H}^n$ and $j = 1, ..., N_k$.
$\{\xi_{1,p}, ..., \xi_{N_k,p}\}$ is an orthonormal basis of ${}_H\bigwedge^k_p$. Let us also refer to $\{\xi_1, ..., \xi_{N_k}\}$ as to a left invariant moving frame in ${}_H\bigwedge^k$.

Remark 2.3.24. Given a left invariant moving frame $\{\xi_1, ..., \xi_{N_k}\}$ of ${}_H\bigwedge^k$, and hence a dual moving frame $\{\xi_1^\natural, ..., \xi_{N_k}^\natural\}$ of ${}_H\bigwedge_k$, both $\mathcal{D}^{k,m}_{\mathbb{H}}(K)$ and $\mathcal{D}^m_{\mathbb{H},k}(K)$ can be identified with $(C^k_{\mathbb{H}}(K))^{N_k}$, and endowed with the induced norms $\|\cdot\|_m$.
The family of norms $\|\cdot\|_m$, $m = 0, 1, ...$ induces as usual a structure of Fréchet space in $\mathcal{D}^k_{\mathbb{H}}(K)$ and $\mathcal{D}_{\mathbb{H},k}(K)$

Let us recall a well-know result of algebraic geometry, see [90, 96].

Lemma 2.3.25. *Let us consider the algebraic operator* L *on horizontal* k*-differential forms*

$$L := \begin{cases} \mathcal{D}^k_{H\mathbb{H}} \to \mathcal{D}^{k+2}_{H\mathbb{H}} \\ \alpha \mapsto d\theta \wedge \alpha \end{cases}$$

L *is injective if* $k \le n-1$ *and* L *is surjective if* $k \ge n-1$*.*

Remark 2.3.26. If $k = n-1$ L is an isomorphism.

2.4 Rumin Complex

Let us expose now the results of Rumin [90]. Following [8, 58], we can obtain from ${}_H\bigwedge^*$ a complex of intrinsic differential forms that fits the structure of $\mathbb{H}^n$ in the same way as De Rham complex does for usual differential forms in Euclidean spaces in [90].

Let us show now that the spaces of integrable covectors are canonically isomorphic to the spaces defined by Rumin.

Definition 2.4.1. *Let us define $\mathcal{I}^*$ as the graded ideal of differential forms generated by θ, i.e.*

$$\mathcal{I}^* := \left\{ \beta \wedge \theta + \gamma \wedge d\theta : \beta, \gamma \in \bigwedge\nolimits^* \mathfrak{h} \right\}$$

and let us define $\mathcal{J}^*$ as the annihilator of $\mathcal{I}^*$, i.e.

$$\mathcal{J}^* := \left\{ \alpha \in \bigwedge\nolimits^* \mathfrak{h} : \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0 \right\}.$$

Remark 2.4.2. $\mathcal{I}^*$ and $\mathcal{J}^*$ are graded, that is

$$\mathcal{I}^* = \bigoplus_{k=1}^{2n+1} \mathcal{I}^k \text{ and } \mathcal{J}^* = \bigoplus_{k=1}^{2n+1} \mathcal{J}^k$$

where

$$\mathcal{I}^k := \left\{ \beta \wedge \theta + \gamma \wedge d\theta : \beta \in \bigwedge\nolimits^{k-1} \mathfrak{h}, \gamma \in \bigwedge\nolimits^{k-2} \mathfrak{h} \right\}$$

$$\mathcal{J}^k := \left\{ \alpha \in \bigwedge\nolimits^k \mathfrak{h} : \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0 \right\}.$$

For $1 \leq k \leq n$ we have $\mathcal{I}^{2n+1-k} = \bigwedge^{2n+1-k} \mathfrak{h}$ and $\mathcal{J}^k = 0$.

Let us define

$$\ker \mathcal{I}^k := \left\{ v \in \bigwedge\nolimits_k \mathfrak{h} : \langle \varphi | v \rangle = 0 \ \forall \varphi \in \mathcal{I}^k \right\}$$

and let us define analogously $\ker J^{2n+1-k}$. The following identities, or natural isomorphisms, hold.

Theorem 2.4.3. *For $1 \leq k \leq n$*

$$ {}_H\bigwedge\nolimits_k = \ker \mathcal{I}^k \quad \text{ and } \quad {}_H\bigwedge\nolimits_{2n+1-k} \simeq \frac{\bigwedge_{2n+1-k} \mathfrak{h}}{\ker J^{2n+1-k}} \tag{2.23}$$

$$ {}_H\bigwedge\nolimits^k \simeq \frac{\bigwedge^k \mathfrak{h}}{\ker I^k} \quad \text{ and } \quad {}_H\bigwedge\nolimits^{2n+1-k} = \mathcal{J}^{2n+1-k} \tag{2.24}$$

Proof. (see [58]) Let us prove the first equality in (2.23). If $v \in \bigwedge_k \mathfrak{h}$ the condition $\langle \gamma \wedge d\theta | v \rangle = 0$ for all $\beta \in \bigwedge^{k-1} \mathfrak{h}$ implies $v \in \bigwedge_k \mathfrak{h}_1$, hence we get

$$\ker \mathcal{I}^k = \left\{ v \in \bigwedge_k \mathfrak{h}_1 : \langle \gamma \wedge d\theta | v \rangle = 0 \ \forall \gamma \in \bigwedge^{k-2} \mathfrak{h} \right\}.$$

We conclude by the equivalence of **i** and **iii** in Theorem 2.3.16.

To prove the second one in (2.23) recall that, by definition 2.3.12,

$$_H\bigwedge_{2n+1-k} = *_H\bigwedge_k = * \ker \mathcal{I}^k.$$

Moreover,

$$\ker \mathcal{I}^k = \left\{ v \in \bigwedge_k \mathfrak{h} : \langle \varphi^\natural, v \rangle = 0 \, \forall \varphi \in \mathcal{I}^k \right\}.$$

Hence

$$*(\ker \mathcal{I}^k) = \left\{ v \in \bigwedge_{2n+1-k} \mathfrak{h} : \langle *\varphi^\natural, v \rangle = 0 \, \forall \varphi \in \mathcal{I}^k \right\}. \tag{2.25}$$

Now notice that

$$\varphi \in \mathcal{I}^k \qquad \Longleftrightarrow \qquad *\varphi^\natural \in \ker \mathcal{J}^{2n+1-k}. \tag{2.26}$$

Indeed $*\varphi^\natural \in \ker \mathcal{J}^{2n+1-k} \iff$

$$\iff \langle \psi | * \varphi^\natural \rangle = 0, \, \forall \psi \in \mathcal{J}^{2n+1-k} \iff \langle *\psi | \varphi^\natural \rangle = 0, \, \forall \psi \in \mathcal{J}^{2n+1-k}$$

hence

$$\iff \langle \alpha | \varphi^\natural \rangle = 0, \forall \alpha \in *(\mathcal{J}^{2n+1-k}) = (\mathcal{I}^k)^\perp \iff \langle \alpha, \varphi \rangle = 0, \forall \alpha \in (\mathcal{I}^k)^\perp \iff \varphi \in \mathcal{I}^k.$$

Finally, from (2.25) and (2.26) it follows

$$* \left({}_H\bigwedge_k \right) \simeq * \left(\ker \mathcal{I}^k \right) = \left\{ v \in \bigwedge_{2n+1-k} \mathfrak{h} : \langle \psi, v \rangle = 0 \, \forall \psi \in \ker \mathcal{J}^{2n+1-k} \right\} =$$

$$= \left(\ker \mathcal{J}^{2n+1-k} \right)^\perp \simeq \frac{\bigwedge_{2n+1-k} \mathfrak{h}}{\ker \mathcal{J}^{2n+1-k}}.$$

This concludes the proof of the second part of (2.23).
Let us prove (2.24). Recall that for $1 \leq k \leq 2n+1$, ${}_H\bigwedge^k = \bigwedge^1 ({}_H\bigwedge_k)$. Given that for any two finite dimensional vector spaces V and W with V subspace of W, it holds that

$$\bigwedge^1 \left(\frac{W}{V} \right) \simeq \ker(V) \quad \text{and} \quad \bigwedge^1 V \simeq \frac{\bigwedge^1 W}{\ker(V)},$$

we have for $k = 1, ..., n$

$$\bigwedge^1 \left(\ker \mathcal{I}^k\right) \simeq \frac{\bigwedge^1 \bigwedge_k \mathfrak{h}}{\ker\left(\ker \mathcal{I}^k\right)} \simeq \frac{\bigwedge^k \mathfrak{h}}{\ker \mathcal{I}^k},$$

and for $k = n+1, ..., 2n+1$

$$\bigwedge^1 \left(\frac{\bigwedge_k \mathfrak{h}}{\ker \mathcal{J}^k}\right) \simeq \ker\left(\ker \mathcal{J}^k\right) = \mathcal{J}^k.$$

$\square$

Remark 2.4.4. Let us write explicitly an isomorphism realizing (2.24). For $1 \leq k \leq n$ denote by

$$R :_H \bigwedge^k \to \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$$

the map defined by $R\alpha := [\alpha]$, where $[\alpha]$ is the equivalence class of α. Then denote by

$$P : \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} \to_H \bigwedge^k$$

the map $[\alpha] \to \pi(\alpha)$ that associates with a class $[\alpha]$ the orthogonal projection $\pi(\alpha)$ in $\bigwedge^k \mathfrak{h}$ of a representative of $[\alpha]$ on the orthogonal complement $\mathcal{I}_k^\perp$ of the linear space

$$\mathcal{I}_k := \left\{\beta \wedge \theta + \gamma \wedge d\theta : \, \beta \in \bigwedge^{k-1} \mathfrak{h}, \, \gamma \in \bigwedge^{k-2} \mathfrak{h}\right\}.$$

Note that this definition does not depend on the representative chosen.
Let us note that $PR\alpha = \alpha$ for any $\alpha \in_H \bigwedge^k$. Moreover, if $[\alpha] \in \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$ then $\alpha - \pi(\alpha) \in \mathcal{I}_k$, so that $[\alpha] = [\pi(\alpha)] = [P[\alpha]]$, and hence $RP[\alpha] = \alpha$ for any $[\alpha] \in \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$.
Let us now show that $\mathcal{I}_k^\perp =_H \bigwedge^k$. If $\alpha = (v_1 \wedge ... \wedge v_k)^\natural \in_H \bigwedge^k$ is a simple k-covector, then for any $\beta = \sum_{J=(j_1,...,j_{k-1})} \beta_J \theta_{j_1} \wedge ... \wedge \theta_{j_{k-1}} \in \bigwedge^{k-1} \mathfrak{h}_1$ we have, recalling the notation $\theta = \theta_{2n+1}$

$$\langle \beta \wedge \theta, \alpha \rangle = \sum_J \beta_J \langle \theta_{j_1} \wedge ... \wedge \theta_{j_{k-1}} \wedge \theta_{2n+1} | v_1 \wedge ... \wedge v_k \rangle =$$

$$= \sum_J \beta_J \det\left(\langle \theta_{j_i} | v_l \rangle\right) = 0$$

for $\langle \theta_{2n+1} | v_l \rangle = 0$ for $l = 1, ..., k$. Moreover for any $\gamma \in \bigwedge^{k-2} \mathfrak{h}$,

$$\langle \gamma \wedge d\theta, \alpha \rangle = \langle \gamma \wedge d\theta | v_1 \wedge ... \wedge v_k \rangle = 0,$$

by Theorem 2.3.16. Thus $\alpha \in \mathcal{I}_k^\perp$ and hence ${}_H\bigwedge^k$ is a linear subspace of $\mathcal{I}_k^\perp$. On the other hand, by (2.24), both ${}_H\bigwedge^k$ and $\mathcal{I}_k^\perp$ have the same dimension $\dim \bigwedge^k \mathfrak{h} - \dim \mathcal{I}_k$, and hence they coincide. $\square$

Remark 2.4.5. Let us finally observe that

$$\text{if } k \le n \qquad \mathcal{J}^k = \{0\},$$

$$\text{if } k \ge n+1 \qquad \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} = \{0\}.$$

Let us study the Rumin complex, cfr. [8, 58, 90].

Theorem 2.4.6. *There exists a second order operator $D : \frac{\bigwedge^n \mathfrak{h}}{\mathcal{I}^n} \to \mathcal{J}^{n+1}$ such that the contact-complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(\mathbb{H}^n) \xrightarrow{d_c} \frac{\bigwedge^1 \mathfrak{h}}{\mathcal{I}^1} \xrightarrow{d_c} \dots \xrightarrow{d_c} \frac{\bigwedge^n \mathfrak{h}}{\mathcal{I}^n} \xrightarrow{D}$$

$$\xrightarrow{D} \mathcal{J}^{n+1} \xrightarrow{d_c} \mathcal{J}^{n+2} \xrightarrow{d_c} \dots \xrightarrow{d_c} \mathcal{J}^{2n+1} \xrightarrow{d_c} 0$$

has the same cohomology as De Rham complex, where d_c is a first-order operator that depends only by horizontal derivatives.

Remark 2.4.7. If $n+1 \le k \le 2n+1$ d_c is the usual exterior differential. If $1 \le k \le n-1$ d_c is the operator on the quotient spaces $\frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$ such that, if $[\alpha] \in \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$

$$d_c[\alpha] = [d\alpha].$$

By Theorem 2.4.3 $\frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$ is isomorphic to ${}_H\bigwedge^k$, let us also define the differential operator $d_c' := P d_c R$ that make the following diagram to be commutative:

$$\begin{array}{ccc}
{}_H\bigwedge^k & \xrightarrow{d_c'} & {}_H\bigwedge^{k+1} \\
R \downarrow & & \uparrow P \\
\frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} & \xrightarrow{d_c} & \frac{\bigwedge^{k+1} \mathfrak{h}}{\mathcal{I}^{k+1}}
\end{array}$$

Let us write an explicit form of the operator d_c' with respect to a left invariant orthonormal moving frame $\{\xi_1, \dots, \xi_{N_k}\}$ of ${}_H\bigwedge^k$, obviously in the case $1 \le$

$k \leq n-1$. By linearity, let for instance f be a smooth function, and let $i \in \{1, ..., N_k\}$ be fixed. We have

$$d_c'(f\xi_i) = Pd_cR(f\xi_i) = P[d(f\xi_i)] = P(df \wedge \xi_i) + fP(d\xi_i).$$

$\xi_i \in_H \bigwedge^k \mathfrak{h}_1$ and hence it is a linear combination (with constant coefficients determined by left translation at the origin) of $dp_1, ..., dp_{2n}$. Thus $d\xi_i = 0$. Thus, because $df = d_{\mathbb{H}} f + (Tf)\theta$, we get

$$d_c'(f\xi_i)(p) = P[d_{\mathbb{H}} f \wedge \xi_i](p) = \sum_{j=1}^{2n} (W_j f(p)) P[dp_j \wedge \xi_i](p) =$$

$$= \sum_{j=1}^{2n} (W_j f(p)) \Lambda^{k+1} d\tau_{p^{-1}} \left(P\left[\Lambda^{k+1} d\tau_p (dp_j \wedge \xi_i)\right](e)\right) =$$

$$= \sum_{j=1}^{2n} (W_j f(p)) \left(\Lambda^{k+1} d\tau_{p^{-1}} \left(P\left[dp_j \wedge \xi_{i,e}\right](e)\right)\right).$$

If $\{\eta_1, ..., \eta_{N_{k+1}}\}$ is a left invariant orthonormal moving frame of $_H \bigwedge^{k+1}$, then there exist real constants $c^l_{i,j,k}$ such that

$$P[dp_j \wedge \xi_{i,e}](e) = \sum_l c^l_{i,j,k} \eta_{l,e}$$

and hence

$$d_c'(f\xi_i)(p) = \sum_l \left(\sum_{j=1}^{2n} c^l_{i,j,k} (W_j f(p)) \right) \eta_{l,p}. \tag{2.27}$$

Remark 2.4.8. We give an explicit representation of the operator D.
Let $[\beta] \in \dfrac{\bigwedge^n \mathfrak{h}}{\mathcal{I}^n}$. There exists an unique lifting $\widetilde{\beta}$ such that $d\widetilde{\beta} \in \mathcal{J}^{n+1}$, i.e. $d\widetilde{\beta} \wedge \theta = 0$ and $d\widetilde{\beta} \wedge d\theta = 0$. Indeed in the class $[\beta]$ there exists always a purely horizontal element, that we can still denote by β, because any form $\beta \in \bigwedge^n \mathfrak{h}$ can be written as $\beta = \beta_H + \theta \wedge \beta_T$, with $\beta_H, \beta_T \in \mathcal{D}^*_{H\mathbb{H}}$ purely horizontal. Let us choose

$$\widetilde{\beta} := \beta - \theta \wedge L^{-1}(d_{\mathbb{H}}\beta),$$

where L is the operator defined in Lemma 2.3.25. Then (cfr. [8, 90]) the operator $D :_H \bigwedge^n \longrightarrow_H \bigwedge^{n+1}$ is defined setting $D\beta := d\widetilde{\beta}$, and so

$$D\beta = \theta \wedge \left(\mathcal{L}_T \beta + d_{\mathbb{H}} L^{-1}(d_{\mathbb{H}}\beta)\right)$$

where $\mathcal{L}_T$ is the Lie derivative along T.

Remark 2.4.9. Thanks to Theorem 2.4.3 and remark 2.4.7 and 2.4.8, recalling definition 2.3.21 we can rewrite the Rumin complex as

$$0 \longrightarrow \mathcal{D}^0_{\mathbb{H}}(\mathbb{H}^n) \longrightarrow \mathcal{D}^1_{\mathbb{H}}(\mathbb{H}^n) \xrightarrow{d'_c} \ldots \xrightarrow{d'_c} \mathcal{D}^n_{\mathbb{H}}(\mathbb{H}^n) \xrightarrow{D}$$

$$\xrightarrow{D} \mathcal{D}^{n+1}_{\mathbb{H}}(\mathbb{H}^n) \xrightarrow{d_c} \ldots \xrightarrow{d_c} \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathbb{H}^n) \xrightarrow{d_c} 0$$

see [8, 58].

We can give the definitions of $\nabla_{\mathbb{H}} f$ and $\mathrm{div}_{\mathbb{H}} F$ in an alternative way respect to definition 2.2.8, using the differential forms' language, recalling that ${}_H\bigwedge_1 \equiv \mathfrak{h}_1 \equiv H\mathbb{H}^n$.

Definition 2.4.10. *Let $\Omega \in \mathbb{H}^n$ be an open set and $f \in C^1_{\mathbb{H}}(\Omega)$; let us define $\nabla_{\mathbb{H}} f$ as the horizontal vector field*

$$\nabla_{\mathbb{H}} f := (d'_c f)^{\natural} \tag{2.28}$$

Definition 2.4.11. *Let $\Omega \in \mathbb{H}^n$ be an open set and let $F \in C^1_{\mathbb{H}}(\Omega, H\mathbb{H}^n)$; let us define*

$$\mathrm{div}_{\mathbb{H}} F := \left(*d'_c(*F^{\natural})\right)^{\natural} \tag{2.29}$$

Remark 2.4.12. Because $\{W_1, ..., W_{2n}\}$ is a left invariant orthonormal moving frame of ${}_H\bigwedge_1$ and a horizontal vector field F can be written in the form $F := \sum_{j=1}^{2n} F_j W_j$, F can be identified with the vector-valued function $(F_1, ..., F_{2n})$. Thus

$$\mathrm{div}_{\mathbb{H}} F = \sum_{j=1}^{2n} W_j F_j$$

Definition 2.4.13. *Let F be a smooth section of ${}_H\bigwedge_1$, let us define*

$$\mathrm{curl}_{\mathbb{H}} F := \left(DF^{\natural}\right)^{\natural} \qquad \text{if } n = 1 \tag{2.30}$$

$$\mathrm{curl}_{\mathbb{H}} F := \left(d'_c F^{\natural}\right)^{\natural} \qquad \text{if } n \geq 2 \tag{2.31}$$

Remark 2.4.14. Let us give an explicit representation of $\mathrm{curl}_{\mathbb{H}} F$.
If $n = 1$, an orthonormal left invariant moving frame of ${}_H\bigwedge^2$ is given by $\{\theta_2 \wedge \theta_3, -\theta_1 \wedge \theta_3\}$, see remark 2.3.17. By definition we have that

$$d_{\mathbb{H}} F^{\natural} = (W_2 F_1)\theta_2 \wedge \theta_1 + (W_1 F_2)\theta_1 \wedge \theta_2 = (W_1 F_2 - W_2 F_1)\theta_1 \wedge \theta_2.$$

Since $d\theta_3 := -\theta_1 \wedge \theta_2$, it follows that

$$L^{-1}\left(d_{\mathbb{H}} F^\natural\right) = -(W_1F_2 - W_2F_1).$$

On the other hand

$$\mathcal{L}_T F^\natural = (W_1W_2 - W_2W_1)F_1\theta_1 + (W_1W_2 - W_2W_1)F_2\theta_2$$

and so

$$\begin{aligned} DF^\natural = \theta_3 \wedge \mathcal{L}_T F^\natural + d_{\mathbb{H}} L^{-1}(d_{\mathbb{H}} F^\natural) = \theta_3 \wedge [(W_1W_2 - W_2W_1)F_1\theta_1 + (W_1W_2 - W_2W_1)F_2\theta_2 + \\ -W_1(W_1F_2 - W_2F_1)\theta_1 - W_2(W_1F_2 - W_2F_1)\theta_2] = \\ = \theta_3 \wedge [(2W_1W_2F_1 - W_2W_1F_1 - W_1^2F_2)\theta_1 + (W_1W_2F_2 - 2W_2W_1F_2 - W_2^2F_1)\theta_2] = \\ = (2W_1W_2F_1 - W_2W_1F_1 - W_1^2F_2)(-\theta_1 \wedge \theta_3) + (W_1W_2F_2 - 2W_2W_1F_2 - W_2^2F_1)(\theta_2 \wedge \theta_3). \end{aligned}$$

Thus $\mathrm{curl}_{\mathbb{H}} F$ can be identified with the second order vector-valued operator

$$(F_1, F_2) \longmapsto (2W_1W_2F_1 - W_2W_1F_1 - W_1^2F_2\, ,\, W_1W_2F_2 - 2W_2W_1F_2 - W_2^2F_1). \tag{2.32}$$

If $n \geq 2$, let us denote by π_0 the orthogonal projection in $\bigwedge^2 \mathfrak{h}_1$ along the linear space Θ spanned by $d\theta_{2n+1}$. Then by remark 2.4.7

$$d_c' F^\natural = \pi_0(d_{\mathbb{H}} F^\natural).$$

We compute

$$d_{\mathbb{H}} F^\natural = d_{\mathbb{H}}\left(\sum_{j=1}^{2n} F_j\theta_j\right) = \sum_{j=1}^{2n}\sum_{i=1}^{2n} W_iF_j(\theta_i \wedge \theta_j) =$$

$$= \sum_{1\leq i<j\leq 2n} (W_iF_j - W_jF_i)\theta_i \wedge \theta_j =: \sum_{1\leq i<j\leq 2n} F_{i,j}\theta_i \wedge \theta_j.$$

Recalling that $d\theta_{2n+1} = -\sum_{k=1}^{n} \theta_k \wedge \theta_{k+n}$ and that

$$\{\theta_i \wedge \theta_j\}_{\substack{1\leq i<j\leq 2n \\ j\neq i+n}} \cup \left\{\frac{1}{\sqrt{2}}(\theta_h \wedge \theta_{h+n} - \theta_{h+1} \wedge \theta_{h+1+n})\right\}_{h=1,\dots,n-1}$$

is an orthonormal basis of $\bigwedge^2 \mathfrak{h}_1 \cap \Theta^\perp =_H \bigwedge^2$ (see remark 2.3.17)

$$\pi_0\left(d_H F^\natural\right) = \sum_{\substack{1\leq i<j\leq 2n \\ j\neq i+n}} F_{i,j}\theta_i \wedge \theta_j + \frac{1}{\sqrt{2}}\sum_{h=1}^{n-1}(F_{h,h+n} + F_{h+1,h+1+n})(\theta_h \wedge \theta_{h+n} - \theta_{h+1} \wedge \theta_{h+1+n}).$$

Thus $\mathrm{curl}_{\mathbb{H}} F$ can be identified with the first order vector valued operator

$$(F_1, \dots, F_{2n}) \longmapsto \left(\dots, F_{i,j}, \dots, \frac{1}{\sqrt{2}}(F_{h,h+n} - F_{h+1,h+1+n}), \dots \right)$$

with $1 \le i < j \le 2n$, $j \ne i+n$ and $h = 1, ..., n-1$.

Remark 2.4.15. Let us notice that $\mathrm{curl}_{\mathbb{H}} :_H \bigwedge^1 \to_H \bigwedge^2$, then $\mathrm{curl}_{\mathbb{H}} F \in {}_H \bigwedge^2$. It follows by Rumin Theorem 2.4.6 that $\mathrm{curl}_{\mathbb{H}}$ is a second order operator if $n = 1$ and a first order operator if $n \ge 2$.

The Rumin Theorem 2.4.6 yields an exactitudes' result of the 1-differential forms in $\mathbb{H}^n$. Similarly to the classical Poincarè Lemma in Euclidean setting, we have that if $F = (F_1, ..., F_{2n})$ with $F_j \in \mathcal{D}'(\Omega)$, where $\Omega \subset \mathbb{H}^n$ is open and simply connected, there exists $f \in \mathcal{D}'(\Omega)$ such that

$$\nabla_{\mathbb{H}} f = F \quad \text{if and only if} \quad \mathrm{curl}_{\mathbb{H}} F = 0, \tag{2.33}$$

where the equalities have to be understood in distributional sense.
We can give an alternative proof of this result, that does not use the differential forms' language and that is obtained by the commutation of the vector fields $W_j = X_j, Y_j, T$. If $n = 1$ this equivalence is proved in [59].

Theorem 2.4.16. *Let $\Omega \subseteq \mathbb{H}^n$ be a simply connected open set and let $F = (F_1, ..., F_{2n})$ with $F_j \in \mathcal{D}'(\Omega)$ $j = 1, ..., n$. Then the following conditions are equivalent*

i *there exists $f \in \mathcal{D}'(\Omega)$ such that*

$$\nabla_{\mathbb{H}} f = F \qquad \text{in } \Omega \tag{2.34}$$

in distributional sense.

ii *If $n = 1$*

$$TF_1 = X_1^2 F_2 - X_1 Y_1 F_1 \qquad \text{and} \qquad TF_2 = Y_1 X_1 F_2 - Y_1^2 F_1 \tag{2.35}$$

in distributional sense.
If $n \ge 2$, for each $i, j = 1, ..., n$

$$X_i F_j = X_j F_i, \qquad X_i F_{j+n} = Y_j F_i, \qquad Y_i F_{j+n} = Y_j F_{i+n} \qquad \text{with } i \ne j \tag{2.36}$$

$$X_j F_{j+n} - Y_j F_j = X_i F_{i+n} - Y_i F_i \tag{2.37}$$

in distributional sense.

Proof. **i⇒ii**: If $n = 1$ we have $TX_1f = X_1Tf$ and $TY_1f = Y_1Tf$. Since $T = X_1Y_1 - Y_1X_1$ we have

$$T(X_1f) = X_1(X_1Y_1 - Y_1X_1)f \quad \text{and} \quad T(Y_1f) = Y_1(X_1Y_1 - Y_1X_1)f.$$

If $\exists f$ distributional solution of (2.34) we have therefore

$$TF_1 = X_1^2F_2 - X_1Y_1F_1 \quad \text{and} \quad TF_2 = Y_1X_1F_2 - Y_1^2F_1.$$

If $n \geq 2$ let us consider the vector fields X_i, X_j, Y_i and Y_j with $i \neq j$. This fields are commutative, i.e. $X_iX_j = X_jX_i$, $X_iY_j = Y_jX_i$ and $Y_iY_j = Y_jY_i$. Let f be a distributional solution of (2.34), by $X_iX_jf = X_jX_if$, $X_iY_jf = Y_jX_if$ and $Y_iY_jf = Y_jY_if$ in distributional sense, we obtain (2.36).

We obtain the condition (2.37) from the commutation of X_j and Y_j $\forall\, j = 1, ..., n$. In fact we have $X_jY_j - Y_jX_j = T$. But, for $i \neq j$ we have $X_iY_i - Y_iX_i = T$ too. Therefore we have in distributional sense

$$X_jY_jf - Y_jX_jf = Tf = X_iY_if - Y_iX_if,$$

if f is a distributional solution of (2.34) we obtain (2.37).

ii⇒i: In the following the derivatives must be understood in distributional sense.

Step 1. Let us show that the problem (2.34) is equivalent to an Euclidean type problem. Let us define for fixed i $\forall j = 1, ..., n$ the vector field $\mathcal{F} \in \mathcal{D}'(\Omega, \mathbb{R}^{2n+1})$

$$\begin{cases} \mathcal{F}_j = F_j + \frac{1}{2}x_{j+n}(X_iF_{i+n} - Y_iF_i) \\ \mathcal{F}_{j+n} = F_{j+n} - \frac{1}{2}x_j(X_iF_{i+n} - Y_iF_i) \\ \mathcal{F}_{2n+1} = X_iF_{i+n} - Y_iF_i \end{cases} \tag{2.38}$$

If $f \in \mathcal{D}'(\Omega)$ is a distributional solution of (2.34) we have

$$\begin{aligned} \mathcal{F}_j &= X_jf + \frac{1}{2}x_{j+n}Tf = \frac{\partial f}{\partial x_j} \\ \mathcal{F}_{j+n} &= X_{j+n}f - \frac{1}{2}x_jTf = \frac{\partial f}{\partial x_{j+n}} \\ \mathcal{F}_{2n+1} &= X_iY_if - Y_iX_if = Tf = \frac{\partial f}{\partial t} \end{aligned}$$

We obtain that f satisfies

$$\nabla f = \mathcal{F} \tag{2.39}$$

On the other side, let $\mathcal{F}$ be defined as in (2.38) and let f such that (2.39) holds. By definition

$$\frac{\partial f}{\partial t} = \mathcal{F}_{2n+1} = X_iF_{i+n} - Y_iF_i$$

and

$$\mathcal{F}_j = F_j + \tfrac{1}{2}x_{j+n}\mathcal{F}_{2n+1}$$
$$\mathcal{F}_{j+n} = F_{j+n} - \tfrac{1}{2}x_j\mathcal{F}_{2n+1}$$

Because f solves (2.39) by assumption, we have

$$F_j = \frac{\partial f}{\partial x_j} - \frac{1}{2}x_{j+n}\frac{\partial f}{\partial t} = X_j f$$
$$F_{j+n} = \frac{\partial f}{\partial x_{j+n}} + \frac{1}{2}x_j\frac{\partial f}{\partial t} = Y_j f$$

and so we obtain that (2.34) and (2.39) are equivalent.

By classical Poincaré lemma the existence of distributional solution f for the problem (2.39) is equivalent to ask, for $i, j = 1, ..., 2n$,

$$\frac{\partial \mathcal{F}_j}{\partial t} = \frac{\partial \mathcal{F}_{2n+1}}{\partial x_j} \tag{2.40}$$

$$\frac{\partial \mathcal{F}_j}{\partial x_i} = \frac{\partial \mathcal{F}_i}{\partial x_j} \tag{2.41}$$

In the following we are going to prove that the condition (2.35) when $n = 1$ and the conditions (2.36),(2.37) when $n \geq 2$ infer (2.40),(2.41).

Step 2. If $n = 1$ let us consider the first condition (2.35)

$$X_1^2F_2 - X_1Y_1F_1 = TF_1$$

That is

$$X_1(X_1F_2 - Y_1F_1) - \frac{\partial F_1}{\partial t} = 0,$$

therefore

$$\frac{\partial}{\partial x}(X_1F_2 - Y_1F_1) - \frac{1}{2}y\frac{\partial}{\partial t}(X_1F_2 - Y_1F_1) - \frac{\partial F_1}{\partial t} = 0$$

$$\frac{\partial}{\partial x}(X_1F_2 - Y_1F_1) - \frac{\partial}{\partial t}\left(F_1 + \frac{1}{2}y(X_1F_2 - Y_1F_1)\right) = 0$$

and so we have the conditions (2.40)

$$\frac{\partial \mathcal{F}_3}{\partial x} - \frac{\partial \mathcal{F}_1}{\partial t} = 0.$$

Analogously, from

$$Y_1X_1F_2 - Y_1^2F_1 = TF_2$$

we obtain
$$\frac{\partial \mathcal{F}_3}{\partial y} - \frac{\partial \mathcal{F}_2}{\partial t} = 0.$$
By the conditions (2.40) we obtain immediately (2.41) by Schwarz Theorem.

