GENERALIZED UNIT NORMAL BUNDLES

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Abstract. We define and discuss a generalized notion of unit normal bundle in the context of rectifiable currents. In particular, we get results aimed at the application of direct methods to variational problems in a geometric measure-theoretic setting.

1. Introduction

Geometric measure theory plays a fundamental role in the process of development of theoretical machinery aimed to deal with geometric variational problems. A lot of work has been produced since the early seminal papers by De Giorgi's and Federer's schools and many open questions within the existing theory give life to the research on this subject.

The purpose of our paper is that of providing some tools to go into "unexplored territories along the boundary" between geometric measure theory and differential geometry, and carry out a process aimed to bridge the gap between the neighbouring regions. An example of application which we consider significant of how this approach can be fruitful, has been given recently in [14], where generalized Gauss graphs theory has been used to extend a compactness result by Langer [19], concerning sets of hypersurfaces subject to an uniform boundedness condition on the volumes and on the L^p norm of the second fundamental forms (if p is sufficiently large). Our interest in this subject is justified by the abundance of applications of unit normal bundles in the general literature and, in particular, along the "boundary" mentioned above. One can easily verify the substance of this assertion by giving a look at [6, 7, 8, 18, 24], just to cite a few main references.

This is not the first time that unit normal bundles have been considered in the context of geometric measure theory. Martina Zähle defined normal bundle currents for sets of positive reach and used them to represent curvature measures [25, 26] and to handle problems related to integral geometry. In particular, in a joined work with W. Rother [22], a simplified proof of a principal kinematic formula is provided. Unfortunately, the lack of compactness or closure results for the Zähle's generalized unit normal bundles seems to make their application to many variational problems in a geometric measure-theoretic setting rather hard to realize.

Now let us briefly describe the plan of the paper. Most of the notation and terminology is collected in section §2. In §3 we compute explicitly the multivector orienting a smooth unit normal bundle. The results led naturally to Definition 4.1, where we introduce the set $\mathrm{GUNB}_n(\mathbf{R}^N_x)$ of "generalized unit normal bundles of order n". It turns out immediately that such a set enjoys some nice properties as

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that of being closed with respect to weak convergence of currents and that of including any current carried by a measurable subset of the classical unit normal bundle over a smooth d-dimensional surface embedded in $\mathbf{R}^N_x$, provided d does not exceed n. Then, after some preliminaries from linear algebra, we state a coarea formula (Proposition 4.3) and a structure theorem (Proposition 4.4), summarizing the main explicit geometric properties induced on a generalized unit normal bundle by the axioms of Definition 4.1. As a first consequence we show that in the codimension one case an integral current is a generalized unit normal bundle if and only if it is a generalized Gauss graph (Proposition 4.5). Another corollary asserts that if a sequence of generalized unit normal bundles converges weakly, then the measures associated to the respective surfaces in $\mathbf{R}^N_x$ converge in the weak* sense (Proposition 5.1). be found in In section §6 we succeed in proving a non-cancellation result (Proposition 6.3), which could appear a little surprising in that it is well-known that cancellation phenomenons arise very frequently during a passage to the limit. Roughly, it asserts that if the surfaces corresponding to each member of a weakly convergent sequence in $\mathrm{GUNB}_n(\mathbf{R}^N_x)$ contain a given measurable set, then so do also the surface corresponding to the limit. A similar fact occurs also in the context of generalized Gauss graphs ([12, Proposition 5.1]), but in the present context we need a further result about slicing (Proposition 6.1 and Proposition 6.2) to overcome some complications related to coarea. In view of applications to variational problems and in order to get more general results, following a similar path to [14], we open §7 by defining special subclasses of $\mathrm{GUNB}_n(\mathbf{R}^N_x)$ (Definition 7.1). It follows a subsection about functionals, where we give examples of energies involving well-known curvature notions, e.g. the Chern-Lashof curvature (Remark 7.3). Then a couple of theorems about closure, semicontinuity and compactness are proved to hold for sets, in such special classes of generalized unit normal bundles, subject to a certain uniform boundedness condition on the energies (Proposition 7.3 and Proposition 7.4). Section §8 is devoted to a couple of applications. The first one concerns an existence result for a variational problem. In the second one we prove that the weak limit of unit normal bundle currents induced by regular surfaces has to be nice, provided areas and certain integral norms of Chern-Lashof curvature (of the appropriate order) and of Weingarten map remain bounded.

2. General notation

For the main notation about geometric measure theory, we refer the reader to the classical literature on this subject [16, 21, 23]. A $\mathcal{H}^h$−measurable countably h-rectifiable set is simply said to be a $\mathcal{H}^h$-rectifiable set. Given an open subset U of an euclidean space, let $\mathcal{D}^h(U)$ denote the family of smooth h-forms with compact support in U. The set of integer multiplicity rectifiable h-currents in U is indicated with $\mathcal{R}_h(U)$. Recall that an element $T \in \mathcal{R}_h(U)$ is determined by a $\mathcal{H}^h$-rectifiable subset R of U (carrier), a $\mathcal{H}^h \llcorner R$-measurable simple multivector field η (orientation) and a locally $\mathcal{H}^h \llcorner R$-integrable integer-valued function θ (multiplicity). We shall write $T = [\![R, \eta, \theta]\!]$.

Throughout the paper, while dealing with generalized unit normal bundles of surfaces, we have to keep distinguished the N-dimensional euclidean space where the surfaces live from the one where the normals live. We shall denote them by $\mathbf{R}^N_x$ and $\mathbf{R}^N_y$, where the subscripts x and y are the respective variable names. The standard orthonormal bases of $\mathbf{R}^N_x$ and $\mathbf{R}^N_y$ will be denoted by

$$\{e_1, \dots, e_N\} \qquad \text{and} \qquad \{\varepsilon_1, \dots, \varepsilon_N\}$$

while, according to the tradition,

$$\{dx^1, \dots, dx^N\} \qquad \text{and} \qquad \{dy^1, \dots, dy^N\}$$

will be the corresponding dual bases. Given a k-dimensional linear subspace E of $\mathbf{R}^N_x$ or of $\mathbf{R}^N_y$, then the unit sphere in E and its volume will be indicated, respectively, with $\mathbf{S}^{k-1}(E)$ and $\mathbf{s}_{k-1}$.

Our conventions about multi-indices and multivectors are standard. Given a couple of positive integers H and h such that $H \geq h \geq 0$, we set

$$I(H,h) := \begin{cases} \{\delta = (\delta_1, \dots, \delta_h) \mid 1 \leq \delta_1 < \delta_2 < \dots < \delta_h \leq H\} & \text{if } h > 0 \\ \emptyset & \text{if } h = 0. \end{cases}$$

If $\alpha \in I(H,h)$, then $\sigma(\alpha,\bar{\alpha})$ is the sign of the permutation of $(1,2,\dots,H)$ into $(\alpha,\bar{\alpha})$. The short notation for products will be often used. For example:

$$e_\alpha := e_{\alpha_1} \wedge \dots \wedge e_{\alpha_d}, \qquad dx^\beta := dx^{\beta_1} \wedge \dots \wedge dx^{\beta_k}, \qquad \kappa_\gamma := \kappa_{\gamma_1} \cdots \kappa_{\gamma_n}.$$

Given a non zero simple multivector τ, then E_τ will be the linear subspace associated to τ (i.e. the enveloping subspace of τ, according to [20]).

As usual, $*$ is the Hodge star isomorphism in $\mathbf{R}^N_x$:

$$*e_\alpha = \sigma(\alpha,\bar{\alpha}) e_{\overline{\alpha}}, \qquad \alpha \in I(N,h).$$

We will indicate every usual inner product and dual pairing with the same notation $\langle \cdot \mid \cdot \rangle$. The corresponding length function will be

$$|\cdot| := \sqrt{\langle \cdot \mid \cdot \rangle}.$$

The $d-$th stratum $(d = 0,1,\dots,h)$ of

$$\zeta \in \wedge_h(\mathbf{R}^N_x \times \mathbf{R}^N_y) \qquad \big(\omega \in \mathcal{D}^h(\mathbf{R}^N_x \times \mathbf{R}^N_y)\big)$$

is given by

$$\zeta_{(d)} := \sum_{\substack{\gamma \in I(N,d) \\ \delta \in I(N,h-d)}} \langle \zeta \mid e_\gamma \wedge \varepsilon_\delta \rangle \, e_\gamma \wedge \varepsilon_\delta \qquad \Bigg(\omega_{(d)} := \sum_{\substack{\gamma \in I(N,d) \\ \delta \in I(N,h-d)}} \langle \omega \mid dx^\gamma \wedge dy^\delta \rangle \, dx^\gamma \wedge dy^\delta \Bigg)$$

and hence also the d-th stratum of $T = [\![R,\eta,\theta]\!] \in \mathcal{R}_h(\mathbf{R}^N_x \times \mathbf{R}^N_y)$ can be defined as follows

$$T_{(d)}(\omega) := \int_R \langle \eta_{(d)} \mid \omega \rangle \theta d\mathcal{H}^h = \int_R \langle \eta \mid \omega_{(d)} \rangle \theta d\mathcal{H}^h = T(\omega_{(d)}) \tag{2.1}$$

for all $\omega \in \mathcal{D}^h(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. Regarding T and the strata $T_{(h)}$ as vector measures, we get

$$|T_{(h)}| = |\eta_{(h)}| \theta \, \mathcal{H}^h \llcorner R = |\eta_{(h)}| \, |T|$$

where $|\cdot|$ denotes, in this case, the total variation.

