Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations

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Abstract

We study singular perturbations of optimal stochastic control problems and differential games arising in the dimension reduction of system with multiple time scales. We analyze the uniform convergence of the value functions via the associated Hamilton-Jacobi-Bellman-Isaacs equations, in the framework of viscosity solutions. The crucial properties of ergodicity and stabilization to a constant that the Hamiltonian must possess are formulated as differential games with ergodic cost criteria. They are studied under various different assumptions and with PDE as well as control-theoretic methods. We construct also an explicit example where the convergence is not uniform. Finally we give some applications to the periodic homogenization of Hamilton-Jacobi equations with non-coercive Hamiltonian and of some degenerate parabolic PDEs.

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CHAPTER 1

Introduction and statement of the problem

1.1. Introduction

Consider the controlled system with a small parameter $\varepsilon > 0$

$$
\begin{aligned}
dx_s &= f(x_s, y_s, \alpha_s)\, ds + \sigma(x_s, y_s, \alpha_s)\, dW_s, \quad x_0 = x \in \mathbb{R}^n, \\
dy_s &= \tfrac{1}{\varepsilon} g(x_s, y_s, \alpha_s)\, ds + \tfrac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s, \alpha_s)\, dW_s, \quad y_0 = y \in \mathbb{R}^m
\end{aligned} \tag{1.1}
$$

where W_s is a r-dimensional Brownian motion, and the optimal control problem of minimizing the cost functional

$$
J(t, x, y, \alpha) := E_{(x,y)} \left[\int_0^t l(x_s, y_s, \alpha_s)\, ds + h(x_t, y_t) \right],
$$

as α varies in the set of admissible control functions $\mathcal{A}(t)$. It is a model of systems where some state variables, y_s here, evolve at a much faster time scale than the other variables, x_s. Passing to the limit as $\varepsilon \to 0+$ is a classical singular perturbation problem. Its solution leads to the elimination of the state variables y and the reduction of the dimension of the system from $n + m$ to n. Of course the limit control problem keeps some informations on the fast part of the system.

There is a large mathematical and engineering literature on singular perturbation problems in control, both in the deterministic ($\sigma \equiv 0,\ \tau \equiv 0$) and in the stochastic case. General references are the books [**O'M74, KKO86, Ben88, Kus90, YZ98, KP03**]. The survey paper [**Nai02**] lists more than 450 references, and we will not try to review all the different results and methods. We will mention the mathematical contributions most related to this paper.

We begin with the methods that aim at deriving directly an explicit description of the limit system. The first approach is the order reduction method originated in the work of Levinson and Tichonov on ODEs and extended to deterministic control systems by several authors, see Kokotović, Khalil & O'Reilly [**KKO86**], Bensoussan [**Ben88**], Dontchev & Zolezzi [**DZ93**], Veliov [**Vel97**], and the references therein. It works when the limit of the fast dynamics is the algebraic equation $g(x_s, y_s, \alpha_s) = 0$ and the stationary points of the fast dynamics are attractive. For deterministic systems with more general asymptotic behavior of the fast variables the classical averaging method for ODEs of Krylov and Bogolyubov was developed to the theory of limit occupational measures for control systems by Artstein, Gaitsgory, Leizarowitz, and others [**Gai92, AG97, Art99, GL99, Lei02, QW03, Gai04**], see also the references therein. Stochastic systems with uncontrolled fast dynamics ($g = g(x, y), \tau = \tau(x, y)$) were studied by Bensoussan [**Ben88**], Kushner [**Kus90**], Bielecki & Stettner [**BS89**], see also the references therein. The controlled case appears much more difficult and some results were obtained only in last ten years by Kabanov & Pergamenshchikov [**KP97, KP03**] and Borkar & Gaitsgory [**BG05**].

A different approach to the singular perturbation problem consists of studying the limit as $\varepsilon \to 0+$ of the value function

$$u^\varepsilon(t,x,y) := \inf_{\alpha \in \mathcal{A}(t)} J(t,x,y,\alpha)$$

and characterizing it as the unique solution of a limiting Hamilton-Jacobi-Bellman (briefly, HJB) equation. It evolved through the work of P.-L. Lions [**Lio82**], Jensen & Lions [**JL84**], Gaitsgory [**Gai96**], Bagagiolo and the second author [**BB98**], Artstein and Gaitsgory [**AG00**], and the authors of this paper [**AB01**]; the last reference also treated for the first time stochastic systems with controlled non-linear fast dynamics, for non-degenerate diffusions and in some degenerate cases. This approach starts from the HJB equation in $\mathbb{R}^{n+m}$ satisfied by u^ε, that in the deterministic case is of first order