Step 3. If $n \geq 2$ let us prove that conditions (2.36),(2.37) imply
$$X_jX_iF_{i+n} - X_jY_iF_i = TF_j \text{ and } Y_jX_iF_{i+n} - Y_jY_iF_i = TF_{j+n} \;\; \forall\, i,j = 1,...,n \tag{2.42}$$
Let us consider the identity of (2.36) in distributional sense
$$\begin{aligned} X_jF_{i+n} &= Y_iF_j \\ X_jF_i &= X_iF_j \end{aligned}$$
with $i,j = 1,...,n$, $i \neq j$, i.e.
$$\begin{aligned} \int_\Omega F_{i+n}X_j\psi\, d\mathcal{L}^{2n+1} &= \int_\Omega F_jY_i\psi\, d\mathcal{L}^{2n+1} \\ \int_\Omega F_iX_j\psi\, d\mathcal{L}^{2n+1} &= \int_\Omega F_jX_i\psi\, d\mathcal{L}^{2n+1} \end{aligned}$$
$\forall\, \psi \in C_c^\infty(\Omega)$. Let us choose $\psi = X_i\varphi$ in the first equation and $\psi = Y_i\varphi$ in the second one, with $\varphi \in C_c^\infty(\Omega)$, we find
$$\begin{aligned} \int_\Omega F_{i+n}X_jX_i\varphi\, d\mathcal{L}^{2n+1} &= \int_\Omega F_jY_iX_i\varphi\, d\mathcal{L}^{2n+1} \\ \int_\Omega F_iX_jY_i\varphi\, d\mathcal{L}^{2n+1} &= \int_\Omega F_jX_iY_i\varphi\, d\mathcal{L}^{2n+1} \end{aligned} \tag{2.43}$$
Therefore $\forall\, \varphi \in C_c^\infty(\Omega)$
$$\int_\Omega (F_{i+n}X_iX_j\varphi - F_iY_iX_j\varphi)\; d\mathcal{L}^{2n+1} = \int_\Omega (F_jX_iY_i\varphi - F_jY_iX_i\varphi)\; d\mathcal{L}^{2n+1}$$
that is in distributional sense
$$X_jX_iF_{i+n} - X_jY_iF_i = TF_j. \tag{2.44}$$
By (2.37) we have by (2.44)
$$X_j(X_jF_{j+n} - Y_jF_j) = TF_j \tag{2.45}$$
(2.44) and (2.45) infer a part of the conditions (2.42)
$$X_jX_iF_{i+n} - X_jY_iF_i = TF_j \qquad \forall\, i,j = 1,...,n$$

If we consider the identities of (2.36)

$$\begin{array}{l} Y_j F_{i+n} = Y_i F_{j+n} \\ Y_j F_i = X_i F_{j+n} \end{array}$$

we obtain in the same way the remaining conditions (2.42).

Step 4. Let us consider the conditions (2.42): for $j = 1, ..., n$ and i fixed we have

$$X_j X_i F_{i+n} - X_j Y_i F_i = T F_j$$

That is

$$X_j (X_i F_{i+n} - Y_i F_i) - \frac{\partial F_j}{\partial t} = 0,$$

therefore

$$\frac{\partial}{\partial x_j}(X_i F_{i+n} - Y_i F_i) - \frac{1}{2} x_{j+n} \frac{\partial}{\partial t}(X_i F_{i+n} - Y_i F_i) - \frac{\partial F_j}{\partial t} = 0$$

$$\frac{\partial}{\partial x_j}(X_i F_{i+n} - Y_i F_i) - \frac{\partial}{\partial t}\left(F_j + \frac{1}{2} x_{j+n}(X_i F_{i+n} - Y_i F_i)\right) = 0$$

and so we have the conditions (2.40)

$$\frac{\partial \mathcal{F}_{2n+1}}{\partial x_j} - \frac{\partial \mathcal{F}_j}{\partial t} = 0.$$

Analogously, from

$$Y_j X_i F_{i+n} - Y_j Y_i F_i = T F_{j+n}$$

we obtain

$$\frac{\partial \mathcal{F}_{2n+1}}{\partial x_{j+n}} - \frac{\partial \mathcal{F}_{j+n}}{\partial t} = 0.$$

Step 5. Now let us consider the conditions (2.36)

$$X_i F_j - X_j F_i = 0 \quad \forall\, i, j = 1, ..., n, \ i \neq j.$$

We have

$$\frac{\partial F_j}{\partial x_i} - \frac{1}{2} x_{i+n} \frac{\partial F_j}{\partial t} - \frac{1}{4} x_{i+n} x_{j+n} \frac{\partial \mathcal{F}_{2n+1}}{\partial t} - \frac{\partial F_i}{\partial x_j} + \frac{1}{2} x_{j+n} \frac{\partial F_i}{\partial t} + \frac{1}{4} x_{j+n} x_{i+n} \frac{\partial \mathcal{F}_{2n+1}}{\partial t} = 0$$

Therefore

$$\frac{\partial F_j}{\partial x_i} + \frac{1}{2} x_{j+n} \frac{\partial}{\partial t}\left(F_i + \frac{1}{2} x_{i+n} \mathcal{F}_{2n+1}\right) - \frac{\partial F_i}{\partial x_j} - \frac{1}{2} x_{i+n} \frac{\partial}{\partial t}\left(F_j + \frac{1}{2} x_{j+n} \mathcal{F}_{2n+1}\right) = 0$$

$$\frac{\partial F_j}{\partial x_i} + \frac{1}{2}x_{j+n}\frac{\partial \mathcal{F}_i}{\partial t} - \frac{\partial F_i}{\partial x_j} - \frac{1}{2}x_{i+n}\frac{\partial \mathcal{F}_j}{\partial t} = 0$$

For the conditions (2.40), that we have proved in step 3, we get

$$\frac{\partial F_j}{\partial x_i} + \frac{1}{2}x_{j+n}\frac{\partial \mathcal{F}_{2n+1}}{\partial x_i} - \frac{\partial F_i}{\partial x_j} - \frac{1}{2}x_{i+n}\frac{\partial \mathcal{F}_{2n+1}}{\partial x_j} = 0$$

Then, if $i, j = 1, ..., n \ i \neq j$,

$$\frac{\partial}{\partial x_i}\left(F_j + \frac{1}{2}x_{j+n}\mathcal{F}_{2n+1}\right) - \frac{\partial}{\partial x_j}\left(F_i + \frac{1}{2}x_{i+n}\mathcal{F}_{2n+1}\right) = 0$$

and so we have the conditions (2.41) $\forall\, i, j = 1, ..., n,\ \ i \neq j$

$$\frac{\partial \mathcal{F}_j}{\partial x_i} - \frac{\partial \mathcal{F}_i}{\partial x_j} = 0.$$

By using the remaining identities of (2.36)

$$X_i F_{j+n} - Y_j F_i = 0, \qquad Y_i F_{j+n} - Y_j F_{i+n} = 0$$

we find $\dfrac{\partial \mathcal{F}_{j+n}}{\partial x_i} - \dfrac{\partial \mathcal{F}_i}{\partial x_{j+n}} = 0$ and $\dfrac{\partial \mathcal{F}_{j+n}}{\partial x_{i+n}} - \dfrac{\partial \mathcal{F}_{i+n}}{\partial x_{j+n}} = 0$

Step 6. Let us consider the conditions (2.37) $\forall\, j = 1, ..., n$ and i fixed as in (2.38)

$$X_j F_{j+n} - Y_j F_j = X_i F_{i+n} - Y_i F_i = \mathcal{F}_{2n+1}.$$

Therefore

$$\frac{\partial F_{j+n}}{\partial x_j} - \frac{1}{2}x_{j+n}\frac{\partial F_{j+n}}{\partial t} + \frac{1}{4}x_{j+n}x_j\frac{\partial \mathcal{F}_{2n+1}}{\partial t} - \frac{\partial F_j}{\partial x_{j+n}} - \frac{1}{2}x_j\frac{\partial F_j}{\partial t} - \frac{1}{4}x_j x_{j+n}\frac{\partial \mathcal{F}_{2n+1}}{\partial t} = \mathcal{F}_{2n+1}$$

$$\frac{\partial F_{j+n}}{\partial x_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}\left(F_j + \frac{1}{2}x_{j+n}\mathcal{F}_{2n+1}\right) - \frac{\partial F_j}{\partial x_{j+n}} - \frac{1}{2}x_{j+n}\frac{\partial}{\partial t}\left(F_{j+n} - \frac{1}{2}x_j\mathcal{F}_{2n+1}\right) = \mathcal{F}_{2n+1}$$

$$\frac{\partial F_{j+n}}{\partial x_j} - \frac{1}{2}\mathcal{F}_{2n+1} - \frac{1}{2}x_j\frac{\partial \mathcal{F}_j}{\partial t} - \frac{\partial F_j}{\partial x_{j+n}} - \frac{1}{2}\mathcal{F}_{2n+1} - \frac{1}{2}x_{j+n}\frac{\partial \mathcal{F}_{j+n}}{\partial t} = 0$$

By identities (2.40) already proved at the step 3, we obtain

$$\frac{\partial F_{j+n}}{\partial x_j} - \frac{1}{2}\mathcal{F}_{2n+1} - \frac{1}{2}x_j\frac{\partial \mathcal{F}_{2n+1}}{\partial x_j} - \frac{\partial F_j}{\partial x_{j+n}} - \frac{1}{2}\mathcal{F}_{2n+1} - \frac{1}{2}x_{j+n}\frac{\partial \mathcal{F}_{2n+1}}{\partial x_{j+n}} = 0$$

Therefore

$$\frac{\partial}{\partial x_j}\left(F_{j+n} - \frac{1}{2}x_j\mathcal{F}_{2n+1}\right) - \frac{\partial}{\partial x_{j+n}}\left(F_j + \frac{1}{2}x_{j+n}\mathcal{F}_{2n+1}\right) = 0$$

from which (2.41) in the last case

$$\frac{\partial \mathcal{F}_{j+n}}{\partial x_j} - \frac{\partial \mathcal{F}_j}{\partial x_{j+n}} = 0.$$

□

Chapter 3

Intrinsic Regular Hypersurfaces in the Heisenberg group

In this chapter we introduce the main object of the thesis: the $\mathbb{H}$-regular hypersurfaces. Let us recall that in the Euclidean setting $\mathbb{R}^n$, a $\mathbf{C}^1$-hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a notion was easily transposed in [55] to the Heisenberg group. We shall say that $S \subset \mathbb{H}^n$ is an intrinsic $\mathbb{H}$-*regular hypersurface* if it is locally defined as the zero level set of $f \in C^1_{\mathbb{H}}(\mathbb{H}^n)$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on S (see Definition 3.1.1). These hypersurfaces can have an extremely bad behavior from the Euclidean viewpoint (see [69]), nevertheless they turn out to be regular with respect to the intrinsic geometry.
This definition of $\mathbb{H}$-regularity yields an Implicit Function Theorem, proved in [55] for the Heisenberg group and in [56] for a general Carnot group (see also [32] for an extension to a CC metric space). By this Theorem an $\mathbb{H}$-regular hypersurfaces could be seen as a X_1-*graph*, namely there exists a continuous parametrization $\Phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{H}^n$ of S, see definition 3.1.12 and Theorem 3.1.13.
In section 3.2, 3.3 we recall the main results of [4, 94] and some improvements of these contained in [18]: the ∇^{ϕ}-differentiability and the description of the normal of an $\mathbb{H}$-regular hypersurface through a non linear partial differential equation: the Burgers' equation. We study in particular the role of the ∇^{ϕ}-exponential maps and the concept of broad* solution of the system $\nabla^{\phi}\phi = w$. Let us underline the characterization given in Theorems 3.2.12 and 3.3.9: $S = \Phi(\omega) = G^1_{\mathbb{H},\phi}(\omega)$ is an $\mathbb{H}$-regular surface if and only if the distribution $\nabla^{\phi}\phi$ is represented by a function $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ and there exists a family $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ such that, for any open set ω', we have

$$\phi_\epsilon \to \phi \text{ and } \nabla^{\phi_\epsilon}\phi_\epsilon \to w \text{ uniformly in } \omega'.$$

We give finally in Theorem 3.3.12 an important original improvement of the compactness Theorem 5.9 of [4], with an Hölder continuous regularity result for broad* solutions.
The last section 3.4 is devoted to the exposition of the results obtained in [20] in collaboration with Davide Vittone. We give negative answers to some questions about the parametrization Φ of $\mathbb{H}$-regular hypersurface. $\Psi : (\mathbb{R}^2, \rho) \to (\mathbb{H}^1, d_\infty)$ cannot be bi-Lipschitz where $\rho = ((x,z),(x',z')) := |x-x'| + |z-z'|^{1/2}$ (result obtained with the help of G. Citti and Z. Balogh) and $\Phi : (\mathbb{R}^{2n}, d) \to (\mathbb{H}^n, d_\infty)$ cannot belong to any Sobolev class of metric-space valued functions when $d = \|\cdot\|$ and $d = d_{\infty\,|\omega}$.

3.1 Intrinsic Regular Hypersurfaces and Implicit Functions Theorem

In this section we study some fundamental definitions and results following [4, 55].

Definition 3.1.1. *We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurface if for every $P \in S$ there exist an open ball $U_\infty(P,r)$ and a function $f \in C^1_{\mathbb{H}}(U_\infty(P,r))$ such that*

i $S \cap U_\infty(P,r) = \{Q \in U_\infty(P,r) : f(Q) = 0\}$;

ii $\nabla_{\mathbb{H}} f(P) \neq 0$.

We will denote with $\nu_S(P)$ the horizontal normal to S at a point $P \in S$, i.e. the unit vector

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|_P}$$

and with $T^g_{\mathbb{H}} S(P)$ the tangent group to S at P, i.e. the proper subgroup of $\mathbb{H}^n$ defined by

$$T^g_{\mathbb{H}} S(P) := \{Q : \langle\, \nabla_{\mathbb{H}}(f \circ \tau_P)(0),\, \pi_0(Q))\,\rangle_0 = 0\}.$$

Finally, we use the notation $T_{\mathbb{H}} S(P)$ for the tangent plane to S at P, i.e. the lateral $P \cdot T^g_{\mathbb{H}} S(P)$.

Definition 3.1.2. *Let $S \subset \mathbb{H}^n$ be an hypersurface, i.e. a submanifold of topological codimension 1. A point $P \in S$ is said to be characteristic for S if the Euclidean tangent plane to S at P coincides with the horizontal fiber $\mathrm{H}\mathbb{H}^n_P$.*

Remark 3.1.3. From an intrinsic viewpoint the characteristic points are singular points for S. If S is an $\mathbb{H}$-regular hypersurfaces, every $P \in S$ is not characteristic because $\nabla_{\mathbb{H}} f(P) \neq 0$.

The classes of euclidean regular hypersurfaces and of $\mathbb{H}$-regular hypersurfaces are disjoint, in the sense that there are $\mathbb{H}$-regular surfaces in $\mathbb{H}^1 \simeq \mathbb{R}^3$ that are (Euclidean) fractal sets (see [69]), and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$-regular hypersurfaces (see [55], Remark 6.2). An euclidean regular hypersurfaces is $\mathbb{H}$-regular provided it has no characteristic points.

Example 3.1.4. In $\mathbb{R}^3 \simeq \mathbb{H}^1$ the Euclidean plane $O := \{(x, y, t) \in \mathbb{H}^1 : t = 0\}$ is Euclidean regular while it is not $\mathbb{H}$-regular at the origin: it is enough to observe that $O \backslash \{0\}$ is $\mathbb{H}$-regular and its horizontal normal $\nu_{O\backslash\{0\}} = \dfrac{(y, -x)}{\sqrt{x^2 + y^2}}$ cannot be extended at the origin. Let us notice that regular Euclidean hypersurfaces whose points never are characteristic are $\mathbb{H}$-regular.

Example 3.1.5. $S := \{(x, y, t) \in \mathbb{H}^1 : \ f(x, y, t) = x - \sqrt{x^4 + y^4 + t^2} = 0\}$ is $\mathbb{H}$-regular in a neighborhood of 0 but not C^1 regular at the origin.

Remark 3.1.6. $\mathbb{H}$-regular hypersurfaces could be extremely bad from an Euclidean point of view: indeed, there are examples (see [69]) of $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$ which look as Euclidean fractal sets of Hausdorff dimension $5/2$ in $\mathbb{R}^3$:

Theorem 3.1.7. *There exists an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^1$ such that*

$$\mathcal{H}^{\frac{5-\varepsilon}{2}}(S) > 0 \qquad \forall\, \varepsilon \in \ (0, 1)$$

In particular, S is not 2-Euclidean rectifiable.

Let us recall two fundamental results about intrinsic regular hypersurfaces: their proofs can be found in [55].

Theorem 3.1.8. [Blow-up Theorem] *Let Ω be an open set in $\mathbb{H}^n$ and let $E \subset \mathbb{H}^n$ be such that $\partial E \cap \Omega = S \cap \Omega$ where $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurface. If $P_0 \in S$ and $r > 0$ put*

$$E_{P_0,r} := \delta_{1/r}(P_0^{-1} \cdot E) = \{P \in \mathbb{H}^n : \delta_r(P_0^{-1} \cdot P) \in E\}.$$

Then there is a $c(n) > 0$ such that

i $\displaystyle\lim_{r\to 0} |\partial E_{P_0,r}|_{\mathbb{H}}(U_\infty(0, 1)) = \lim_{r\to 0} \frac{|\partial E|_{\mathbb{H}}(U_\infty(P_0, r))}{r^{2n+1}} = \mathcal{H}^{2n}(T^g_{\mathbb{H}} S(P_0) \cap U_\infty(0, 1)) = c(n);$

ii $|\partial E|_{\mathbb{H}} \llcorner \Omega = c(n)\, \mathcal{S}_\infty^{Q-1} \llcorner (S \cap \Omega)$.

Theorem 3.1.9. [Whitney Extension Theorem] *Let $F \subset \mathbb{H}^n$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to H\mathbb{H}^n$ be two continuous functions. We set*

$$R(Q,P) := \frac{f(Q) - f(P) - \langle k(P), \pi_P(P^{-1} \cdot Q)\rangle_P}{d(P,Q)},$$

and, if $K \subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(Q,P)| : P, Q \in K,\ 0 < d_\infty(P,Q) < \delta\}.$$

If $\rho_K(\delta) \to 0$ as $\delta \to 0$ for every compact set $K \subset F$, then there exist $\tilde{f} : \mathbb{H}^n \to \mathbb{R}$, $\tilde{f} \in C^1_{\mathbb{H}}(\mathbb{H}^n)$ such that $\tilde{f}_{|F} \equiv f$ and $\nabla_{\mathbb{H}} \tilde{f}_{|F} \equiv k$.

Let us introduce some useful subspaces of $\mathfrak{h}_n$ (here $\widehat{X}_j$ means that in an enumeration we omit X_j):

$$\begin{aligned}
&\mathfrak{o} := \mathrm{span}\{X_1, \dots, X_{2n}\};\\
&\mathfrak{v}_j := \mathrm{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}, T\} \quad (1 \le j \le 2n);\\
&\mathfrak{o}_j := \mathrm{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}\} \quad (1 \le j \le 2n);\\
&\mathfrak{l}_j := \mathrm{span}\{X_j\} \quad (1 \le j \le 2n);\\
&\mathfrak{z} := \mathrm{span}\{T\}
\end{aligned}$$

and let $\pi_{\mathfrak{o}}, \pi_{\mathfrak{v}_j}, \pi_{\mathfrak{o}_j}, \pi_{\mathfrak{l}_j}, \pi_{\mathfrak{z}}$ be the projections of $\mathfrak{h}_n$ onto $\mathfrak{o}, \mathfrak{v}_j, \mathfrak{o}_j, \mathfrak{l}_j$ and $\mathfrak{z}$ respectively. Define the following subsets of $\mathbb{H}^n$:

$$\begin{aligned}
&\mathbb{O} := \exp(\mathfrak{o}) = \{P \in \mathbb{H}^n : p_{2n+1} = 0\};\\
&\mathbb{V}_j := \exp(\mathfrak{v}_j) = \{P \in \mathbb{H}^n : p_j = 0\};\\
&\mathbb{O}_j := \exp(\mathfrak{o}_j) = \mathbb{O} \cap \mathbb{V}_j = \{P \in \mathbb{H}^n : p_j = p_{2n+1} = 0\};\\
&\mathbb{L}_j := \exp(\mathfrak{l}_j) = \{P \in \mathbb{H}^n : p_i = 0\ \forall i \ne j\};\\
&\mathbb{T} := \exp(\mathfrak{z}) = \{P \in \mathbb{H}^n : p_1 = \dots = p_{2n} = 0\}.
\end{aligned}$$

and let $\pi_{\mathbb{O}}, \pi_{\mathbb{V}_j}, \pi_{\mathbb{O}_j}, \pi_{\mathbb{L}_j}$ and $\pi_{\mathbb{T}}$ be the maps defined by $\exp \circ \pi_{\mathfrak{o}} \circ \exp^{-1}$, $\exp \circ \pi_{\mathfrak{v}_j} \circ \exp^{-1}$ and so on; we will refer to them as orthogonal projections of $\mathbb{H}^n$ on $\mathbb{O}, \mathbb{V}_j, \mathbb{O}_j, \mathbb{L}_j$ and $\mathbb{T}$.

The following properties of these projections are straightforward:

Proposition 3.1.10. *For any $P, Q \in \mathbb{H}^n$ we have*

$$\begin{aligned}
&\pi_{\mathbb{O}_1}(P) = \pi_{\mathbb{O}} \circ \pi_{\mathbb{V}_1}(P) = \pi_{\mathbb{V}_1} \circ \pi_{\mathbb{O}}(P)\\
&\pi_{\mathbb{O}_1}(P \cdot Q) = \pi_{\mathbb{O}_1}(\pi_{\mathbb{O}_1}(P) \cdot \pi_{\mathbb{O}_1}(Q))\\
&\pi_{\mathbb{T}}(P \cdot Q) = \pi_{\mathbb{T}}(P) \cdot \pi_{\mathbb{T}}(Q) \cdot \pi_{\mathbb{T}}(\pi_{\mathbb{O}}(P) \cdot \pi_{\mathbb{O}}(Q))\\
&\|\pi_{\mathbb{M}}(P)\|_\infty \le \|P\|_\infty \quad \forall \mathbb{M} \in \{\mathbb{O}, \mathbb{O}_1, \mathbb{V}_1, \mathbb{L}_1, \mathbb{T}\}
\end{aligned}$$

Remark 3.1.11. Let us observe that $\mathbb{T}$ is the center of the group, and that only $\mathbb{T}, \mathbb{L}_j$ and $\mathbb{V}_j$ are subgroups; $\mathbb{O}_j$ is a subgroup only if $n = 1$ (because in this case it coincides with $\mathbb{L}_j$), while $\mathbb{O}$ is never a subgroup. We agree to indicate with αe_j the point $\exp(\alpha X_j) \in \mathbb{L}_j$; then for each $P \in \mathbb{H}^n$ there is a unique way to write P in the form $P_{\mathbb{V}_j} \cdot P_{\mathbb{L}_j}$ for points $P_{\mathbb{V}_j} \in \mathbb{V}_j, P_{\mathbb{L}_j} \in \mathbb{L}_j$: it is sufficient to take $P_{\mathbb{L}_j} = p_j e_j$ and $P_{\mathbb{V}_j} = P \cdot P_{\mathbb{L}_j}^{-1} \in \mathbb{V}_j$.

There is a natural identification between $\mathbb{V}_j$ and $\mathbb{R}^{2n}$ given by a diffeomorphism

$$\iota : \mathbb{R}^{2n} \longrightarrow \mathbb{V}_j \subset \mathbb{H}^n \tag{3.1}$$

Without loss of generality we can assume $j = 1$ and define when $n = 1$ as

$$\iota(\eta, \tau) = (0, \eta, \tau), \tag{3.2}$$

while for $n \geq 2$ and $(\eta, v, \tau) \in \mathbb{R}^{2n} \equiv \mathbb{R}_\eta \times \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau$ ι is defined as

$$\iota((\eta, v, \tau)) = (0, v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau), \tag{3.3}$$

where $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n})$. In this way we can introduce the notion of intrinsic graph in $\mathbb{H}^n$.

Definition 3.1.12. *A set $S \subset \mathbb{H}^n$ is an X_1-graph if there is a function $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ such that $S = G^1_{\mathbb{H},\phi}(\omega) := \{\iota(A) \cdot \phi(A)e_1 : A \in \omega\}$.*

More generally, after fixing an identification $\iota_j : \mathbb{R}^{2n} \to \mathbb{V}_j$, for $j = 2, \dots, 2n$ we can define X_j-graphs as those subsets S of $\mathbb{H}^n$ for which there exists a function $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ such that $S = \{\iota_j(A) \cdot \phi(A)e_j : A \in \omega\}$. Moreover the notion of X_1- graph is not a pointless generalization. For instance there are $\mathbb{H}$-regular X_1- graphs in $\mathbb{H}^1$ which are not Euclidean graphs (see [58], Example 3.9).

Let us recall the following results proved in [55].

Theorem 3.1.13. [Implicit Function Theorem] *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in C^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f(0) > 0$, $f(0) = 0$. Let*

$$\begin{aligned} E &:= \{(z,t) \in \Omega : f(z,t) < 0\} \\ S &:= \{(z,t) \in \Omega : f(z,t) = 0\}; \end{aligned}$$

then there exist a connected open neighborhood $\mathcal{U}$ of 0 such that

E has finite $\mathbb{H}$-perimeter in $\mathcal{U}$;
$\partial E \cap \mathcal{U} = S \cap \mathcal{U}$;
$\nu_E(P) = -\nabla_{\mathbb{H}} f(P)/|\nabla_{\mathbb{H}} f(P)|_P = \nu_S(P)$ for all $P \in S \cap \mathcal{U}$.

Moreover there exists a unique continuous function $\phi : \omega := [-\delta, \delta] \times [-\delta, \delta]^{2n-2} \times [-\delta^2, \delta^2] \subset \mathbb{R}^{2n} \to [-h, h]$ *such that* $S \cap \overline{\mathcal{U}} = \Phi(\omega)$*, where* $\delta, h > 0$ *and* Φ *is the map defined as* $\Phi(\eta, v, \tau) = \iota(\eta, v, \tau) \cdot \phi(\eta, v, \tau)e_1$*,* $(\eta, v, \tau) \in \omega$*; given explicitly by*

$$\begin{aligned} &\Phi(\eta, v, \tau) = \Big(\phi(\eta, v, \tau), v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau - \frac{\eta}{2}\phi(\eta, v, \tau)\Big) && \textit{if } n \geq 2 \\ &\Phi(\eta, \tau) = \Big(\phi(\eta, \tau), \eta, \tau - \frac{\eta}{2}\phi(\eta, \tau)\Big) && \textit{if } n = 1. \end{aligned} \tag{3.4}$$

The $\mathbb{H}$*-perimeter has the integral representation*

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_\omega \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\Phi(A))\, d\mathcal{L}^{2n}(A). \tag{3.5}$$

Remark 3.1.14. We can rewrite Theorem 3.1.13 as follow: Let $S = \{f = 0\}$ be the level set of a $C^1_{\mathbb{H}}$ function f such that $X_1 f > 0$. Then, locally on S, there exists a unique continuous map $\phi : \omega \subset \mathbb{R}^{2n} \simeq \mathbb{V}_1 \to \mathbb{R}$ such that $S = \Phi(\omega)$, where Φ is defined by

$$\Phi(A) := \exp(\phi(A)X_1)(\iota(A)) \tag{3.6}$$

with $A \in \omega$. Moreover, Φ turns out to be an homeomorphism.

Notice the formal analogy of Theorem 3.1.13 with classical Implicit Function Theorem in $\mathbb{R}^n$: in that setting, in fact, a C^1 surface was seen as graph of a function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. The construction of the graph of g works on these terms: start from a point A on $\mathbb{R}^{n-1}$ ($\mathbb{R}^{n-1}$ plays the role of $\mathbb{V}_1$ - again, a maximal subgroup of $\mathbb{R}^n$) and follow the orthogonal direction (the analogous of our X_1 direction) for a length $g(A)$. The point you reach is the graph of g over A: exactly what done in (3.6). Compare also figure 3.1

Let us recall by [69] a regularity result for the parametrization Φ of an $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$.

Theorem 3.1.15. *Let* $S \subset \mathbb{H}^1$ *be an* $\mathbb{H}$*-regular surface. Let* $\Phi : \omega \subset \mathbb{R}^2 \to \mathbb{H}^1$ *be the locally parametrization of* S *in* (3.4)*, then for each* $P_0 \in S$ *there exist* $\delta, L > 0$ *and an open neighborhood* $\mathcal{U}$ *of* P_0 *such that*

$$\Phi\big([-\delta, \delta] \times [-\delta^2, \delta^2]\big) = S \cap \bar{\mathcal{U}}$$

$$d_\infty\left(\Phi(u), \Phi(v)\right) \leq L|u - v|^{\frac{1}{2}} \qquad \forall\, u, v \in [-\delta, \delta] \times [-\delta^2, \delta^2]. \tag{3.7}$$

Moreover the $\mathbb{H}$*-regular surface* $S = \{(x, y, t) \in \mathbb{H}^1 :\ x = 0\}$ *cannot be locally parametrized by means of any Hölder continuous map of order* $\frac{1}{2} < \alpha \leq 1$*.*

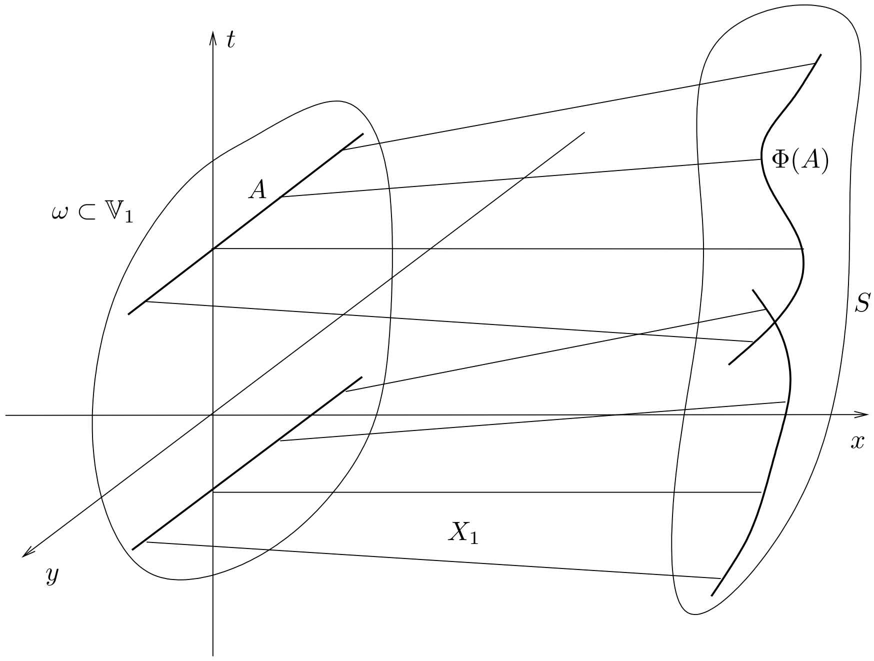


Figure 3.1: Intrinsic graphs.

Since $\mathbb{V}_1$ is a subgroup of $\mathbb{H}^n$ closed with respect to the dilations in (2.2), $\mathbb{R}^{2n}$ can be endowed through the identification ι by a structure of homogeneous group in the sense of Folland and Stein (see [53]), i.e. we can define a group law in $\mathbb{R}^{2n}$

$$A \star B := \iota^{-1}(\iota(A) \cdot \iota(B)) \quad A, B \in \mathbb{R}^{2n} \tag{3.8}$$

and a family of intrinsic dilations $\delta^\star_\lambda : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(\lambda > 0)$

$$\delta^\star_\lambda(A) := \iota^{-1}(\delta_\lambda(\iota(A)) \quad \in \mathbb{R}^{2n} \tag{3.9}$$

such that $(\mathbb{R}^{2n}, \star, \delta^\star_\lambda)$ turns out to be a homogeneous group.

Explicitly, if $n > 1$ and $A = (\eta, v, \tau), B = (\eta', v', \tau') \in \mathbb{R}^{2n}$ we have

$$A \star B = (\eta + \eta', v + v', \tau + \tau' + \mathfrak{c}(v', v)) \tag{3.10}$$

where

$$\mathfrak{c}(v', v) = +\frac{1}{2}\sum_{j=2}^{n}(v_{n+j}v'_j - v_j v'_{n+j}) \tag{3.11}$$

with $v = (v_2, \dots, v_n, v_{n+2}, \dots v_{2n})$, $v' = (v'_2, \dots, v'_n, v'_{n+2}, \dots v'_{2n})$. If $n = 1$ and $A = (\eta, \tau), B = (\eta', \tau') \in \mathbb{R}^2$ we have

$$A \star B = (\eta + \eta', \tau + \tau'). \tag{3.12}$$

The dilations become explicitly

$$\begin{array}{ll} \delta^\star_\lambda(\eta, v, \tau) = (\lambda\eta, \lambda v, \lambda^2\tau) & \text{for } n \geq 2 \\ \delta^\star_\lambda(\eta, \tau) = (\lambda\eta, \lambda^2\tau) & \text{for } n = 1. \end{array}$$

Notice that in both cases the induced group structure is the one arising from direct product $\mathbb{R} \times \mathbb{R}$ if $n = 1$, and $\mathbb{R} \times \mathbb{H}^{n-1}$ if $n > 1$, via the identification $\mathbb{R}^{2n} = \mathbb{R}_\eta \times (\mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau) = \mathbb{R} \times \mathbb{H}^{n-1}$.