To simplify the formulae, throughout the paper we will often omit to mention the obvious isometric immersions

$$\mathbf{R}^N_x \hookrightarrow \mathbf{R}^N_x \times \mathbf{R}^N_y, \qquad \mathbf{R}^N_y \hookrightarrow \mathbf{R}^N_x \times \mathbf{R}^N_y.$$

Finally, consider the obvious isometric isomorphism

$$\Phi : \mathbf{R}^N_x \to \mathbf{R}^N_y \qquad \big(\Phi(e_j) := \varepsilon_j\big).$$

Then, for the sake of simplicity, we set

$$[\cdot]^\bullet := \Phi, \qquad [\cdot]_\bullet := \Phi^{-1}.$$

and extend this convention to multivectors and multicovectors, e.g. $[e_\beta]^\bullet = \varepsilon_\beta$, $[\varepsilon_\beta]_\bullet = e_\beta$ and $[dx^\beta]^\bullet = dy^\beta$.

3. The current carried by a measurable subset of a smooth unit normal bundle

Given a submanifold M of $\mathbf{R}^N_x$, let us denote its unit normal bundle by $\mathbf{B}$. Recall that $\mathbf{B}$ is canonically oriented and has dimension $N-1$, whatever the dimension of M ([6, §6.7.21], [8, §7]). Then we can consider a $\mathcal{H}^{N-1}$-measurable subset E of B and the current

$$T := [\![E, \eta_{|E}, 1]\!] \tag{3.1}$$

where η is the unit simple $(N-1)$-vector orienting $\mathbf{B}$.

Let us proceed to the computation of η at $(x,y) \in \mathbf{B}$, in terms of principal directions $\{\tau_i\}$ and curvatures $\{\kappa_i\}$ of M at x with respect to y. First of all, let d be the dimension of M and let $h : U \subset \mathbf{R}^d \to \mathbf{R}^N_x$ be a local parametrization of M around x, where U is an open set containing 0 which is mapped by h to x. Without loss of generality, we can also suppose

$$\frac{\partial h}{\partial u^i}(0) = \tau_i.$$

Then, denoting the codimension of M with k (i.e. $k := N - d$) and choosing an orthonormal family $\{\nu_1, \dots, \nu_k\}$ of smooth normal vector fields, defined in a neighbourhood of x, such that

$$\nu_k(x) = y_\bullet \qquad \text{and} \qquad \nu_1(x) \wedge \cdots \wedge \nu_k(x) = *(\tau_1 \wedge \cdots \wedge \tau_d),$$

we can define the following local parametrization of $\mathbf{B}$ around (x,y)

$$H(u,t) := h(u) + \sum_{j=1}^{k-1} t^j \left[\nu_j\big(h(u)\big)\right]^\bullet + \sqrt{1-|t|^2}\left[\nu_k\big(h(u)\big)\right]^\bullet, \quad (u,t) \in U \times V$$

where V is a neighbourhood of 0 in $\mathbf{R}^{k-1}$ (compare [6, (6.7.11)]). Hence, denoting the normal component by $[\,\cdot\,]^\perp$, we find

$$dH_{(0,0)}\left(\frac{\partial}{\partial u^i}\right) = \tau_i + [d\nu_{k,x}(\tau_i)]^\bullet = \tau_i - \kappa_i \tau_i^\bullet + \left\{[d\nu_{k,x}(\tau_i)]^\perp\right\}^\bullet$$

i.e.

$$dH_{(0,0)}\left(\frac{\partial}{\partial u^i}\right) = \tau_i - \kappa_i \tau_i^\bullet + \sum_{j=1}^{k-1} \langle d\nu_{k,x}(\tau_i) | \nu_j(x)\rangle [\nu_j(x)]^\bullet$$

in that $\langle \nu_k | \nu_k \rangle \equiv 1$. Since one has also $dH_{(0,0)}\left(\frac{\partial}{\partial t^i}\right) = [\nu_i(x)]^\bullet$, we get easily

$$\begin{aligned}
\zeta :=& \wedge_{N-1}\, dH_{(0,0)}\left(\frac{\partial}{\partial u^1} \wedge \cdots \wedge \frac{\partial}{\partial u^d} \wedge \frac{\partial}{\partial t^1} \wedge \cdots \wedge \frac{\partial}{\partial t^{k-1}}\right) \\
=& (\tau_1 - \kappa_1 \tau_1^\bullet) \wedge \cdots \wedge (\tau_d - \kappa_d \tau_d^\bullet) \wedge [\nu_1(x)]^\bullet \wedge \cdots \wedge [\nu_{k-1}(x)]^\bullet \\
=& (-1)^{k-1} \tau \wedge (\nu^\bullet \llcorner y) + \sum_{j=0}^{d-1} \sum_{\alpha \in I(d,j)} (-1)^{N-1-j} \sigma(\alpha, \overline{\alpha}) \kappa_{\overline{\alpha}} \tau_\alpha \wedge \tau_{\overline{\alpha}}^\bullet \wedge (\nu^\bullet \llcorner y)
\end{aligned}$$

where $\tau := \tau_1 \wedge \cdots \wedge \tau_d$ and $\nu := \nu_1(x) \wedge \cdots \wedge \nu_k(x)$. Hence we see that the j-th stratum of $\eta(x,y) = \zeta/|\zeta|$ has to be zero provided j exceeds d, i.e. (omitting the argument (x,y))

$$\eta = \eta_{(0)} + \eta_{(1)} + \eta_{(2)} + \cdots + \eta_{(d)} \tag{3.2}$$

while the expressions of the other strata are given by:

$$\eta_{(j)} = \frac{\zeta_{(j)}}{|\zeta|} = \frac{(-1)^{N-1-j} \sum_{\alpha \in I(d,j)} \sigma(\alpha, \overline{\alpha}) \kappa_{\overline{\alpha}} \tau_\alpha \wedge \tau_{\overline{\alpha}}^\bullet \wedge (\nu^\bullet \llcorner y)}{\sqrt{1 + \sum_{i=1}^d \sum_{\alpha \in I(d,i)} \kappa_\alpha^2}}$$

where $j = 0, 1, \dots, d-1$, and

$$\eta_{(d)} = \frac{\zeta_{(d)}}{|\zeta|} = \frac{(-1)^{k-1}\tau \wedge (\nu^\bullet \llcorner y)}{\sqrt{1+\sum_{i=1}^d \sum_{\alpha\in I(d,i)} \kappa_\alpha^2}}. \tag{3.3}$$

The following proposition provides, in particular, a rule to select the canonical orientation and motivates axiom (iii) in Definition 4.1 below.

Proposition 3.1. *Let φ be the 1-form in $\mathbf{R}^N_x \times \mathbf{R}^N_y$ defined by*

$$\varphi(x,y) := \sum_{i=1}^N y^i \, dy^i.$$

Then, for all $j \in \{0,1,\dots,d\}$ and $\omega \in \mathcal{D}^j(\mathbf{R}^N_x)$, one has

$$\big\langle \zeta \big| \omega \wedge [(*\omega)^\bullet \llcorner \varphi]\big\rangle = \big\langle \zeta_{(j)} \big| \omega \wedge [(*\omega)^\bullet \llcorner \varphi]\big\rangle = (-1)^{N+d+1} \sum_{\alpha\in I(d,j)} \kappa_{\overline{\alpha}} \langle \tau_\alpha|\omega\rangle^2.$$

In particular

$$\big\langle \zeta \big| \omega \wedge [(*\omega)^\bullet \llcorner \varphi]\big\rangle = \big\langle \zeta_{(d)} \big| \omega \wedge [(*\omega)^\bullet \llcorner \varphi]\big\rangle = (-1)^{N+d+1} \langle \tau|\omega\rangle^2$$

provided $\omega \in \mathcal{D}^d(\mathbf{R}^N_x)$.

Proof. Since

$$*\tau_\alpha = \sigma(\alpha,\overline{\alpha})\, \tau_{\overline{\alpha}} \wedge \nu$$

for all $\alpha \in I(d,j)$, we obtain

$$\begin{aligned}
\big\langle \tau_\alpha \wedge \tau_{\overline{\alpha}}^\bullet \wedge (\nu^\bullet \llcorner y) \big| \omega \wedge [(*\omega)^\bullet \llcorner y]\big\rangle &= \langle \tau_\alpha|\omega\rangle \big\langle \tau_{\overline{\alpha}}^\bullet \wedge (\nu^\bullet \llcorner y) \big| (*\omega)^\bullet \llcorner y\big\rangle \\
&= (-1)^{d+j} \langle \tau_\alpha|\omega\rangle \big\langle (\tau_{\overline{\alpha}} \wedge \nu) \llcorner y_\bullet \big| *\omega \llcorner y_\bullet \big\rangle \\
&= (-1)^{d+j} \langle \tau_\alpha|\omega\rangle \big\langle \tau_{\overline{\alpha}} \wedge \nu \big| *\omega \big\rangle \\
&= (-1)^{d+j} \sigma(\alpha,\overline{\alpha}) \langle \tau_\alpha|\omega\rangle^2.
\end{aligned}$$

Hence the statement follows immediately by recalling the expression of ζ computed above. □

Remark 3.1. Let M be a countable union of pairwise disjoint (in measure) submanifolds M_j of $\mathbf{R}^N_x$. Denote the unit normal bundle of M_j by B_j and set $B := \cup_j B_j$. The sets M and B will be referred to as a "regular agglomeration of surfaces" and the "unit normal bundle of M", respectively. Then the definition of T can be obviously extended to the case when E is a $\mathcal{H}^{N-1}$-measurable subset of B, even if the dimensions of the M_j are not homogeneous, provided E has locally finite measure. We will say that T is "induced by E" and the maximal dimension of the surfaces will be called "order" of T (or of the agglomeration M). Note that if T_j is the current induced by $E \cap B_j$, then one has $T = \sum_j T_j$.