$$u_t^\varepsilon + \max_{\alpha \in A} \left\{ -f(x,y,\alpha) \cdot D_x u^\varepsilon - g(x,y,\alpha) \cdot \frac{D_y u^\varepsilon}{\varepsilon} - l(x,y,\alpha) \right\} = 0,$$

and in the stochastic case is of second order

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0, \tag{1.2}$$

where $H = \max_{\alpha \in A} L^\alpha$ and L^α is the generator of the process in (1.1) with the constant control α and $\varepsilon = 1$. One expects that the limit $u(t,x)$ does not depend on y and solves a PDE in $\mathbb{R}^n$ governed by an *effective Hamiltonian* $\overline{H}$. It turns out that $\overline{H}$ is the value of an ergodic control problem in $\mathbb{R}^m$ for the fast subsystem with frozen slow variable x and $\varepsilon = 1$. Once this is found, one tries to prove that the limit of u^ε solves the effective PDE

$$u_t + \overline{H}\,(x, D_x u, D_{xx} u) = 0. \tag{1.3}$$

If this PDE, with suitable initial conditions, has at most one solution, then we have a characterization of the limit $u(t,x)$ and a way to compute it, at least in principle, by solving a lower dimensional PDE. The theory of viscosity solutions for first order and for second order, degenerate parabolic, fully nonlinear equations is the natural framework for this approach. The ideas and methods for homogenization problems initiated by Lions, Papanicolaou & Varadhan [**LPV86**] and Evans [**Eva89, Eva92**] turn out to be particularly useful.

The PDE approach to singular perturbations just described was put in abstract form in our paper [**AB03**] for general degenerate parabolic equations (1.2) with H satisfying some natural structural conditions, but not necessarily of the Bellman form $\max_\alpha L^\alpha$, and with initial conditions

$$u^\varepsilon(0,x,y) = h(x,y). \tag{1.4}$$

All the data were assumed $\mathbb{Z}^m$-periodic in y. We singled out two crucial properties for the convergence of u^ε. The first was named *ergodicity* of H, and states that the solution of the degenerate parabolic PDE in $\mathbb{R}^m$

$$w_t + H(x,y,p,D_y w, X, D^2_{yy} w, 0) = 0 \quad \text{in } (0,+\infty) \times \mathbb{R}^m, \qquad w(0,y) = 0,$$

with frozen $x, p \in \mathbb{R}^n$ and X $n \times n$ symmetric matrix, has a limit as $t \to \infty$ independent of y. Then this limit is the candidate effective Hamiltonian $\overline{H}(x,p,X)$. The second property concerns the pair (H,h) and is called *stabilization to a constant.*

If $H'(x,p,q,Y)$ denotes the homogeneous part of H with respect to the entries $q = D_y u$ and $Y = D^2_{yy}u$, the stabilization property states that the solution to

$$w_t + H'(x,y,D_y w, D^2_{yy}w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad w(0,y) = h(x,y),$$

with $x \in \mathbb{R}^n$ frozen, has a limit as $t \to \infty$ independent of y. Then this limit is the candidate *effective initial data* $\overline{h}(x)$. Note that the last PDE is again degenerate parabolic and m-dimensional, and it is also homogeneous. The main theorem of [**AB03**] stated that, if these two properties hold, then the weak viscosity semilimits of u^ε satisfy (1.3) and

$$u(0,x) = \overline{h}(x). \tag{1.5}$$

That paper gave also some examples where the effective Cauchy problem (1.3) (1.5) satisfies the comparison principle among viscosity sub- and supersolutions, and therefore u^ε converge locally uniformly to its unique solution. Note that this theory is designed to apply to the Isaacs Hamiltonians

$$H = \min_{\beta\in B}\max_{\alpha\in A} L^{\alpha,\beta} \quad \text{or} \quad H = \max_{\alpha\in A}\min_{\beta\in B} L^{\alpha,\beta},$$

where each $L^{\alpha,\beta}$ is the generator of a diffusion process. Therefore it allows to treat singular perturbation problems for zero-sum two-person *differential games*, deterministic and stochastic. For these problems the system is controlled by two players