We define a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ as a homomorphism which is also positively homogeneous of degree 1 with respect to the dilations, i.e. $L \circ \delta^\star_\lambda = \lambda L$. The following Proposition comes from Proposition 5.4 in [55]:

Proposition 3.1.16. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}$ be a $\star$-linear functional; then there is a unique vector $w_L \in \mathbb{R}^{2n-1}$ such that $L(A) = \langle A, w_L \rangle$, where we intend that*

$$\begin{array}{ll} \langle A, w_L \rangle = \eta w_{L n+1} + \displaystyle\sum_{\substack{j=2 \\ j \neq n+1}}^{2n} v_j w_{Lj} & \text{if } n \geq 2, w_L = (w_{L2}, \dots, w_{L2n}) \text{ and } A = (\eta, v, \tau) \\ \langle A, w_L \rangle = \eta w_{L2} & \text{if } n = 1, w_L = w_{L2} \text{ and } A = (\eta, \tau). \end{array}$$

Conversely, through the previous formulas we can associate to each $w \in \mathbb{R}^{2n-1}$ a unique $\star$-linear functional L_w.

Observe that the choice of the enumeration of the components of w_L has been made in order to be coherent with the one made for the components of v and with the fact that η is the $(n + 1)$-th coordinate of $\iota(A)$.

For $n \geq 2$ the tangent space of $\mathbb{V}_1$ is linearly generated by the restrictions of $X_2, ..., Y_1, ..., Y_n, T$ to $\mathbb{V}_1$ and so we can define the vector fields $\widetilde{X}_2, ..., \widetilde{Y}_1, ..., \widetilde{Y}_n, \widetilde{T}$ on $\mathbb{R}^{2n}$ given by

$$\widetilde{X}_j := (\iota^{-1})_* X_j, \qquad \widetilde{Y}_j := (\iota^{-1})_* Y_j, \qquad \widetilde{T} := (\iota^{-1})_* T,$$

where $(\iota^{-1})_*$ is the usual push forward of vector fields after the diffeomorphism ι^{-1}. In coordinates, for $j = 2, ..., n$,

$$\widetilde{X}_j(\eta, v, \tau) = \frac{\partial}{\partial v_j} - \frac{v_{j+n}}{2} \frac{\partial}{\partial t}, \tag{3.13}$$

$$\widetilde{Y}_1 = \frac{\partial}{\partial \eta},$$

$$\widetilde{Y}_j(\eta, v, \tau) = \frac{\partial}{\partial v_{j+n}} + \frac{v_j}{2} \frac{\partial}{\partial t},$$

$$\widetilde{T} = \frac{\partial}{\partial \tau}.$$

If $n = 1$ the tangent space to $\mathbb{V}_1$ is generated by the restriction of Y_1 and T to $\mathbb{V}_1$, we can so define

$$\widetilde{Y}_1(\eta, \tau) := (\iota^{-1})_* Y_1 = \frac{\partial}{\partial \eta},$$

$$\widetilde{T}(\eta, \tau) := (\iota^{-1})_* T = \frac{\partial}{\partial \tau}.$$

It follows from definition that $\widetilde{X}_j, \widetilde{Y}_j, \widetilde{T}$ are $\star$-left-invariant.

Let us introduce the nonlinear differential operator

$$C^1(\omega) \ni \phi \to \mathfrak{B}\phi, \tag{3.14}$$

where $\mathfrak{B}\phi$ is a Burgers' type operator which can be represented in distributional form as

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} + \frac{1}{2}\frac{\partial \phi^2}{\partial \tau}. \tag{3.15}$$

With this notations let us provide an improvement of Theorem 3.1.13:

Theorem 3.1.17. *Under the same assumption of Theorem* 3.1.13, *let* $\widetilde{X}_j, \widetilde{Y}_j$ *be the vector fields defined in* (3.13) *and let* $\mathfrak{B}\phi$ *the distribution in* (3.15) *on* ω, *where* ϕ *and* ω *are given by Theorem* 3.1.13. *Then if* $n = 1$

$$\mathfrak{B}\phi = -\frac{Y_1 f}{X_1 f} \circ \Phi,$$

if $n \geq 2$

$$\widetilde{X}_j \phi = -\frac{X_j f}{X_1 f} \circ \Phi, \quad \widetilde{Y}_j \phi = -\frac{Y_j f}{X_1 f} \circ \Phi, \quad \mathfrak{B}\phi = -\frac{Y_1 f}{X_1 f} \circ \Phi$$

where the equalities must be understood in distributional sense on ω.
Moreover, the $\mathbb{H}$*-perimeter has the integral representation*

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = c(n)\mathcal{S}_\infty^{\mathcal{Q}-1} \llcorner (S \cap \mathcal{U}) = \int_\omega \sqrt{1 + (\mathfrak{B}\phi)^2 + \sum_{j=2}^n [|\widetilde{X}_j \phi|^2 + |\widetilde{Y}_j \phi|^2]}\, d\mathcal{L}^{2n}. \tag{3.16}$$

If $n = 1$ *we have simply*

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = c(1)\mathcal{S}_\infty^{\mathcal{Q}-1} \llcorner (S \cap \mathcal{U}) = \int_\omega \sqrt{1 + |\mathfrak{B}\phi|^2}\, d\eta\, d\tau.$$

In agreement with (3.13) and Theorem 3.1.17 let $\phi : \omega \to \mathbb{R}$ be a given function; we will indicate with ∇^ϕ the family of first-order operators $(\nabla^\phi_2, \dots, \nabla^\phi_{2n})$ defined for $n \geq 2$ by

$$\nabla^\phi_j := \begin{cases} \widetilde{X}_j = \dfrac{\partial}{\partial v_j} - \dfrac{v_{j+n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } 2 \leq j \leq n \\ \widetilde{Y}_1 + \phi \widetilde{T} = \dfrac{\partial}{\partial \eta} + \phi \dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \widetilde{Y}_{j-n} = \dfrac{\partial}{\partial v_j} + \dfrac{v_{j-n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \leq j \leq 2n, \end{cases} \tag{3.17}$$

while for $n = 1$ we put $\nabla^\phi = \nabla^\phi_2 := \widetilde{Y}_1 + \phi \widetilde{T} = \dfrac{\partial}{\partial \eta} + \phi \dfrac{\partial}{\partial \tau}$.
We will denote $\nabla^\phi_{n+1} := W^\phi$ and $\widetilde{\nabla}_{\mathbb{H}} := (\widetilde{X}_2, ..., \widetilde{X}_n, \widetilde{Y}_2, ..., \widetilde{Y}_n)$.

3.2 ∇^ϕ-differentiability

In this section we will recall the main results of [4, 94]: the ∇^ϕ-differentiability and its relation with $\mathbb{H}$-regular hypersurfaces. For the proof see [4, 94].

Let ω be an open, connected and bounded subset of $\mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$ if $n \geq 1$, of $\mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n = 1$ and let $\phi : \omega \to \mathbb{R}$ be a given function. Moreover let us define if $n \geq 2$ and $A_0 = (\eta_0, v_0, t_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$

$$\begin{aligned} I_r(A_0) &:= \{(\eta, v, \tau) \in \mathbb{R}^{2n} : \, |\eta - \eta_0| < r, \, |v - v_0| < r, \, |\tau - \tau_0| < r\} = \\ &= (\eta_0 - r, \eta_0 + r) \times U(v_0, r) \times (\tau_0 - r, \tau_0 + r) \end{aligned}$$

where $U(v_0, r)$ denotes the Euclidean open ball centered at v_0 with radius $r > 0$ in $\mathbb{R}^{2n-2}$, and if $n = 1$ and $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$

$$I_r(A_0) := \{(\eta, \tau) \in \mathbb{R}^2 : \, |\eta - \eta_0| < r, \, |\tau - \tau_0| < r\} = (\eta_0 - r, \eta_0 + r) \times (\tau_0 - r, \tau_0 + r)$$

Definition 3.2.1. *For $A, B \in \omega$ we define the graph distance*

$$\rho_\phi(A, B) := \|\pi_{\mathbb{O}_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty + \|\pi_{\mathbb{T}}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty \tag{3.18}$$

Explicitly, if $n \geq 2$ and $A = (\eta, v, \tau), \; B = (\eta', v', \tau') \in \omega$ we have

$$\rho_\phi(A, B) = |(\eta', v') - (\eta, v)| + \left| \tau' - \tau - \frac{1}{2}(\phi(B) + \phi(A))(\eta' - \eta) + \mathfrak{c}(v, v') \right|^{1/2};$$

if $n = 1$ and $A = (\eta, \tau), B' = (\eta', \tau') \in \omega$ we have

$$\rho_\phi(A, B) = |\eta' - \eta| + \left| \tau' - \tau - \frac{1}{2}(\phi(B) + \phi(A))(\eta' - \eta) \right|^{1/2}.$$

Proposition 3.2.2. *If there is an $L > 0$ such that*

$$|\phi(A) - \phi(B)| \leq L\, \rho_\phi(A, B) \tag{3.19}$$

for all $A, B \in \omega$, then the quantity ρ_ϕ in (3.18) is a quasimetric on ω, id est

i $\rho_\phi(A, B) = 0 \iff A = B$;

ii $\rho_\phi(A, B) = \rho_\phi(B, A)$;

iii *there exists $q > 1$ such that $\rho_\phi(A, B) \leq q\, [\, \rho_\phi(A, C) + \rho_\phi(C, B)\,]$*

for all $A, B, C \in \omega$.

Remark 3.2.3. The distance ρ_ϕ is equivalent to the metric d_∞ restricted to the graph S, i.e. there exists a constant $C > 0$ such that

$$\frac{1}{C}\, \rho_\phi(A, B) \leq d_\infty\left(\Phi(A), \Phi(B)\right) \leq C \rho_\phi(A, B) \qquad \forall\, A, B \in \omega.$$

Definition 3.2.4. *Let $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ be a fixed continuous function, and let $A \in \omega$ and $\psi : \omega \to \mathbb{R}$ be given.*

i *We say that ψ is ∇^ϕ-differentiable at A if there is a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ such that*

$$\lim_{B \to A} \frac{|\psi(B) - \psi(A) - L(A^{-1} \star B)|}{\rho_\phi(A, B)} = 0. \tag{3.20}$$

ii *We say that ψ is uniformly ∇^ϕ-differentiable at A if there is a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ such that, if we define*

$$M_\phi(\psi, A, L, r) := \sup_{\substack{B, B' \in I_r(A) \\ B \neq B'}} \left\{ \frac{|\psi(B') - \psi(B) - L(B^{-1} \star B')|}{\rho_\phi(B, B')} \right\} \tag{3.21}$$

then $\lim_{r \downarrow 0} M_\phi(\psi, A, L, r) = 0$.

Remark 3.2.5. If ψ is ∇^ϕ-differentiable at A, then it is continuous at A. Indeed, if $L : \mathbb{R}^{2n} \to \mathbb{R}$ is such that (3.20) holds and w_L is as in Proposition 3.1.16, then for any $B \in \omega$

$$\psi(B) - \psi(A) = \frac{\psi(B) - \psi(A) - \langle w_L, A^{-1} \star B \rangle}{\rho_\phi(A, B)} \cdot \rho_\phi(A, B) + \langle w_L, A^{-1} \star B \rangle$$

and we deduce the continuity of ψ at A from the ∇^ϕ-differentiability at A together with the fact that $\rho_\phi(A, B)$ is bounded near A.

Remark 3.2.6. We stress the fact that if $\psi : \omega \to \mathbb{R}$ is uniformly ∇^ϕ-differentiable at $A \in \omega$, then ψ is Lipschitz continuous (between the spaces (ω, ρ_ϕ) and $(\mathbb{R}, d_{eucl})$) in a neighborhood of A; in fact there exist $C, r > 0$ such that

$$\frac{|\psi(B) - \psi(A) - L(A^{-1} \star B)|}{\rho_\phi(A, B)} \leq C$$

for all $B \in I_r(A)$, whence

$$|\psi(B) - \psi(A)| \leq |\langle w_L, A^{-1} \star B\rangle| + C\rho_\phi(A, B) \leq (|w_L| + C)\rho_\phi(A, B)$$

We will indicate the $\star$-linear functional L such that (3.20) holds by $d_{\nabla^\phi}\psi(A)$; we will call the vector w_L the ∇^ϕ-differential of ψ at A, and we will indicate it by $\nabla^\phi\psi(A)$, writing $\nabla^\phi_j\psi(A)$ for w_{Lj}, $j = 2, \dots, 2n$. These definitions are well posed because of the following

Lemma 3.2.7. *Let $\phi, \psi : \omega \to \mathbb{R}$ be such that ψ is ∇^ϕ-differentiable at $A \in \omega$, and let L be a $\star$-linear functional such that* (3.20) *holds; then L is unique.*

Uniformly ∇^ϕ-differentiable functions have continuous ∇^ϕ-differentials:

Proposition 3.2.8. *Let $\phi, \psi : \omega \to \mathbb{R}$ be two continuous functions; suppose that there exists an $A \in \omega$ such that ψ in uniformly ∇^ϕ-differentiable at A and that ψ is ∇^ϕ-differentiable in an open neighborhood $\mathcal{U}$ of A. Then $\nabla^\phi\psi : \mathcal{U} \to \mathbb{R}^{2n-1}$ is continuous at A.*

Remark 3.2.9. The inverse proposition could be false: $\nabla^\phi\psi \in C^0(\omega)$, does not infer that ψ is ∇^ϕ-differentiable, as we can see this counterexample done with V. Magnani.
Let $\omega := (-\delta, \delta) \times (-\delta, \delta)$ $\phi \equiv 0$ on ω and $\psi = \eta + g(\tau)$ with $g \notin C^0(-\delta, \delta)$, then $\nabla^\phi\psi = W^\phi\psi = \psi_\eta + 0 = 1 \in C^0(\omega)$, but $\psi \notin C^0(\omega)$ and therefore ψ is not ∇^ϕ-differentiable.

Proposition 3.2.10. *Let $\phi, \psi : \omega \to \mathbb{R}$ be continuous functions such that ψ is ∇^ϕ-differentiable at a point $A = (\eta, v, \tau) \in \omega$ (respectively $A = (\eta, \tau)$ if $n = 1$). For $j = 2, \dots, 2n$ let $\gamma_j : [-\delta, \delta] \to \omega$ be a C^1-integral curve of the vector field ∇^ϕ_j with $\gamma_j(0) = A$ and such that the map*

$$[-\delta, \delta] \ni s \longmapsto \phi(\gamma_j(s)) \in \mathbb{R}$$

is of class C^1. Then we have

$$\lim_{s\to 0} \frac{\psi(\gamma_j(s)) - \psi(\gamma_j(0))}{s} = \nabla^\phi_j\psi(A). \tag{3.22}$$

Theorem 3.2.11. *Let $\phi, \psi \in C^1(\omega)$; then ψ is uniformly ∇^ϕ-differentiable at A for all $A \in \omega$ and*

$$\nabla^\phi \psi(A) = \left(\widetilde{X}_2 \psi, \dots, \widetilde{X}_n \psi, \frac{\partial \psi}{\partial \eta} + \phi \frac{\partial \psi}{\partial \tau}, \widetilde{Y}_2 \psi, \dots, \widetilde{Y}_n \psi \right)(A)$$

for all $A \in \omega$. In particular, $\nabla^\phi \psi : \omega \to \mathbb{R}^{2n-1}$ is continuous.

Let us recall now the main Theorem of [4] and [94], that characterize the relations between $\mathbb{H}$-regular graphs and ∇^ϕ-differentiability:

Theorem 3.2.12. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function and let $\Phi : \omega \to \mathbb{H}^n$ be the function defined by $\Phi(A) := \iota(A) \cdot \phi(A) e_1$. Let $S := \Phi(\omega)$. Then the following conditions are equivalent:*

i *S is an $\mathbb{H}$-regular surface and $\nu_S^{(1)}(P) < 0$ for all $P \in S$, where $\nu_S(P) = \left(\nu_S^{(1)}(P), \dots, \nu_S^{(2n)}(P)\right)$ is the horizontal normal to S at a point $P \in S$;*

ii *ϕ is uniformly ∇^ϕ-differentiable at any $A \in \omega$ and the vector function $\nabla^\phi \phi : \omega \to \mathbb{R}^{2n-1}$ is continuous.*

Moreover, for every $P \in S$

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right)(\Phi^{-1}(P)), \tag{3.23}$$

and

$$\mathcal{S}_\infty^{Q-1}(S) = c(n) \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n} \tag{3.24}$$

where $\mathcal{L}^{2n}$ denotes the Lebesgue $2n$-dimensional measure on $\mathbb{R}^{2n}$ and $c(n)$ a positive constant depending only on n.

Remark 3.2.13. The parametrization ϕ of an $\mathbb{H}$-regular hypersurface S is regular in the sense of uniform ∇^ϕ-differentiability along the directions $\widetilde{X}_j$, $\widetilde{Y}_1 + \phi \widetilde{T}$, $\widetilde{Y}_j$. These vector fields possess a precise relationship with the structure of S, since they corresponds to the horizontal directions of S. Suppose for a moment ϕ to be C^1; then the intersection of the tangent space to S with the horizontal layer is a $2n-1$ dimensional space, since the construction in (3.6) prevents the occurrence of characteristic points. The images on the surface of this family of vector fields form a basis of the intersection of tangent and horizontal space.

3.3 ∇^{ϕ}-exponential maps and characterization of the uniform ∇^{ϕ}-differentiability

Let us recall some results of [4, 18, 94], that will be crucial in our work:

Definition 3.3.1. *Let $\phi : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ be continuous functions. We call ϕ a broad* solution of the system*

$$\nabla^{\phi}\phi = w \qquad \text{in } \omega \tag{3.25}$$

if for every $A \in \omega$, $\forall\, j = 2, ..., 2n$ there exists a map, we will call exponential map,

$$\exp(\cdot\nabla_j^{\phi})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$$

where $0 < \delta_2 < \delta_1$, such that if $\gamma_j^B(s) = \exp(s\nabla_j^{\phi})(B)$,

(E.1) $\gamma_j^B \in C^1([-\delta_2, \delta_2])$

(E.2) $\begin{cases} \dot{\gamma}_j^B = \nabla_j^{\phi} \circ \gamma_j^B \\ \gamma_j^B(0) = B \end{cases}$

(E.3) $\phi\left(\gamma_j^B(s)\right) - \phi\left(\gamma_j^B(0)\right) = \displaystyle\int_0^s w_j\left(\gamma_j^B(r)\right)\, dr$

$\forall\, B \in I_{\delta_2}(A)$, $\forall\, j = 2, ..., 2n$.

Remark 3.3.2. When $n = 1$, then $\nabla^{\phi}\phi = \mathfrak{B}\phi = \dfrac{\partial\phi}{\partial\eta} + \phi\dfrac{\partial\phi}{\partial\tau}$ and this definition extends the notion of broad solution for the Burgers equation $\mathfrak{B}\phi = w$ given in [21] (see Definition 2.3) provided $\phi, w : \omega \to \mathbb{R}$ are (locally) Lipschitz continuous. In our case ϕ and w are supposed to be only continuous, then the classical theory of solutions for ODEs breaks down and the notion of broad solution does not apply (see [49] for an interesting account of this subject and its recent developments).

Remark 3.3.3. Notice that if the exponential maps of ∇^{ϕ} at A exist, then the map

$$[-\delta_2, \delta_2] \ni s \longmapsto \phi\left(\exp_A\left(s\nabla_j^{\phi}\right)(B)\right)$$

is of class C^1 for each $j \in \{2, \dots, 2n\}$ and each $B \in I_{\delta_2}(A)$.

Remark 3.3.4. Observe that, because of the left invariance of the fields $\widetilde{X}_j$, for $j \neq n+1$ one must have

$$\exp_A\left(s\nabla_j^\phi\right)(B) = B \star \iota^{-1}\left(\exp s\widetilde{X}_j\right) = B \star \iota^{-1}(s\, e_j). \tag{3.26}$$

Moreover, if there are the exponential maps of ∇^ϕ at A (in particular there are w_j as in $(E.3)$), then for any $\lambda = (\lambda_2, \dots, \lambda_n, \lambda_{n+2}, \dots, \lambda_{2n}) \in \mathbb{R}^{2n-2}$ there exists also an exponential map for the field $\sum \lambda_j \nabla_j^\phi$, i.e. there are two continuous maps $\gamma_\lambda : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$ (with, possibly, a $\delta_2 > 0$ smaller than the one in $(E.2)$, depending on λ) and $w_\lambda : \omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\dot{\gamma_\lambda}(s,B) = \sum \lambda_j \nabla_j^\phi(\gamma_\lambda(s,B))\\
&\gamma_\lambda(0,B) = B\\
&\phi(\gamma_\lambda(s,B)) - \phi(\gamma_\lambda(0,B)) = \int_0^s w_\lambda(\gamma(r,B))\, dr
\end{aligned}$$

In fact, it is sufficient to take $\gamma_\lambda(s,B) := B \star (0, s\lambda, 0)$ and $w_\lambda := \sum \lambda_j w_j$.

The following Lemma provides sufficient conditions to guarantee the existence of exponential maps of ∇^ϕ.

Lemma 3.3.5. *Let $\phi : \omega \to \mathbb{R}$ be continuous, and suppose that*

i *there exists $w \in C^0(\omega)$ such that, in distributional sense,*

$$\begin{aligned}
&w = (w_2, \dots, w_{2n}) = \left(\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{X}_{n+2}\phi, \dots, \widetilde{X}_{2n}\phi\right) && \text{if } n \geq 2\\
&w = \mathfrak{B}\phi && \text{if } n = 1
\end{aligned}$$

ii *there is a family of functions $\{\phi_\epsilon\}_{\epsilon>0} \subset C^1(\omega)$ such that for each $\omega' \Subset \omega$ we have*

$$\phi_\epsilon \to \phi,\ \nabla^{\phi_\epsilon}\phi_\epsilon \to w \quad \text{uniformly on } \overline{\omega'}.$$

Then for each $A \in \omega$ there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$, there exists $\exp_A(s\nabla_j^\phi)(B) \in \overline{I_{\delta_1}(A)} \Subset \omega$ for all $(s,B) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$; moreover,

$$w_j(B) = \frac{d}{ds}\phi\left(\exp_A(s\nabla_j^\phi)(B)\right)_{|s=0}$$

for each $B \in I_{\delta_2}(A)$.

Remark 3.3.6. Let us explicitly stress that both the uniqueness and the global continuity of the exponential maps

$$\exp(\cdot\nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)}$$

are not guaranteed provided only ϕ, w are continuous. Indeed notice that Definition 3.3.1 is not asking the exponential maps to be continuous in the parameter B, see for instance the following example of [94], Remark 4.34.

Example 3.3.7. Let us consider the function $\mathbb{R}^2 \to \mathbb{R}$

$$\phi(\eta,\tau) := \begin{cases} -\dfrac{\tau^\alpha}{1-\alpha} & \text{if } \tau \geq 0 \\ 0 & \text{if } \tau < 0 \end{cases} \tag{3.27}$$

For $\frac{1}{2} < \alpha < 1$ the X_1-graph of ϕ is an $\mathbb{H}$-regular surfaces (see Corollary 3.3.10) and

$$\nabla^\phi\phi(\eta,\tau) = W^\phi\phi(\eta,\tau) = \begin{cases} \dfrac{\alpha}{(1-\alpha)^2}\tau^{2\alpha-1} & \text{if } \tau \geq 0 \\ 0 & \text{if } \tau < 0 \end{cases}$$

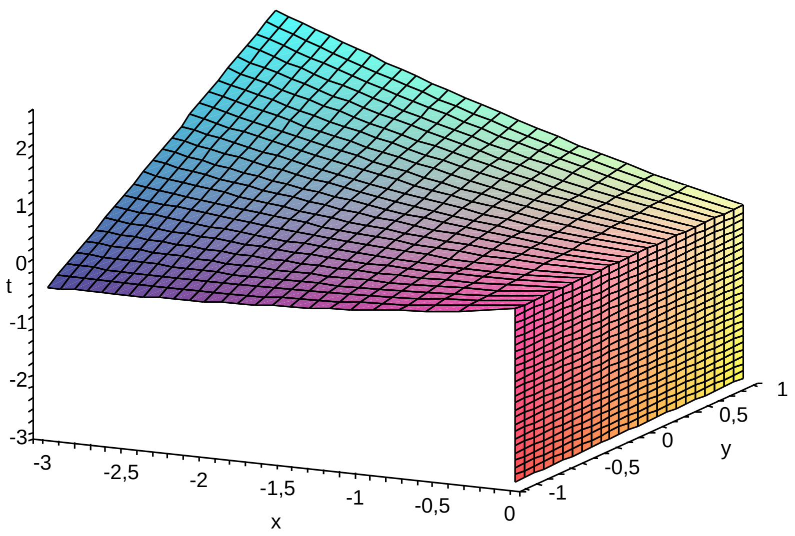


Figure 3.2: The intrinsic graph of the map ϕ in (3.27).

S is the union of two C^1 surfaces

$$S_+ := \big\{(x,y,t) \in \mathbb{H}^n : x = -\tfrac{1}{1-\alpha}\big(t + \tfrac{1}{2}xy\big)^\alpha\big\} \quad \text{and} \quad S_- := \big\{(0,y,t) \in \mathbb{H}^1 : t < 0\big\}$$

glued together along the line $L := \{(0,y,0) : y \in \mathbb{R}\}$. The surface S is not C^1 since the tangent planes to S_+, S_- at a point $(0,y,0)$ are different (see figure 3.2).

Let us notice that the only possible definition of exponential maps is

$$\exp_0(xW^\phi)(0,z) = \begin{cases} \Big(x, (z^{1-\alpha} - x)^{\frac{1}{1-\alpha}}\Big) & \text{if } x \geq 0 \text{ and } z > 0 \\ (x,z) & \text{if } x \geq 0 \text{ and } z < 0 \end{cases}$$

which is not continuous since

$$\lim_{z\to 0^+} \exp_0(xW^\phi)(0,z) = \Big(x, |x|^{\frac{1}{1-\alpha}}\Big) \neq (x,0) = \lim_{z\to 0^-} \exp_0(xW^\phi)(0,z)$$

for any $x < 0$. □

As in Euclidean spaces the gradient of a function is the vector composed by the derivatives along the exponentials of the vectors of the canonical basis, the ∇^ϕ-differential is the vector made by the derivatives along the exponentials of ∇^ϕ, for the proof see [4, 94].

Theorem 3.3.8. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function such that, for a certain $A \in \omega$, the following conditions are fulfilled:*

i *there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$ there exist a family of exponential maps*

$$\exp_A\big(s\nabla_j^\phi\big) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)}.$$

ii *for each $\omega' \Subset \omega$*

$$\lim_{r\to 0^+} \sup\left\{\frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} : B', B \in \overline{\omega'},\ 0 < |B' - B| \leq r\right\} = 0.$$

Then ϕ is uniformly ∇^ϕ-differentiable at A and

$$\big(\nabla_j^\phi \phi\big)(A) = \frac{d}{ds}\phi\big(\exp_A(s\nabla_j^\phi)(A)\big)_{|s=0}.$$

Let us now recall one of the main result of [4, 94], that is fundamental in our work and that give the characterization of the uniform ∇^ϕ-differentiability.

Theorem 3.3.9. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then the following conditions are equivalent:*

i *ϕ is uniformly ∇^ϕ-differentiable at A for each $A \in \omega$;*

ii *the distribution $\nabla^\phi \phi$ is represented by a function $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ and there exists a family $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ such that, for any open set $\omega' \Subset \omega$, we have*

$$\phi_\epsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\epsilon} \phi_\epsilon \to w \text{ uniformly in } \omega'. \tag{3.28}$$

Moreover, for every open set $\omega' \Subset \omega$

$$\lim_{r\to 0^+} \sup \left\{ \frac{|\phi(A) - \phi(B)|}{\sqrt{|A - B|}} : A, B \in \omega', 0 < |A - B| < r \right\} = 0. \tag{3.29}$$

An interesting application of Theorem 3.3.9 provides a simple way to exhibit $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$ which are not Euclidean regular, see [94].

Corollary 3.3.10. *Let $\phi : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which depends only on τ, i.e. $\phi = \phi(\tau) : I \to \mathbb{R}$ for a certain open interval $I \subseteq \mathbb{R}$, and suppose that $\phi^2 : I \to \mathbb{R}$ is of class C^1. Then ϕ is uniformly ∇^ϕ-differentiable at A for every $A \in \omega$ and*

$$\nabla^\phi \phi(A) = \frac{1}{2}(\phi^2)'(A).$$

In particular $\nabla^\phi \phi$ is continuous and ϕ parametrizes an $\mathbb{H}$-regular surface in $\mathbb{H}^1$.

The condition (3.29) is named in the literature little Hölder continuity of order $\frac{1}{2}$. Due to the fact it will be needed in the sequel, let us introduce the associated spaces (see, for instance, [75]).

Definition 3.3.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set open set.*

i *If $\alpha \in (0, 1)$ then let us denote by $h^\alpha(\overline{\Omega})$ the set of functions $f \in C^0(\overline{\Omega})$ such that*

$$\lim_{r\to 0} L_\alpha(\bar{\Omega}, f, r) = 0$$

where

$$L_\alpha(f, \overline{\Omega}, r) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in \overline{\Omega},\ 0 < |x - y| < r \right\} \tag{3.30}$$

We will denote by $L_0(f, \overline{\Omega}, r)$ the modulus of continuity of a function $f \in C^0(\overline{\Omega})$, i.e. the quantity in (3.30) with $\alpha = 0$.

ii *Let us denote by $h^\alpha_{loc}(\Omega)$ the set of function $f \in C^0(\Omega)$ such that $f \in h^\alpha\left(\overline{\Omega'}\right)$ for each open set $\Omega' \Subset \Omega$.*

iii *If $f \in Lip(\Omega)$ let us denote*

$$L_1(f, \overline{\Omega}) := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in \overline{\Omega},\, x \neq y\right\} \tag{3.31}$$

Let us show the following fundamental Hölder continuous regularity result for broad* solutions which extends a previous one given in Theorem 5.8 of [4] for C^1 regular solution ϕ of (3.25).

Theorem 3.3.12. *Let us assume that $\phi : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ are continuous functions and that ϕ is a broad* solution of the system $\nabla^\phi \phi = w$ in ω. Then for each $A_0 \in \omega$ there exist $0 < r_2 < r_1$ and a function $\alpha : (0, +\infty) \to [0, +\infty)$, which depends only on A_0, $\|\phi\|_{L^\infty(I_{r_1}(A_0))}$, $\|w\|_{L^\infty(I_{r_1}(A_0))}$ and on the modulus of continuity of w_{n+1} on $I_{r_1}(A_0)$, such that $\lim_{r\to 0} \alpha(r) = 0$ and*

$$L_{\frac{1}{2}}\left(\phi, \overline{I_{r_2}(A_0)}, r\right) = \sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in \overline{I_{r_2}(A_0)},\, 0 < |A - B| \leq r\right\} \leq \alpha(r) \tag{3.32}$$

for all $r \in (0, r_2)$.

Before the proof of Theorem 3.3.12 let us introduce a key preliminary result.

Lemma 3.3.13. *Let $Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $Q_2 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ with $0 < \delta_2 < \delta_1$. Let $f_i \in C^0(Q_1)$ $(i = 1, 2)$ and $x : Q_2 \to [\tau_0 - \delta_1, \tau_0 + \delta_1]$ be given such that*

i *$x(\cdot, \tau) \in C^2([-\delta_2, \delta_2]) \quad \forall \tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$;*

ii

$$\begin{cases} \dfrac{d^i}{ds^i} x(s, \tau) = f_i(s, x(s, \tau)) & (i = 1, 2) \\ x(0, \tau) = \tau \end{cases}$$

$\forall s \in [-\delta_2, \delta_2]$, $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$.

Then

$$L_{\frac{1}{2}}(g, [\tau_0 - \delta_2, \tau_0 + \delta_2], r) \leq \max\left\{r^{1/4}, 2\sqrt{2\, L_0(f_2, Q_1, r + 2\, c_0\, r^{1/4})}\right\} \tag{3.33}$$

for each $r \in (0, r_0)$*, where* $g(\tau) := f_1(0,\tau)$*,* $c_0 := 2\,\|f_1\|_{L^\infty(Q_1)}$*,* $0 < r_0 < \frac{\delta_2^4}{16}$*. Moreover if* $f_2 \in Lip(Q_1)$ *and* $L_1 = L_1(f_2, Q_1)$ *then*

$$L_1(g, [\tau_0 - \delta_2, \tau_0 + \delta_2]) \le \frac{2}{\delta_2} \tag{3.34}$$

Proof. First let us prove (3.33). Let us denote

$$\beta(r) := L_0(f_2, Q_1, r), \quad \alpha(r) := \max\left\{ r^{1/4}, 2\sqrt{2\,\beta(r + 2\,c_0\,r^{1/4})} \right\}$$

if $r \ge 0$. Let us observe that

$$\frac{\beta\left(r + \frac{2\,c_0\,\sqrt{r}}{\alpha(r)}\right)}{\alpha(r)^2} \le \frac{1}{8} \quad \forall r > 0\,. \tag{3.35}$$

Indeed since $\alpha(r) \ge r^{1/4}$ then $\frac{\sqrt{r}}{\alpha(r)} \le r^{1/4}$. Therefore

$$\beta\left(r + \frac{2\,c_0\,\sqrt{r}}{\alpha(r)}\right) \le \beta\left(r + 2\,c_0\,r^{1/4}\right) \le \frac{\alpha(r)^2}{8}\,.$$

Let us introduce the curves

$$\gamma_\tau(s) := (s, x(s,\tau))$$

if $s \in [-\delta_2, \delta_2]$. By assumptions **i** and **ii** we can represent each $x(\cdot,\tau)$ for each $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$ as

$$\begin{aligned} x(s,\tau) &= \tau + \int_0^s f_1(\gamma_\tau(\sigma))\,d\sigma \\ &= \tau + f_1(0,\tau)\,s + \int_0^s (s-\sigma) f_2(\gamma_\tau(\sigma))\,d\sigma \quad \forall s \in [-\delta_2, \delta_2]\,. \end{aligned} \tag{3.36}$$

By the first equality in (3.36) we get

$$|x(s,\tau) - x(s,\tau')| \le |\tau - \tau'| + c_0\,|s| \quad \forall s \in [-\delta_2,\delta_2], \tau,\tau' \in [\tau_0 - \delta_2, \tau_0 + \delta_2] \tag{3.37}$$

and then

$$|f_2(\gamma_\tau(\sigma)) - f_2(\gamma_{\tau'}(\sigma))| \le \beta(|\gamma_\tau(\sigma) - \gamma_{\tau'}(\sigma)|) \le \beta(|\tau - \tau'| + c_0\,|s|) \tag{3.38}$$

for each $|\sigma| \le |s|$ and $\tau, \tau' \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$.