4. Generalized unit normal bundles

4.1. **The definition.** The discussion of Section §3 is quite illuminating about the definition we are going to state. Actually, as we shall see in the structure theorem, the main geometric properties characterizing the current (3.1) will continue to hold for the generalized items defined below.

Definition 4.1. A current $T \in \mathcal{R}_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$ is said to be a "generalized unit normal bundle (of order n with base space $\mathbf{R}^N_x$)" if it is carried by a subset of $\mathbf{R}^N_x \times \mathbf{S}^{N-1}(\mathbf{R}^N_y)$ and the following conditions are satisfied:

(i) $T_{(n+1)} = T_{(n+2)} = \ldots = T_{(N-1)} = 0$;
(ii) $T \llcorner \varphi_\bullet = 0$, where φ is defined as in Proposition 3.1;
(iii) for all $\omega \in \mathcal{D}^n(\mathbf{R}^N_x)$ and for all nonnegative functions $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$,

$$(-1)^{N+n+1} T\,(g\,\omega \wedge [(*\omega)^\bullet \llcorner \varphi]) \geq 0.$$

The set of generalized unit normal bundles (of order n with base space $\mathbf{R}^N_x$) will be denoted by $\mathrm{GUNB}_n(\mathbf{R}^N_x)$.

Remark 4.1. Let M, B, E and T be as in Remark 3.1. Then, recalling the formulas obtained in Section §3, it is easy to see that T belongs to $\mathrm{GUNB}_n(\mathbf{R}^N_x)$, provided the order d of M does not exceed n. This point is worth some more details with reference to axiom (iii). First of all, note that if $d < n$ then the condition is trivially verified (the first member in the inequality is identically zero!), even if the bundle is equipped with a noncanonical orientation. In the more interesting case $d = n$, the axiom could result not satisfied if the orientation is chosen arbitrarily. More precisely, if B^* denotes the unit normal bundle of the agglomeration formed by the n-dimensional surfaces of M, then Proposition 3.1 shows that axiom (iii) is satisfied if and only B^* is oriented canonically, while we are free again to choose any orientation $B \backslash B^*$. As we shall see below, the same consideration continue to hold in the generalized setting (compare Remark 4.4).

Remark 4.2. If $m < n$, then $\mathrm{GUNB}_m(\mathbf{R}^N_x)$ is a proper subset of

$$\mathcal{T} := \{T \in \mathrm{GUNB}_n(\mathbf{R}^N_x) | T_{(m+1)} = \ldots = T_{(n)} = 0\}.$$

For example any current carried by the unit normal bundle of a m-dimensional smooth surface M and equipped with the noncanonical orientation belongs to the residual set $\mathcal{T} \backslash \mathrm{GUNB}_m(\mathbf{R}^N_x)$.

Remark 4.3. The definition is closed with respect to weak convergence of currents, i.e. if $\{T_j\} \subset \mathrm{GUNB}_n(\mathbf{R}^N_x)$ converges weakly to $T \in \mathcal{R}_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$, then $T \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$. In particular the closure of axiom (i) follows from (2.1). Observe that, passing to the limit, phenomenons of "dimension dropping" (roughly speaking!) occur frequently, even in the special case when the T_j are induced by measurable subsets of smooth unit normal bundles. For example, given the cylindric surfaces in $\mathbf{R}^3_x$

$$M_j := \left\{x \in \mathbf{R}^3_x \,|\, (x^1)^2 + (x^2)^2 = 1/j^2,\ x^3 \in]0,1[\right\}$$

let T_j be the generalized unit normal bundle induced by the graph of the outward Gauss map of M_j. Then $\{T_j\}$ converges weakly to the current induced by the unit normal bundle of the segment $\{(0,0)\} \times]0,1[$ in $\mathbf{R}^3_x$. Similarly the generalized unit normal bundles induced by the graphs of the outward Gauss maps of shrinking spherical surfaces centered in a point $P \in \mathbf{R}^3_x$ converge weakly to the normal bundle current induced by the unit normal bundle of $\{P\}$. When dimension dropping occurs, apparently, we have no more explicit orientation conditions on the limit. However we are inclined to believe that nice orientation qualities could be inherited from the limit. Note that the closure property conjectured in Remark 4.4 below, would represent a result in favour of this argument.

4.2. **Projection and coarea.** In preparation for the structure theorem, we state some preliminary results. Let V be a vector subspace or $\mathbf{R}^N_x \times \mathbf{R}^N_y$ of dimension h and consider the standard orthogonal projection operator

$$\pi : \mathbf{R}^N_x \times \mathbf{R}^N_y \to \mathbf{R}^N_x.$$

Moreover let η be an orientation of V, i.e.

$$\eta = u_1 \wedge \cdots \wedge u_h$$

where $\{u_j\}$ is an orthonormal basis of V. In the following couple of propositions we show, in particular, how to deduce the rank and the coarea factor of $\pi_{|V}$ from the orientation η.

Lemma 4.1. *Assume* $\dim(\pi V) = d$. *Then one can choose the orthonormal basis* $\{u_j\}$ *in such a way that*

1. $\{u_{d+1}, \ldots, u_h\}$ *is an orthonormal basis of* $\ker(\pi_{|V})$, *while* $\{\pi u_1, \ldots, \pi u_d\}$ *is a basis of* πV*;*
2. *one has*

$$\begin{cases} \eta_{(d)} = \wedge_d \pi(u_1 \wedge \cdots \wedge u_d) \wedge u_{d+1} \wedge \cdots \wedge u_h \\ \eta_{(d+1)} = \cdots \eta_{(h)} = 0. \end{cases}$$

Moreover $|\eta_{(d)}|$ *coincides with the* d *dimensional Jacobian of* $\pi_{|V}$, *i.e.*

$$|\eta_{(d)}| = \sqrt{\det\left[\pi_{|V} \circ (\pi_{|V})^t\right]}. \tag{4.1}$$

Proof. Statement (1) is an easy consequence of

$$\dim\left[\ker(\pi_{|V})\right] = \dim V - \dim(\pi V) = h - d.$$

It follows that

$$\eta = (\pi u_1 + v_1) \wedge \cdots \wedge (\pi u_d + v_d) \wedge u_{d+1} \wedge \cdots \wedge u_h$$

where $v_i := u_i - \pi u_i \in \mathbf{R}^N_y$, which implies (2). Hence we get (4.1):

$$|\eta_{(d)}| = |\wedge_d \pi\,(u_1 \wedge \ldots \wedge u_d)| = \sqrt{\sum_{\alpha \in I(h,d)} |\wedge_d \pi\,(u_\alpha)|^2} = \sqrt{\det\left[\pi_{|V} \circ (\pi_{|V})^t\right]}.$$

□

Lemma 4.2. *The conditions*

$$\eta_{(d)} \neq 0 \qquad \textit{and} \qquad \eta_{(d+1)} = \cdots = \eta_{(h)} = 0 \tag{4.2}$$

are satisfied if and only if $\dim(\pi V) = d$.

Proof. Lemma 4.1 yields the "if" implication. To get the converse, let us assume condition (4.2) and denote $\dim(\pi V)$ by d'. Then

$$\eta_{(d')} \neq 0 \qquad \text{and} \qquad \eta_{(d'+1)} = \cdots = \eta_{(h)} = 0$$

whence, comparing with (4.2), we obtain $d' = d$. □

Now we can apply the previous results to study the coarea of $\pi_{|R}$, where R is a $\mathcal{H}^h$-rectifiable subset of $\mathbf{R}^N_x \times \mathbf{R}^N_y$ oriented by a $\mathcal{H}^h$-measurable field η of unit simple h-vectors. Define

$$R^{(d)} := \left\{(x,y) \in R \;\middle|\; \eta_{(d)}(x,y) \neq 0, \quad \eta_{(d+1)}(x,y) = \cdots = \eta_{(h)}(x,y) = 0\right\}$$

for $d = 0, 1, \ldots, h$.