$$\begin{aligned} dx_s &= f(x_s,y_s,\alpha_s,\beta_s)\,ds + \sigma(x_s,y_s,\alpha_s,\beta_s)\,dW_s, \quad x_0 = x \in \mathbb{R}^n,\\ dy_s &= \tfrac{1}{\varepsilon}g(x_s,y_s,\alpha_s,\beta_s)\,ds + \tfrac{1}{\sqrt{\varepsilon}}\tau(x_s,y_s,\alpha_s,\beta_s)\,dW_s, \quad y_0 = y \in \mathbb{R}^m, \end{aligned} \tag{1.6}$$

and the cost functional

$$J(t,x,y,\alpha,\beta) := E_{(x,y)}\left[\int_0^t l(x_s,y_s,\alpha_s,\beta_s)\,ds + h(x_t,y_t)\right],$$

is minimized over α_s by the first player and maximized over α_s by the second. The convergence result of [**AB03**] is precisely stated and slightly extended in Section 2.3.

The purpose of the present paper is to provide a reference framework for the study of singular perturbations with PDE methods in the generality of stochastic differential games, by complementing the abstract theory of [**AB03**] with several sets of conditions that make it work successfully. The main part concerns the properties of ergodicity and stabilization. First of all, in Sections 2.1 and 2.2, we reformulate both properties and the definitions of $\overline{H}$ and $\overline{h}$ in terms of differential games with ergodic-type cost criteria for the *fast subsystem*

$$dy_s = g(x,y_s,\alpha_s,\beta_s)ds + \tau(x,y_s,\alpha_s,\beta_s)dW_s, \quad y_0 = y,$$

with frozen x (see also [**AB07**] for more on this issue). We also give another PDE characterization of ergodicity and stabilization in terms of the validity of a strong maximum principle and of the equicontinuity of some value functions. In Chapter 3 we analyze the uncontrolled case (g,τ independent of α,β) and show the connections with the classical ergodic theory. The special case of a hypoelliptic diffusion is studied in Chapter 5 by purely analytic methods; similar results were obtained by Ichihara & Kunita [**IK74**] using probabilistic methods. In Chapter 4 we prove that for uniformly non-degenerate matrices τ the Hamiltonian is ergodic and stabilizing, by PDE methods following Evans [**Eva92**] and Arisawa & Lions

[**AL98**]. Chapter 6 is devoted to controllability conditions on the fast subsystem that ensure ergodicity, extending earlier work in the deterministic single-player case by Arisawa [**Ari97, Ari98**], Grüne [**Gru98**], and Artstein & Gaitsgory [**AG00**], and to their variants that give also the stabilization property; the game theoretic formulation is crucial here. In Chapter 7 we exploit the periodicity of the data and use non-resonance conditions; here we must assume that only one player is active, so the ergodicity is known from Arisawa & Lions [**AL98**], whereas the stabilization requires a strengthened form of non-resonance.

The next issue is proving results of uniform convergence for the value functions u^ε under explicit conditions on the data. This is done by analyzing the regularity of the effective Hamiltonian and checking whether it implies comparison and uniqueness for the Cauchy problem (1.3) (1.5). Although $\overline{H}$ is automatically continuous and degenerate elliptic, and $\overline{h}$ is continuous, to get the regularity in x needed for the comparison principle one must impose further conditions. We show in Chapter 8 that this is not a merely technical issue. In fact, we construct an explicit example with H ergodic and (H, h) stabilizing, but with $\overline{H}$ not Lipschitz continuous at one point. We analyze the pointwise limit of u^ε and show that it is discontinuous at the same point. This is another main result of the paper, because it was unexpected (see Artstein [**Art04**] for a related discussion). As for the sufficient conditions for uniform convergence, in addition to ergodicity and stabilization, an easy one is the independence of the fast dynamics g, τ from the slow variable x. In some special cases we prove an explicit formula for $\overline{H}$ that can be used to check directly the desired regularity, see Chapters 3, 5, and 6. In general, we look at the *true cell problem*

$$H(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2\chi, 0) = \overline{H}(\overline{x}, y, \overline{p}, \overline{X}) \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic},$$

and study the regularity of the *corrector* χ, which can be exploited to get further properties of $\overline{H}$. This is done in Chapters 4 and 6.

Another question that we address is the nature of the *effective control problem* or *effective differential game*, that is, a system in $\mathbb{R}^n$ and a payoff whose value function is the solution of the effective Cauchy problem (1.3) (1.5). Of course $\overline{h}$ is the effective terminal cost, and one can always construct from $\overline{H}$ a system and a running cost that do the job. However, we are interested in a formula representing the effective system and cost in terms of the data as directly as possible. In some special cases we can indeed give such explicit formulas: see Section 3.4 for the case of uncontrolled fast variables, Section 6.3 for a fast subsystem with suitable controllability properties, and Section 6.5 for a formula derived from the reduction order method recalled above. To go further one should extend the theory of limit occupational measures to differential games, but this is a completely open problem at the moment.