In particular by the second equality in (3.36) and (3.38), for $0 \le s \le \delta_2$,

$$x(s,\tau) - x(s,\tau') \le \tau_1 - \tau_2 + (g(\tau) - g(\tau'))s + \beta(|\tau - \tau'| + c_0\,|s|)\,s^2 \tag{3.39}$$

for each $\tau, \tau' \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$.

By contradiction, let us assume there exist $\tau_0 - \delta_2 \leq \tau_2 < \tau_1 \leq \tau_0 + \delta_2$, $0 < \bar{r} < r_0$ such that

$$0 < |\tau_1 - \tau_2| \leq \bar{r} \tag{3.40}$$

$$\frac{|g(\tau_1) - g(\tau_2)|}{\sqrt{\tau_1 - \tau_2}} > \alpha(\bar{r})\,. \tag{3.41}$$

By (3.41) we get

$$g(\tau_1) - g(\tau_2) < -\alpha(\bar{r})\,\sqrt{\tau_1 - \tau_2} \tag{3.42}$$

or

$$g(\tau_1) - g(\tau_2) > \alpha(\bar{r})\,\sqrt{\tau_1 - \tau_2} \tag{3.43}$$

Let us prove now that if (3.42) holds then there exists $0 < s^* < \delta_2$ such that

$$x(s^*, \tau_1) = x(s^*, \tau_2). \tag{3.44}$$

Let $\bar{s} := 2\,\dfrac{\sqrt{\tau_1 - \tau_2}}{\alpha(\bar{r})}$ then

$$\bar{s} \in [0, \delta_2], \quad x(\bar{s}, \tau_1) - x(\bar{s}, \tau_2) < 0\,. \tag{3.45}$$

Indeed by (3.40) and the definition of α, $\bar{s} \leq 2\,\frac{\sqrt{\tau_1-\tau_2}}{\alpha(|\tau_1-\tau_2|)} \leq 2\,(\tau_1 - \tau_2)^{1/4} \leq 2\,\bar{r}^{1/4} \leq 2\,r_0^{1/4} \leq \delta_2$. On the other hand by (3.39) (with $s = \bar{s}$, $\tau = \tau_1$, $\tau' = \tau_2$), (3.42) and (3.35)

$$\begin{aligned}
x(\bar{s}, \tau_1) - x(\bar{s}, \tau_2) &\leq \tau_1 - \tau_2 - 2\,(\tau_1 - \tau_2) + 4\,\frac{\beta(|\tau_1 - \tau_2| + c_0\,\bar{s})}{\alpha(\bar{r})^2}\,(\tau_1 - \tau_2) \\
&= (\tau_1 - \tau_2)\left(-1 + 4\,\frac{\beta(\bar{r} + 2\,c_0\,\sqrt{\bar{r}}/\alpha(\bar{r}))}{\alpha(\bar{r})^2}\right) \\
&\leq -\frac{1}{2}\,(\tau_1 - \tau_2) < 0\,.
\end{aligned}$$

Then (3.45) follows. Let

$$s^* := \sup\{s \in [0, \delta_2] :\, x(s, \tau_1) > x(s, \tau_2)\} \tag{3.46}$$

then by (3.44) $0 < s^* < \bar{s} \leq \delta_2$ and it satisfies (3.44).

If (3.43) holds let us consider

$$f_1^*(\eta, \tau) = -f_1(-\eta, \tau), \quad f_2^*(\eta, \tau) = f_2(-\eta, \tau) \quad (\eta, \tau) \in Q_1$$

$$x^*(s, \tau) = x(-s, \tau), (s, \tau) \in Q_2\,,$$

$$g^*(\tau) = -f_1(0, \tau) \quad \tau \in [\tau_0 - \delta_1, \tau_0 + \delta_1]\,.$$

Then since in this case

$$\frac{d^i}{ds^i}x^*(s,\tau) = f_i^*(s,x^*(s,\tau)) \ \text{ if } |s| \le \delta_2,\ \tau \in [\tau_0-\delta_1,\tau_0+\delta_1], \quad (i=1,2)$$

$$g^*(\tau_1) - g^*(\tau_2) < -\alpha(\bar r)\sqrt{\tau_1-\tau_2}$$

we can repeat the argument above, getting that there exist $-\delta_2 < s* < 0$ such that (3.44) still holds. Let us prove now that

$$f_1(\gamma_{\tau_1}(s*)) \ne f_1(\gamma_{\tau_2}(s*)), \tag{3.47}$$

then a contradiction and (3.33) will follow. Indeed, for instance, let us assume (3.42). Then by (3.36) and (3.38)

$$f_1(\gamma_{\tau_1}(s*)) - f_1(\gamma_{\tau_2}(s*)) = g(\tau_1) - g(\tau_2) + \int_0^{s*} (f_2(\gamma_{\tau_1}(\sigma)) - f_2(\gamma_{\tau_2}(\sigma)))\,d\sigma \le$$

$$\le g(\tau_1) - g(\tau_2) + \beta(|\tau_1-\tau_2| + c_0\, s*)\, s* \le g(\tau_1) - g(\tau_2) + \beta(|\tau_1-\tau_2| + c_0\,\bar s)\,\bar s$$

$$\le -\alpha(\bar r)\sqrt{\tau_1-\tau_2} + 2\,\frac{\beta(|\tau_1-\tau_2| + c_0\,\bar s)}{\alpha(\bar r)}\sqrt{\tau_1-\tau_2} \le$$

$$\le -\alpha(\bar r)\sqrt{\tau_1-\tau_2} + 2\,\frac{\beta(\bar r + 2\,c_0\sqrt{\bar r}/\alpha(\bar r))}{\alpha(\bar r)}\sqrt{\tau_1-\tau_2} =$$

$$= 2\,\alpha(\bar r)\sqrt{\tau_1-\tau_2}\left[-\frac{1}{2} + \frac{\beta(\bar r + 2\,c_0\sqrt{\bar r}/\alpha(\bar r))}{\alpha(\bar r)^2}\right].$$

By (3.35) we get that

$$f_1(\gamma_{\tau_1}(s*)) - f_1(\gamma_{\tau_2}(s*))) < 0$$

and (3.47) follows.

Let us prove now (3.34). The proof's scheme partially follows the previous one. By contradiction, let us assume there exist $\tau_0-\delta_2 \le \tau_2 < \tau_1 \le \tau_0+\delta_2$ such that

$$\frac{|g(\tau_1)-g(\tau_2)|}{\tau_1-\tau_2} > \frac{2}{\delta_2}$$

For instance let us assume that

$$K := \frac{g(\tau_1)-g(\tau_2)}{\tau_1-\tau_2} < -\frac{2}{\delta_2} \tag{3.48}$$

otherwise we can argue as before to reduce to this case. Then we have only to prove there exists $0 < s^* < \delta_2$ such that (3.44) holds. Indeed we can apply now the classical uniqueness result for ODEs' solutions with Lipschitz continuous data to the Cauchy problem

$$\begin{cases} \frac{d^2}{ds^2}y(s) = \ f_2(s, y(s)) \\ y(s^*) = \tau^*, \ \frac{d}{ds}y(s^*) = \ f_1(s^*, \tau^*) \end{cases}$$

where $\tau^* = \ x(s^*, \tau_1) = \ x(s^*, \tau_2)$ and we get a contradiction.

Let s^* be as in (3.46), then $0 < s^* \leq \delta_2$. Because $f_2 \in Lip(Q_1)$, by the second equality in (3.36) and (ii), for $0 \leq s \leq \delta_2$,

$$x(s, \tau_1) - x(s, \tau_2) \leq \tau_1 - \tau_2 + (g(\tau_1) - g(\tau_2))s + L_1 s \int_0^s |x(\sigma, \tau_1) - x(\sigma, \tau_2)| d\sigma. \tag{3.49}$$

Let us prove (3.44). Let

$$u(s) := \int_0^s (x(\sigma, \tau_1) - x(\sigma, \tau_2)) d\sigma$$

if $0 \leq s \leq s^*$, then by (3.49)

$$\frac{d}{ds}u(s) \leq a(s) + b(s)u(s) \quad 0 \leq s \leq s^*$$

with $a(s) := \tau_1 - \tau_2 + (g(\tau_1) - g(\tau_2))s$, $b(s) = L_1\, s$. By applying Gronwall's inequality (see, for instance, [51], appendices B.2 j) we get if $0 \leq s \leq s^*$

$$0 \leq \int_0^s (x(\sigma, \tau_1) - x(\sigma, \tau_2)) d\sigma = u(s) \leq \exp\left(\int_0^s b(\sigma) d\sigma\right) \cdot \left[u(0) + \int_0^s a(\sigma) d\sigma\right] = \tag{3.50}$$

$$= \exp\left(L_1 \frac{s^2}{2}\right) \left[(\tau_1 - \tau_2)s + \frac{g(\tau_1) - g(\tau_2)}{2} s^2\right] = \exp\left(L_1 \frac{s^2}{2}\right) (\tau_1 - \tau_2)\, s \left(1 + \frac{K}{2}\, s\right).$$

Let $\bar{s} := -2/K$ and notice that by (3.48) $0 < \bar{s} < \delta_2$. Then by (3.50) we infer $0 < s^* \leq \bar{s} < \delta_2$ and (3.44) follows. □

Remark 3.3.14. Notice that in order to get (3.34) we have actually exploited the weak assumption

$$|f_2(\eta, \tau) - f_2(\eta, \tau')| \leq \ L_1 |\tau - \tau'| \quad \forall \eta \in [-\delta_2, \delta_2],\ \tau \in [\tau_0 - \delta_1, \tau_0 + \delta_1]$$

instead of $f_2 \in Lip(Q_1)$.

Proof of Theorem 3.3.12. Let $A_0 = (\eta_0, \tau_0) \in \omega$ if $n = 1$ and $A_0 = (\eta_0, v_0, \tau_0) \in \omega$ if $n \geq 2$. Then since ϕ is a broad* solution of (3.25) there exists family of exponential maps at A_0

$$\exp_{A_0}(\cdot\nabla_j^{\phi})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset \omega \tag{3.51}$$

where $0 < \delta_2 < \delta_1$ and $j = 2, \ldots, 2n$ satisfying $(E.1)$, $(E.2)$ and $(E.3)$.

Let us denote $I_1 := \overline{I_{\delta_1}(A_0)}$, $I_2 := \overline{I_{\delta_2}(A_0)}$, $K := \sup_{A \in I_1} |A|, M := \|\phi\|_{L^\infty(I_1)}, N := \|\nabla^\phi \phi\|_{L^\infty(I_1)}$; let $\beta(r) := L_0(w_{n+1}, I_1, r)$ be the modulus of continuity of w_{n+1} on I_1.

Let $A = (\eta, \tau) \in I_2$ if $n = 1$ and $A = (\eta, v, \tau) \in I_2$ if $n \geq 2$ and let us denote by $\gamma_A(s) = \gamma_{n+1}^A(s) = \exp_{A_0}(s\nabla_{n+1}^{\phi})(A)$ if $s \in [-\delta_2, \delta_2]$. Let $\gamma_A(s) = (\eta + s, \tau_A(s))$ if $n = 1$ and $\gamma_A(s) = (\eta + s, v, \tau_A(s))$ if $n \geq 2$. Then τ_A satisfies

$$\begin{cases} \dfrac{d^2}{ds^2}\tau_A(s) = \dfrac{d}{ds}[\phi(\gamma_A(s))] = w_{n+1}(\gamma_A(s)) & \forall\, s \in [-\delta_2, \delta_2] \\ \tau_A(0) = \tau, \quad \dfrac{d}{ds}\tau_A(0) = \phi(A) \end{cases} \tag{3.52}$$

Let us observe also that

$$\exp_{A_0}(\cdot\nabla_{n+1}^{\phi})(\cdot) : [-r_2, r_2] \times \overline{I_{r_2}(A_0)} \to \overline{I_{\delta_2}(A_0)} = I_2 \tag{3.53}$$

provided

$$r_2 < \frac{\delta_2}{M + 2}\,. \tag{3.54}$$

Indeed if $(s, A) \in [-r_2, r_2] \times \overline{I_{r_2}(A_0)}$ then by (3.51) and $(E.2)$

$$\gamma_A(s) - A_0 = \begin{cases} (\eta - \eta_0 + s, \tau_A(s) - \tau_0) & \text{if}\, n = 1 \\ (\eta - \eta_0 + s, v - v_0, \tau_A(s) - \tau_0) & \text{if}\, n \geq 2 \end{cases} \in \overline{I_{\delta_2}(0)}$$

provided (3.54) holds.

First let us consider the case $n = 1$ and divide the proof in three steps.

Step 1. Let us prove that

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, \tau), B = (\eta, \tau') \in I_2, 0 < |A - B| \leq r\right\} \leq \alpha_1(r) \tag{3.55}$$

for every $r \in (0, r_0)$ where

$$\alpha_1(r) := \max\left\{r^{1/4}, \sqrt{L_0(w_{n+1}, I_1, r + 2\,M\,r^{1/4})}\right\}, \quad 0 < r_0 < \frac{\delta_2^4}{16}\,. \tag{3.56}$$

Let $A = (\eta, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ and let $x(s, \tau) := \tau_A(s)$ if $|s| \le \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta}(s, \tau) := \phi(\eta + s, \tau)$, $f_{2,\eta}(s, \tau) := w_2(\eta + s, \tau)$, $g_\eta(\tau) = \phi(\eta, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$ is fixed. By (3.52) and since

$$\|f_{1,\eta}\|_{L^\infty(Q_1)} \le M, \;\; L_0(f_{2,\eta}, Q_1, r) \le L_0(w_2, I_1, r) \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$$

we can apply (3.33) of Lemma 3.3.13 and (3.55) follows.

Step 2. Let us prove that

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, \tau), B = (\eta', \tau) \in \overline{I_{r_2}(A_0)}, 0 < |A - B| \le r\right\} \le \alpha_2(r) \tag{3.57}$$

for every $r \in (0, r_2)$ where

$$\alpha_2(r) := \sqrt{M}\,\alpha_1(M\,r) + N\sqrt{r}, \quad 0 < r_2 < \min\left\{\frac{\delta_2}{M + 2}, \frac{r_0}{M}\right\} \tag{3.58}$$

and $\alpha_1(r)$ and r_0 are the quantities in (3.56).

Suppose on the contrary there exist $\bar A = (\bar\eta', \bar\tau), \bar B = (\bar\eta, \bar\tau) \in \overline{I_{r_2}(A_0)}$, $0 < \bar r \le r_2$ such that $0 < |\bar A - \bar B| \le \bar r$ and

$$\frac{|\phi(\bar A) - \phi(\bar B)|}{|\bar A - \bar B|^{1/2}} > \sqrt{M}\,\alpha_1(M\,\bar r) + N\,\sqrt{\bar r}\,. \tag{3.59}$$

Let $\bar C := \gamma_{\bar A}(\bar\eta - \bar\eta') = (\bar\eta, \bar\tau')$ and let us notice that $\bar C \in I_2$ because (3.53) and (3.54). Moreover

$$|\bar\tau' - \bar\tau| = \left|\int_0^{\bar\eta - \bar\eta'} \phi(\gamma_{\bar A}(\sigma))\, d\sigma\right| \le M\,|\bar\eta - \bar\eta'|\,. \tag{3.60}$$

On the other hand by (3.59) and $(E.3)$

$$\begin{aligned} |\phi(\bar B) - \phi(\bar C)| &\ge |\phi(\bar B) - \phi(\bar A)| - |\phi(\bar A) - \phi(\bar C)| \\ &\ge \left[\sqrt{M}\,\alpha_1(M\,\bar r) + N\,\sqrt{\bar r} - N\,\sqrt{|\bar\eta - \bar\eta'|}\,\right]\sqrt{|\eta - \eta'|} \\ &\ge \sqrt{M}\,\alpha_1(M\,\bar r)\,\sqrt{|\bar\eta - \bar\eta'|} \end{aligned} \tag{3.61}$$

Let us notice that $\bar\tau \neq \bar\tau'$. If not $\bar C = (\bar\eta, \bar\tau') = (\bar\eta, \bar\tau) = \bar B$ and since $\alpha_1(r) > 0\,\forall r > 0$ by (3.61) $M = 0$. Therefore $\phi \equiv 0$ in I_1 and then a contradiction by (3.59).

By (3.61) and (3.60) we get $\bar B = (\bar\eta, \bar\tau)$, $\bar C = (\bar\eta, \bar\tau') \in I_2$ and

$$\frac{|\phi(\bar B) - \phi(\bar C)|}{\sqrt{|\bar B - \bar C|}} \geq \alpha_1(M\,\bar r)$$

with $0 < |\bar B - \bar C| = |\bar\tau - \bar\tau'| \leq M\,\bar r \leq M\,r_2 \leq r_0$ and then a contradiction for step 1.

Step 3. Let $A = (\eta, \tau)$, $B = (\eta', \tau') \in \overline{I_{r_2}(A_0)}$ with $0 < |A - B| \leq r$ then

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leq \frac{|\phi(\eta, \tau) - \phi(\eta', \tau)|}{|\eta - \eta'|^{1/2}} + \frac{|\phi(\eta', \tau) - \phi(\eta, \tau')|}{|\tau - \tau'|^{1/2}}. \tag{3.62}$$

By steps 1, 2 and (3.62) we get the thesis by choosing $r_1 = \delta_1$, r_2 as in (3.58) and $\alpha(r) = \alpha_1(r) + \alpha_2(r)$ where $\alpha_1(r)$ and $\alpha_2(r)$ are respectively defined in (3.56) and (3.58).

Let us consider now the case $n \geq 2$. Let $\widehat{\cdot} : \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau \to \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ the projection defined as $\widehat{(\eta, v, \tau)} = (\eta, \tau)$. Let us notice that $\widehat{I_r(A)} = I_r(\hat A)$ for each $A \in \mathbb{R}^{2n}$. For fixed $v \in \overline{U(v_0, \delta_1)}$ let us define

$$\phi_v(\eta, \tau) := \phi(\eta, v, \tau), \quad w_v(\eta, \tau) := w_{n+1}(\eta, v, \tau) \quad \text{if} \quad (\eta, \tau) \in \overline{I_{\delta_1}(\hat A_0)}$$

and notice that

$$\widehat{\exp_{A_0}(s\nabla^\phi_{n+1})(A)} = \exp_{\hat A_0}(s\nabla_2^{\phi_v})(\hat A) \quad s \in [-\delta_2, \delta_2]$$

for each $A \in \overline{I_{\delta_2}(A_0)}$ where $\exp_{A_0}(\cdot\nabla^\phi_{n+1})(\cdot)$ is the exponential map in (3.51) with $j = n + 1$. In particular

$$\exp_{\hat A_0}(\cdot\nabla_2^{\phi_v})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(\hat A_0)} \to \overline{I_{\delta_1}(\hat A_0)}$$

and it satisfies $(E.1)$, $(E.2)$ and $(E.3)$ in the case n=1 with $w_2 = w_v$. Moreover

$$\begin{gathered} M_v := \|\phi_v\|_{L^\infty(I_{\delta_1}(\hat A_0))} \leq M, \qquad N_v := \|w_v\|_{L^\infty(I_{\delta_1}(\hat A_0))} \leq N, \\ L_0(w_v, \overline{I_{\delta_1}(\hat A_0)}, r) \leq L_0(w_{n+1}, \overline{I_{\delta_1}(A_0)}, r) \end{gathered} \tag{3.63}$$

for each $v \in \overline{U(v_0, \delta_1)}$ and $r > 0$. Therefore we can apply the case $n = 1$ and by (3.63) we get

$$\sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, v, \tau), B = (\eta', v, \tau') \in \overline{I_{r_2}(A_0)}, 0 < |A - B| \le r \right\} \le \alpha_3(r) \tag{3.64}$$

for each $r \in (0, r_2)$ where $\alpha_3(r) = \alpha_1(r) + \alpha_2(r)$ and $\alpha_1(r)$ is defined in (3.56), $\alpha_2(r)$ and r_2 are defined in (3.58).

In order to achieve the proof we can follow the argument given in step 5 of the proof of Theorem 5.8 in [4]. Indeed we can carry out the same estimates and we get

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \le N\, |A - B|^{1/2} + \left(\frac{K}{2} + 2 \right) \alpha_3(|A - B|)$$

for each A, $B \in \overline{I_{r_2}(A_0)}$ and $0 < |A - B| \le r_2$ and we have done. □

3.4 Negative answers to questions of good parametrization

A problem raised in [55], directly related to the theory of rectifiability in the Heisenberg group (see also [1, 58, 85, 91]), is the following one: is it possible to see $\mathbb{H}$-regular hypersurfaces as bi-Lipschitz deformations of a given "model" metric space? Here, by bi-Lipschitz we mean Lipschitz continuous maps with Lipschitz continuous inverse map. In [34], D. R. Cole and S. Pauls have proved that, in the setting of the first Heisenberg group $\mathbb{H}^1$, any noncharacteristic C^1-surface S can be locally parametrized by means of a Lipschitz homeomorphism defined on an open subset of the plane $\mathbb{R}^2_{x,z}$ endowed with the "parabolic" distance ϱ defined by

$$\varrho\big((x, z), (x', z')\big) := |x - x'| + |z - z'|^{1/2} ;$$

this space can be naturally identified with the subgroup $\mathbb{V}_1 \subset \mathbb{H}^1$ endowed with the restriction of d_∞. We are able to show that Cole-Pauls homeomorphism is indeed bi-Lipschitz continuous, cfr. [20, 94].

Theorem 3.4.1. *Let S be a C^1 surface; then for any non characteristic point $P \in S$ there is a Lipschitz continuous mapping*

$$\Psi : (\mathcal{A}, \varrho) \longrightarrow (\mathcal{U}, d_\infty) ,$$

from an open set $\mathcal{A} \subset \mathbb{R}^2$ to a neighbourhood $\mathcal{U}$ of P in S, with Lipschitz inverse map Ψ^{-1}.

Proof. It is not restricting to suppose that $P = 0$ and that a neighbourhood $\mathcal{U} \subset S$ of 0 is parametrized by a C^1 function $\phi : \omega \to \mathbb{R}$ with $\phi(0) = 0$, where $\omega \subset (\mathbb{R}^2, \rho_\phi)$. Recalling that in $\mathbb{H}^1$ $\nabla^\phi = W^\phi$, let us introduce the map

$$\begin{aligned} \psi \ &: \ \mathcal{A} \to \omega \\ &(x, z) \mapsto \exp(xW^\phi)(0, z) \end{aligned}$$

which is defined, possibly restricting ω, on a proper open set $\mathcal{A} \subset \mathbb{R}^2$. It is not difficult to notice that the Lipschitz homeomorphism $\Psi : \mathcal{A} \to \mathcal{U}$ introduced by D. R. Cole and S. Pauls is such that $\Psi = \Phi \circ \psi$. Since Φ is a (ρ_ϕ-d_∞) bi-Lipschitz homeomorphism, it will be sufficient to show that the inverse map ψ^{-1} is (ρ_ϕ-ϱ)-Lipschitz continuous.

To this aim, for any $A = (\eta, \tau) \in \omega$ let us introduce the curve z_A solution to the ODE

$$z_A(\eta) = \tau, \qquad \dot{z}_A(s) = \phi(s, z_A(s))\,.$$

It is immediate to see that $\psi^{-1}(A) = \psi^{-1}(\eta, \tau) = (\eta, z_A(0))$. The Lipschitz estimate we need to prove is therefore

$$|\eta' - \eta| + |z_B(0) - z_A(0)|^{1/2} \le c\,\rho_\phi(A, B) \qquad \forall\ A = (\eta, \tau), B = (\eta', \tau') \in \omega\,.$$

If $\eta' = \eta$ we have

$$|z_B(0) - z_A(0)| = \left| \tau' - \tau + \int_\eta^0 [\phi(s, z_B(s)) - \phi(s, z_A(s))]ds \right| \le$$

$$\le |\tau' - \tau| + c_1 \int_\eta^0 |z_B(s) - z_A(s)|ds$$

and by Gronwall's lemma one conclude that

$$\varrho(\psi^{-1}(A), \psi^{-1}(B)) = |z_B(0) - z_A(0)|^{1/2} \le c_2|\tau' - \tau|^{1/2} = \rho_\phi(A, B)\,.$$

If $\eta' \neq \eta$ we define $C := \exp((\eta - \eta')W^\phi)(B) = (\eta, \tau'')$. We refer to [4, Theorem 3.8] for the proof of the inequality $|\tau'' - \tau'|^{1/2} \le c_3\rho_\phi(A, B)$; with this in our hands we can conclude in a stroke since

$$\begin{aligned} \varrho(\psi^{-1}(A), \psi^{-1}(B)) &= |\eta' - \eta| + |z_B(0) - z_A(0)|^{1/2} \\ &= |\eta' - \eta| + |z_C(0) - z_A(0)|^{1/2} \\ &\le |\eta' - \eta| + c_2|\tau'' - \tau|^{1/2} \\ &\le c\,\rho_\phi(A, B)\,. \end{aligned}$$

□

Theorem 3.4.1 fails to hold for general $\mathbb{H}$-regular surfaces: the counterexample is provided exactly by the intrinsic graph $S := \Phi(\omega) = \overline{S_+ \cup S_-}$ of the map ϕ in example 3.3.7. Notice that $(\mathbb{R}^2, \varrho)$ is not connected by curves of finite length, while $(S_+ \cup L, d_\infty)$ is. This happens because the upper halfplane in $\mathbb{V}_1 \simeq \mathbb{R}^2$ is connected by means of the exponential curves of W^ϕ, namely by the curves $\{c_w^+\}_{w\in\mathbb{R}}$ and c_0^- in (3.65) and (3.66).

Theorem 3.4.2. *There exists an $\mathbb{H}$-regular hypersurface $S \subset \mathbb{H}^1$ and a point $P \in S$ such that, for any open set $\mathcal{A} \subset \mathbb{R}^2$ and any neighborhood $\mathcal{U}$ of P on S, there cannot exists a bi-Lipschitz map $\Psi : (\mathcal{A}, \varrho) \to (\mathcal{U}, d_\infty)$.*

Theorem 3.4.2 follows immediately from the next result: in the latter, however, we follow a slightly different path from the one outlined above, thus proving a stronger statement.

Theorem 3.4.3. *Let S be the $\mathbb{H}$-regular surface given by the X_1-graph of the map ϕ in example* 3.3.7 *with $\frac{1}{2} < \alpha < 1$, and suppose that*

$$\Psi : (\mathcal{A}, \varrho) \longrightarrow (\mathcal{U}, d_\infty)$$

is a Lipschitz continuous and surjective map from an open set $\mathcal{A} \subset \mathbb{R}^2$ to a neighborhood $\mathcal{U}$ of 0 in S. Then Ψ is not an homeomorphism; in particular, it cannot be bi-Lipschitz.

Proof. Step 1: horizontal curves on S. For any fixed z the curve $\gamma_z := \Psi(\cdot, z) : \mathbb{R} \to \mathbb{H}^1$ is Lipschitz continuous; in particular (see [83]) it must be horizontal, i.e. absolutely continuous and such that $\dot\gamma_z \in \mathfrak{g}_1$ almost everywhere. Since γ_z lies on S, and the latter is the union of two C^1 surfaces, γ_z must be contained in (a piece of) an integral curve of the vector field

$$Y_1 + (W^\phi \phi \circ \Phi^{-1}) X_1 \,,$$

which is (up to a normalization) the unique vector field which is both horizontal and tangent to S; thanks to proposition 3.1.17, $\Phi^{-1}(\gamma_z)$ must be (a piece of) an integral curve of W^ϕ in $\mathbb{R}^2$.

Let us investigate the behaviour of the integral curves of W^ϕ, i.e. the solutions of the Cauchy problem

$$c'(s) = W^\phi(c(s)) = \partial_\eta + \phi(c(s))\partial_\tau \,.$$

More precisely, if $c(s) = (c_\eta(s), c_\tau(s))$ we have

$$c'_\eta = 1 \qquad \text{and} \qquad c'_\tau = -\frac{c_\tau^\alpha}{1-\alpha} \,.$$

Lipschitz regularity of the coefficients of the ODE is violated at points $(\eta, 0)$, therefore we cannot expect uniqueness of solutions whenever $c_\tau = 0$. By standard considerations on this kind of problem we can divide the solutions of the ODE into two families $\{c_w^+\}_{w\in\mathbb{R}}$ and $\{c_\zeta^-\}_{\zeta\leq 0}$:

$$c_w^+(s) = \begin{cases} (s, (w-s)^{\frac{1}{1-\alpha}}) & \text{if } s \leq w \\ (s, 0) & \text{if } s > w \end{cases} \tag{3.65}$$

$$c_\zeta^-(s) = (s, \zeta). \tag{3.66}$$

Notice that for a given curve c_w^+ the parameter w denotes the point $(w, 0)$ where it meets the horizontal axis η, cfr. also figure 3.3. We will also write c_w^{++} to denote the restriction of c_w^+ to $(-\infty, w]$, i.e. the part of c_w^+ lying in the upper halfplane. The upper (closed) halfplane is connected by means of c_0^- and of paths of type c_w^+.

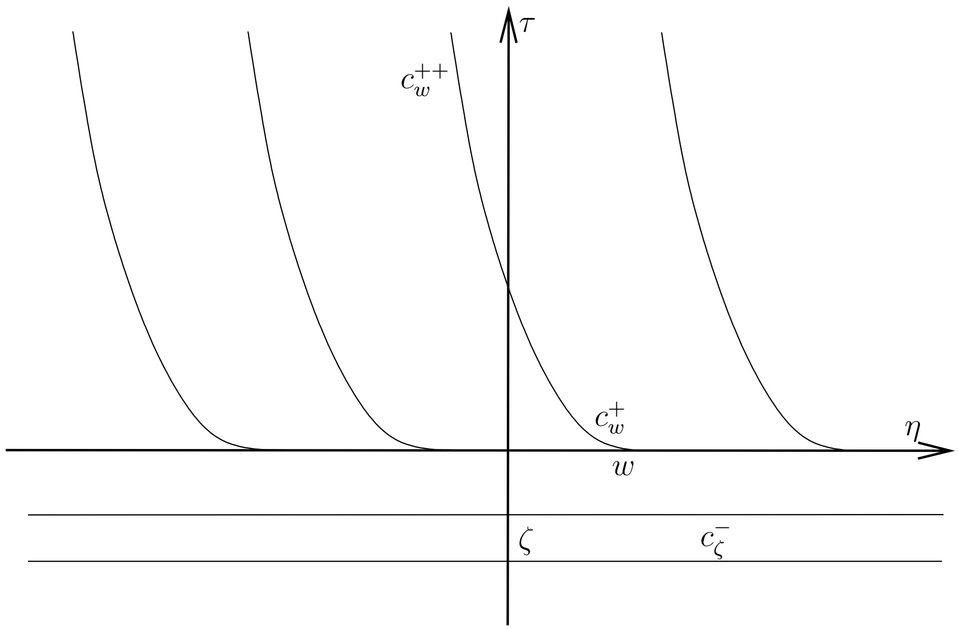


Figure 3.3: Exponential lines of W^ϕ.

Step 2: a curve passing through 0. It will not be restrictive to suppose $\Psi(0,0) = 0 \in S$ and $\mathcal{U} = \Phi\big((-\delta,\delta)^2\big)$ for some $\delta > 0$. Let us denote by ψ the (ϱ-ρ_ϕ)-Lipschitz induced map $\Phi^{-1} \circ \Psi : \mathcal{A} \to (-\delta,\delta)^2$, which is surjective and such that $\psi(0,0) = (0,0)$; suppose by contradiction that it is also an homeomorphism. Then the set

$$K := \psi^{-1}\{(0,\tau) : \tau \in [0, \delta/2]\}$$

is a compact subset of $\mathcal{A}$, and so for sufficiently small $r > 0$

$$\{(x+h,z) : (x,z) \in K, -r \leq h \leq r\} \subset \mathcal{A}\,. \tag{3.67}$$

Let us set

$$\begin{array}{l} r_+ := \sup\{x > 0 : \psi(x,0) \in \mathbb{R} \times \{0\}\} \geq 0 \\ r_- := \inf\{x < 0 : \psi(x,0) \in \mathbb{R} \times \{0\} \leq 0\,. \end{array}$$

One cannot have $r_+ = r_- = 0$; indeed, this would imply

$$\{\psi(x,0) : x > 0\} \subset \mathrm{Im}\ c_0^{++}\backslash\{0\} \qquad \text{and} \qquad \{\psi(x,0) : x < 0\} \subset \mathrm{Im}\ c_0^{++}\backslash\{0\}\,,$$

and by continuity ($\psi(0,0) = 0$) we obtain

$$\{\psi(x,0) : x > 0\} \cap \{\psi(x,0) : x < 0\} \neq \emptyset$$

i.e. ψ is not injective, a contradiction.