Proposition 4.3 (coarea)**.** *The following facts hold:*

1. *The* $R^{(d)}$ *are pairwise disjoint* $\mathcal{H}^d$*-rectifiable sets and*
$$R = R^{(0)} \cup R^{(1)} \cup \cdots \cup R^{(h)};$$
2. *Denoting with* $\operatorname{ap} J_d(\pi_{|R^{(d)}})$ *the approximate* d *dimensional Jacobian of* $\pi_{|R^{(d)}}$ *(see* [16, Theorem 3.2.22]*), one has*
$$\operatorname{ap} J_d(\pi_{|R^{(d)}}) = |\eta_{(d)}| \neq 0$$
$\mathcal{H}^h \llcorner R^{(d)}-$*a.e.;*
3. *The projection* $\pi R^{(d)}$ *is a* $\mathcal{H}^d$*-rectifiable subset of* $\mathbf{R}^N_x$*;*
4. *For* $\mathcal{H}^d \llcorner (\pi R^{(d)})$ *almost all* x*, the set* $\pi^{-1}(x) \cap R^{(d)}$ *is* $\mathcal{H}^{h-d}$*-rectifiable and the coarea formula*
$$\int_{R^{(d)}} g \, d\mathcal{H}^h = \int_{\pi R^{(d)}} \left(\int_{\pi^{-1}(x) \cap R^{(d)}} \frac{g}{|\eta_{(d)}|} d\mathcal{H}^{h-d} \right) d\mathcal{H}^d(x)$$
holds, where g *is any* $\overline{\mathbf{R}}$ *valued function such that* $g/|\eta_{(d)}|$ *is* $\mathcal{H}^d \llcorner R^{(d)}-$*integrable.*

Proof. The first assertion is obvious, while the second one follows readily from Lemma 4.1. If we denote by $V(x,y)$ the tangent space for R at (x,y), then Lemma 4.2 yields
$$\dim(\pi V(x,y)) = d$$
for $\mathcal{H}^h \llcorner R^{(d)}$ almost all (x,y). Hence we get assertion (3). Finally (4) follows from (2), (3) and [16, Theorem 3.2.22]. □

4.3. **The structure of a generalized unit normal bundle.** In the following theorem, some main properties of smooth unit normal bundles are extended to the generalized items. In particular, a formula similar to (3.3) is proved. Consider
$$T = [\![R, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$$
and let $d \in \{0, 1, \ldots, n\}$. By Proposition 4.3 there exists a $\mathcal{H}^d$-measurable field τ of unit simple $d-$vectors in $\mathbf{R}^N_x$ orienting $\pi R^{(d)}$.

Proposition 4.4 (structure)**.** *We have the following results:*

1. *At* $\mathcal{H}^{N-1}$ *almost every point of* R*, one has*
$$\eta = \eta_{(0)} + \eta_{(1)} + \cdots + \eta_{(n)},$$
just as in the smooth case, compare (3.2)*. Hence also the decomposition*
$$R = R^{(0)} \cup R^{(1)} \cup \cdots \cup R^{(n)}$$
holds;
2. *For* $\mathcal{H}^{N-1} \llcorner R^{(d)}$ *almost all* (x,y)*, the vector* $y_\bullet$ *is orthogonal to* $E_{\tau(x)}$*;*
3. *For* $\mathcal{H}^d \llcorner (\pi R^{(d)})$ *almost all* x*,* $\pi^{-1}(x) \cap R^{(d)}$ *is a* $\mathcal{H}^{N-d-1}$*-rectifiable subset of*
$$\mathbf{S}_T(x) := \{x\} \times \left[\mathbf{S}^{N-d-1}\left(E^\perp_{\tau(x)} \right) \right]^\bullet$$
oriented by the field $(x,y) \mapsto (*\tau(x)^\bullet) \llcorner y$*;*
4. *There exists a* $\mathcal{H}^{N-1}$*-measurable function* $\sigma_d : R^{(d)} \to \{\pm 1\}$ *such that the equality*
$$\eta_{(d)}(x,y) = \sigma_d(x,y) |\eta_{(d)}(x,y)| \, \tau(x) \wedge \left[\left(* \, \tau(x)^\bullet \right) \llcorner y \right]$$
holds at $\mathcal{H}^{N-1} \llcorner R^{(d)}$ *almost all* (x,y)*. In the special case* $d = n$*, one has* $\sigma_n \equiv (-1)^{N+n+1}$*.*

Proof. (1) It is an immediate consequence of Definition 4.1(i), formula (2.1) and Proposition 4.3(1).

(2) Recalling Lemma 4.1 and Lemma 4.2, we find that, for $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost all (x,y), there exists an orthonormal basis $\{u_j\}$ of $E_{\eta(x,y)}$ such that the set $\{u_{d+1}, \dots, u_{N-1}\}$ is an orthonormal basis for the kernel of $\pi_{|E_{\eta(x,y)}}$, $\{\pi u_1, \dots, \pi u_d\}$ is a basis of $\pi(E_{\eta(x,y)}) = E_{\tau(x)}$ and

$$\eta_{(d)}(x,y) = \wedge_d \pi(u_1 \wedge \dots \wedge u_d) \wedge u_{d+1} \wedge \dots \wedge u_{N-1}.$$

It follows that (omitting the arguments x and y)

$$\eta_{(d)} = \pm |\eta_{(d)}| \, \tau \wedge u_{d+1} \wedge \dots \wedge u_{N-1}. \tag{4.3}$$

On the other hand, by axiom (ii) of Definition 4.1, we have also

$$\langle \eta | \varphi_\bullet \wedge \omega \rangle = 0$$

almost everywhere with respect to $\mathcal{H}^{N-1} \llcorner R$, for all $\omega \in \mathcal{D}^{N-2}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. Thus we find

$$\langle \tau | y_\bullet \wedge A \rangle \langle u_{d+1} \wedge \dots \wedge u_{N-1} | B \rangle = \langle \tau \wedge u_{d+1} \wedge \dots \wedge u_{N-1} | y_\bullet \wedge A \wedge B \rangle = 0$$

for all $A \in \wedge_{d-1}(\mathbf{R}^N_x)$ and $B \in \wedge_{N-d-1}(\mathbf{R}^N_y)$, at $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost all (x,y). We get immediately that $\tau(x) \llcorner y_\bullet = 0$, for $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost all (x,y).

(3) It derives easily from (2) and Proposition 4.3(4).

(4) First of all, let us prove that

$$\zeta(x,y) := (*\tau(x)^\bullet) \llcorner y = \pm u_{d+1} \wedge \dots \wedge u_{N-1} \tag{4.4}$$

for $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost all (x,y), where the u_j are chosen like above. Indeed

$$E_{\zeta(x,y)} \subset \ker \left(\pi_{|E_{\eta(x,y)}} \right). \tag{4.5}$$

for $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost all (x,y), by (3). Since $E_{\zeta(x,y)}$ and $\ker(\pi_{|E_{\eta(x,y)}})$ have the same dimension, the equality has to hold in (4.5). Hence (4.4) follows readily, by recalling that $\{u_{d+1}, \dots, u_{N-1}\}$ is an orthonormal basis of $\ker(\pi_{|E_{\eta(x,y)}})$. Now the formulas (4.3) and (4.4) yield

$$\eta_{(d)} = \sigma_d \, |\eta_{(d)}| \, \tau \wedge \zeta \tag{4.6}$$

where σ_d is a $\mathcal{H}^{N-1} \llcorner R^{(d)}$-measurable function which takes values in $\{\pm 1\}$. In order to compute σ_n, let us recall axiom (iii) of Definition 4.1. We find

$$(-1)^{N+n+1} \langle \eta \,|\, A \wedge ((*A)^\bullet \llcorner y) \rangle \geq 0$$

for all $A \in \wedge_n(\mathbf{R}^N_x)$ and $\mathcal{H}^{N-1} \llcorner R^{(n)}$ almost everywhere, whence

$$\begin{aligned} 0 &\leq (-1)^{N+n+1} \sigma_n |\eta_{(n)}| \, \langle \tau \wedge \zeta \big| A \wedge ((*A)^\bullet \llcorner y) \rangle \\ &= (-1)^{N+n+1} \sigma_n |\eta_{(n)}| \, \langle \tau | A \rangle \, \langle \zeta \big| (*A)^\bullet \llcorner y \rangle \\ &= (-1)^{N+n+1} \sigma_n |\eta_{(n)}| \, \langle \tau | A \rangle \, \langle y_\bullet \wedge ((*\tau) \llcorner y_\bullet) \,|\, *A \rangle \end{aligned}$$

for all $A \in \wedge_n(\mathbf{R}^N_x)$ and $\mathcal{H}^{N-1} \llcorner R^{(n)}$ almost everywhere, by (4.6). Since

$$y_\bullet \wedge ((*\tau) \llcorner y_\bullet) = *\tau$$

by (2), we get

$$0 \leq (-1)^{N+n+1} \sigma_n |\eta_{(n)}| \, \langle \tau | A \rangle^2$$

for all $A \in \wedge_n(\mathbf{R}^N_x)$ and $\mathcal{H}^{N-1} \llcorner R^{(n)}$ almost everywhere, which concludes the proof. $\square$

Remark 4.4. Assertion (4) in Proposition 4.4 shows that axiom (iii) of Definition 4.1 induces "orientation properties" only on $R^{(n)}$, i.e. the part of carrier where the projection has maximal rank. In fact T will continue to be a generalized unit normal bundle of order n even after that a new, arbitrary, orientation for $R\backslash R^{(n)}$ has been chosen in place of the old one (compare Remark 4.1). Obviously one can get a "richer" definition by equipping every $R^{(d)}$ with the canonical orientation, i.e. by replacing axiom (iii) with the following stronger one that is surely satisfied by every current T such as that defined in Remark 3.1:

(iii*) The inequality

$$(-1)^{N+d+1}\big(T \llcorner R^{(d)}\big)\,(g\,\omega \wedge [(*\omega)^\bullet \llcorner \varphi]) \geq 0$$

holds for all nonnegative functions $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$ and $\omega \in \mathcal{D}^d(\mathbf{R}^N_x)$, with $d = 0, 1, \ldots, n$.