The last issue we consider, in Chapter 9, is the application of our results to the periodic homogenization of Bellman-Isaacs PDEs, that is, passing to the limit in the Cauchy problems

$$v_t^\varepsilon + G\left(x, \frac{x}{\varepsilon}, D_x v^\varepsilon\right) = 0, \quad v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right)$$

and

$$v_t^\varepsilon + F\left(x, \frac{x}{\varepsilon}, D_{xx}^2 v^\varepsilon\right) = 0, \quad v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right),$$

with F (degenerate) elliptic. This gives informations on the corresponding control problems and differential games for systems in highly oscillating media, i.e.,

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right), \qquad x_0 = x$$

and, respectively,

$$dx_s = \sigma\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right) dW_s, \qquad x_0 = x,$$

and with oscillating costs

$$E_x\left[\int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)\right].$$

If we set $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$, we find for $u^\varepsilon(t,x,y)$ a singularly perturbed PDE in $\mathbb{R}^{2n}$ of the form (1.2) with initial data (1.4), to which we can apply the previous theory. This leads to several new homogenization results. In the first order case the classical assumption is the coercivity of G in the $p = D_x v$ variables [**LPV86, Eva92, AB01**], and we deal with two sets of much weaker conditions. In the second order case the main novelties are the oscillating initial data and an example involving a hypoelliptic operator. In the companion paper with Marchi [**ABM07**] we combine this method with a regular perturbation argument to cover the homogenization of general parabolic equations depending on first and second derivatives (see also [**AB01**] for some special case).

Throughout the paper we assume, as in [**AB03**], that all data are $\mathbb{Z}^m$-periodic in y, that is, the fast variables live on an m-dimensional torus. This assumption is convenient to avoid boundary conditions on the fast variables and therefore reduce the technicalities in the assumptions and in the proofs. Most of our results can be extended to compact manifolds without boundary, and also to the case with boundary by imposing and treating boundary conditions such as Neumann or state constraints. For instance, the case of deterministic control with the fast variables constrained in the closure of an open bounded set with Lipschitz boundary was studied in [**AB01**].

Let us point out the main additions that this paper makes to the existing literature. First of all it gives a general unified method for studying singular perturbations for deterministic and stochastic systems, and for one as well as two competing controllers. Usually the assumptions and the methods are quite different in the deterministic and in the stochastic setting. And not much is known on singularly perturbed differential games: for deterministic systems there are results by Gaitsgory [**Gai96**] and by Subbotina [**Sub96, Sub99, Sub00, Sub01**], with little overlapping with ours, and there is some literature on discrete-time Markov games, but we are not aware of any paper dealing with singular perturbations for games whose dynamics are described by diffusion processes.

Another improvement is the generality of the terminal cost $h(x,y)$ depending also on the fast variables y. This produces two mathematical difficulties: finding the effective terminal cost, and dealing with a boundary layer at time $t = 0$. For this reason most of our results are new even in the case of a single player, cfr. [**AB01**]. For homogenization problems the oscillating initial data $h\left(x, \frac{x}{\varepsilon}\right)$ is also largely new, cfr. [**BOFM92, JKO94**] for earlier results. In the first order case some different non-coercive Hamiltonians were considered recently by Birindelli &

Wigniolle [**BW03**] and Barles [**Bar07**], see also Gomes [**Gom07**]. The homogenization theory for equations involving hypoelliptic operators treated up to now stationary variational equations on the Heisenberg group, see Biroli, Mosco & Tchou [**BMT96**] and Franchi & Tesi [**FT02**]. We refer to Lions & Souganidis [**LS05**] and its bibliography for other recent advances in the homogenization of fully nonlinear PDEs, and to the survey of Evans [**Eva04**] for the connections with the KAM theory of Hamiltonian systems.