Step 3: conclusion. Therefore, one between r_+ and r_- is nonzero: by substituting ψ with $\psi'(x,z) := \psi(-x,z)$ if necessary, we can suppose that $r_+ > 0$. One has

$$\{\psi(x,0) : 0 \leq x \leq r_+\} \subset \mathbb{R} \times \{0\},$$

otherwise the curve $\psi(\cdot,0)|_{[0,r_+]}$ would "leave" the horizontal axis $\mathbb{R} \times \{0\}$ and then "return" on it after some time. This could be possible only by covering forward and then backward a piece of some c_w^{++}, and contradicting in particular the injectivity of ψ. We can choose $r \in (0, r_+)$ such that (3.67) holds. Set $A := \psi(r,0) = (\overline{\eta}, 0)$; by continuity one must have

$$\begin{array}{ll} [0,\overline{\eta}] \times \{0\} \subset \{\psi(x,0) : 0 \leq x < r\} & \text{if } \overline{\eta} > 0 \\ [\overline{\eta},0] \times \{0\} \subset \{\psi(x,0) : 0 \leq x < r\} & \text{if } \overline{\eta} < 0. \end{array} \tag{3.68}$$

Since $A \neq 0$ (i.e. $\overline{\eta} \neq 0$) we easily find an $\epsilon > 0$ such that

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset,$$

where (cfr. figure 3.4)

$$\mathcal{V}_1 := \bigcup_{0<w<\epsilon} \mathrm{Im}\ c_w^{++} \qquad \text{and} \qquad \mathcal{V}_2 := \bigcup_{\overline{\eta}-\epsilon<w<\overline{\eta}+\epsilon} \mathrm{Im}\ c_w^{++}\,.$$

Notice that $A \in \mathcal{V}_2$, since $A \in c_{\overline{\eta}}^{++}$. Now, it is not difficult to prove that, in order join a point $A_1 \in \mathcal{V}_1$ with a point $A_2 \in \mathcal{V}_2$ by following only exponential lines of W^ϕ, one must cover the whole segment I defined by

$$I := [\epsilon, \overline{\eta} - \epsilon] \times \{0\} \ \text{ if } \ \overline{\eta} > 0, \qquad I := [\overline{\eta} + \epsilon, 0] \times \{0\} \ \text{ if } \ \overline{\eta} < 0\,.$$

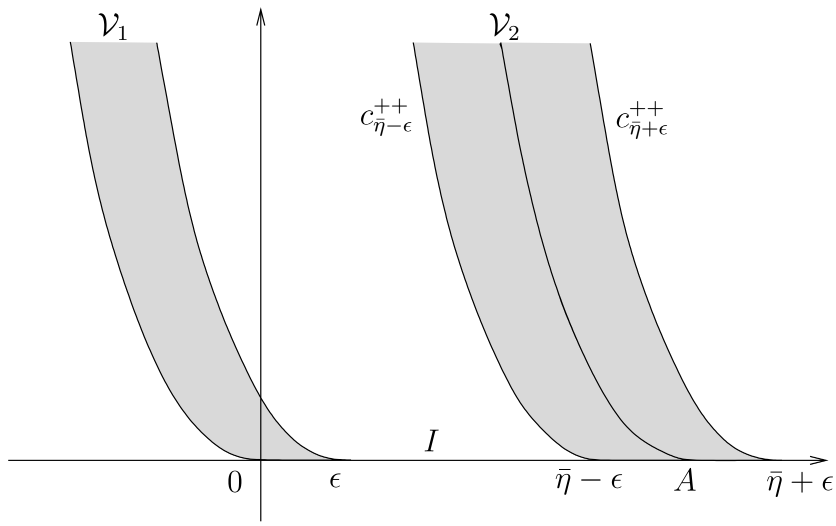


Figure 3.4: The sets $\mathcal{V}_1$ and $\mathcal{V}_2$ and the interval I.

Setting $(x_\tau, z_\tau) := \psi^{-1}(0,\tau)$, one can notice that

$$\lim_{\tau\to 0} \psi(x_\tau + r, z_\tau) = \psi(r,0) = A.$$

For sufficiently small $\tau > 0$ the curve $\psi(\cdot, z_\tau)$ joins $A_1 := (0,\tau) \in \mathcal{V}_1$ to $A_2 := \psi(x_\tau + r, z_\tau)$ following only exponentials of W^ϕ; moreover, A_2 must belong to $\mathcal{V}_2$. This implies that $I \subset \mathrm{Im}\ \psi(\cdot, z_\tau)$; since (see (3.68)) we have also $I \subset \mathrm{Im}\ \psi(\cdot, 0)$, this would contradict the injectivity of ψ provided we are able to choose a sufficiently small τ such that $z_\tau \neq 0$. Were this not possible, there would exist $\lambda > 0$ such that $\psi^{-1}(0,\tau) = (x_\tau, 0)$ for any $\tau \in [0,\lambda]$, i.e.

$$\{0\} \times [0,\lambda] \subset \mathrm{Im}\, \psi(\cdot, 0)\,.$$

Therefore the image $\Phi(\{0\} \times [0,\lambda])$ would be a horizontal curve, while it can be easily checked that this is not the case. A contradiction arises and the proof is completed. □

Remark 3.4.4. Since $\Phi : (\omega, \rho_\phi) \to (\mathcal{U}, d_\infty)$ is bi-Lipschitz, see remark 3.2.3, the problem of Theorems 3.4.1 and 3.4.2 is equivalent of finding a bi-Lipschitz mapping $\psi : (\mathcal{A}, \rho) \to (\omega, \rho_\phi)$.

Remark 3.4.5. In the spirit of Federer's approach to rectifiability (see [52]) it would be interesting to understand if $\mathbb{H}$-regular surfaces can be seen as Lipschitz images of the parabolic plane. In this sense, Theorem 3.4.3 essentially says that one cannot expect injectivity of the parametrization, since the images on S of horizontal lines $(\cdot, \tau) \subset (\mathbb{R}^2, \varrho)$ are forced to meet.

The surface S can be locally parametrized by means of Lipschitz images of the parabolic plane. This clearly follows by Theorem 3.4.1 for neighbourhoods of points in $S_+ \cup S_-$. For points $P \in L$, it will be sufficient to observe that $P^{-1} \cdot S = S$ (thus reducing to the case $P = 0$) and to show that the map

$$\psi(x,z) := \begin{cases} c^+_{z-1}(x) & \text{if } z > 0 \\ c^-_z(x) & \text{if } z \leq 0\,, \end{cases}$$

is (ϱ-ρ_ϕ)-Lipschitz continuous from a neighbourhood of $(0,0)$ to a neighbourhood of $(0,0)$ in $\mathbb{V}_1 \simeq \mathbb{R}^2$. Explicitely, we have

$$\psi(x,z) = \begin{cases} (x, (z-1-x)^{1/1-\alpha}) & \text{if } z > 0 \text{ and } x \leq z-1 \\ (x,0) & \text{if } z > 0 \text{ and } x \geq z-1 \\ (x,z) & \text{if } z \leq 0\,. \end{cases}$$

Clearly, ψ is not injective, as $\psi(0,z) = (0,0)$ for any $z \in [0,1]$.

It is not difficult (see [4]) to show that exponential curves of W^ϕ are (locally) Lipschitz continuous with respect to ρ_ϕ; in particular

$$\rho_\phi(\psi(x_1,z),\psi(x_2,z)) \leq c|x_1 - x_2|$$

for $(x_1,z),(x_2,z)$ in a neighbourhood of $(0,0)$. It will therefore be sufficient to prove that

$$\rho_\phi(\psi(x,z_1),\psi(x,z_2)) \leq C|z_1 - z_2|^{1/2}$$

for some $C > 0$ and any $(x,z_1),(x,z_2)$ in a neighbourhood of $(0,0)$. We have several cases to take into account. If $z_1, z_2 > 0$, $x \leq z_1 - 1$ and $x \leq z_2 - 1$ then

$$\rho_\phi(\psi(x,z_1),\psi(x,z_2)) = |(z_1-1-x)^{1/1-\alpha} - (z_2-1-x)^{1/1-\alpha}|^{1/2} \leq C|z_1-z_2|^{1/2}\,,$$

where we used that $s \mapsto s^{1/1-\alpha}$ is locally Lipschitz continuous since $1/(1-\alpha) > 2$. If $z_1, z_2 > 0$ and $z_2 - 1 \leq x \leq z_1 - 1$ then

$$\rho_\phi(\psi(x,z_1),\psi(x,z_2)) = (z_1-1-x)^{1/2(1-\alpha)} \leq (z_1-z_2)^{1/2(1-\alpha)} \leq C|z_1-z_2|^{1/2}\,.$$

If $z_1 > 0,\ z_2 \leq 0$ and $x \leq z_1 - 1$ we can restrict to $x \geq -1$ to get

$$\rho_\phi(\psi(x,z_1),\psi(x,z_2)) = ((z_1-1-x)^{1/1-\alpha} - z_2)^{1/2} \leq (z_1^{1/1-\alpha} - z_2)^{1/2} \leq (Cz_1 - (C \vee 1)z_2)^{1/2}\,.$$

The remaining cases $z_1 > 0,\ z_2 \leq 0,\ x > z_1 - 1$ and $z_1, z_2 \leq 0$ are easy to handle.

Remark 3.4.6. The problem of finding bi-Lipschitz (or even just Lipschitz) parametrizations of $\mathbb{H}$-regular surfaces in $\mathbb{H}^n$, $n \geq 2$, is still open even for smooth hypersurfaces. The model space should be $\mathbb{R} \times \mathbb{H}^{n-1} \simeq \mathbb{R}^{2n}$ endowed with the product distance

$$\tilde{\varrho}\big((x, A), (x', A')\big) := |x - x'| + d_\infty(A, A') \qquad (x, A), (x', A') \in \mathbb{R} \times \mathbb{H}^{n-1}\,.$$

It can be easily seen that this distance is equivalent to the restriction of d_∞ to $\mathbb{R}^{2n} \simeq \mathbb{V}_1$, as both of them are homogeneous and left invariant on $\mathbb{V}_1$.

Moreover, it is still not clear whether the statement of Theorem 3.4.2 extends to the higher dimensional case $n \geq 2$: namely, if there exist $\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$ that are not bi-Lipschitz equivalent to $\mathbb{R} \times \mathbb{H}^{n-1}$. Notice, for istance, that $\mathbb{R} \times \mathbb{H}^{n-1}$ is connected by means of finite length curves; the same happens for any $\mathbb{H}$-regular surface, the subgroup $\mathbb{V}_1$ being always connected by integral curves of $\mathrm{span}\{X_2, \ldots, X_n, W^\phi, Y_2, \ldots, Y_n\}$ when $n \geq 2$ (see [4, 32]).

An other problem of regularity of the parametrization of $\mathbb{H}$-regular hypersurfaces S is studied in [69]. They show that each $\mathbb{H}$-regular surfaces $S \subset \mathbb{H}^1$ can be locally parameterized by means a Hölder continuous map of order $\frac{1}{2}$, see Theorem 3.1.15. At least in the $\mathbb{H}^1$ case (see Remark 4.3 therein) it was conjectured that the parametrization $\Phi : (\omega, d) \to (S, d_\infty)$ should belong to $W^{1,4}_m((\omega, d), (\mathbb{H}^n, d_c))$, where the distance on ω is the Euclidean one on $\mathbb{V}_1 \simeq \mathbb{R}^2$. Therefore it would be interesting to investigate the regularity of Φ with respect to some "fixed" distance d on ω; in this sense, the question risen in [69] was to understand whether the map Φ belongs to some Sobolev class of maps between metric spaces.

To answer this question, let us recall the definition of Sobolev space $W^{1,p}(M, N)$ for (M, d_M) and (N, d_N) metric space, see [5], that is equivalent to the classical in the euclidean setting.

Definition 3.4.7. *Let $(M, d_M), (N, d_N)$ be metric spaces and let μ a non-negative Borel measure, finite on bounded subsets of M. For $p \in [1, +\infty]$ let us define $W^{1,p}_m(M, \mu, N)$as the space of all functions $u : M \to N$ (with the identification $u \equiv v$ if $u = v$ μ-a.e.) with the following property:*
$\exists\, g \in L^p(M, d\mu)$ $\exists\, E \subset M$ such that $g \geq 0$, $\mu(E) = 0$ and

$$d_N(u(A), u(B)) \leq d_M(A, B)(g(A) + g(B)) \quad \forall\, A, B \in M \setminus E. \tag{3.69}$$

Such a map g is called upper gradient of u. In the Heisenberg setting we shall denote $W^{1,p}_m(\omega, \mathbb{H}^n) \equiv W^{1,p}_m(\omega, \mathcal{L}^{2n}, \mathbb{H}^n)$. We are able to show that the intrinsic parametrization (3.4) of $\mathbb{H}$-regular surfaces does not belong to any

Sobolev class of metric-space valued functions between $(\mathbb{R}^{2n}, d)$ and $(\mathbb{H}^n, d_\infty)$ when $d = \|\cdot\|$ is the Euclidean metric and when $d = d_{\infty\,|\omega}$ if ϕ is not identically vanishing, where $d_{\infty|\omega}(A, B) := d_\infty(\iota(A), \iota(B))\ \forall\ A, B \in \omega$.

Theorem 3.4.8. *The parametrization $\Phi : \omega \subset \mathbb{R}^{2n} \to S$ of an $\mathbb{H}$-regular hypersurface S does not belong to $W^{1,p}_m((\omega, d), (\mathbb{H}^n, d_\infty))$ for any $1 \le p \le +\infty$ when $d = \|\cdot\|$ is the Euclidean distance on ω. The same result holds when $d = d_{\infty|\omega}$ on ω provided Φ is not the inclusion map $\omega \hookrightarrow \mathbb{H}^n$ (i.e. if $\phi \not\equiv 0$).*

Theorem 3.4.8 follows by Theorem 3.4.9 in the case $d = \|\cdot\|$ and by Theorem 3.4.11 in the case $d = d_{\infty|\omega}$.

Theorem 3.4.9. *Let $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function; then there cannot exist a function $g \in L^p_{loc}(\omega)$ such that*

$$d_\infty(\Phi(A), \Phi(B)) \le (g(A) + g(B))\|A - B\|\,.$$

Theorem 3.4.9 immediately follows from the following

Lemma 3.4.10. *Under the same hypothesis on ϕ of Theorem 3.4.9; there exists no measurable function $g : \omega \to \overline{\mathbb{R}}$ such that*

i *g is $\mathcal{L}^{2n}$-a.e. finite;*

ii *for any $A = (\eta, v, \tau), A' = (\eta', v', \tau') \in \omega$ it holds*

$$\left|\tau' - \tau - \frac{1}{2}(\phi(A') + \phi(A))(\eta' - \eta) + \mathfrak{c}(v', v)\right|^{1/2} \le (g(A)+g(A'))\|A-A'\|\,.$$

As usually, we use the notation $\mathfrak{c}(v', v) := \frac{1}{2}\sum_{j=2}^n (v_{n+j}v'_j - v_j v'_{n+j})$.

Proof. We reason by contradiction. Since $\mathcal{L}^{2n}\big(\{|g| < +\infty\}\big) > 0$ there exist $\bar\eta, \bar v$ such that

$$\mathcal{L}^1\big(\{\tau \in \mathbb{R} : (\bar\eta, \bar v, \tau) \in \omega,\ |g(\bar\eta, \bar v, \tau)| < \infty\}\big) > 0\,.$$

In particular there is $M \in \mathbb{R}$ with $\mathcal{L}^1(E_M) > 0$, where E_M is defined by

$$E_M := \{\tau \in \mathbb{R} : (\bar\eta, \bar v, \tau) \in \omega,\ |g(\bar\eta, \bar v, \tau)| \le M\}.$$

Let us choose a Lebesgue point $\bar\tau \in E_M$; in particular $|g(\bar\eta, \bar v, \bar\tau)| \le M$ and there exists a sequence $\{\tau_j\}_{j\in\mathbb{N}} \subset E_M$ with $\tau_j \to \bar\tau$. Let us then exploit condition **ii** for points $A = (\bar\eta, \bar v, \bar\tau)$ and $A_j = (\bar\eta, \bar v, \tau_j)$ to get

$$\begin{aligned} |\tau_j - \bar\tau|^{1/2} &\le (g(A) + g(A_j))\|A - A_j\| \\ &\le 2M|\tau_j - \bar\tau| \end{aligned}$$

and so

$$\frac{1}{|\tau_j - \bar\tau|^{1/2}} \le 2M.$$

We then let $j \to \infty$ to obtain a contradiction. □

Similarly, we have the following

Theorem 3.4.11. *Let $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous, non identically vanishing function; then there cannot exist a function $g \in L^p_{loc}(\omega)$ such that*

$$d_\infty(\Phi(a), \Phi(B)) \le (g(A) + g(B)) d_{\infty\,|\omega}(A, B)\,.$$

Theorem 3.4.11 immediately follows from the following

Lemma 3.4.12. *Under the same hypothesis on ϕ of Theorem 3.4.11; there exists no measurable function $g : \omega \to \overline{\mathbb{R}}$ such that*

i *g is $\mathcal{L}^{2n}$-a.e. finite;*

ii *for any $A = (\eta, v, \tau), A' = (\eta', v', \tau') \in \omega$ it holds*

$$\left|\tau' - \tau - \frac{1}{2}(\phi(A') + \phi(A))(\eta' - \eta) + \mathfrak{c}(v', v)\right|^{1/2} \le (g(A)+g(A')) d_{\infty\,|\omega}(A, A')\,.$$

Proof. Since $\mathcal{L}^{2n}\big(\{\phi \ne 0\} \cap \{|g| < +\infty\}\big) > 0$ there exist $\bar v, \bar\tau$ such that

$$\mathcal{L}^1\big(\{\eta \in \mathbb{R} : (\eta, \bar v, \bar\tau) \in \omega, \phi(\eta, \bar v, \bar\tau) \ne 0, |g(\eta, \bar v, \bar\tau)| < \infty\}\big) > 0\,.$$

In particular there is $M \in \mathbb{R}$ with $\mathcal{L}^1(E_M) > 0$, where E_M is defined by

$$E_M := \{\eta \in \mathbb{R} : (\eta, \bar v, \bar\tau) \in \omega, \phi(\eta, \bar v, \bar\tau) \ne 0, |g(\eta, \bar v, \bar\tau)| \le M\}.$$

Let us choose a Lebesgue point $\bar\eta \in E_M$; in particular, $\phi(\bar\eta, \bar v, \bar\tau) \ne 0$, $|g(\bar\eta, \bar v, \bar\tau)| \le M$ and there exists a sequence $\{\eta_j\}_{j\in\mathbb{N}} \subset E_M$ with $\eta_j \to \bar\eta$. Let us then exploit condition **ii** for points $A = (\bar\eta, \bar v, \bar\tau)$ and $A_j = (\eta_j, \bar v, \bar\tau)$ to get

$$\begin{aligned}\left|-\frac{1}{2}(\phi(A) + \phi(A_j))(\eta_j - \bar\eta)\right|^{1/2} &\le (g(A) + g(A_j)) d_{\infty\,|\omega}(A, A_j) \\ &\le 2M|\eta_j - \bar\eta|\end{aligned}$$

whence

$$|\phi(A) + \phi(A_j)| \le 8M^2 |\eta_j - \bar\eta|^{1/2}\,.$$

We then let $j \to \infty$ to obtain $\phi(A) = 0$, a contradiction. □

Remark 3.4.13. Let us notice that, if $\phi : (\omega, d_{\infty\,|\omega}) \to (\mathbb{H}^n, d_\infty)$ is identically vanishing and ω is bounded, then $\phi \in W^{1,p}(\omega, \mathbb{H}^n)$. Indeed we have

$$\Phi(\eta, v, \tau) = \iota(\eta, v, \tau) \cdot 0e_1 = (0, v_2, ..., v_n, \eta, v_{n+2}, ..., v_{2n}, \tau).$$

If $g \equiv 1$ then $g \in L^p(\omega)$ because ω is bounded and, for $A = (\eta^A, v^A, \tau^A),\ B = (\eta^B, v^B, \tau^B) \in \omega$ we have

$$d_\infty(\Phi(A), \Phi(B)) = d_\infty\left(\iota(\eta^A, v^A, \tau^A), \iota(\eta^B, v^B, \tau^B)\right) = d_{\infty\,|\omega}(A, B).$$

Chapter 4

Intrinsic regular graphs vs. Weak Solution of non linear first-order PDEs

In this chapter we expose the main results of [18, 19], written in collaboration with F. Serra Cassano. We are going to study the links between $\mathbb{H}$-regular intrinsic graphs and suitable notions of weak-solution for a system of non linear first-order PDEs.

In section 4.1 we establish the relationship between $\mathbb{H}$-regular graphs and the notion of *broad* solution* for

$$\nabla^{\phi}\phi = w \qquad \text{in} \quad \omega \tag{4.1}$$

and we give an important characterization of the functions $\phi : \omega \to \mathbb{R}$ for which $S = \Phi(\omega) = G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular. More precisely we will prove that $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular if and only if ϕ is a broad* solution of (4.1) (Theorem 4.1.1).

When $n = 1$ the notion of broad* solution extends the classical notion of broad solution for Burgers's equation through characteristic curves provided ϕ and w are locally Lipschitz continuous (see Definition 1.4.9 and Remark 3.3.2). In our case ϕ and w are supposed to be only continuous then the classical theory of solutions for ODEs breaks down and the notion of broad solution does not apply (see [49] for an interesting account of this subject and its recent developments). On the other hand broad* solutions ϕ of (4.1) can be constructed with a continuous datum w in such a way their intrinsic graph $S = G^1_{\mathbb{H},\phi}(\omega)$ looks as a fractal set from the Euclidean point of view (see Remark 4.1.6).

In section 4.2 we prove an important relationship between weak solution of (4.1) and $\mathbb{H}$-regular graphs, which characterize them. In analogy to euclidean case, if $\phi, w \in C^0(\omega)$, then ϕ is a distributional solution of (4.1) if and only if $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular. Indeed we conclude that the notions of distributional, entropy and broad* solution of (4.1) are the same in the case of ϕ continuous.

In section 4.3 we face the problem of existence and uniqueness of the broad* solution of the system (4.1). By proposition 1.2.9, Theorem 3.3.9 and Kružhkov's global uniqueness result for entropy solutions of conservation laws (see Theorem 1.3.3), we can infer a local uniqueness result for broad* solution of (4.1) uniformly bounded in ω provided initial value conditions (see Theorem 4.3.1). Using the duality between $\mathbb{H}$-regular graphs and the notion of broad* solution of (4.1) proved in section 4.1, we can see this uniqueness result by a geometrical point of view: in $\mathbb{H}^1$ if two $\mathbb{H}$-regular surfaces S_1, S_2 have a common vertical curve and the same horizontal normal, then $S_1 = S_2$. To obtain the same result in $\mathbb{H}^n$ it is enough that the $\mathbb{H}$-regular surfaces have only a common point and the same horizontal normal. As far as the existence of broad* solutions for (4.1) is concerned we will prove that there doesn't always exist for any assigned datum w, but only for suitable one. If $n \geq 2$ we will give a characterization in order to be a regular solution of (4.1) with regular datum w, indeed we write in Theorem 4.3.5 compatibility's conditions (among the regular components of the datum w) equivalent to the existence of broad* solution of (4.1), as pointed out in Remark 4.3.6.

In section 4.4 we will study the Euclidean regularity of a $\mathbb{H}$-regular graph $S = G^1_{\mathbb{H},\phi}(\omega)$ through the regularity of its intrinsic gradient $\nabla^\phi \phi$. In the main result we will prove that ϕ is locally Lipschitz continuous whenever $S = G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular and the $(n+1)$-th component $W^\phi \phi$ of the intrinsic gradient $\nabla^\phi \phi$ is locally Lipschitz continuous (see Theorem 4.4.1). Moreover a regularizing effect of the intrinsic gradient $\nabla^\phi \phi$ when $n \geq 2$ is stressed by an higher regularity result which fails if $n = 1$ (see Corollary 4.4.6 and Remark 4.4.7). More precisely if $\nabla^\phi \phi = w \in Lip(\omega, \mathbb{R}^{2n-1})$ a.e. in ω, then $\phi \in C^1(\omega)$, see Theorem 4.4.8.

4.1 $\mathbb{H}$-Regular Hypersurfaces and Weak Solutions of Non Linear First-Order PDEs

We can provide a characterization of $\mathbb{H}$-regular hypersurfaces of $\mathbb{H}^n$ in term of broad* solutions of the system (4.1) (see also [32], Theorem 1.4).

Theorem 4.1.1. *Let $\omega \subseteq \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$ be continuous functions. Then the following conditions are equivalent:*

i

$$\phi \text{ is a broad* solution of the system } \nabla^{\phi}\phi = w \quad \text{in } \omega\,; \tag{4.2}$$

ii $S = G^1_{\mathbb{H},\phi}(\omega)$ *is* $\mathbb{H}$*-regular and* $\nu_S^{(1)}(P) < 0$ *for all* $P \in S$*, where we denote with* $\nu_S(P) = \left(\nu_S^{(1)}(P), \dots, \nu_S^{(2n)}(P)\right)$ *the horizontal normal to* S *at a point* $P \in S$*. Moreover*

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1+|\nabla^{\phi}\phi|^2}}, \frac{\nabla^{\phi}\phi}{\sqrt{1+|\nabla^{\phi}\phi|^2}}\right)\left(\Phi^{-1}(P)\right)$$

$\forall\, P \in S$ *where* $\nabla^{\phi}\phi$ *denotes the intrinsic gradient of* Φ*.*

The proof of Theorem 4.1.1 relies on a preliminary result. The following is given in [4] though not explicitly stated.

Lemma 4.1.2. *Let* ϕ, w *be as in Theorem* 4.1.1*. Then the thesis of Theorem* 4.1.1 *holds provided that the condition*

$$\phi \in h^{\frac{1}{2}}_{loc}(\omega) \tag{4.3}$$

is also required in the statement **i**.

Proof. **i** $\Rightarrow$ **ii** The thesis follows at once by Theorems 3.2.12 and 3.3.8.
ii $\Rightarrow$ **i**: By Theorems 3.2.12 and 3.3.9 we get (4.3) holds and there is a family $(\phi_\epsilon)_\epsilon \subset C^1(\omega)$ such that

$$\phi_\epsilon \to \phi, \qquad \nabla^{\phi_\epsilon}\phi_\epsilon \to \nabla^{\phi}\phi \tag{4.4}$$

uniformly on the compact sets contained in ω. Finally by (4.4) and Lemma 3.3.5, we get (4.2). □

Proof of Theorem 4.1.1. We have only to prove that the assumption (4.3) can be omitted. It follows by the Hölder continuous regularity result for broad* solutions of Theorem 3.3.12. □

Remark 4.1.3. Let us explicitly point out that the characterization of ℍ-regular intrinsic graphs in Theorem 4.1.1 is not contained in [4] (see Theorems 3.2.12 and 3.3.9). Indeed the results contained in [4] yield the thesis of Theorem 4.1.1 provided the additional assumption that ϕ is little Hölder continuous of order 1/2 (see Lemma 4.1.2). Here the key step to the proof of Theorem 4.1.1 will be to gain 1/2-little Hölder continuity when ϕ is supposed to be only a (continuous) broad* solution of the system (4.1) (see Theorem 3.3.12).

By Theorem 4.1.1 we get that each (classical) Lipschitz continuous pointwise solution of the system $\nabla^\phi \phi = w$ with w continuous induces a $\mathbb{H}$-regular X^1-graph. More precisely

Corollary 4.1.4. *Let $\phi \in Lip_{loc}(\omega)$, $w \in C^0(\omega; \mathbb{R}^{2n-1})$ such that $\nabla^\phi \phi = w$ a.e. in ω. Then $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular. In particular $G^1_{\mathbb{H},\phi}(\omega)$ turns out to be $\mathbb{H}$-regular when $\phi \in C^1(\omega)$.*

Proof. By Theorem 4.1.1 we have only to prove (4.2). Let us pick $A \in \omega$, then by classical basic ODE theory there exists $0 < \delta_2 < \delta_1$ such that for each $B \in I_{\delta_2}(A)$, $\forall\, j = 2, ..., n$ there is an unique classical solution

$$\gamma_j^B : [-\delta_2, \delta_2] \to \overline{I_{\delta_1}(A)} \Subset \omega$$

of the Cauchy problem

$$\begin{cases} \dot{\gamma}_j^B(s) = \nabla_j^\phi \left(\gamma_j^B(s)\right) & \forall s \in [-\delta_2, \delta_2] \\ \gamma_j^B(0) = B. \end{cases}$$

Thus $(E.1)$ and $(E.2)$ of Definition 3.3.1 follow. On the other hand since $\phi \in Lip_{loc}(\omega)$ by the chain rule $[-\delta_2, \delta_2] \ni s \to \phi\left(\gamma_j^B(s)\right)$ is differentiable a.e. and

$$\frac{d}{ds}\phi\left(\gamma_j^B(s)\right) = w_j\left(\gamma_j^B(s)\right) \quad \text{a.e. } s \in [-\delta_2, \delta_2]\,.$$

Therefore $(E.3)$ follows too. □

Corollary 4.1.5. *Let $\phi \in C^0(\omega)$ and $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$. Let us assume that ϕ is a broad* solution of* (4.1)*. Then ϕ is also a distributional solution, i.e. for each $\varphi \in C_c^\infty(\omega)$*

$$\int_\omega \phi \widetilde{X}_i \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_i \varphi \, d\mathcal{L}^{2n} \quad \forall\, i \neq n+1 \tag{4.5}$$

and

$$\int_\omega \left(\phi \frac{\partial \varphi}{\partial \eta} + \frac{1}{2}\phi^2 \frac{\partial \varphi}{\partial \tau}\right) d\mathcal{L}^{2n} = -\int_\omega w_{n+1} \varphi \, d\mathcal{L}^{2n} \tag{4.6}$$

Proof. By Theorems 4.1.1 and 3.3.9 there exists a family $(\phi_\epsilon)_\epsilon \subset C^1(\omega)$ such that for each open set $\omega' \Subset \omega$

$$\phi_\epsilon \to \phi, \quad \nabla^{\phi_\epsilon} \phi_\epsilon \to w \qquad \text{uniformly in}\, \omega'.$$

Thus integrating by parts we get (4.5) and (4.6). □

Remark 4.1.6. Corollary 4.1.4 yields that $\mathbb{H}$-regular graphs could not be C^1 Euclidean regular. Actually in $\mathbb{H}^1 \simeq \mathbb{R}^3$ there are examples of $\mathbb{H}$-regular graphs $S = G^1_{\mathbb{H},\phi}(\omega)$ such that $\mathcal{H}^{2+\epsilon}(S) > 0\ \forall\, 0 < \epsilon < \frac{1}{2}$ (see [69]), i.e. S looks as a fractal set in $\mathbb{R}^3$ from Euclidean metric point of view. In particular by Theorem 4.1.1 the defining function $\phi : \omega \to \mathbb{R}$ of the graph is a broad* solution of the system $\nabla^\phi \phi = w$ in ω for a suitable continuous function $w : \omega \to \mathbb{R}$. Let us stress that, since S is not a 2-rectifiable set from Euclidean metric point of view, $\phi \notin BV_{loc}(\omega)$, where $BV_{loc}(\omega)$ denotes the set of the functions of locally bounded variations in ω (see also [4], Corollary 5.10). Moreover arguing as in [69] a similar $\mathbb{H}$-regular graph can be constructed in $\mathbb{H}^n$ with $n \geq 2$.

4.2 ℍ-Regular Hypersurfaces and Continuous Solutions of Non Linear First-Order PDEs

Now we are ready to state a new characterization of $\mathbb{H}$-regular graphs $\Phi(\omega)$, see [19].

Theorem 4.2.1. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

i *The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface and $\nu^1_S(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu^1_S(P), \dots, \nu^{2n}_S(P))$ is the horizontal normal to S at P.*

ii *There exists $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of the system* (4.1).

Remark 4.2.2. The characterization given in Theorem 4.2.1 is the exact counterpart of the distributional one in the Euclidean setting. Namely a function $\phi \in C^1(\omega)$ can be understood as a continuous distributional solution of $\nabla\phi = w$ in ω, provided $w \in C^0(\omega; \mathbb{R}^m)$ and $\omega \subset \mathbb{R}^m$ open set.

Remark 4.2.3. Let us observe that the strong approximation assumption $\nabla^{\phi_\epsilon}\phi_\epsilon \to \nabla^\phi\phi$ of Theorem 3.3.9 is not required in the statement of Theorem 4.2.1 **ii**. Its equivalence to the statement of Theorem 3.3.9 is not immediate. Our strategy will be to prove the equivalence between Theorem 4.2.1 and Theorem 4.1.1, i.e. to prove that each continuous distributional solution of the system (4.1) is a broad* solution.

On the other hand we do not know whether the approximation $\nabla^{\phi_\epsilon}\phi_\epsilon \to \nabla^\phi\phi$ can be directly obtained by recoursing to technical devices like mollification or approximation by vanishing viscosity of the continuous distributional solutions of the system (4.1). A very deep study of vanishing viscosity solutions with bounded variation of nonlinear hyperbolic systems has been carried out in [15] (see also the remark in [15], section 1.3). This study does not seem to apply to our context where the solution is supposed to be only continuous.