In particular, if we assume (iii*) insted of (iii) and proceed just as in the computation of σ_n, then we can derive the expression of σ_d for all d, i.e. $\sigma_d \equiv (-1)^{N+d+1}$. Moreover, under such an assumption, one can easily verify that the "residual set" of Remark 4.2 is empty.

Unfortunately we do not know if such a modified definition enjoys the property (desirable, in the variational context) of being closed with respect to weak convergence of currents, even if we are in favour. That is the reason why we prefer to leave Definition 4.1 unchanged.

Now we can prove that, in codimension one, the notions of generalized unit normal bundle and generalized Gauss graph coincide. For basic facts and notation about generalized Gauss graphs, we refer to [5, 11] (further details and applications can be found in [1, 2, 3, 4, 10, 12, 13, 14, 15]).

Proposition 4.5. *Let the codimension be one, i.e. $N = n+1$. Then an integral current is a generalized Gauss graph if and only if it is a generalized unit normal bundle.*

Proof. We have to prove that Definition 4.1 is equivalent to [11, Definition 3.1].

First of all, recall from [17] that (in codimension one) the boundary condition (iii) of [11, Definition 3.1] can be dropped because it follows from the other axioms. Moreover, note that the orthogonality conditions (ii) of the same definition and (ii) of Definition 4.1 coincide trivially.

Let $T = [\![R, \eta, \theta]\!]$ be a generalized Gauss graph and show that it has to be a generalized unit normal bundle. Indeed we already know that T verifies axioms (i) and (ii) of Definition 4.1. Moreover [11, Proposition 4.1(2)] yields the equality

$$\eta_{(n)} = (-1)^n |\eta_{(n)}| * y_\bullet$$

almost everywhere with respect to $\mathcal{H}^n \llcorner R$, whence

$$\langle \eta | \omega \wedge [(*\omega)^\bullet \llcorner \varphi]\rangle = (-1)^n |\eta_{(n)}| \big((*\omega)^\bullet \llcorner \varphi\big) \langle *y_\bullet | \omega\rangle = |\eta_{(n)}| \langle *y_\bullet | \omega\rangle^2$$

almost everywhere with respect to $\mathcal{H}^n \llcorner R$ and for all $\omega \in \mathcal{D}^n(\mathbf{R}^N_x)$. This shows that T satisfies also (iii) of Definition 4.1.

To prove the converse assertion, consider a generalized unit normal bundle $T = [\![R, \eta, \theta]\!]$. Assertions (2) and (4) of Proposition 4.4 imply

$$\eta_{(n)} = |\eta_{(n)}|\,\langle *\tau^\bullet | y\rangle \tau = (-1)^n |\eta_{(n)}|\,\langle \tau | * y_\bullet\rangle \tau = (-1)^n |\eta_{(n)}| * y$$

almost everywhere with respect to $\mathcal{H}^n \llcorner R$. Then T satisfies axiom (iv) of [11, Definition 3.1], by [11, Proposition 4.1(2)]. Recalling the remarks at the beginning of this proof, wa can conclude that T is a generalized Gauss graph. $\square$

5. A convergence result

An essential role is played in the sequel by the total variation of the highest order stratum of a generalized unit normal bundle. In particular, the following proposition will prove to be very useful.

Proposition 5.1. *Let* $\{T_j\} \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$ *be such that*

$$T_j \rightharpoonup T \in \mathcal{R}_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$$

(hence $T \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$*). Then*

$$|T_{j_{(n)}}| \to |T_{(n)}|$$

in the weak sense of measures.*

Remark 5.1 (rough interpretation of the result). Given

$$T = [\![R, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x),$$

the geometric interpretation of the measure $|T_{(n)}|$ can be easily derived from coarea formula. In fact Proposition 4.3(4) implies

$$\int g\, d|T_{(n)}| = \int_{\pi R} \left(\int_{\pi^{-1}(x) \cap R^{(n)}} g(x,y)\theta(x,y)\, d\mathcal{H}^{N-n-1}(y) \right) d\mathcal{H}^n(x)$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. Recalling that y lives on a compact set, the unit sphere, we get

$$\int g\, d|T_{(n)}| = \int_{\pi R} g\tilde{\theta}\, d\mathcal{H}^n \tag{5.1}$$

for all $g \in C_c(\mathbf{R}^N_x)$, i.e.

$$\pi_{\#}|T_{(n)}| = \tilde{\theta}\, \mathcal{H}^n \llcorner (\pi R) = \tilde{\theta}\, \mathcal{H}^n \llcorner (\pi R^{(n)})$$

where $\tilde{\theta}$ is the $\mathcal{H}^n \llcorner (\pi R)$-measurable function defined as follows

$$\tilde{\theta}(x) := \int_{\pi^{-1}(x) \cap R^{(n)}} \theta(x,y)\, d\mathcal{H}^{N-n-1}(y).$$

In particular, if T is induced by the unit normal bundle of a regular agglomeration M (i.e. it is defined as in Remark 3.1 with $E = B$) then one has $\tilde{\theta}(x) = \mathbf{s}_{N-n-1}$ for $\mathcal{H}^n \llcorner (\pi R)$ almost all x. Hence we get

$$\pi_{\#}|T_{(n)}| = \mathbf{s}_{N-n-1}\mathcal{H}^n \llcorner M = \mathbf{s}_{N-n-1}\mathcal{H}^n \llcorner M^*$$

where M^* denotes the subagglomeration of $n-$dimensional surfaces. Now, on the basis of these considerations, we can give the following rough interpretion of our convergence result: if $T_j \rightharpoonup T$ then the measures carried by the highest dimension agglomerations of surfaces corresponding to the T_j converge in the weak* sense to the measure carried by the highest dimension agglomeration of surfaces corresponding to T.

Remark 5.2. By virtue of Proposition 4.5, if the codimension is one, Proposition 5.1 coincides with [5, Proposition 2.8] (or with [11, Proposition 4.2]).

The proof of Proposition 5.1 is a trivial consequence of the following lemma. Consider the $(N-1)$-form in $\mathbf{R}^N_x \times \mathbf{R}^N_y$

$$\Omega_{d,N} := (-1)^{Nd+N+1} \sum_{\beta \in I(N, N-d-1)} \left[(*\varphi_\bullet) \llcorner dx^\beta \right] \wedge dy^\beta$$

where φ is the 1-form defined in Proposition 3.1, i.e. $\varphi(x,y) := \sum_{i=1}^N y^i\, dy^i$. Then

Lemma 5.2. *Given $T = [\![R, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$ and $d \in \{0, 1, \ldots, n\}$, the following formula holds*

$$\langle \eta | \Omega_{d,N} \rangle = (-1)^{N+d+1} \sigma_d |\eta_{(d)}|, \qquad \mathcal{H}^{N-1} \llcorner R^{(d)}\text{-}a.e. \tag{5.2}$$

i.e.

$$\big(T \llcorner R^{(d)}\big)(g\, \Omega_{d,N}) = (-1)^{N+d+1} \int g \sigma_d \, d\big(|T_{(d)}| \llcorner R^{(d)}\big)$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. In the special case $d = n$ one has

$$T(g \Omega_{n,N}) = \int g \, d|T_{(n)}| \tag{5.3}$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$.

Proof. By Proposition 4.4(4) we get

$$\begin{aligned}
(-1)^{Nd+N+1} \sigma_d \langle \eta | \Omega_{d,N} \rangle &= (-1)^{Nd+N+1} \sigma_d \langle \eta_{(d)} | \Omega_{d,N} \rangle \\
&= |\eta_{(d)}| \sum_{\beta \in I(N, N-d-1)} \langle \tau \wedge ((*\tau)^{\bullet} \llcorner y) \,|\, [(*y_{\bullet}) \llcorner e_\beta] \wedge \varepsilon_\beta \rangle \\
&= |\eta_{(d)}| \sum_{\beta \in I(N, N-d-1)} \langle \tau | (*y_{\bullet}) \llcorner e_\beta \rangle \langle (*\tau)^{\bullet} \llcorner y | \varepsilon_\beta \rangle \\
&= (-1)^{Nd} |\eta_{(d)}| \sum_{\beta \in I(N, N-d-1)} \langle (*y_{\bullet}) \llcorner \tau | e_\beta \rangle \langle (*\tau)^{\bullet} \llcorner y | \varepsilon_\beta \rangle \\
&= (-1)^{Nd} |\eta_{(d)}| \langle (*y_{\bullet}) \llcorner \tau | (*\tau) \llcorner y_{\bullet} \rangle
\end{aligned}$$

$\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost everywhere. Then (5.2) follows readily from the equality

$$\begin{aligned}
\tau \wedge \big((*\tau) \llcorner y_{\bullet}\big) &= (-1)^d \big[\tau \wedge (*\tau)\big] \llcorner y_{\bullet} = (-1)^d (e_1 \wedge \cdots \wedge e_N) \llcorner y_{\bullet} \\
&= (-1)^d * y_{\bullet}.
\end{aligned}$$

which holds $\mathcal{H}^{N-1} \llcorner R^{(d)}$ almost everywhere, by Proposition 4.4(2). In order to prove (5.3), note that

$$\big(T \llcorner R^{(j)}\big)(g\, \Omega_{n,N}) = \int g \, d\big(|T_{(n)}| \llcorner R^{(j)}\big) = 0$$

whenever $j < n$ and for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$, by definition of $R^{(j)}$. Hence, recalling also Proposition 4.4(1), we obtain

$$T(g \Omega_{n,N}) = \big(T \llcorner R^{(n)}\big)(g\, \Omega_{n,N}) \qquad \text{and} \qquad \int g \, d|T_{(n)}| = \int g \, d\big(|T_{(n)}| \llcorner R^{(n)}\big)$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. The conclusion follows immediately from (5.2) and Proposition 4.4(4). ☐

6. A non-cancellation result for weakly convergent sequences of generalized unit normal bundles

Cancellation phenomenons arise very frequently during a passage to the limit, whereby Proposition 6.3 below could appear a little surprising. Indeed it states, roughly, that if the agglomerations of surfaces corresponding to each member of a weakly convergent sequence in $\mathrm{GUNB}_n(\mathbf{R}^N_x)$ contain a $\mathcal{H}^n$-measurable set, then so do also the agglomeration of surfaces corresponding to the limit. In the case of codimension one, such a result follows immediately from Proposition 4.5 and [12, Proposition 5.1], while, in the general case, the simple argument used in [12] to deal with generalized Gauss graphs is not sufficient to get the proof in this context. What we should need is a slicing theorem for mappings taking values in a rectifiable set but, unfortunately, we do not know if the usual slicing theory [16, §4.3] can be extended to such mappings (we expect it can!). However we succeed in overcoming this obstacle by obtaining a couple of results, Proposition 6.1 and Proposition 6.2, which apply successfully to the proof of Proposition 6.3. Before to state them, let us introduce some notation.

Let $T = [\![R, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$. Then, by recalling Proposition 4.3 and Proposition 4.4, we can define the "fiber current at x"

$$T_x^{(n)} := [\![\pi^{-1}(x) \cap R^{(n)}, (-1)^{N+n+1} (*\tau(x)^\bullet) \llcorner y, \theta_x]\!] \in \mathcal{R}_{N-n-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$$

for $\mathcal{H}^n \llcorner (\pi R)$ almost all x, where $\theta_x(y) := \theta(x,y)$ and τ is defined as in Proposition 4.4 (with $d = n$). Recall that, with respect to the measure $\mathcal{H}^n$, the set πR is equivalent to $\pi R^{(n)}$.

Lemma 6.1. *The following formulas hold:*

1. *For all $\mu = \mu(x) \in \mathcal{D}^n(\mathbf{R}^N_x)$ and $\lambda = \lambda(x,y) \in \mathcal{D}^{N-n-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$,*
$$T(\mu \wedge \lambda) = \int_{\pi R} \langle \tau(x) | \mu(x) \rangle T_x^{(n)}(\lambda)\, d\mathcal{H}^n(x);$$
2. *For all $\mu = \mu(x) \in \mathcal{D}^n(\mathbf{R}^N_x)$ and $\gamma = \gamma(x,y) \in \mathcal{D}^{N-n-2}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$,*
$$(-1)^n\, \partial T(\mu \wedge \gamma) = \int_{\pi R} \langle \tau(x) | \mu(x) \rangle \partial T_x^{(n)}(\gamma)\, d\mathcal{H}^n(x).$$

Proof. By Proposition 4.3 and Proposition 4.4 we get

$$\begin{aligned} T(\mu \wedge \lambda) =& \int_R \langle \eta | \mu \wedge \lambda\ rangle \theta\, d\mathcal{H}^{N-1} = \int_{R^{(n)}} \langle \eta_{(n)} | \mu \wedge \lambda \rangle \theta\, d\mathcal{H}^{N-1} \\ =& \int_{\pi R} \left(\int_{\pi^{-1}(x) \cap R^{(n)}} \left\langle \frac{\eta_{(n)}}{|\eta_{(n)}|} \middle| \mu \wedge \lambda \right\rangle \theta_x d\mathcal{H}^{N-n-1} \right) d\mathcal{H}^n(x) \\ =& \int_{\pi R} \langle \tau | \mu \rangle \left(\int_{\pi^{-1}(x) \cap R^{(n)}} \langle (-1)^{N+n+1} (*\tau(x)^\bullet) \llcorner y | \lambda \rangle\, \theta_x d\mathcal{H}^{N-n-1} \right) d\mathcal{H}^n(x) \end{aligned}$$

i.e. just the first formula. As for the second one, it can be obtained from (1) by taking $\lambda = d\gamma$. We find

$$\int_{\pi R} \langle \tau(x) | \mu(x) \rangle T_x^{(n)}(d\gamma)\, d\mathcal{H}^n(x) = T(\mu \wedge d\gamma) = (-1)^n \left[T\left(d(\mu \wedge \gamma) \right) - T(d\mu \wedge \gamma) \right].$$

and hence we conclude observing that the equality $\langle \eta | d\mu \wedge \gamma \rangle = 0$ has to hold $|T|$−a.e., by Proposition 4.4(1). □

Now we are ready to prove the following result.

Lemma 6.2. *Let*

$$\partial T(\mu \wedge \gamma) = 0$$

for all $\mu \in \mathcal{D}^n(\mathbf{R}^N_x), \gamma \in \mathcal{D}^{N-n-2}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$ (note that this condition is verified, for example, whenever $\partial T = 0$). Then, for $\mathcal{H}^n \llcorner (\pi R)$ almost all x, the multiplicity function θ_x assumes a constant value c_x and

$$T_x = [\![\mathbf{S}_T(x), (-1)^{N+n+1} (*\tau(x)^\bullet) \llcorner y, c_x]\!]$$

where $\mathbf{S}_T(x)$ and τ are defined as in Proposition 4.4 (with $d = n$).

Proof. First of all, Lemma 6.1(2) implies that $\partial T_x = 0$ at $\mathcal{H}^n \llcorner (\pi R)$ almost all x. Then the assertion follows from a standard constancy argument, by recalling Proposition 4.4. □

Now we are ready to prove the announced result.

Proposition 6.3. *Let*

$$T_j = [\![R_j, \eta_j, \theta_j]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x), \qquad T = [\![R, \eta, \theta]\!] \in \mathcal{R}_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$$

be such that

(i) $\partial T_j = 0$;
(ii) $T_j \rightharpoonup T$ *(hence* $T \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$ *and* $\partial T = 0$*)*;
(iii) E *is a* $\mathcal{H}^n$*-measurable subset of* $\mathbf{R}^N_x$ *such that* $E \subset \pi R_j$*, for all* j.

Then one has also $E \subset \pi R$.

Proof. Proposition 5.1 and assumption (ii) yield

$$\int g \, d|T_{(n)}| = \lim_j \int g \, d|T_{j\,(n)}|$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. Hence, by recalling (5.1), we get

$$\int_{\pi R} g\tilde{\theta} \, d\mathcal{H}^n = \lim_j \int_{\pi R_j} g\tilde{\theta}_j \, d\mathcal{H}^n \tag{6.1}$$

for all $g \in C_c(\mathbf{R}^N_x \times \mathbf{R}^N_y)$. But assumption (i) and Lemma 6.2 imply

$$\begin{aligned}\tilde{\theta}_j(x) &= \int_{\pi^{-1}(x)\cap R_j^{(n)}} \theta_j(x,y) \, d\mathcal{H}^{N-n-1}(y) \geq \mathcal{H}^{N-n-1}\left(\pi^{-1}(x) \cap R_j^{(n)}\right) \\ &= \mathcal{H}^{N-n-1}(\mathbf{S}_{T_j}(x)) = \mathbf{s}_{N-n-1}\end{aligned}$$

at $\mathcal{H}^n \llcorner (\pi R_j)$ almost all x, for all j. Then, by (6.1) and hypothesis (iii), we obtain

$$\int_{\pi R} g\tilde{\theta} d\mathcal{H}^n \geq \mathbf{s}_{N-n-1} \int_E g \, d\mathcal{H}^n$$

for all nonnegative functions $g \in C_c(\mathbf{R}^N_x)$. The conclusion follows immediately from the arbitrariness of g. □

7. Special classes of generalized unit normal bundles. Functionals. Semicontinuity and compactness results

We shall follow the same path as in [13] and [14].