Proof of Theorem 4.2.1. **i** $\Rightarrow$ **ii**: It follows at once by Theorems 3.2.12 and 3.3.9.

ii$\Rightarrow$ **i**: Our strategy is to prove that each continuous distributional solutions of (4.1) is a broad* solution. Then the thesis will follow by Theorem 4.1.1. We will divide the proof in two steps.

Step 1. Let us assume $n = 1$. In this case ϕ is a distributional solution of Burgers' equation

$$\frac{\partial}{\partial \eta} u + \frac{1}{2} \frac{\partial}{\partial \tau}(u^2) = g \quad \text{in}\, \omega. \tag{4.7}$$

Let us fix $A_0 = (\eta_0, \tau_0) \in \omega$ and let $I_{2\delta_1}(A_0) \Subset \omega$. Let $M = \sup_{I_{2\delta_1}(A_0)} |\phi|$ and $\delta_2 = \min\left\{\frac{\delta_1}{4}, \frac{\delta_1}{2M}\right\}$, then Peano's Theorem yields that $\forall\, A = (\eta, \tau) \in \overline{I_{\delta_2}(A_0)}$ there exists a function $\xi^A \in C^1([-\delta_2, \delta_2])$ such that

$$\gamma^A(s) := (\eta + s,\, \tau + \xi^A(s)) \in I_{\delta_1}(A_0) \quad \forall s \in [-\delta_2, \delta_2]\,, \tag{4.8}$$

and ξ^A is a solution of the Cauchy problem

$$\begin{cases} \dot{\xi}(s) = u(s, \xi(s)) \\ \xi(0) = 0 \end{cases} \tag{4.9}$$

where $u : (-\delta_1, \delta_1) \times (-\delta_1, \delta_1) \to \mathbb{R}$ is the function $u(\tilde{\eta}, \tilde{\tau}) := \phi(\eta + \tilde{\eta},\, \tau + \tilde{\tau})$. On the other hand u is a continuous distributional solution of (4.7) with $g(\tilde{\eta}, \tilde{\tau}) := w_2(\eta + \tilde{\eta},\, \tau + \tilde{\tau})$ in $\omega = (-\delta_1, \delta_1) \times (-\delta_1, \delta_1)$. By Theorem 1.4.15, we have that $\big(\xi^A(s), \nu^A(s)\big)$ satisfies on $[-\delta_2, \delta_2]$ the system of ODEs

$$\begin{cases} \dot{\xi}^A(s) = \nu^A(s) \\ \dot{\nu}^A(s) = g(s, \xi^A(s)) \end{cases} \tag{4.10}$$

where $\nu^A(s) = u(s, \xi^A(s))$. In particular ν^A and $\dot{\xi}^A \in C^1([-\delta_2, \delta_2])$. Therefore the curve $\gamma^A : [-\delta_2, \delta_2] \to I_{\delta_1}(A_0)$ satisfies $(E.1)$, $(E.2)$ and $(E.3)$ for each $A \in I_{\delta_2}(A_0)$ and we are done.

Step 2. Let us assume $n \geq 2$. Let us notice that the $2n - 1$ vector fields $\widetilde{X}_2, \dots, \widetilde{X}_n, \widetilde{Y}_2, \dots, \widetilde{Y}_n, \widetilde{T}$ is the canonical basis of the Lie algebra $\mathfrak{h}_{n-1}$ associated to $\mathbb{H}^{n-1} \simeq \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau$. A point $p \in \mathbb{H}^{n-1}$ can be denoted by means of the usual identification with $\mathbb{R}^{2n-1}$ by $p = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n}, \tau)$. We will characterize the exponential maps of definition 3.3.1 as integral curves respectively of the $2(n-1)$ horizontal vector fields on $\mathbb{H}^{n-1}$, $\widetilde{X}_2, \dots, \widetilde{X}_n$, $\widetilde{Y}_2, \dots, \widetilde{Y}_n$, and of the vector field on $\mathbb{R}^2$

$$\frac{\partial}{\partial \eta} + \phi(\cdot, v_2, \dots, v_n, v_{n+2}, \dots, v_{2n}, \cdot)\,\frac{\partial}{\partial \tau}.$$

Let $A_0 = (\eta^0, v^0, \tau^0) \in \omega$ and let us fix $r > 0$ such that

$$I_r(A_0) = (\eta^0 - r, \eta^0 + r) \times U(v^0, r) \times (\tau^0 - r, \tau^0 + r) \Subset \omega \tag{4.11}$$

where $v^0 = (v^0_2, \dots, v^0_n, v^0_{n+2}, \dots, v^0_{2n}) \in \mathbb{R}^{2n-2}$ and $U(v^0, r)$ denotes the open ball in $\mathbb{R}^{2n-2}$ centered at v^0 with radius r. Let us denote

$$I_{\mathbb{H}^{n-1},r}(v^0, \tau^0) := U(v^0, r) \times (\tau^0 - r, \tau^0 + r)$$

$$I_{\mathbb{R}^2,r}(\eta^0, \tau^0) := (\eta^0 - r, \eta^0 + r) \times (\tau^0 - r, \tau^0 + r).$$

Let $\psi : I_r(A_0) \to \mathbb{R}$ be a given function then, for fixed $\bar\eta \in (\eta^0 - r, \eta^0 + r)$ and $\bar v \in U(v^0, r)$, let us denote the functions $\psi_{1,\bar\eta} : U(v^0, r) \times (\tau^0 - r, \tau^0 + r) \subset \mathbb{H}^{n-1} \to \mathbb{R}$, $\psi_{2,\bar v} : (\eta^0 1 - r, \eta^0 + r) \times (\tau^0 - r, \tau^0 + r) \subset \mathbb{R}^2 \to \mathbb{R}$ as

$$\psi_{1,\bar\eta}(v, \tau) := \psi(\bar\eta, v, \tau), \quad \psi_{2,\bar v}(\eta, \tau) := \psi(\eta, \bar v, \tau) \tag{4.12}$$

Let us observe now that, when $j \neq n+1$ and $A = (\eta, v, \tau)$, a C^1 curve $\gamma : [-\delta_2, \delta_2] \to \overline{I_{\delta_1}(A_0)}$ satisfies $(E.2)$ and $(E.3)$ iff

$$\gamma(s) = (\eta, \exp(s\,\widetilde{X}_j)(v, \tau)) \quad \forall s \in [-\delta_2, \delta_2]\,, \tag{4.13}$$

$[-\delta_2, \delta_2] \ni s \to \phi_{1,\eta}(\exp(s\,\widetilde{X}_j)(v, \tau))$ is C^1 and

$$\frac{d}{ds}\phi_{1,\eta}\big(\exp(s\,\widetilde{X}_j)(v, \tau)\big) = w_{j,1,\eta}\big(\exp(s\,\widetilde{X}_j)(v, \tau)\big) \quad \forall s \in [-\delta_2, \delta_2]\,, \tag{4.14}$$

Whereas, when $j = n+1$ and $A = (\eta, v, \tau)$, $\gamma : [-\delta_2, \delta_2] \to \overline{I_{\delta_1}(A_0)}$ satisfies $(E.2)$ and $(E.3)$ iff there exists a C^1 function $\xi : [-\delta_2, \delta_2] \to [\tau^0 - \delta_1, \tau^0 + \delta_1]$ such that

$$\gamma(s) = (\eta + s, v, \tau + \xi(s)), \quad \begin{cases} \dot\xi(s) = \phi_{2,v}(\eta + s, \tau + \xi(s)) \\ \xi(0) = 0 \end{cases} \quad \forall s \in [-\delta_2, \delta_2] \tag{4.15}$$

$[-\delta_2, \delta_2] \ni s \to \phi_{2,v}\big(\eta + s, \tau + \xi(s)\big)$ is C^1 and

$$\frac{d}{ds}\phi_{2,v}\big(\eta + s, \tau + \xi(s)\big) = w_{j,2,v}\big(\eta + s, \tau + \xi(s)\big) \quad \forall s \in [-\delta_2, \delta_2]\,, \tag{4.16}$$

where $\phi_{i,\eta}$, $w_{j,i,\eta}$, $\phi_{i,v}$ and $w_{j,i,v}$ are the functions defined in (4.12) respectively with $\psi \equiv \phi$ and $\psi \equiv w_j$.

By (4.11) and selecting test functions in (4.5) and (4.6) respectively of the type

$$\varphi(\eta, v_2, \ldots, v_n, v_{n+2}, \ldots, v_{2n}, \tau) = \varphi_1(\eta)\, \varphi_2(v_2, \ldots, v_n, v_{n+2}, \ldots, v_{2n}, \tau)$$

with $\mathrm{supp}(\varphi_1) \Subset (\eta^0 - r, \eta^0 + r)$, $\mathrm{supp}(\varphi_2) \Subset I_{\mathbb{H}^{n-1},r}(v^0, \tau^0) := U(v^0, r) \times (\tau^0 - r, \tau^0 + r)$ and

$$\varphi(\eta, v_2, \ldots, v_n, v_{n+2}, \ldots, v_{2n}, \tau) = \varphi_1(\eta, \tau)\, \varphi_2(v_2, \ldots, v_n, v_{n+2}, \ldots, v_{2n})$$

with $\mathrm{supp}(\varphi_1) \Subset I_{\mathbb{R}^2,r}(\eta^0, \tau^0) := (\eta^0 - r, \eta^0 + r) \times (\tau^0 - r, \tau^0 + r)$, $\mathrm{supp}(\varphi_2) \Subset U(v^0, r)$ we get

$$\int_{I_{\mathbb{H}^{n-1},r}(v^0,\tau^0)} \phi_{1,\eta}\, \widetilde{X}_j \varphi \, d\mathcal{L}^{2n-1} = -\int_{I_{\mathbb{H}^{n-1},r}(v^0,\tau^0)} w_{j,1,\eta}\, \varphi \, d\mathcal{L}^{2n-1} \quad \forall\, j \neq n+1 \tag{4.17}$$

for each $\varphi \in C_c^\infty(I_{\mathbb{H}^{n-1},r}(v^0, t^0))$ and $\eta \in (\eta^0 - r, \eta^0 + r)$,

$$\int_{I_{\mathbb{R}^2,r}(\eta^0,\tau^0)} \left(\phi_{2,v} \frac{\partial \varphi}{\partial \eta} + \frac{1}{2} \phi_{2,v}^2 \frac{\partial \varphi}{\partial \tau} \right) d\mathcal{L}^2 = -\int_{I_{\mathbb{R}^2,r}(\eta^0,\tau^0)} w_{n+1,2,v}\, \varphi \, d\mathcal{L}^2 \tag{4.18}$$

for each $\varphi \in C_c^\infty\left(I_{\mathbb{R}^2,r}(\eta^0, \tau^0)\right)$ and $v \in U(v^0, r)$. Let us notice now that (4.17) means, by definition, that $\phi_{1,\eta} \in C^1_{\mathbb{H}}\left(I_{\mathbb{H}^{n-1},r}(v^0, \tau^0)\right)$ with respect to the horizontal differentiable structure of $\mathbb{H}^{n-1}$, for each $\eta \in (\eta^0 - r, \eta^0 + r)$. Meanwhile (4.18) and the previous step 1 yield that the intrinsic graph $\Phi\left(I_{\mathbb{R}^2,r}(\eta^0, \tau^0)\right) \subset \mathbb{H}^1$ induced by the function $\phi \equiv \phi_{2,v}$ is $\mathbb{H}$-regular for each $v \in U(v^0, r)$.

We have to prove to achieve the proof the existence, for every $j = 2, \ldots, 2n$, of an exponential map

$$\exp_{A_0}(\cdot \nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset \omega\,.$$

Namely for fixed $A \in \overline{I_{\delta_2}(A_0)}$ the existence of a curve $\gamma^A(s) := \exp_{A_0}\left(s\, \nabla_j^\phi\right)(A)$ satisfying (4.13) and (4.14) when $j \neq n+1$, and (4.15) and (4.16) when $j = n+1$.

If $j \neq n+1$, by classical ODEs' results, there exist $0 < \delta_2 < \delta_1$ such that

$$\exp(\cdot\, \widetilde{X}_j)(\cdot) : \; [-\delta_2, \delta_2] \times \overline{I_{\mathbb{H}^{n-1},\delta_2}(v^0, \tau^0)} \to \overline{I_{\mathbb{H}^{n-1},\delta_1}(v^0, \tau^0)}\,.$$

Let $A = (\eta, v, \tau) \in \overline{I_{\delta_2}(A_0)}$. Then $\gamma^A(s) := \big(\eta, \exp(s\,\widetilde{X}_j)(v,\tau)\big)$ satisfies (4.13) by construction and (4.14) by proposition 2.2.15 applied with $\Omega \equiv I_{\mathbb{H}^{n-1},\delta_1}(v^0,\tau^0)$, $f \equiv \phi_{1,\eta}$ and $g_j \equiv w_{j,1,\eta}$, since $\phi_{1,\eta} \in C^1_{\mathbb{H}}\big(I_{\mathbb{H}^{n-1},\delta_1}(v^0,\tau^0)\big)$.

If $j = n+1$, let $I_{2\delta_1}(A_0) \Subset \omega$ and let $M = \sup_{I_{2\delta_1}(A_0)} |\phi|$ and $\delta_2 = \min\left\{\frac{\delta_1}{4}, \frac{\delta_1}{2\,M}\right\}$. Then we can repeat verbatim the construction of the step 1 to the function $\phi_{2,v} : I_{\mathbb{R}^2, 2\,\delta_1}(\eta^0,\tau^0) \to \mathbb{R}$ for each $v \in \overline{U(v^0,\delta_2)}$. Indeed let $A = (\eta, v, \tau) \in \overline{I_{\delta_2}(A_0)}$, then there exists a C^1 function $\xi : [-\delta_2,\delta_2] \to \mathbb{R}$ such that the curve $\gamma^A(s) := (\eta + s, v, \tau + \xi(s)) \in I_{\delta_1}(A_0)\,\forall s \in [-\delta_2,\delta_2]$ and ξ is a solution of the Cauchy problem (4.9) with $u(\tilde\eta,\tilde\tau) = \phi_{2,v}(\eta+\tilde\eta, \tau+\tilde\tau)$ if $(\tilde\eta,\tilde\tau) \in (-\delta_1,\delta_1)\times(-\delta_1,\delta_1)$, for each $v \in \overline{U(v^0,\delta_2)}$. In particular γ^A satisfies (4.15). Moreover also (4.16) holds since the function $\nu^A(s) := u(s,\xi(s)) = \phi_{2,v}(\eta + s, \tau + \xi(s)) \in Lip([-\delta_2,\delta_2])$ and satisfies the system (4.10) with $g(\tilde\eta.\tilde\tau) := w_{n+1,2,v}(\eta+\tilde\eta, \tau+\tilde\tau)$, for each $v \in \overline{U(v^0,\delta_2)}$. □

Let us stress the following link between $\mathbb{H}$-regular hypersurfaces and entropy solution of Burgers' equation. This implication is automatically contained in Theorem 4.2.1.

Proposition 4.2.4. *Let $\omega = (-r_0,r_0)\times(-r_0,r_0)$. Let us assume that $S = G^1_{\mathbb{H},\phi}(\omega) \subset \mathbb{H}^1$ is $\mathbb{H}$-regular and let $w := \nabla^\phi\phi \in C^0(\omega)$. Then ϕ is an entropy solution of the initial value problem*

$$\begin{cases} u_\eta + \left(\frac{u^2}{2}\right)_\tau = w & \text{in}\,(0,r_0)\times(-r_0,r_0) \\ u(0,\tau) = \phi(0,\tau) & \forall\,\tau \in [-r_0,r_0] \end{cases}$$

Proof. By Theorems 3.2.12 and 3.3.9 there exists a family $(\phi_\epsilon) \subset C^1(\omega)$ such that, for any open set $\omega' \Subset \omega$, we have

$$\phi_\epsilon \to \phi \quad \nabla^{\phi_\epsilon}\phi_\epsilon = W^{\phi_\epsilon}\phi_\epsilon \to w \qquad \text{uniformly in}\,\omega'. \tag{4.19}$$

Let us define $u_\epsilon(\eta,\tau) := \phi_\epsilon(\eta,\tau)$, $g_\epsilon(\eta,\tau) := W^{\phi_\epsilon}\phi_\epsilon(\eta,\tau)$ for $(\eta,\tau) \in (0,r_0)\times(-r_0,r_0)$, then by (4.19) and Proposition 1.2.9 we have done. □

Remark 4.2.5. Theorem 4.2.1 yields that the notions of distributional, entropy and broad* solution of (4.1) are the same in the case of ϕ continuous. Indeed we have the following link:

i ϕ distributional solution $\iff$ $\Phi(\omega)$ $\mathbb{H}$-regular $\iff$ ϕ broad* solution, by Theorems 4.1.1 and 4.2.1.

ii ϕ distributional solution $\Longrightarrow$ ϕ broad* solution, by Theorem 1.4.17.

iii ϕ entropy solution $\Longrightarrow$ ϕ distributional solution, by defintion 1.2.4.

iv ϕ broad* solution $\Longrightarrow$ $\Phi(\omega)$ $\mathbb{H}$-regular $\Longrightarrow$ ϕ entropy solution, by Theorem 4.1.1 and proposition 4.2.4.

4.3 $\nabla^\phi \phi = w$: uniqueness and existence

We are going now to study the local uniqueness in $h_M^{\frac{1}{2}}(\overline{\omega})$ and existence in $C^2(\overline{\omega})$ of broad* solution of the system $\nabla^\phi \phi = w$. Let us begin with the problem of the uniqueness.

Theorem 4.3.1. *Let $M > 0$, $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2(n-1)} \times \mathbb{R}_\tau$ if $n \geq 2$, $r_0 > 0$, $w = (w_2, ..., w_{2n}) \in C^0\left(I_{r_0}(A_0); \mathbb{R}^{2n-1}\right)$ be given. Let $\phi_i \in C^0\left(\overline{I_{r_0}(A_0)}\right)$ verifying*

$$|\phi_i(A)| \leq M \quad \forall A \in \overline{I_{r_0}(A_0)} \qquad (i = 1, 2).$$

i *Let $n = 1$, $\phi_0 \in C^0([\tau_0 - r_0, \tau_0 + r_0])$, let ϕ_i $(i = 1, 2)$ be two broad* solutions of the initial value problem*

$$\begin{cases} W^\phi \phi = w & \text{in } I_{r_0}(A_0) \\ \phi(\eta_0, \tau) = \phi_0(\tau) & \forall \tau \in [\tau_0 - r_0, \tau_0 + r_0] \end{cases} \tag{4.20}$$

Then $\phi_1 = \phi_2$ in $I_r(A_0)$ if $0 < r < \dfrac{r_0}{1 + M}$.

ii *Let $n \geq 2$, $\alpha \in \mathbb{R}$ let ϕ_i $(i = 1, 2)$ be two broad* solutions of the initial value problem*

$$\begin{cases} \nabla^\phi \phi = w & \text{in } I_{r_0}(A_0) \\ \phi(A_0) = \alpha \end{cases} \tag{4.21}$$

Then $\phi_1 = \phi_2$ in $I_r(A_0)$ if $0 < r < \dfrac{r_0}{1 + M}$.

Remark 4.3.2. It is well-known that the uniqueness fails for the problem

$$\begin{cases} W^\phi \phi = 0 & \text{in } I_{r_0}((0,0)) \\ \phi(\eta, 0) = 0 & \forall \tau \in [-r_0, r_0] \end{cases} . \tag{4.22}$$

Indeed, for instance, the functions $\phi_1 := 0$ and $\phi_2(\eta, \tau) := \dfrac{\tau}{\eta + c}$ with $c \in \mathbb{R} \setminus \{0\}$ are broad* solutions of (4.22) for r_0 small enough.

Proof. **i** First let us observe without loss of generality we can assume that $A_0 = (0, 0)$. Otherwise let us consider $\phi^*(\eta, \tau) = \phi(\eta - \eta_0, \tau - \tau_0)$ and the associated initial value problem.

$$\begin{cases} W^{\phi^*} \phi^* = w^* & \text{in } I_{r_0}((0,0)) \\ \phi^*(0, \tau) = \phi_0^*(\tau) & \forall \tau \in [-r_0, r_0] \end{cases} \tag{4.23}$$

where $w^*(\eta,\tau) = w(\eta-\eta_0,\tau-\tau_0)$, $\phi_0^*(\tau) = \phi_0(\tau-\tau_0)$, $(\eta,\tau) \in I_{r_0}((0,0))$, $\tau \in [-r_0, r_0]$. Then it is easy to see by definition that ϕ is a broad* solution of (4.20) if and only if ϕ^* is a broad* solution of (4.23).

Let ϕ_i, $i = 1, 2$, be two broad* solutions of the problem (4.20). Then by Theorem 4.1.1 $S_i = G^1_{\mathbb{H},\phi_i}\left(\overline{I_{r_0}((0,0))}\right)$ are $\mathbb{H}$-regular if $\omega = I_{r_0}((0,0))$. On the other hand by Proposition 4.2.4 ϕ_i are entropy solutions of the problem

$$\begin{cases} u_\eta + u u_\tau = g & \text{in } (0,r_0)\times(-r_0,r_0) \\ u(0,\tau) = \phi_0(\tau) & \forall \tau \in [-r_0,r_0] \end{cases} \tag{4.24}$$

with $g(\eta,\tau) = w(\eta,\tau)$. Thus Corollary 1.3.4 yields that for $r < \dfrac{r_0}{1+M}$

$$\phi_1 = \phi_2 \qquad \mathcal{L}^2 - \text{a.e. in } (0,r)\times(-r,r)$$

and by the continuity of ϕ_i we get

$$\phi_1 = \phi_2 \qquad \text{in } (0,r)\times(-r,r). \tag{4.25}$$

On the other hand it is easy to see, arguing in the same way, that $u_i(\eta,\tau) = -\phi_i(-\eta,\tau)$ $(\eta,\tau) \in [0,r_0]\times[-r_0,r_0]$ still turns out to be entropy solutions of the problem (4.24) with $g(\eta,\tau) = w(-\eta,\tau)$ $(\eta,\tau) \in (0,r_0)\times(-r_0,r_0)$, then

$$\phi_1 = \phi_2 \qquad \text{in } (-r,0)\times(-r,r). \tag{4.26}$$

Thus by (4.25) and (4.26) we achieve the proof.

ii Arguing as before we can assume that $A_0 = (0, v_0, 0)$. Let ϕ_i, $i = 1, 2$, be broad* solutions of (4.21) with $n \geq 2$. Let us fix $\eta \in (-r_0, r_0)$ and let us define

$$f_i^{(\eta)}(v,\tau) = \phi_i(\eta,v,\tau) \qquad (v,\tau) \in U(v_0,r_0)\times(-r_0,r_0).$$

By Theorems 4.1.1, 3.2.12 and 3.3.9 there exist two families $(\phi_{i,\epsilon})_\epsilon \subset C^1(I_r(A_0))$ such that

$$\phi_{i,\epsilon} \to \phi_i, \quad \nabla^{\phi_{i,\epsilon}}\phi_{i,\epsilon} \to w \quad \text{uniformly in } \overline{I_r(A_0)} \tag{4.27}$$

for every $0 < r < r_0$. In particular from (4.27), for a fixed $\eta \in (-r,r)$

$$\widetilde{\nabla}_{\mathbb{H}} f_i^{(\eta)} = \hat{w}_{n+1}(\eta,\cdot,\cdot) \quad \text{in } U(v_0,r)\times(-r,r) \tag{4.28}$$

in distributional sense. By linearity, if $f^{(\eta)}(v,\tau) := f_1^{(\eta)}(v,\tau) - f_2^{(\eta)}(v,\tau)$

$$\widetilde{\nabla}_{\mathbb{H}} f^{(\eta)} = 0 \quad \text{in } U(v_0,r)\times(-r,r) \tag{4.29}$$

in distributional sense. By (4.29) and Lemma 2.2.12 there exists a function $\psi = \psi(\eta) : (-r, r) \to \mathbb{R}$ such that

$$\phi_2(\eta, v, \tau) = \psi(\eta) + \phi_1(\eta, v, \tau) \quad \forall(\eta, v, \tau) \in I_r(A_0). \tag{4.30}$$

Since $\phi_i(A_0) = \alpha$, $i = 1, 2$, by (4.30) we get that $\psi(0) = 0$. Then

$$\phi_0^{(v)} := \phi_1(0, v, \tau) = \phi_2(0, v, \tau) \quad \forall\, (v, \tau) \in U(v_0, r) \times (-r, r). \tag{4.31}$$

Let us fix now $v \in U(v_0, r)$ and let us define

$$u_i \equiv u_i^{(v)}(\eta, \tau) := \phi_i(\eta, v, \tau) \quad (\eta, \tau) \in (0, r) \times (-r, r).$$

In order to achieve the proof it is enough to show that u_i, $i = 1, 2$, are entropy solutions of the initial value problem

$$\begin{cases} u_\eta + uu_\tau = g & \text{in } (0, \tau) \times (-r_0, r_0) \\ u(0, \tau) = \phi_0^{(v)}(\tau) & \forall \tau \in [-r_0, r_0] \end{cases} \tag{4.32}$$

where $g(\eta, \tau) := w_{n+1}(\eta, v, \tau)$. Indeed by Corollary 1.3.4 and arguing as before we can conclude that $\phi_1 = \phi_2$ in $I_r(A_0)$. For fixed $v \in U(v_0, r)$ let, for $i = 1, 2$,

$$u_{i,\epsilon}(\eta, \tau) := \phi_{i,\epsilon}(\eta, v, \tau) \quad (\eta, \tau) \in [0, r] \times [-r, r],$$
$$g_{i,\epsilon}(\eta, \tau) := \nabla^{\phi_{i,\epsilon}} \phi_{i,\epsilon}(\eta, v, \tau) \quad (\eta, \tau) \in [0, r] \times [-r, r].$$

By (4.27) and Proposition 1.2.9 we infer at once that u_i are entropy solutions of the problem (4.32) and we have done. $\square$

Theorem 4.3.1 yields the following local uniqueness result for $\mathbb{H}$-regular graphs with a prescribed horizontal normal.

Corollary 4.3.3. *Let M, $r_0 > 0$, $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2(n-1)} \times \mathbb{R}_\tau$ if $n \geq 2$. Let $w = (w_2, \ldots, w_{2n}) : I_{r_0}(A_0) \to \mathbb{R}^{2n-1}$ be continuous and let us denote*

$$\nu(A) := \left(-\frac{1}{\sqrt{1 + |w(A)|^2}}, \frac{w(A)}{\sqrt{1 + |w(A)|^2}} \right) \quad A \in I_{r_0}(A_0)\,.$$

Let $\Gamma_0 \subset \mathbb{H}^n$, $P_0 \in \mathbb{H}^n$ be

$$\Gamma_0 := \{(0, \eta_0, \tau) \cdot \phi_0(\tau)\, e_1 : \tau \in [\tau_0 - r_0, \tau_0 + r_0]\}\,, \quad P_0 = (0, \eta_0, \tau_0) \cdot \phi_0(\tau_0) e_1$$

if $n = 1$ where $\phi_0 \in C^0([\tau_0 - r_0, \tau_0 + r_0])$ is given, and

$$\Gamma_0 := \{\, P_0\}, \quad P_0 = (0, \eta_0, v_0, \tau_0) \cdot \alpha\, e_1$$

if $n \geq 2$ *where* $\alpha \in \mathbb{R}$ *is given. Let* $S_i = G^1_{\mathbb{H},\phi_i}(I_{r_0}(A_0)) \subset \mathbb{H}^n$ $(i = 1, 2)$ *be two* $\mathbb{H}$*-regular graphs such that*

$$|\phi_i(A)| \leq M \quad \forall A \in I_{r_0}(A_0)\ (i = 1, 2)\,,$$

$$\nu_{S_i}(P) = \nu\left(\Phi_i^{-1}(P)\right) \quad \forall P \in S_i \cap U_\infty(P_0, r_0), \quad \Gamma_0 \cap U_\infty(P_0, r_0) \subset S_i \cap U_\infty(P_0, r_0)$$

where $\Phi_i : I_{r_0}(A_0) \to \mathbb{H}^n$ *is the parameterization in* (3.4) *with* $\phi \equiv \phi_i$*. Then, if* $0 < r < \dfrac{r_0}{1+M}$,

$$\phi_1 = \phi_2 \quad in\ I_r(A_0)$$

Now let us deal with the problem of the local existence of broad* solutions for the system (4.1). We are going to prove there are broad* solution of the problems (4.20) and (4.21) for arbitrary initial value conditions for suitable data w.

Theorem 4.3.4. *Let* $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ *if* $n = 1$*,* $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2(n-1)} \times \mathbb{R}_\tau$ *if* $n \geq 2$*.*

i *Let* $n = 1$*,* $\phi_0 \in h^{\frac{1}{2}}([\tau_0 - r_0, \tau_0 + r_0])$*,* $w_0 \in C^0([\tau - 0 - r_0, \tau_0 + r_0])$ *be given. Then there exist* $\phi, w \in C^0\left(\overline{I_{r_0}(A_0)}\right)$ *such that* ϕ *is a broad* solution of the initial value problem* (4.20) *for* r_0 *small enough and* $w \equiv w_0$ *on* $[\tau_0 - r_0, \tau_0 + r_0]$*.*

ii *Let* $n \geq 2$*,* $\alpha \in \mathbb{R}$*,* $w^0 = (w_2^0, ..., w_{2n}^0) \in \mathbb{R}^{2n-1}$ *be given. Then the function*

$$\phi(\eta, v, \tau) = \alpha + w^0_{n+1} \cdot (\eta - \eta_0) + \sum_{\substack{i=2 \\ i \neq n+1}}^{2n} w_i^0 \cdot (v_i - v_i^0) \qquad (\eta, v, \tau) \in \mathbb{R}^{2n}$$

is a broad solution of the problem* (4.21) *with* $w = w^0$*.*

Proof. **i** Let us observe that, arguing as in the proof of Theorem 4.3.1 **i** we can assume that $A_0 = (0, 0)$. With the notation of Theorem 3.1.9 let

$$F := \{(\phi_0(\tau), 0, \tau) : \tau \in [-r_0, r_0]\}$$

$f \equiv 0$, $\quad k : F \to H\mathbb{H}^1 \simeq \mathbb{R}^2$, $\quad k(x, y, t) := \left(1, -w_0\left(y, t + \dfrac{xy}{2}\right)\right)$ if $(x, y, t) \in F$.

Let $Q = (\phi_0(\tau'), 0, \tau')$, $P = (\phi_0(\tau), 0, \tau)$ with $\tau \neq \tau' \in [-r_0, r_0]$, then

$$|R(Q, P)| = \frac{|f(Q) - f(P) - \langle k(P), \pi_p(P^{-1} \cdot Q)\rangle_P|}{d_\infty(P, Q)} = \tag{4.33}$$

$$= \frac{|-(\phi_0(\tau')-\phi_0(\tau))+w_0(0,\tau)\cdot 0|}{\max\left\{|\phi(\tau')-\phi(\tau)|,\ \sqrt{|\tau'-\tau|}\right\}} \le \frac{|\phi_0(\tau')-\phi_0(\tau)|}{\sqrt{|\tau'-\tau|}}$$

Since $\phi_0 \in h^{\frac{1}{2}}([-r_0,+r_0])$, for compact set $K \subseteq F$, by (4.33) we get

$$\lim_{\delta\to 0^+} \rho_K(\delta) = 0.$$

Then by Whitney's extension Theorem 3.1.9 there exists $\tilde f : \mathbb{H}^1 \to \mathbb{R}$, $\tilde f \in C^1_{\mathbb{H}}(\mathbb{H}^1)$ such that

$$\tilde f = 0 \quad \text{and} \quad \nabla_{\mathbb{H}} \tilde f = k \quad \text{in } F. \tag{4.34}$$

Let $P_0 := (\phi_0(0,0),0,0) \in F$, $g(P) := \tilde f(P_0 \cdot P)$ for $P \in \mathbb{H}^1$, $S = \{P \in \mathbb{H}^1 : g(P) = 0\}$. Since $g \in C^1_{\mathbb{H}}(\mathbb{H}^1)$, $0 \in S$, $X_1 g(0) = 1$ by the Implicit Function Theorem 3.1.13 and Proposition 3.1.17 there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{H}^1$ of 0 such that

$$S \cap \mathcal{U} \text{ is } \mathbb{H}\text{-regular.} \tag{4.35}$$

Moreover there exist $\delta > 0$ and an unique continuous function $\widetilde\phi : \widetilde I = [-\delta,\delta] \times [-\delta^2,\delta^2] \to \mathbb{R}$ such that

$$\widetilde\Phi\left(\widetilde I\right) = G^1_{\mathbb{H},\widetilde\phi}\left(\widetilde I\right) = S \cap \overline{\mathcal{U}} \tag{4.36}$$

if $\widetilde\Phi(\tilde\eta,\tilde\tau) = (0,\tilde\eta,\tilde\tau)\cdot\widetilde\phi(\tilde\eta,\tilde\tau)e_1$ with $(\tilde\eta,\tilde\tau) \in \widetilde I$ and

$$\mathfrak{B}\widetilde\phi = \widetilde w \quad \text{in } \widetilde I \tag{4.37}$$

in distributional sense, where

$$\widetilde w(\tilde\eta,\tilde\tau) = \left(-\frac{Y_1 g}{X_1 g}\circ\Phi\right)(\tilde\eta,\tilde\tau) = -\frac{Y_1\tilde f}{X_1\tilde f}\left(P_0\cdot\widetilde\Phi(\tilde\eta,\tilde\tau)\right).$$

Let us perform now the change of variable $\psi : \widetilde I \to \mathbb{R}^2$

$$\psi(\tilde\eta,\tilde\tau) = (\tilde\eta,\tilde\tau+\phi_0(0)\tilde\eta) = (\eta,\tau)$$

and let $I := \psi\left(\widetilde I\right)$. Let us define

$$\phi(\eta,\tau) := \phi_0(0,0) + \widetilde\phi(\eta,\tau-\phi_0(0)\eta) \quad (\eta,\tau)\in I.$$

Then by (4.36)

$$S_0 := \tau_{P_0}\left(S\cap\overline{\mathcal{U}}\right) = \tau_{P_0}\left(G^1_{\mathbb{H},\widetilde\phi}(\widetilde I)\right) = G^1_{\mathbb{H},\phi}(I). \tag{4.38}$$

Let $r_0 > 0$ so small such that $\overline{I_{r_0}(0,0)} \subset I$. By (4.34), (4.35) and (4.38) we get that

$$\phi(0,\tau) = \phi_0(\tau) \quad \forall \tau \in [-r_0, r_0], \tag{4.39}$$

$$G^1_{\mathbb{H},\phi}(I_{r_0}(0,0)) \,\text{is}\, \mathbb{H}\text{-regular}. \tag{4.40}$$

On the other hand it is easy to see that by (4.37)

$$\mathfrak{B}\phi = w \quad \text{in}\, I_{r_0}(0,0) \tag{4.41}$$

in distributional sense, where

$$w(\eta,\tau) = \widetilde{w}\left(\psi^{-1}(\eta,\tau)\right) \qquad (\eta,\tau) \in I_{r_0}(0,0).$$

Thus by (4.40) and (4.41) and Theorem 3.3.9 we get

$$\phi \in h^{\frac{1}{2}}_{loc}\left(I_{r_0}(0,0)\right) \tag{4.42}$$

and

$$W^\phi \phi = w \quad \text{in}\, I_{r_0}(0,0). \tag{4.43}$$

Finally by (4.39), (4.40), (4.43) and Theorem 4.1.1 we get that ϕ is a broad* solution of $W^\phi\phi = w$ in ω.

ii It is an easy calculation. $\square$

Now we are going to see that in the case $n \geq 2$ there are C^2-regular solutions of the system $\nabla^\phi \phi = w = (w_2, ..., w_{2n})$ in ω provided compatibility's conditions among the components w_i.