7.1. **λ-special generalized unit normal bundles.** In order to get more general results, it is convenient to grade the set of generalized unit normal bundles into suitable subclasses. Let us recall some notation. First of all, for $m \leq n+1$, we set

$$J(n,m) := \left\{\lambda = (\lambda_1, \dots, \lambda_m) \,\middle|\, 0 \leq \lambda_1 \leq \cdots \leq \lambda_m \leq n\right\}.$$

Then, given $T = [\![R, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$ and $\lambda \in J(n,m)$, let

$$\eta^\lambda := \sum_{j=1}^m \eta_{(\lambda_j)}, \qquad T^\lambda := \eta^\lambda |T|.$$

Definition 7.1. When the condition $|T^\lambda| \ll |T_{(n)}|$ occurs, we say that T is a "λ-special generalized unit normal bundle". The set of λ-special generalized unit normal bundles is denoted with $\mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x)$. If $\lambda = (0,1,2,\dots,n)$ then the symbol λ is dropped, i.e. T is simply called "special generalized unit normal bundle" and $\mathrm{SGUNB}_n(\mathbf{R}^N_x)$ is the set of special generalized unit normal bundles.

In the following couple of remarks and in the subsequent proposition, we state some facts which help us to interpret the absolute continuity condition characterizing a λ-special generalized unit normal bundle.

Remark 7.1. ¿From the formulas obtained in section §3, it follows immediately that if M is a regular agglomeration of n-dimensional surfaces then the current T defined in Remark 3.1 belongs to $\mathrm{SGUNB}_n(\mathbf{R}^N_x)$.

Remark 7.2. Assertion (1) of Proposition 4.4 shows that $T \llcorner R^{(n)} \in \mathrm{SGUNB}_n(\mathbf{R}^N_x)$, while the lack of canonical orientation for $T \llcorner R^{(d)}$ (with $d < n$, compare Remark 4.4) does not allow, in general, to have

$$T \llcorner R^{(d)} \in \mathrm{SGUNB}_d(\mathbf{R}^N_x), \text{ for all } d.$$

Note that such a fact occurs under the strongest definition of generalized unit normal bundle, i.e. once axiom (iii) in Definition 4.1 has been replaced by axiom (iii*).

Proposition 7.1. *For all $\lambda \in J(n,m)$ one has*

$$\mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x) = \{T = [\![R,\eta,\theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x) \,|\, T^\lambda = T^\lambda \llcorner R^{(n)}\}.$$

In particular

$$\mathrm{SGUNB}_n(\mathbf{R}^N_x) = \{T = [\![R,\eta,\theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x) \,|\, T = T \llcorner R^{(n)}\}.$$

Proof. Consider $T = [\![R,\eta,\theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$. Then from $|T^\lambda| \ll |T_{(n)}|$ we get $|T^\lambda|(R^{(i)}) = 0$, for all $i < n$. Hence, recalling also Proposition 4.4(1), the equality $T^\lambda = T^\lambda \llcorner R^{(n)}$ follows.

On the other hand, if T satisfies $T^\lambda = T^\lambda \llcorner R^{(n)}$, then one has also

$$|T^\lambda| = |T^\lambda \llcorner R^{(n)}| = |T^\lambda| \llcorner R^{(n)} \leq |T| \llcorner R^{(n)} \ll |T_{(n)}|$$

which concludes the proof. □

The next result follows at once from Proposition 4.5. For the definition of λ-special generalized Gauss graph, the rader can refer to [14].

Proposition 7.2. *Let $N = n+1$. Then an integral current is a λ-special generalized Gauss graph if and only if it is a λ-special generalized unit normal bundle.*

7.2. **Functionals.** Let a continuous function

$$F(x,y;q) : \mathbf{R}^N_x \times \mathbf{R}^N_y \times \wedge_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y) \to [0,+\infty]$$

be given and let $\lambda \in J(n,m)$. Then we can define a functional

$$\mathcal{G}^\lambda_F : \mathrm{GUNB}_n(\mathbf{R}^N_x) \to [0,+\infty]$$

as follows ($T = [\![R,\eta,\theta]\!]$)

$$\mathcal{G}^\lambda_F(T) := \int F\left(x,y;\frac{\eta^\lambda(x,y)}{|\eta_{(n)}(x,y)|}\right) d|T_{(n)}|(x,y).$$

Note that

$$\begin{aligned}\mathcal{G}^\lambda_F(T) &= \int_{\pi R^{(n)}} \left[\int_{\pi^{-1}(x)\cap R^{(n)}} F\left(x,y;\frac{\eta^\lambda(x,y)}{|\eta_{(n)}(x,y)|}\right) \theta(x,y)\, d\mathcal{H}^{N-n-1}(y)\right] d\mathcal{H}^n(x)\\ &= \int_{\pi R} \left[\int_{\pi^{-1}(x)\cap R^{(n)}} F\left(x,y;\frac{\eta^\lambda(x,y)}{|\eta_{(n)}(x,y)|}\right) \theta(x,y)\, d\mathcal{H}^{N-n-1}(y)\right] d\mathcal{H}^n(x)\end{aligned}$$

by (3) and (4) of Proposition 4.3 and Proposition 4.4(1). Note that if T is induced by a measurable subset of the unit normal bundle of a regular agglomeration of surfaces then the formulas obtained in §3 yield the expression of $\mathcal{G}^\lambda_F(T)$ in terms of principal curvatures.

Example 7.1. Given $\lambda \in J(n,m)$ and $p \geq 0$, let

$$F(x,y;q) := F_p^\lambda(x,y;q) := \sum_{j=1}^m |q_{(\lambda_j)}|^p$$

and denote the correspondent functional simply by $\mathcal{F}_p^\lambda$, where the symbol λ will be omitted if $\lambda = (0,1,2,\dots,n)$. Since

$$\mathcal{F}_p^\lambda = \sum_{j=1}^m \mathcal{F}_p^{(\lambda_j)} \tag{7.1}$$

we can conveniently restrict our discussion to the functionals $\mathcal{F}_p^{(i)}$ for which the formula obtained above becomes

$$\mathcal{F}_p^{(i)}(T) = \int_{\pi R} \left[\int_{\pi^{-1}(x)\cap R^{(n)}} \left(\frac{|\eta_{(i)}|}{|\eta_{(n)}|} \right)^p \theta \, d\mathcal{H}^{N-n-1} \right] d\mathcal{H}^n(x). \tag{7.2}$$

Note that if $p \geq 1$ then a constant $c(m,p)$ has to exist such that $F_p^\lambda(x,y;q) \geq c|q^\lambda|^p$. It follows that

$$\mathcal{F}_p(T) \geq c \int \left| \frac{\eta}{|\eta_{(n)}|} \right|^p d|T_{(n)}| \geq c\,|T|(R^{(n)})$$

whence, recalling Proposition 7.1, we get in particular

$$\mathcal{F}_p(T) \geq c\,\mathbf{M}(T) \tag{7.3}$$

for all $T \in \mathrm{SGUNB}_n(\mathbf{R}_x^N)$.

Remark 7.3. Let M and B be respectilvely a n-dimensional submanifold of $\mathbf{R}_x^n$ and its unit normal bundle. Then denote with T_M the generalized unit normal bundle induced by B and consider the fibre over x in B, i.e. the set

$$\mathbf{S}_x := \mathbf{S}_{T_M}(x) = \{x\} \times \left[\mathbf{S}^{N-n-1}(\mathbf{N}_x M)\right]^\bullet.$$

Invoking (7.2) and the formulas of §3, we get

$$\mathcal{F}_p^{(i)}(T_M) = \int_M \left[\int_{\mathbf{S}_x} \left(\sum_{\alpha \in I(n,n-i)} \kappa_\alpha^2 \right)^{\frac{p}{2}} d\mathcal{H}^{N-n-1} \right] d\mathcal{H}^n(x)$$

for $i = 0,1,2,\dots,n-1$, while $\mathcal{F}_p^{(n)}(T_M) = \mathbf{s}_{N-n-1}\mathcal{H}^n(M)$. In view of next section, it is convenient to define the following couple of functionals:

$$\mathcal{B}_p(M) := \left[\mathcal{F}_p^{(n-1)}(T_M)\right]^{\frac{1}{p}}, \qquad \mathcal{C}_p(M) := \frac{1}{\mathbf{s}_{N-n-1}} \mathcal{F}_p^{(0)}(T_M).$$

Note that

$$\mathcal{B}_p(M) = \left[\int_M \left(\int_{\mathbf{S}_x} l_M^p \, d\mathcal{H}^{N-n-1} \right) d\mathcal{H}^n(x) \right]^{\frac{1}{p}}, \qquad \mathcal{C}_p(M) = \int_M K_M^{(p)} \, d\mathcal{H}^n.$$

where

$$l_M(x,y) := \sqrt{\sum_{j=1}^n \kappa_j(x,y)^2}$$

denotes the usual norm of the Weingarten map and

$$K_M^{(p)}(x) := \frac{1}{\mathbf{s}_{N-n-1}} \int_{\mathbf{S}_x} \left|\kappa_1(x,y)\kappa_2(x,y)\cdots\kappa_n(x,y)\right|^p d\mathcal{H}^{N-n-1}(y),$$

is the Chern-Lashof curvature of order p, which coincides with the usual Chern-Lashof curvature, provided $p = 1$. A great deal of work around the problems related to such kind of functionals has been produced by differential geometers since the Chern-Lashof curvature has been defined in [9] (e.g. [7, Chapter 5], [8, 24]).

Before stating the results about closure, compactness and semicontinuity, we need to introduce a special class of integrands (compare [13, 14]).

Definition 7.2. A function

$$F(x,y;q) : \mathbf{R}^N_x \times \mathbf{R}^N_y \times \wedge_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y) \to [0,+\infty]$$

will be said to be a "λ-standard integrand" if

(i) It is continuous;
(ii) It is convex with respect to q^λ;
(iii) There exists a continuous function $g(x,y;t) : \mathbf{R}^N_x \times \mathbf{R}^N_y \times [0,+\infty) \to [0,+\infty)$, nondecreasing in t, with $g(x,y;t) \to +\infty$ locally uniformly in (x,y) as $t \to +\infty$, and such that $F(x,y;q^\lambda) \geq g(x,y;|q^\lambda|)|q^\lambda|$ for all $(x,y;q)$.

Note that F^λ_p of Example 7.1 is a λ-standard integrand, provided $p > 1$ (with $g(x,y;t) = ct^{p-1}$). We are ready to enunciate the first theorem about closure and semicontinuity.

Proposition 7.3 (closure and semicontinuity)**.** *Let F be a λ-standard integrand (e.g. $F = F^\lambda_p$ with $p > 1$) and consider $\{T_j\} \subset \mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x)$ such that*

1. $T_j \rightharpoonup T \in \mathcal{R}_{N-1}(\mathbf{R}^N_x \times \mathbf{R}^N_y)$;
2. $\sup_j \mathcal{G}^\lambda_F(T_j) < +\infty$.

Then one has

$$T \in \mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x) \qquad \textit{and} \qquad \mathcal{G}^\lambda_F(T) \leq \liminf_j \mathcal{G}^\lambda_F(T_j).$$

Proof. Set

$$\mu_j := |T_{j\,(n)}|, \qquad \mu := |T_{(n)}|, \qquad f_j := \frac{\eta^\lambda_j}{|\eta_{j\,(n)}|}$$

and proceed just as in the proof of [13, Theorem 4.1] (compare also [14, Theorem 4.2]). Note that here Proposition 5.1 plays the fundamental role played by Proposition 3.2 in [13]. ☐

We get immediately the following corollary about compactness.

Proposition 7.4 (compactness)**.** *If F is a λ-standard integrand (e.g. $F = F^\lambda_p$ with $p > 1$) and $\{T_j\}$ is a sequence in $\mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x)$ satisfying*

$$\sup_j \mathcal{G}^\lambda_F(T_j) + \sup_j \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < +\infty \tag{7.4}$$

then there exist a subsequence $\{T_{j'}\}$ and $T \in \mathrm{SGUNB}^\lambda_n(\mathbf{R}^N_x)$ such that

$$T_{j'} \rightharpoonup T, \qquad \mathcal{G}^\lambda_F(T) \leq \lim_{j'} \mathcal{G}^\lambda_F(T_{j'}) = \liminf_j \mathcal{G}^\lambda_F(T_j).$$

Remark 7.4. In the special case $\lambda = (0,1,2,\dots,n)$, the previous proposition gives a compactness result for $\mathrm{SGUNB}_n(\mathbf{R}^N_x)$, by Definition 7.1. Moreover, if the integrand F is chosen such that $F \geq cF^\lambda_p$, where c is a positive constant (e.g. $F = F^\lambda_p$ with $p > 1$), then the condition (7.4) can be replaced with

$$\sup_j \mathcal{G}^\lambda_F(T_j) + \sup_j \mathbf{M}(\partial T_j) < +\infty$$

by estimate (7.3).

8. Two applications

This section is devoted to give a couple of applications. In the first one we prove an existence result for a variational problem. The second application shows that the weak limit of currents induced by unit normal bundles of submanifolds has to be nice, provided areas and certain integral norms of Chern-Lashof curvature (of the appropriate order) and of Weingarten map remain bounded.

Proposition 8.1 (first application). *Consider a $\mathcal{H}^n$-rectifiable set E in $\mathbf{R}_x^N$ and let*

$$\mathcal{G} : \mathrm{GUNB}_n(\mathbf{R}_x^N) \to [0, +\infty]$$

be a functional defined through a $(0, 1, \ldots, n)$-standard integrand. Assume that $\mathcal{G}$ is coercive in

$$\mathcal{I} := \left\{T = [\![R, \eta, \theta]\!] \in \mathrm{SGUNB}_n(\mathbf{R}_x^N) \mid \partial T = 0,\, \pi R \supset E\right\}$$

i.e. every subset S of $\mathcal{I}$ such that $\mathcal{G}(S)$ is bounded, has a subsequence converging weakly in $\mathrm{GUNB}_n(\mathbf{R}_x^N)$. Then $\mathcal{G}_{|\mathcal{I}}$ has a minimizer.

Remark 8.1. If $p > 1$, the functional $\mathcal{F}_p$ satisfies the hypotheses of Proposition 8.1. In particular, the coerciveness is a consequence of (7.3) and compactness for integral currents ([16, Theorem 4.2.17], [21, Theorem 5.5], [23, Theorem 32.2]).

Proof. We can suppose

$$\inf \mathcal{G}_{|\mathcal{I}} < +\infty$$

otherwise the statement is trivially true. Since $\mathcal{G}$ is coercive in $\mathcal{I}$, every minimizing sequence must have a subsequence converging weakly to some element T in $\mathrm{GUNB}_n(\mathbf{R}_x^N)$. By the closure result proved in Proposition 7.3 and recalling Proposition 6.3, we obtain $T \in \mathcal{I}$. Then the minimality of T follows from the semicontinuity of $\mathcal{G}$ asserted in Proposition 7.3. ☐

Before to consider the second application, in which we will prove the good qualities of functionals $\mathcal{B}_q$ and $\mathcal{C}_q$ (recall the notation of Remark 7.3) with respect to coerciveness, we need a lemma.

Lemma 8.2. *The following statements hold for every n-dimensional regular surface M in $\mathbf{R}_x^N$:*

1. *For $i = 1, \ldots, n-1$ and $p \geq 0$,*
$$\mathcal{F}_p^{(i)}(T_M) \leq c(n,p)\mathcal{B}_{p(n-i)}(M)^{p(n-i)};$$
2. *If $q > p > 0$, then*
$$\mathcal{B}_p(M) \leq c(N,p,q)\mathcal{H}^n(M)^{\frac{1}{p}-\frac{1}{q}}\mathcal{B}_q(M);$$
3. *If $n \geq p(n-1) > 0$, then*
$$\mathcal{B}_{p(n-1)}(M) \leq c(N,p)\mathcal{H}^n(M)^{\frac{1}{p(n-1)}-\frac{1}{n}}\|L_M\|_{\mathcal{L}^n(M)}$$
where L_M denotes the length of the second fundamental form of M (compare [7, §29.4]).

Proof. Assertion (1) follows readily from the inequality

$$\sum_{\alpha \in I(n,n-i)} \kappa_\alpha^2 \leq \sum_{\alpha \in I(n,n-i)} \left(\kappa_1^2 + \cdots + \kappa_n^2\right)^{n-i} \leq c(n)\left(\kappa_1^2 + \cdots + \kappa_n^2\right)^{n-i}$$

by recalling Remark 7.3. As for assertion (2), it can be derived by applying Hölder inequality twice. Finally, (2) and [7, Lemma 29.4.3] yield easily statement (3). ☐

Proposition 8.3 (second application). *Let $p \geq 1$ and consider a sequence $\{M_j\}$ of n-dimensional surfaces in $\mathbf{R}^N_x$ such that*

$$\sup_j \mathbf{M}(\partial T_{M_j}) < +\infty, \quad \sup_j \mathcal{H}^n(M_j) < +\infty, \quad \sup_j \mathcal{C}_p(M_j) < +\infty$$

and

$$\sup_j \mathcal{B}_{p(n-1)}(M_j) < +\infty \qquad or \qquad \sup_j \|L_{M_j}\|_{\mathcal{L}^n(M_j)} < +\infty.$$

Then there exists a subsequence $\{M_{j'}\}$ and $T = [\![G, \eta, \theta]\!] \in \mathrm{GUNB}_n(\mathbf{R}^N_x)$ such that

$$T_{M_{j'}} \rightharpoonup T.$$

The generalized unit normal bundle T is special, provided $p > 1$. If $\partial T_{M_{j'}} = 0$ and a $\mathcal{H}^n$-rectifiable set E is contained in $M_{j'}$, for all j', one has also $E \subset \pi G$.

Proof. ¿From Lemma 8.2 and Remark 7.3, recalling (7.1), we get

$$\sup_j \mathcal{F}_p(T_{M_j}) < +\infty.$$

Since the T_{M_j} are special generalized Gauss graphs, by Remark 7.1, we can invoke (7.3) and get the first assertion by the compactness theorem for integral currents ([16, Theorem 4.2.17], [21, Theorem 5.5], [23, Theorem 32.2]). Proposition 7.3, as well as Proposition 7.4 or Remark 7.4, shows that T is special whenever $p > 1$. The last statement is a trivial consequence of Proposition 6.3. □

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