Theorem 4.3.5. *Let us denote $\omega = (\eta_0 - r_0, \eta_0 + r_0) \times \hat{\omega}$ where $\hat{\omega} \subseteq \mathbb{R}^{2(n-1)}$ is an open set and $P_0 = (\eta_0, v_0, \tau_0)$ and $r > 0$. Let $w = (w_2, ..., w_{n+1}, ..., w_{2n}) \in C^2(\omega; \mathbb{R}^{2n-1})$, $n \geq 2$. Let us define*

$$\psi(\eta, v, \tau) := \left(\widetilde{X}_2 w_{2+n} - \widetilde{Y}_2 w_2\right)(\eta, v, \tau)$$

$$E(\eta, v, \tau) = e^{-\int_{\eta_0}^{\eta} \psi(\eta', v, \tau)\, d\eta'}$$

$$I(\eta, v, \tau) = \int_{\eta_0}^{\eta} \frac{w_{n+1}(\eta', v, \tau)}{E(\eta', v, \tau)}\, d\eta'$$

$$E_1(\eta, v, \tau) = E(\eta, v, \tau) I(\eta, v, \tau)$$

$$\underline{a} = (a_2, ..., a_n, a_{n+2}, ..., a_{2n}) \qquad a_j = \frac{\widetilde{X}_j E}{E}$$

$$\underline{b} = (b_2, ..., b_n, b_{n+2}, ..., b_{2n}) \qquad b_j = \frac{w_j - \widetilde{X}_j E_1}{E}$$

where $\eta_0 \in \mathbb{R}$ is fixed and $\hat{w}_{n+1} := (w_2, ..., w_n, w_{n+2}, ..., w_{2n})$. Then the following statements are equivalent:

i *There exists* $\phi \in C^2(\omega)$ *solution of* (4.1).

ii *There exists* $C \in C^2(\hat\omega)$ *such that*

$$\widetilde{\nabla}_{\mathbb{H}} C(v,\tau) = \hat{w}_{n+1}(\eta_0, v, \tau) \qquad \forall\,(v,\tau) \in \hat\omega \tag{4.44}$$

and

$$\underline{a}(\eta, v, \tau)C(v,\tau) = \underline{b}(\eta, v, \tau) - \underline{b}(\eta_0, v, \tau). \tag{4.45}$$

$\forall \eta \in (\eta_0 - r, \eta_0 + r)$, $\forall (v,\tau) \in \hat\omega$. *Moreover* ϕ *and* C *are linked by the relation*

$$\phi(\eta, v, \tau) = E_1(\eta, v, \tau) + E(\eta, v, \tau)C(v,\tau). \tag{4.46}$$

Proof. It is not restrictive to assume $\eta_0 = 0$.

i $\Rightarrow$ **ii** Let us assume that there exists $\phi \in C^2(\omega)$ such that $\nabla^\phi \phi = w$ in ω. Let us observe that

$$\frac{\partial \phi}{\partial \tau} = \widetilde{T}\phi = [\widetilde{X}_2 \widetilde{Y}_2 - \widetilde{Y}_2 \widetilde{X}_2]\phi = \widetilde{X}_2 w_{2+n} - \widetilde{Y}_2 w_2 =: \psi. \tag{4.47}$$

Thus we can linearize the system getting

$$\widetilde{\nabla}_{\mathbb{H}} \phi = \hat{w}_{n+1} \qquad \text{in}\,\omega \tag{4.48}$$

$$\frac{\partial \phi}{\partial \eta} + \phi\psi = w_{n+1} \qquad \text{in}\,\omega \tag{4.49}$$

For fixed $(v,\tau) \in \hat\omega$, by the uniqueness of linear ODE (4.49), we can represent ϕ as

$$\phi(\eta, v, \tau) = E_1(\eta, v, \tau) + E(\eta, v, \tau)\phi(0, v, \tau) \tag{4.50}$$

Let us denote

$$C(v,\tau) := \phi(0, v, \tau) \qquad (v,\tau) \in \hat\omega$$

and let us prove (4.45). By (4.48) and (4.50) we get that

$$\hat{w}_{n+1} = \widetilde{\nabla}_{\mathbb{H}}(E_1 + EC) = \widetilde{\nabla}_{\mathbb{H}} E_1 + \widetilde{\nabla}_{\mathbb{H}} E \cdot C + E\widetilde{\nabla}_{\mathbb{H}} C$$

and then $\forall\,(\eta, v, \tau) \in \omega$

$$\widetilde{\nabla}_{\mathbb{H}} C(v,\tau) + \underline{a}(\eta, v, \tau)C(v,\tau) = \underline{b}(\eta, v, \tau). \tag{4.51}$$

By choosing $\eta = 0$, since $\underline{b}(0, v, \tau) = \hat{w}_{n+1}(0, v, \tau)$ and $\underline{a}(0, v, \tau) \equiv 0$ we get at once (4.44) and (4.45).

ii $\Rightarrow$ **i** Let us assume that there exists $C \in C^2(\hat\omega)$ such that (4.44) and (4.45) hold. Let us define ϕ as in (4.50) with $C(\eta,\tau) \equiv \phi(0, \eta, \tau)$, then it is easy to verify that $\nabla^\phi \phi = w$ in ω. $\square$

Remark 4.3.6. Let us explicitly point out the system (4.1) differs from system (2.34). For instance, let us assume that $w \in C^2(\mathbb{R}^{2n}, \mathbb{R}^{2n-1})$ such that

$$\hat{w}_{n+1}(\eta, v, \tau) \equiv 0 \qquad \text{in } \omega := \mathbb{R}^{2n} \tag{4.52}$$

and

$$w_{n+1}(\eta, v, \tau) = w_{n+1}(v, \tau) \tag{4.53}$$

with

$$\widetilde{\nabla}_{\mathbb{H}} w_{n+1} \not\equiv 0 \qquad \text{in } \omega \tag{4.54}$$

Then compatibility's condition (4.44) is satisfied with $C \equiv$ cost in $\hat{\omega} := \mathbb{R}^{2n-1}$ by Lemma 2.2.12. On the other hand since $\psi \equiv 0$ we have $E \equiv 1$, $E_1(\eta, v, \tau) = I(\eta, v, \tau) = (\eta - \eta_0) w_{n+1}(v, \tau)$, $\underline{a} \equiv 0$,
$\underline{b}(\eta, v, \tau) = -(\eta - \eta_0) \widetilde{\nabla}_{\mathbb{H}} w_{n+1}(v, \tau)$. Then by (4.54)

$$\underline{b}(\eta, v, \tau) - \underline{b}(\eta_0, v, \tau) = -\widetilde{\nabla}_{\mathbb{H}} E_1(\eta, v, \tau) = -(\eta - \eta_0) \widetilde{\nabla}_{\mathbb{H}} w_{n+1}(v, \tau) \not\equiv 0.$$

Therefore compatibility's condition (4.45) is not satisfied and by Theorem 4.3.5 there are no C^2 solutions of the system (4.1).

We are going now to give some explicit regular solutions of the system (4.1) in $\mathbb{H}^2$ by means of Theorem 4.3.5. We will assume in the examples below that $\phi \in C^2(\omega)$ is a solution of system (4.1), $\omega = (\eta_0 - r_0, \eta_0 + r_0) \times \hat{\omega} = (\eta_0 - r_0, \eta_0 + r_0) \times U(v_0, r_0) \times (\tau_0 - r_0, \tau_0 + r_0)$, and we will use the same notations of Theorem 4.3.5.

Remark 4.3.7. *Let us assume that $\exists \phi \in C^2(\omega)$ solution of* (4.1)*. If $C(v, \tau) \equiv 0$ in ω then $\underline{b}(\eta, v, \tau) \equiv 0$ in ω.*

Indeed let us notice that by (4.45) we have

$$\underline{b}(\eta, v, \tau) - \underline{b}(0, v, \tau) = a(\eta, v, \tau) C(v, \tau) = 0 \quad \forall (\eta, v, \tau) \in \omega, \tag{4.55}$$

then by (4.44)

$$\widetilde{\nabla}_{\mathbb{H}} C(v, \tau) = \hat{w}_3(\eta_0, v, \tau) \equiv 0 \quad \forall (\eta, v, \tau) \in U(v_0, r_0) \times (\tau_0 - r_0, \tau_0 + r_0)$$

and by (4.46)

$$\phi(\eta, v, \tau) = E_1(\eta, v, \tau) \qquad \text{in } \omega.$$

Let us observe that by definition

$$E(\eta_0, v, \tau) \equiv 1 \qquad E_1(\eta_0, v, \tau) \equiv 0,$$

therefore $\underline{b}(\eta_0, v, \tau) \equiv 0$ and by (4.55) we conclude $\underline{b}(\eta, v, \tau) \equiv 0$ in ω. $\square$

Remark 4.3.8. *Let us assume that* $\underline{a}(\eta, v, \tau) \equiv 0$ *in* ω *and that* $\exists \phi \in C^2(\omega)$ *solution of* (4.1)*, then it is of the type* $\phi(\eta, v, \tau) = \psi(\eta)\tau + k(\eta, v)$*.*

Indeed by the definition of $\underline{a}$

$$0 = \widetilde{\nabla}_{\mathbb{H}} E(\eta, v, \tau) = -E(\eta, v, \tau) \int_{\eta_0}^{\eta} \widetilde{\nabla}_{\mathbb{H}} \psi(\eta, v, \tau)\, d\eta'$$

$\forall \eta \in (\eta_0 - r_0, \eta_0 + r_0)$, $\forall (v, \tau) \in \hat{\omega}$. Since for fixed $(v, \tau) \in \hat{\omega}$ $\widetilde{\nabla}_{\mathbb{H}} \psi(\cdot, v, \tau) \in C^0((\eta_0 - r_0, \eta_0 + r_0); \mathbb{R}^2)$ we can conclude that

$$\widetilde{\nabla}_{\mathbb{H}} \psi \equiv 0 \quad \text{in } \omega \tag{4.56}$$

By (4.56) and Lemma 2.2.12 we get that $\psi = \psi(\eta)$ $\eta \in (\eta_0 - r_0, \eta_0 + r_0)$. Therefore we can conclude by Theorem 4.3.5 and Theorem 2.4.16 there are solution $\phi \in C^2(\omega)$ of the system (4.1) of the type

$$\phi(\eta, v, \tau) = \psi(\eta)\tau + k(\eta, v) \quad \forall (\eta, v, \tau) \in \omega.$$

□

Example 4.3.9. Let us assume now $w = w(\eta, v)$. Let us observe that in this case $\psi = \psi(\eta, v)$, $E = E(\eta, v)$, $E_1 = E_1(\eta, v)$, $\underline{a} = \underline{a}(\eta, v)$ and $\underline{b} = \underline{b}(\eta, v)$. By Theorem 4.3.5 each solution $\phi \in C^2(\omega)$ of the system (4.1) is of the type

$$\phi(\eta, v, \tau) = E_1(\eta, v) + E(\eta, v) C(v, \tau) \tag{4.57}$$

and

$$\widetilde{\nabla}_{\mathbb{H}} C(v, \tau) = \hat{w}_3(\eta_0, v) \tag{4.58}$$

$$\underline{a}(\eta, v) C(v, \tau) = \underline{b}(\eta, v) - \underline{b}(\eta_0, v) \tag{4.59}$$

$\forall \eta \in (\eta_0 - r_0, \eta_0 + r_0)$, $\forall v \in U(v_0, r_0)$, $\forall \tau \in (\tau_0 - r_0, \tau_0 + r_0)$. By Theorem 2.4.16 the condition (4.58) is equivalent to

$$0 = \left(\frac{\partial^2 w_4}{\partial v_2^2} - \frac{\partial^2 w_2}{\partial v_4 \partial v_2} \right)(\eta_0, v) = \left(\frac{\partial^2 w_4}{\partial v_2 \partial v_4} - \frac{\partial^2 w_2}{\partial v_4^2} \right)(\eta_0, v) \tag{4.60}$$

Let us assume now that $\underline{a}(\eta, v) \not\equiv 0$ in ω. Then by (4.59) we get that $C(v, \tau) = C(v)$. Thus by (4.57) $\phi(\eta, v, \tau) = \phi(\eta, v)$ provided (4.60) holds. On the other hand let $\underline{a}(\eta, v) \equiv 0$ in ω. In this case ϕ can depend on τ. For instance, it is immediate to see that

$$\phi(\eta, v, \tau) = \frac{\tau}{\eta + 2} \quad (\eta, v, \tau) \in (-1, 1) \times U(0, 1) \times (-1, 1)$$

is a solution of the system (4.1) with

$$w(\eta, v\tau) := w(\eta, v) = \left(-\frac{v_4}{2(\eta + 2)}, 0, \frac{v_2}{2(\eta + 2)} \right) \quad \forall (\eta, v, \tau) \in \omega.$$

Example 4.3.10. In the same assumption of example 4.3.9, if $w = w(\eta)$ then a solution ϕ of $\nabla^\phi \phi = w$ is such that $\dfrac{\partial \phi}{\partial \tau} = 0$.
Indeed let us observe that, since $w = w(\eta)$, $\psi = \widetilde{X}_2 w_{2+n} - \widetilde{Y}_2 w_2 = 0$. We conclude so $\dfrac{\partial \phi}{\partial \tau} = \psi = 0$.

Example 4.3.11. In the case $w = w(v, \tau)$ we can find $\phi(\eta, v, \tau)$ solutions of $\nabla^\phi \phi = w$ such that $\dfrac{\partial \phi}{\partial \eta} \neq 0$.
Let us assume in $\mathbb{H}^2$

$$w = \left(-\frac{v_4}{2}, \tau, \frac{v_2}{2}\right), \quad \omega = (-1, 1)^4.$$

Then $\phi(\eta, v, \tau) = \tau + e^{-\eta}$ is a solution of $\nabla^\phi \phi = w$ in ω.

Example 4.3.12. In the case $w = w(\eta, \tau)$ we can find $\phi(\eta, v, \tau)$ solution of $\nabla^\phi \phi = w$ such that $\dfrac{\partial \phi}{\partial v_i} \neq 0$ for some $i \in \{2, ..., 2n\}$.
Let us assume in $\mathbb{H}^2$

$$w = (1, 2\eta, 0), \quad \omega = (-1, 1)^4.$$

Then $\phi(\eta, v, \tau) = v_2 + \eta^2$ is a solution of $\nabla^\phi \phi = w$ in ω.

4.4 Euclidean Regularity of $\mathbb{H}$-Regular Graphs

In this section we are going to prove that for a given $\mathbb{H}$-regular graph $G^1_{\mathbb{H},\phi}(\omega)$ the Euclidean Lipschitz regularity of the $(n+1)$-th component $W^\phi \phi$ of its intrinsic gradient $\nabla^\phi \phi$ yields its Euclidean Lipschitz regularity (see Theorem 4.4.1). Moreover when $n \geq 2$ more regularity holds provided each component of the intrinsic gradient $\nabla^\phi \phi$ is locally Lipschitz continuous (see Theorem 4.4.8).

Theorem 4.4.1. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set, let $G^1_{\mathbb{H},\phi}(\omega)$ be $\mathbb{H}$-regular in $\mathbb{H}^n$ and let us assume the $(n+1)$-th component of its intrinsic gradient $W^\phi \phi \in Lip_{loc}(\omega)$. Then $\phi \in Lip_{loc}(\omega)$.*

Remark 4.4.2. **i** Let us point out that Theorem 4.4.1 is sharp. Indeed in [11], Example 2.8, it was considered the function $\phi : \omega := (-1, 1)^2 \to \mathbb{R}$,

$$\phi(\eta, \tau) := \begin{cases} \dfrac{\tau}{\eta + 1} & \tau \geq 0 \\ \dfrac{\tau}{\eta - 1} & \tau < 0 \end{cases}$$

We compute easily that

$$\frac{\partial \phi}{\partial \eta}(\eta,\tau) = \begin{cases} \dfrac{-\tau}{(\eta+1)^2} & \tau \geq 0 \\ \dfrac{-\tau}{(\eta-1)^2} & \tau < 0 \end{cases} \qquad \frac{\partial \phi}{\partial \tau}(\eta,\tau) = \begin{cases} \dfrac{1}{\eta+1} & \tau \geq 0 \\ \dfrac{1}{\eta-1} & \tau < 0 \end{cases}$$

then $\phi \in Lip_{loc}(\omega) \setminus C^1(\omega)$ and its intrinsic gradient $\nabla^\phi \phi = W^\phi \phi \equiv 0 \in Lip(\omega)$. $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular in $\mathbb{H}^1$ by Theorem 4.2.1.

ii Weakening the assumption $W^\phi\phi \in Lip(\omega)$ with $W^\phi\phi \in C^{0,\alpha}(\omega)$ the thesis of Theorem 4.4.1 can fail. For instance, if $n = 1$ by [4] Corollary 5.11 (see also [94]) we can construct for each $\alpha \in \left(\frac{1}{2}, 1\right)$ a function $\phi \in C^{0,\alpha}(\omega)$ such that $G^1_{\mathbb{H},\phi}(\omega)$ is $\mathbb{H}$-regular and $\nabla^\phi\phi = W^\phi\phi \in C^{0,2\alpha-1}(\omega)$, see example 3.3.7, where ϕ is of the type $\phi(\eta,\tau) := C|\tau|^\alpha$ and $W^\phi\phi = \operatorname{sgn}(\tau)\, 2\alpha C|\tau|^{2\alpha-1}$, C constant.

Before the proof of Theorem 4.4.1 we will need some preliminary results. The first key tool for the proof of Theorem 4.1 will be the following one.

Lemma 4.4.3. *Let $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n}$ if $n \geq 2$, $r_0 > 0$, let $\phi : I_{r_0}(A_0) \to \mathbb{R}$ and $w = (w_2, \dots, w_{2n}) : I_{r_0}(A_0) \to \mathbb{R}^{2n-1}$ be given continuous functions. Let us assume*

i *ϕ is a broad* solution of $\nabla^\phi\phi = w$ in $I_{r_0}(A_0)$;*

ii $w_{n+1} \in Lip\left(\overline{I_{r_0}(A_0)}\right)$.

Then for some $0 < r < r_0$, if $n = 1$

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|} : A = (\eta,\tau), B = (\eta,\tau') \in \overline{I_r(A_0)},\ A \neq B\right\} < \infty \tag{4.61}$$

and, if $n \geq 2$

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|} : A = (\eta,v,\tau), B = (\eta,v,\tau') \in \overline{I_r(A_0)},\ A \neq B\right\} < \infty\,. \tag{4.62}$$

Proof. We are going to follow here the same proof's strategy of Theorem 3.3.12.

Since ϕ is a broad* solution there exists a family of exponential maps at A_0

$$\exp_{A_0}(\cdot\nabla_j^{\phi})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset I_{r_0}(A_0) \tag{4.63}$$

where $0 < \delta_2 < \delta_1$ and $j = 2, \dots, 2n$ satisfying $(E.1)$, $(E.2)$ and $(E.3)$.

Let us denote $I_1 := \overline{I_{\delta_1}(A_0)}$, $I_2 := \overline{I_{\delta_2}(A_0)}$. Let $A = (\eta, \tau) \in I_2$ if $n = 1$ and $A = (\eta, v, \tau) \in I_2$ if $n \geq 2$ and let us denote by $\gamma_A(s) = \gamma_{n+1}^A(s) = \exp_{A_0}(sW^{\phi})(A)$ if $s \in [-\delta_2, \delta_2]$. Let $\gamma_A(s) = (\eta + s, \tau_A(s))$ if $n = 1$ and $\gamma_A(s) = (\eta + s, v, \tau_A(s))$ if $n \geq 2$. Then τ_A satisfies

$$\begin{cases} \dfrac{d^2}{ds^2}\tau_A(s) = \dfrac{d}{ds}[\phi(\gamma_A(s))] = w_{n+1}(\gamma_A(s)). \\ \tau_A(0) = \tau, \quad \dfrac{d}{ds}\tau_A(0) = \phi(A) \end{cases} \tag{4.64}$$

First let us consider the case $n = 1$. Let $A = (\eta, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ and let $x(s, \tau) := \tau_A(s)$ if $|s| \leq \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta}(s, \tau) := \phi(\eta + s, \tau)$, $f_{2,\eta}(s, \tau) := w_2(\eta + s, \tau)$, $g_\eta(\tau) = \phi(\eta, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$ is fixed. By (4.64) and since

$$L_1(f_{2,\eta}, [\tau_0 - \delta_1, \tau_0 + \delta_1]) \leq L_1(f_2, \overline{I_1}) < \infty \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$$

we can apply (3.34) of Lemma 3.3.13 and (4.61) follows with $r = \delta_2$.

In the case $n \geq 2$ and $A = (\eta, v, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times \overline{U(v_0, \delta_2)} \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ let $x(s, \tau) := \tau_A(s)$ if $|s| \leq \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta,v}(s, \tau) := \phi(\eta + s, v, \tau)$, $f_{2,\eta,v}(s, v, \tau) := w_{n+1}(\eta + s, v, \tau)$, $g_{\eta,v}(\tau) = \phi(\eta, v, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times \overline{U(v_0, \delta_1)} \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$, $v \in \overline{U(v_0, \delta_2)}$ are fixed. By (4.64) and since

$$L_1(f_{2,\eta,v}, [\tau_0 - \delta_1, \tau_0 + \delta_1]) \leq L_1(f_2, \overline{I_1}) < \infty \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2], \ v \in \overline{U(v_0, \delta_1)}$$

we can argue as before to get (4.62). $\square$

Remark 4.4.4. Actually in order to get (4.61) and (4.62), by Remark 3.3.14 we can weaken the assumption $w_{n+1} \in Lip\left(\overline{I_{r_0}(A_0)}\right)$ in

$$\sup\left\{\frac{|w_{n+1}(A) - w_{n+1}(B)|}{|A - B|} : A = (\eta, \tau), B = (\eta, \tau') \in \overline{I_{r_2}(A_0)},\ A \neq B\right\} < \infty$$

if $n = 1$ and,

$$\sup\left\{\frac{|w_{n+1}(A) - w_{n+1}(B)|}{|A - B|} : A = (\eta, v, \tau), B = (\eta, v, \tau') \in \overline{I_{r_2}(A_0)},\ A \neq B\right\} < \infty$$

if $n \geq 2$.

Proof of Theorem 4.4.1. Let $A_0 \in \omega$ and $r_0 > 0$ be such that $I_{r_0}(A_0) \Subset \omega$. It is sufficient to prove that $\phi \in Lip(I_r(A_0))$ for some $0 < r < r_0$.

Let $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n}$ if $n \geq 2$. Let us observe that by Theorem 4.1.1 ϕ is a broad* solution of the system

$$\nabla^{\phi}\phi = w \quad \text{in}\,\omega := I_{r_0}(A_0)\,. \tag{4.65}$$

Then we can apply Lemma 4.4.3 and, for some $0 < r < r_0$, we get at once that

$$|\phi(\eta,\tau) - \phi(\eta,\tau')| \leq L\,|\tau - \tau'| \quad \forall \eta \in [\eta_0 - r, \eta_0 + r],\ \tau\,,\tau' \in [\tau_0 - r, \tau_0 + r]$$

if $n = 1$ and

$$|\phi(\eta,v,\tau) - \phi(\eta,v,\tau')| \leq L\,|\tau - \tau'| \quad \forall \eta \in [\eta_0 - r, \eta_0 + r],\ v \in \overline{U(v_0,r)},\ \tau\,,\tau' \in [\tau_0 - r, \tau_0 + r]$$

if $n \geq 2$. Notice also that in both cases there exists

$$\frac{\partial\phi}{\partial\tau} = \in L^{\infty}(\omega) \tag{4.66}$$

in distributional sense. Let us prove now that there exists

$$\frac{\partial\phi^2}{\partial\tau} = 2\phi\frac{\partial\phi}{\partial\tau} \ \in L^{\infty}(\omega) \tag{4.67}$$

in distributional sense. Let us fix an open set $\omega' \Subset \omega$ and let $0 < \epsilon < \operatorname{dist}(\omega', \mathbb{R}^{2n} \setminus \omega)$. Then it is well-defined the convolution

$$\phi_\epsilon := (\phi * \rho_\epsilon)(x) \qquad x \in \omega'$$

where $(\rho_\epsilon)_\epsilon$ is a standard family of mollifiers. In particular by (4.67)

$$\phi_\epsilon \in C^1(\overline{\omega'}) \quad \text{and} \quad \frac{\partial\phi_\epsilon}{\partial\tau} = \frac{\partial\phi}{\partial\tau} * \rho_\epsilon \qquad \in \omega' \tag{4.68}$$

$$\phi_\epsilon \to \phi \qquad \frac{\partial\phi_\epsilon}{\partial\tau} \to \frac{\partial\phi}{\partial\tau} \tag{4.69}$$

in $L^p(\omega')\ \forall\, 1 \leq p < \infty$. On the other hand, $\forall \varphi \in C^1_c(\omega')$, by (4.68)

$$\int_{\omega'} \phi_\epsilon^2 \frac{\partial\varphi}{\partial\tau}\, d\mathcal{L}^{2n} = -2\int_{\omega'} \phi_\epsilon \frac{\partial\phi_\epsilon}{\partial\tau}\varphi\, d\mathcal{L}^{2n}$$

and taking the limit as $\epsilon \to 0^+$ we get

$$\int_{\omega'} \phi^2 \frac{\partial\varphi}{\partial\tau}\, d\mathcal{L}^{2n} = -2\int_{\omega'} \phi \frac{\partial\phi}{\partial\tau}\varphi\, d\mathcal{L}^{2n}$$

and we have done. Let us recall now that by Corollary 4.1.5 ϕ is a distributional solution of (4.65) too, i.e. (4.5) and (4.6) hold. By (4.6) and (4.67) there exists

$$\frac{\partial \phi}{\partial \eta} = w_{n+1} - \frac{1}{2}\frac{\partial \phi^2}{\partial \tau} \ \in L^\infty(\omega)\,.$$

Meanwhile by (4.67) and (4.5) we get there exist

$$\frac{\partial \phi}{\partial v_j} = w_j + \frac{v_{j+n}}{2}\frac{\partial \phi}{\partial \tau} \in L^\infty_{loc}(\omega)$$

$$\frac{\partial \phi}{\partial v_{j+n}} = w_{j+n} - \frac{v_j}{2}\frac{\partial \phi}{\partial \tau} \in L^\infty_{loc}(\omega).$$

and we have done.

$\square$

Let us deal now only with the case $n \geq 2$. We will see there is a stronger regularizing effect of the intrinsic gradient $\nabla^\phi \phi$ on ϕ.

Theorem 4.4.5. *Let $\omega \subseteq \mathbb{R}^{2n}$ be an open set with $n \geq 2$, let $\phi : \omega \to \mathbb{R}$, $w = (w_2, ..., w_{n+1}, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$. Let us assume*

i *$\phi \in L^\infty_{loc}(\omega)$, $w_i \in L^\infty_{loc}(\omega)$ $\forall\, i = 2, ..., n$ and, for some $i_0 = 2, ..., n$, there exists*

$$\widetilde{X}_{i_0} w_{i_0+n} - \widetilde{Y}_{i_0} w_{i_0} \ \in L^\infty_{loc}(\omega) \tag{4.70}$$

in distributional sense,

ii *ϕ is a distributional solution of the system* (4.1)*.*

Then $\phi \in Lip_{loc}(\omega)$.

Proof. By **i**

$$\int_\omega \phi \widetilde{X}_{i_0} \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{i_0} \varphi \, d\mathcal{L}^{2n} \qquad \forall \varphi \in C^1_c(\mathbb{R}^{2n}).$$

Let us prove there exists

$$\frac{\partial \phi}{\partial \tau} = \widetilde{X}_{i_0} w_{i_0+n} - \widetilde{Y}_{i_0} w_{i_0} \ \ \in L^\infty_{loc}(\omega) \tag{4.71}$$

in distributional sense. In fact

$$\int_\omega \phi \frac{\partial \varphi}{\partial \tau} \, d\mathcal{L}^{2n} = -\int_\omega \phi \left(\widetilde{X}_{i_0} \widetilde{Y}_{i_0} - \widetilde{Y}_{i_0} \widetilde{X}_{i_0} \right) \varphi \, d\mathcal{L}^{2n} =$$

$$= -\int_\omega \left(-\widetilde{X}_{i_0}\phi\widetilde{Y}_{i_0}\varphi + \widetilde{Y}_{i_0}\phi\widetilde{X}_{i_0}\varphi\right)\, d\mathcal{L}^{2n} =$$

$$= -\int_\omega \left(-w_{i_0}\widetilde{Y}_{i_0}\varphi + w_{i_0+n}\widetilde{X}_{i_0}\varphi\right)\, d\mathcal{L}^{2n} = -\int_\omega \left(\widetilde{Y}_{i_0}w_{i_0} - \widetilde{X}_{i_0}w_{i_0+n}\right)\varphi\, d\mathcal{L}^{2n}.$$

From (4.71) we have for $j = 2, ..., n$ there exist

$$\frac{\partial\phi}{\partial v_j} = w_j + \frac{v_{j+n}}{2}\frac{\partial\phi}{\partial\tau} \in L^\infty_{loc}(\omega)$$

$$\frac{\partial\phi}{\partial v_{j+n}} = w_{j+n} - \frac{v_j}{2}\frac{\partial\phi}{\partial\tau} \in L^\infty_{loc}(\omega).$$

in distributional sense. Arguing now as in (4.67) we get there exists

$$\frac{\partial\phi^2}{\partial\tau} = 2\phi\frac{\partial\phi}{\partial\tau} \ \in L^\infty_{loc}(\omega) \tag{4.72}$$

in distributional sense. Therefore

$$\frac{\partial\phi}{\partial\eta} = w_{n+1} - \frac{1}{2}\frac{\partial\phi^2}{\partial\tau} \ \in L^\infty_{loc}(\omega)$$

and we achieve the proof. □

Through the same techniques exploited in Theorem 4.4.5 we can get a C^k regularity result for distributional solutions of system (4.1).

Corollary 4.4.6. *Under the same assumptions of Theorem* 4.4.5 *let us replace* (4.70) *with*

$$w_j \in C^k(\omega) \tag{4.73}$$

for $j = 2, \dots, 2n$, *and some* $k \geq 1$. *Then* $\phi \in C^k(\omega)$.

Proof. By Theorem 4.4.5, (4.73) and (4.71) we get $\phi \in Lip_{loc}(\omega)$ and there exists

$$\frac{\partial\phi}{\partial\tau} = \widetilde{X}_{i_0}w_{i_0+n} - \widetilde{Y}_{i_0}w_{i_0} \ \in C^{k-1}(\omega) \tag{4.74}$$

in distributional sense. Then arguing as in the proof of Theorem 4.4.5 we get at once there exists for $j = 2, \dots, n$

$$\frac{\partial\phi}{\partial v_j} = w_j + \frac{v_{j+n}}{2}\frac{\partial\phi}{\partial\tau} \in C^{k-1}(\omega) \tag{4.75}$$

$$\frac{\partial\phi}{\partial v_{j+n}} = w_{j+n} - \frac{v_j}{2}\frac{\partial\phi}{\partial\tau} \in C^{k-1}(\omega) \tag{4.76}$$

in distributional sense. In order to achieve the proof we have to show there exists

$$\frac{\partial \phi}{\partial \eta} \in C^{k-1}(\omega) \tag{4.77}$$

in distributional sense. Indeed by (4.74), (4.75), (4.76) and (4.77), through a standard approximation argument by convolution it follows that $\phi \in C^k(\omega)$. Let us prove (4.77). As in (4.72) it is enough to prove there exists

$$\frac{\partial \phi^2}{\partial \tau} = 2\phi \frac{\partial \phi}{\partial \tau} \ \in C^{k-1}(\omega) \tag{4.78}$$

in distributional sense. By induction on k, since $\phi \in C^0(\omega)$, (4.78) follows. $\square$

Remark 4.4.7. Example given in Remark 4.4.2 **i** infers that the thesis of Corollary 4.4.6 fails if $n = 1$.

Finally let us stress an interesting regularity result for the solutions of system (4.1) in the case $n \geq 2$.

Theorem 4.4.8. *Let $n \geq 2$, $\omega = (\eta_0 - r_0, \eta_0 + r_0) \times \hat{\omega}$ where $\hat{\omega} \subseteq \mathbb{H}^{n-1} \simeq \mathbb{R}^{2n-1}_{(v,\tau)}$ is a connected bounded open set and let $\phi \in Lip(\omega)$ and $w = (w_2, \ldots, w_{2n}) \in Lip(\omega; \mathbb{R}^{2n-1})$ such that*

$$\nabla^{\phi}\phi = w \qquad a.e. \ in\ \omega.$$

Then $\phi \in C^1(\omega)$.

Proof. We have only to prove that

$$\frac{\partial \phi}{\partial \tau} \ \in C^0(\omega)\,. \tag{4.79}$$

Indeed by (4.79) arguing as in the proof of Corollary 4.4.6 we get $\phi \in C^1(\omega)$. We will reduce to deal with the linear system $\widetilde{\nabla}_{\mathbb{H}}\phi = \hat{w}_{n+1}$ in ω. Then, without loss of generality, we can suppose that $\omega = \mathbb{R}^{2n}$. Otherwise, for a fixed open set $\omega' \Subset \omega$, let $\chi \in C_c^\infty(\omega)$ a cut- off function such that $\chi \equiv 1$ in ω'. Then we can replace ϕ and $\hat{w}_{n+1}$ with $\phi^* := \chi\,\phi \in Lip(\mathbb{R}^{2n})$ and $\hat{w}^*_{n+1} := (w^*_2, \ldots, w^*_n, w^*_{n+2}, \ldots, w^*_{2n})$ where $w^*_j := \chi\, w_j + \widetilde{X}_j \chi\, \phi \in Lip(\mathbb{R}^{2n})$ if $j = 2, \ldots, n$ and $w^*_j := \chi\, w_j + \widetilde{Y}_j \chi\, \phi \in Lip(\mathbb{R}^{2n})$ otherwise. Moreover we can suppose that $\widetilde{\nabla}_{\mathbb{H}}\phi(A) = \hat{w}_{n+1}(A)$ for all $A \in \mathbb{R}^{2n}$ since w is continuous. We divide the proof in four steps.

Step 1: We observe that there exist

$$\left(\widetilde{X}_j \frac{\partial \phi}{\partial \tau}, \widetilde{Y}_j \frac{\partial \phi}{\partial \tau}\right) = \left(\frac{\partial w_j}{\partial \tau}, \frac{\partial w_{j+n}}{\partial \tau}\right) \ \in \ (L^\infty(\omega))^2 \tag{4.80}$$

$$\left(\widetilde{X}_j\frac{\partial\phi}{\partial\eta},\widetilde{Y}_j\frac{\partial\phi}{\partial\eta}\right)=\left(\frac{\partial w_j}{\partial\eta},\frac{\partial w_{j+n}}{\partial\eta}\right)\in(L^\infty(\omega))^2 \tag{4.81}$$

in distributional sense, for $j=2,...,n$.

Step 2: Fix $\eta\in\mathbb{R}$ and define $u_\eta(v,\tau):=\frac{\partial\phi}{\partial\tau}(\eta,v,\tau)$ for $(v,\tau)\in\mathbb{H}^{n-1}$, then by (4.80) and Theorem 2.2.21 we obtain that $\forall\eta\in\mathbb{R}$

$$u_\eta\in Lip_{\mathbb{H}}(\mathbb{H}^{n-1}) \tag{4.82}$$

where $Lip_{\mathbb{H}}(\mathbb{H}^{n-1})$ denotes the space of intrinsic locally Lipschitz functions in $\mathbb{H}^{n-1}$ with respect to the distance (2.7) d_∞ in $\mathbb{H}^{n-1}\simeq\mathbb{R}^{2n-1}_{(v,\tau)}$ and

$$\left\|\left(\widetilde{X}_ju_\eta,\widetilde{Y}_ju_\eta\right)\right\|_{(L^\infty(\mathbb{H}^{n-1}))^2}\le\left\|\left(\frac{\partial w_j}{\partial\tau},\frac{\partial w_{j+n}}{\partial\tau}\right)\right\|_{(L^\infty(\mathbb{H}^n))^2}<\infty\qquad\forall\eta\in\mathbb{R}. \tag{4.83}$$

Let us observe also that $\frac{\partial\phi}{\partial\tau}(\eta,\cdot,\cdot)\in C^0(\mathbb{H}^{n-1})$ $\forall\eta\in\mathbb{R}$. In fact by (4.82) and Remark 2.2.20 we have that $u_\eta\in Lip_{\mathbb{H}}(\mathbb{H}^{n-1})\subseteq C^0(\mathbb{H}^{n-1})$ and so we have done.

Step 3: Let us prove that that for every fixed $(v,\tau)\in\mathbb{H}^{n-1}$ $\frac{\partial\phi}{\partial\tau}(\cdot,v,\tau)\in C^0(\mathbb{R})$. It is enough to show that if $\eta_h\to\eta_0$ when $h\to\infty$, then $\forall\,(v,\tau)\in\mathbb{H}^{n-1}$

$$\frac{\partial\phi}{\partial\tau}(\eta_h,v,\tau)\to\frac{\partial\phi}{\partial\tau}(\eta_0,v,\tau).$$

Since

$$\begin{cases}\left(\widetilde{X}_j\phi\right)(\eta_h,v,\tau)=w_j(\eta_h,v,\tau)\\ \left(\widetilde{Y}_j\phi\right)(\eta_h,v,\tau)=w_{j+n}(\eta_h,v,\tau)\end{cases}$$

$\forall\,(v,\tau)\in\mathbb{H}^{n-1}$, $\forall\,h\in\mathbb{N}$ and for $j=2,...,n$, then we have $\mathcal{L}^{2n-1}-a.e.\;(v,\tau)\in\mathbb{H}^{n-1}$

$$\frac{\partial\phi}{\partial\tau}(\eta_h,v,\tau)=\left(\widetilde{X}_j\widetilde{Y}_j\phi-\widetilde{Y}_j\widetilde{X}_j\phi\right)(\eta_h,v,\tau)=\left(\widetilde{X}_jw_{j+n}(\eta_h,v,\tau)\right)-\left(\widetilde{Y}_jw_j(\eta_h,v,\tau)\right). \tag{4.84}$$

Let us denote, for $(v,\tau)\in\mathbb{H}^{n-1}$ and a fixed $j\in\{2,...,n\}$,

$$w_h(v,\tau)=\left(\widetilde{X}_jw_{j+n}\right)(\eta_h,v,\tau)-\left(\widetilde{Y}_jw_j\right)(\eta_h,v,\tau).$$

The sequence $(w_h)_h\subseteq L^\infty(\mathbb{H}^{n-1})$ and $\sup_{h\in\mathbb{N}}\|w_h\|_{L^\infty(\mathbb{H}^{n-1})}<\infty$, then for weak*-compactness there exists $w^*\in L^\infty(\mathbb{H}^{n-1})$ such that, up to a subsequence,

$$w_h\to w^*\qquad\text{in } L^\infty(\mathbb{H}^{n-1})-\text{weak}*. \tag{4.85}$$

We show now that $\mathcal{L}^{2n-1} - a.e. \, (v,\tau) \in \mathbb{H}^{n-1}$

$$w^*(v,\tau) = \left(\widetilde{X}_j w_j\right)(\eta_0, v, \tau) - \left(\widetilde{Y}_j w_{j+n}\right)(\eta_0, v, \tau) = \frac{\partial \phi}{\partial \tau}(\eta_0, v, \tau) \tag{4.86}$$

By definition, $\forall \, \varphi \in C^1_c(\mathbb{H}^{n-1})$

$$\int_{\mathbb{H}^{n-1}} w^*(v,\tau)\varphi(v,\tau)\, dv\, d\tau = \lim_{h\to\infty} \int_{\mathbb{H}^{n-1}} w_h(v,\tau)\varphi(v,\tau)\, dv\, d\tau =$$

$$= \lim_{h\to\infty} \int_{\mathbb{H}^{n-1}} \left[\left(\widetilde{X}_j w_{j+n}\right)(\eta_h, v, \tau) - \left(\widetilde{Y}_j w_j\right)(\eta_h, v, \tau) \right] \varphi(v,\tau)\, dv\, d\tau =$$

$$= - \lim_{h\to\infty} \int_{\mathbb{H}^{n-1}} \left[w_{j+n}(\eta_h, v, \tau)\widetilde{X}_j \varphi(v,\tau) - w_j(\eta_h, v, \tau)\widetilde{Y}_j \varphi(v,\tau) \right] dv\, d\tau =$$

$$= - \int_{\mathbb{H}^{n-1}} \left[w_{j+n}(\eta_0, v, \tau)\widetilde{X}_j \varphi(v,\tau) - w_j(\eta_0, v, \tau)\widetilde{Y}_j \varphi(v,\tau) \right] dv\, d\tau =$$

$$= \int_{\mathbb{H}^{n-1}} \left[\left(\widetilde{X}_j w_{j+n}\right)(\eta_0, v, \tau) - \left(\widetilde{Y}_j w_j\right)(\eta_0, v, \tau) \right] \varphi(v,\tau)\, dv\, d\tau =$$

$$= \int_{\mathbb{H}^{n-1}} \frac{\partial \phi}{\partial \tau}(\eta_0, v, \tau)\varphi(v,\tau)\, dv\, d\tau$$

and so we obtain (4.86).
Let us define

$$u_h(v,\tau) := u_{\eta_h}(v,\tau) = \frac{\partial \phi}{\partial \tau}(\eta_h, v, \tau) \qquad (v,\tau) \in \mathbb{H}^{n-1}$$

by (4.84) and (4.86) we obtain

$$u_h \to u_{\eta_0} \qquad \text{in } L^\infty(\mathbb{H}^{n-1})\text{−weak}* \tag{4.87}$$

Moreover by step 1 we obtain that the sequence $(u_h)_h \subseteq Lip_{\mathbb{H}}(\mathbb{H}^{n-1})$ verifies that

$$\sup_{\mathbb{H}^{n-1}} |u_h| \le \sup_{\mathbb{R}^{2n}} \left| \frac{\partial \phi}{\partial \tau} \right| \tag{4.88}$$

$$\exists \, L > 0 : \, |u_h(v,\tau) - u_h(v',\tau')| \le L d_\infty \left((v,\tau),(v',\tau')\right) \quad \forall \, (v,\tau),(v',\tau') \in \mathbb{H}^{n-1}, \forall \, h \in \mathbb{N}. \tag{4.89}$$

By Arzelá-Ascoli's Theorem, up to a subsequence, there exists $u^* \in Lip_{\tilde{\mathbb{H}}}(\mathbb{H}^{n-1})$ such that

$$u_h \to u^* \qquad \text{uniformly on compact sets of } \mathbb{H}^{n-1} \tag{4.90}$$

By uniqueness, (4.87) and (4.90) we obtain that $u_{\eta_0} = u^*$ $\mathcal{L}^{2n-1}$-a.e. in $\mathbb{H}^{n-1}$. Moreover, because $u_{\eta_0}, u^* \in C^0(\mathbb{H}^{n-1})$, we have that

$$\frac{\partial \phi}{\partial \tau}(\eta_0, v, \tau) = u^*(v, \tau) \qquad \forall\, (v, \tau) \in \mathbb{H}^{n-1} \tag{4.91}$$

The thesis follows by (4.90) and (4.91).
Step 4: Let us show (4.79). Let us prove that for each sequence $(\eta_h, v_h, \tau_h)_h \subset \mathbb{R}^{2n}$ such that

$$(\eta_h, v_h, \tau_h) \longrightarrow (\eta_0, v_0, \tau_0) \qquad \text{in } \mathbb{R}^{2n}$$

then

$$\lim_{h\to\infty} \frac{\partial \phi}{\partial \tau}(\eta_h, v_h, \tau_h) = \frac{\partial \phi}{\partial \tau}(\eta_0, v_0, \tau_0).$$

Let us observe that

$$\frac{\partial \phi}{\partial \tau}(\eta_h, v_h, \tau_h) - \frac{\partial \phi}{\partial \tau}(\eta_0, v_0, \tau_0) =$$

$$= \left(\frac{\partial \phi}{\partial \tau}(\eta_h, v_h, \tau_h) - \frac{\partial \phi}{\partial \tau}(\eta_h, v_0, \tau_0)\right) + \left(\frac{\partial \phi}{\partial \tau}(\eta_h, v_0, \tau_0) - \frac{\partial \phi}{\partial \tau}(\eta_0, v_0, \tau_0)\right) =$$

$$= I_h^{(1)} + I_h^{(2)}.$$

By step 2 the exists $L > 0$ such that $\forall\, (v, \tau), (v', \tau') \in \mathbb{H}^{n-1}$, $\forall \eta \in \mathbb{R}$

$$\left|\frac{\partial \phi}{\partial \tau}(\eta, v, \tau) - \frac{\partial \phi}{\partial \tau}(\eta, v', \tau')\right| \le L\, d_\infty\left((v, \tau), (v', \tau')\right).$$

Thus we have there exists $\lim_{h\to\infty} I_h^{(1)} = 0$. For step 3 we have there exists $\lim_{h\to\infty} I_h^{(2)} = 0$ too, and then we have done. □

Remark 4.4.9. Theorem 4.4.8 does not hold in $\mathbb{H}^1$. As counterexample let us consider the function $\phi(\eta, \tau) := \dfrac{\tau}{\eta + \frac{\tau}{|\tau|}}$ of remark 4.4.2. Indeed $\nabla^\phi \phi \equiv 0 \in Lip(\omega)$ but $\phi \notin C^1(\omega)$.

Corollary 4.4.10. *Let $\omega \subseteq \mathbb{R}^{2n}$ be an open set with $n \ge 2$, let $\phi \in L^\infty_{loc}(\omega)$ and $w = (w_2, ..., w_{n+1}, ..., w_{2n}) \in Lip(\omega; \mathbb{R}^{2n-1})$. Let us assume that ϕ is a distributional solution of $\nabla^\phi \phi = w$ in ω. Then $G^1_{\mathbb{H},\phi}(\omega)$ is C^1-regular.*

Proof. Immediate by Theorems 4.4.8 and 4.4.5. □

By the previous regularity results we can get of the following corollary.

Corollary 4.4.11. *Let* $w = (w_2, \dots, w_{2n}) : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ *and let us denote*

$$\nu(A) := \left(-\frac{1}{\sqrt{1+|w(A)|^2}}, \frac{w(A)}{\sqrt{1+|w(A)|^2}} \right) \quad A \in \omega\,.$$

Let $S = G^1_{\mathbb{H},\phi}(\omega) \subset \mathbb{H}^n$ *be a* $\mathbb{H}$*-regular graph such that*

$$\nu_S(P) = \nu(\Phi^{-1}(P)) \quad \forall P \in S \cap U_\infty(P_0, r_0),$$

where $\Phi : \omega \to \mathbb{H}^n$ *is the parameterization in* (3.4) *and* $P_0 = \Phi(A_0)$.

i *If* w_{n+1} *is Lipschitz continuous then* $S \cap U_\infty(P_0, r)$ *is a Lipschitz hypersurface for some* $r < r_0$*, i.e.* $S \cap U_\infty(P_0, r) = \Phi(\omega) \cap U_\infty(P_0, r)$ *and*

$$\Phi : \omega \subset (\mathbb{R}^{2n}, |\cdot|) \to (\mathbb{R}^{2n+1}, |\cdot|)$$

is Lipschitz continuous and one-to-one;

ii *if* $n \geq 2$ *and* w *is Lipschitz continuous then* $S \cap U_\infty(P_0, r)$ *is a* C^1 *hypersurface in* $\mathbb{R}^{2n+1}$ *for some* $r < r_0$*. Moreover if* $w \in C^k(\omega; \mathbb{R}^{2n-1})$ *for some* $k \geq 1$ *then* $S \cap U_\infty(P_0, r)$ *is a* C^k *hypersurface in* $\mathbb{R}^{2n+1}$ *for some* $r < r_0$.

To conclude, let us rewrite some of our results about ℍ-regular hypersurfaces in wiev of study characterizations and regularity of the solutions of Burgers' equation

$$\frac{\partial}{\partial \eta} u + \frac{1}{2} \frac{\partial}{\partial \tau}(u^2) = g \quad \text{in}\, \omega. \tag{4.92}$$

Corollary 4.4.12. *Let* ϕ, $w \in C^0(\omega)$ *and let us assume that* ϕ *is a distributional solution of Burgers' equation* (4.92) *with* $g \equiv w$*. Then*

i *there exists a family* $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ *such that as* $\epsilon \to 0^+$

$$\phi_\epsilon \to \phi \quad \textit{and} \quad W^{\phi_\epsilon}\phi_\epsilon \to w \quad \textit{in} \;\; L^\infty_{loc}(\omega)\,.$$

ii ϕ *is locally little Hölder continuous in* ω *of order* $1/2$*, i.e. for every open set* $\omega' \Subset \omega$

$$\lim_{r \to 0^+} \sup \left\{ \frac{|\phi(A) - \phi(B)|}{\sqrt{|A-B|}} : A, B \in \omega', 0 < |A-B| < r \right\} = 0\,.$$

iii ϕ *is locally Lipschitz continuous in* ω *provided* w *is Lipschitz continuous.*

Proof. Let us notice that ϕ is a continuous ditributional solution of (4.1) in the case $n = 1$, then by Theorem 4.2.1 $S = \Phi(\omega) \subset \mathbb{H}^1$ is an $\mathbb{H}$-regular surface, where Φ is the parametrization induced by ϕ and given in Theorem 3.1.13.
Then thesis **i** follows by Theorem 3.3.9, equation (3.28).
Theorem 4.1.1 yields that ϕ is a broad* solution of (4.1) too, then by Theorem 3.3.13 we obtain thesis **ii**.
Finally thesis **iii** follows by Theorem 4.4.1. □

Remark 4.4.13. The regularity results in Corollary 4.4.12 **ii** and **iii** are sharp. For instance, there are examples of $\mathbb{H}$-regular graphs $\Phi(\omega)$ with ϕ no better than little Hölder continuous of order 1/2 and Lipschitz continuous even if $w \equiv 0$, see for instance Theorem 3.1.15 and example 3.3.7. (see [69], [4] and [18] too).

Bibliography

[1] L.Ambrosio, B.Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Annalen, 318 (2000), 527–555.

[2] L.Ambrosio, B.Kirchheim, *Currents in metric spaces*, Acta Math., 185 (2000), 1–80.

[3] L.Ambrosio, B.Kleiner, E. Le Donne, *Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane*, Preprint, 2008.

[4] L.Ambrosio,F.Serra Cassano,D.Vittone, *Intrinsic Regular Hypersurfaces in Heisenberg Groups,* J. Geom. Anal. 16 (2006), no. 2, 187–232.

[5] L.Ambrosio,P.Tilli, *Topics on Analysis in Metric Spaces,* Oxford Lecture Series in Mathematics and its Applications, 25. Oxford University Press, Oxford, 2004.

[6] B.P.Andreianov, P.Bènilan, S.N.Kružkov, *L^1-Theory of Scalar Conservation Law with Continuous Flux Function,* J. Funct. Anal. 171, (2000) 15–33.

[7] G. Arena, R.Serapioni, *Intrinsic regular submanifolds in Heisenberg groups are differentiable graphs*, Preprint, 2008.

[8] A.Baldi, B.Franchi, M.C.Tesi, *Compensated compactness in the contact complex of Heisenberg Groups,* Indiana Univ. Math. J. 57 (2008), no. 1, 133–185.

[9] Z.Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew Math., 564 (2003), 63–83.

[10] Z.Balogh, H.Höfer-Isenegger, J.T.Tyson, *Lifts of Lipischitz maps and horizontal fractals in the Heisenberg group*, Ergodic Theory Dynam. Systems 26 (2006), no. 3, 621–651.

[11] V.Barone Adesi, F.Serra Cassano, D.Vittone, *The Bernstein problem for intrinsic graphs in Heisenberg groups*, Calc. Var. PDEs, 30 (2007), 17–49.

[12] P.Bènilan, J.Carrillo, P.Wittbold, *Renormalized entropy solutions of scalar conservation laws*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 2, 313–327.

[13] P.Bènilan, S.N.Kružkov, *First-order quasilinear equations with continuous nonlinearities,* Dokl. Akad. Nauk 339 (1994), no. 2, 151–154; translation in Russian Acad. Sci. Dokl. Math. 50 (1995), no. 3, 391–396.

[14] P.Bènilan, S.N.Kružkov, *Conservation Laws with continuous flux functions,* NoDEA 3 (1996), 395–419.

[15] S.Bianchini, A.Bressan, *Vanishing viscosity solutions of nonlinear Hyperbolic systems*, Ann. of Math. 161 (2005), 223-342.

[16] S.Bianchini, Personal communication.

[17] S.Bianchini, *Transport rays, differential inclusions and applications to Hamilton Jacobi equations*, CIME course 2008, Cetraro.

[18] F.Bigolin, F.Serra Cassano, *Intrinsic regualar graphs in Heisenberg groups vs. weak solutions of non linear first-order PDEs*, Preprint, 2008, available at http://cvgmt.sns.it/papers/bigser07/

[19] F.Bigolin, F.Serra Cassano, *Distributional solutions of Burgers' equation and Intrinsic regular graphs in Heisenberg groups*, Preprint, 2008.

[20] F.Bigolin, D.Vittone, *Some remarks about parametrizations of intrinsic regular surfaces in the Heisenberg group*, to appear on Publicacions Matemtiques, available at http://cvgmt.sns.it/papers/bigvit08/

[21] A.Bressan, *Hyperbolic Systems of Conservation Laws, The One-Dimensional Cauchy Problem,* Oxford Lecture Series in Mathematics and its Applications 20, Oxford Univ. Press, 2000.

[22] L. Capogna, G. Citti, M. Manfredini, *Regularity of non-characteristic minimal graphs in the Heisenbreg group* $\mathbb{H}^1$, Preprint, 2008.

[23] L.Capogna, G. Citti, M. Manfredini, *Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups* $\mathbb{H}^n$, $n > 1$, Preprint, 2008.

[24] L.Capogna, D.Danielli, N.Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. 12 (1994), 203–215.

[25] L.Capogna, D.Danielli, S.D.Pauls, J.T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, PM 259, Birkhäuser, 2007.

[26] J.Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom.Funct.Anal. 9 (1999), 428–517.

[27] J.Cheeger, B.Kleiner, *Generalized differential and bi-Lipschitz nonembedding in* L^1, C. R. Math. Acad. Sci. Paris 343 (2006), no. 5, 297–301.

[28] J.H.Cheng, J.F.Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Austr. Math. Soc. 70 (2005), 507–520.

[29] J.H.Cheng, J.F.Hwang, A.Malchiodi, P.Yang, *Minimal surfaces in pseudohermitian geometry and the Bernstein problem in the Heisenberg group*, Ann. Sc.Norm. Pisa Cl. Sci. 1 (2005), 129–177.

[30] J.H.Cheng, J.F.Hwang, A.Malchiodi, P.Yang, *Existence and uniqueness for p-area minimizers in the Heisenberg group*, Math. Ann. 337 (2007), no. 2, 253–293.

[31] J.H.Cheng, J.F.Hwang, P.Yang, *Regularity of* C^1 *surfaces with prescribed p-mean curvature in the Heisenberg group*, Preprint, 2007.

[32] G. Citti, M. Manfredini, *Implicit function Theorem in Carnot-Carathodory spaces*, Commun. Contemp. Math. 8 (2006), no. 5, 657–680.

[33] J.D.Cole, *On a quasi-linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math. 9 225–236 (1951).

[34] D.R. Cole, S.Pauls, $C^{1,1}$ *Hypersurfaces of the Heisenberg Group are* N*-rectifiable*, Houston J. Math. 32 (2006), no. 3, 713–724.

[35] J.G.Conlon, T.P.Liu, *Admissibility Criteria for Hyperbolic Conservation Laws*, Ind. Uni.Math. J. 30, nr. 5 (1981).

[36] L.Corrias, M.Falcone, R.Natalini, *Numerical schemes for conservation laws via Hamilton-Jacobi equations*, Math. Comp. 64 (1995), no. 210, 555–580.

[37] M.G.Crandall, P.L.Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 277 (1983), no. 1.

[38] M.G.Crandall, P.L.Lions, *Two approximations of solutions of Hamilton-Jacobi equations*, Math. Comp. 43 (1984), no. 167, 1–19.

[39] M.G.Crandall, L.C.Evans, P.L.Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 282 (1984), no. 2.

[40] C.M.Dafermos, *Generalized Characteristics and the Structure of Solutions of Hyperbolic Conservation laws*, Ind. U. Math. J. 26 (1977), No.6 1097–1119.

[41] C.M.Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer Verlag, Heidelberg, (2000).

[42] C.M.Dafermos, *Continuous solutions for balance laws*, Ric.Mat 55 (2006), 79–91.

[43] D.Danielli, N.Garofalo, D.M.Nhieu, *A notable family of entire intrinsic minimal graphs in the Heinsenberg group which are not perimeter minimizing*, Amer. J. Math. 130 (2008), no. 2, 317-339.

[44] D.Danielli, N.Garofalo, D.M.Nhieu, *Minimal Surfaces in Carnot groups*, Preprint, 2004.

[45] D.Danielli, N.Garofalo, D.M.Nhieu, S.Pauls, *Instability of grafical strips and a positive answer to the Berstein problem in the Heisenbrg group* $\mathbb{H}^1$, Preprint, 2006.

[46] G.David, S.Semmes, *Fractured Fractals and Broken Dreams. Self-Similar Geometry through Metric and Measure*, Oxford University Press, (1997).

[47] E.De Giorgi, *Un progetto di teoria delle correnti, forme differenziali e varietà non orientate in spazi metrici*, in *Variational Methods, Non Linear Analysys and Differential Equations in Honour of J.P. Cecconi*, (Genova 1993), M.Chicco et al. Eds. ECIG, Genova, 67–71.

[48] E.De Giorgi, *Problema di Plateau generale e funzionali geodetici*, Atti Sem. Mat. Fis. Univ. Modena 43 (1995), 285–292.

[49] De Lellis C., *Ordinary differential equations with rough coefficients and the renormalization Theorem of Ambrosio (d'après Ambrosio, Di Perna, Lions)*, Preprint, 2007, available at http://www.math.unizh.ch/fileadmin/math/preprints/Exp.972.C.DeLellis.pdf

[50] J.P.Dias, P.G.Lefloch, *Some existence results for conservation laws with source-term,* Math. Meth. Appl. Sci. 25 (2002), 1149-1160.

[51] L.C.Evans, *Partial Differential Equations*, Am.Math.Soc., Providence, 1998.

[52] H.Federer, *Geometric Measure Theory*, Springer, 1969.

[53] G.B.Folland, E.M.Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, 1982.

[54] B.Franchi, R.Serapioni, F.Serra Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups,* J. Nonlinear Convex Anal. 7 (2006), no. 3, 423-441.

[55] B.Franchi, R.Serapioni, F.Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math.Ann., 321 (2001), 479–531.

[56] B.Franchi, R.Serapioni, F.Serra Cassano,*Regular hypersurfaces, intrinsic perimeter and Implicit Function Theorem in Carnot groups*, Comm. Analysis and Geometry 11, (2003), 909–944.

[57] B.Franchi, R.Serapioni, F.Serra Cassano, *On the structure of finite perimeter sets in step 2 Carnot groups*, Journal Geometric Analysis 13, (2003), 421–466.

[58] B.Franchi, R.Serapioni, F.Serra Cassano, *Regular submanifolds, graphs and area formula in Heisenberg groups*, Advances in Math. 211 (2007), 157-203.

[59] B.Franchi, N.Tchou, M.C.Tesi, *div-curl type Theorem, H-convergence, and Stokes formula in the Heisenberg group*, Commun. Contemp. Math. 8 (1) (2006), 67–99.

[60] N.Garofalo, D.M.Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., 49 (1996), 1081–1144.

[61] N.Garofalo, S.Pauls, *The Bernstein problem in the Heisenberg group*, Preprint, 2005.

[62] E.Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston, (1984).

[63] M.Gromov: *Filling Riemannian manifolds.* J. Diff. Geom., 18 (1983), 1–147.

[64] M.Gromov, *Carnot-Carathéodory spaces seen from within*, in *Subriemannian Geometry*, Progress in Mathematics, 144 ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel (1996).

[65] P.Hajłasz, P.Koskela, *Sobolev met Poincare*, Mem. AMS, 145 (2000).

[66] J.Heinonen, *Lectures on analysis in metric spaces*, Springer-Verlag, New York, 2001.

[67] E.Hopf, *The Partial Differential Equation* $u_t + uu_x = \nu u_{xx}$, Commun, Pure Appl. Math. 3 (1950), 201–230.

[68] B.Kirchheim, *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. AMS, 121 (1994), 113–123.

[69] B.Kirchheim, F.Serra Cassano, *Rectifiability and parametrization of intrinsic regular surfaces in the Heisenberg group*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) III (2004), 871–896.

[70] S.N.Kružkov, *First order quasilinear equations in several independent variable,* Math. USSR Sb.10 (1970), 217–243.

[71] H.Ishii, *Remarks on existence of viscosity solutions of Hamilton-Jacobi equations,* Bull.Fac.Sci.Engrg. Chuo Univ. 26 (1983), 5–24.

[72] P.D.Lax, *Hyperbolic System of Conservation Laws II*, Commun. Pure Appl. Math. 6 (1953), 231–258.

[73] G.P.Leonardi, S.Rigot, *Isoperimetric sets on Carnot groups*, Houston J.. Math. 3 (2003), 609–637.

[74] P.L.Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, 69. Pitman (Advanced Publishing Program), Boston, Mass.-London, (1982).

[75] A.Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problem,* PNDETA 16, Birkhäuser, (1995).

[76] V.Magnani, *Elements of Geometric Measure Theory on Sub- Riemannian Groups*, Tesi di Perfezionamento, Scuola Normale Superiore, Pisa, (2002).

[77] V.Magnani, D.Vittone, *An intrinsic measure for submanifolds in stratified groups*, to appear on J. Reine Angew Math.

[78] P.Mattila, *Measures with unique tangent measures in metric groups*, Math. Scand. 97 (2005), 298–398.

[79] J.Mitchell, *On Carnot-Carathèodory metrics*, J.Differ. Geom. 21 (1985), 35–45.

[80] R.Montgomery, *A tour of Sub-Riemannian Geometries, their Geodesics and Applications* AMS, Providence (2002).

[81] R.Monti, *Distances, Boundaries and Surface Measures in Carnot-Carathodory spaces*, PhD Thesis, Universit degli Studi di Trento, (2001).

[82] R.Monti, F.Serra Cassano, *Surface Measures in Carnot-Carathéodory spaces*, Calc. Var. Partial Diff. Eq. 13, (2001), 339–376.

[83] P.Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Annals of Mathematics, 129 (1989), 1–60.

[84] P.Pansu, *Geometrie du Group d'Heisenberg*, These pour le titre de Docteur 3ème cycle, Universite Paris VII, (1982).

[85] S.D.Pauls, *A notion of rectifiability modelled on Carnot groups*, Indiana Univ. Math. J. 53 (2004), 49–81.

[86] S.D.Pauls, *Minimal surfaces in the Heisenberg groups*, Geom. Dedicata 104, 201-231 (2004)

[87] S.D.Pauls, *H-minimal graphs of low regularity in* $\mathbb{H}^1$, Comm. Math. Helv. 81 (2006), 337-381.

[88] M.Ritoré, *Area-Stationary Surfaces and isoperimetric Regions in the Heisenberg Group* $\mathbb{H}^1$, Lectures notes Fifth School on Analysis and Geometry in Metric Spaces, Levico Terme, (2007).

[89] M.Ritoré, C.Rosales, *Area-stationary surfaces in the Heisenberg group* $\mathbb{H}^1$, to appear on Adv. Math.

[90] M.Rumin, *Formes diffèrentielles sur les varietès de contact*, J.Differential Geom. 39 (2) (1994), 281–330.

[91] S.Semmes, *On the non existence of bi-Lipschitz parametrization and geometric problems about A_∞ weights*, Revista Matematica Iberoamericana 12 (1996), 337–410.

[92] E.M.Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[93] N.Th.Varopoulos, L.Saloff-Coste, T.Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, 1992.

[94] D. Vittone, *Submanifolds in Carnot groups*, Tesi di Perfezionamento, Scuola Normale Superiore, Pisa, Birkhäuser 2008.

[95] A.I.Volpert, *The space BV and quasilinear equations*, Math. USSR sb. 2 (1967), 225–267.

[96] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.