Ergodic control has independent interest and a large literature. Our contribution is essentially the extension from the case of a single player to games, with a PDE approach owing to Arisawa and Lions [**AL98**]. For diffusion processes we refer also to the books [**Has80, Ben88, Kus90**], to Bielecki & Stettner [**BS89**], Basak, Borkar & Ghosh [**BBG97**], Kurtz & Stockbridge [**KS98**], and the references therein. For deterministic problems the use of viscosity solutions begins with P.-L. Lions [**Lio85**] and Capuzzo-Dolcetta & Menaldi [**CDM88**], see the presentation and the references in [**BCD97**]. Some deterministic differential games with ergodic cost criterion were studied by Fleming & McEneaney [**FM95**] and more recently by Ghosh & Rao [**GR05**], Bettiol [**Bet05**], and the authors [**AB07, Bar**]. The results about stabilization are entirely new, although the methods are inspired by those employed for ergodicity.

Finally, let us mention that the authors and Marchi extended some results of this paper to problems with an arbitrary number of scales [**ABM07, ABM08**]. Several other developments on singular perturbations of control systems and games and on the homogenization of non-coercive HJ equations are in the recent thesis of Terrone [**Ter08**].

1.2. Stochastic differential games and the singular perturbation problem

We are interested in the stochastic differential equation controlled by two players

$$
\begin{aligned}
&dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s,\\
&dy_s = \tfrac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\, ds + \tfrac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \qquad (1.7)\\
&x_0 = x, \quad y_0 = y.
\end{aligned}
$$

for $s > 0$. Here W_s is a r-dimensional Brownian motion, $\alpha_., \beta_.$ are processes taking values, respectively, in the compact sets A and B, and we will restrict them to the admissible controls that we are going to define next.

Let $\Omega_t := \{\omega \in C([0,t]; \mathbb{R}^r) \,:\, w_0 = 0\}$, $\mathcal{F}_s$ be the σ-algebra generated by the brownian paths up to time s in Ω_t, and P_t be the Wiener measure. This is the canonical sample space of (1.7). An *admissible control* $\alpha_.$ for the first player (resp., $\beta_.$ for the second player) on $[0,t]$ is an $\mathcal{F}_s$-progressively measurable process taking values in A (resp., in B). We will write $\alpha \in \mathcal{A}(t)$ (resp., $\beta \in \mathcal{B}(t)$).

We are also given a cost functional on each time interval $[0,t]$ of the form

$$J(t, x, y, \alpha, \beta) := E_{(x,y)} \left[\int_0^t l(x_s, y_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t) \right],$$

where $E_{(x,y)}$ denotes the expectation, $\alpha = \alpha_. \in \mathcal{A}(t)$, $\beta = \beta_. \in \mathcal{B}(t)$, and $y_.$ is the corresponding solution of (1.7). Here l represents a *running cost* for the first player (gain for the second player) and h is the *terminal cost* depending of the position of the system at the final time t. We assume that the first player wants to minimize the cost and the second player seeks to maximize it. Therefore we have a *two-person zero-sum stochastic differential game*.

Notations: we will denote with $\mathbb{M}^{N,K}$ the set of $N \times K$ matrices, and with $BUC(\mathbb{R}^N)$ the set of bounded and uniformly continuous functions $\mathbb{R}^N \to \mathbb{R}$.

Throughout the paper we will suppose the following assumptions on the data that we will refer to as the *standing assumptions*.

$$
\text{(A)} \quad \begin{cases} f, g, \sigma, \tau, l \text{ are bounded uniformly continuous functions in} \\ \mathbb{R}^n \times \mathbb{R}^m \times A \times B \text{ with values, respectively, in } \mathbb{R}^n, \mathbb{R}^m, \mathbb{M}^{n,r}, \\ \mathbb{M}^{m,r}, \text{ and } \mathbb{R}; \\ \\ f(\cdot,\cdot,\alpha,\beta),\ g(\cdot,\cdot,\alpha,\beta),\ \sigma(\cdot,\cdot,\alpha,\beta),\ \tau(\cdot,\cdot,\alpha,\beta) \text{ are Lipschitz continuous;} \\ \\ \text{all the moduli of continuity of } f(\cdot,\cdot,\alpha,\beta),\ g(\cdot,\cdot,\alpha,\beta),\ \sigma(\cdot,\cdot,\alpha,\beta), \\ \tau(\cdot,\cdot,\alpha,\beta), l(\cdot,\cdot,\alpha,\beta) \text{ are uniform with respect to } \alpha \text{ and } \beta; \\ \\ h \in BUC(\mathbb{R}^{n+m}); \\ \\ \text{all the data are periodic in } y, \text{ i.e.,} \\ \varphi(x,y,\alpha,\beta) = \varphi(x, y+k, \alpha, \beta) \text{ for all } k \in \mathbb{Z}^m \text{ and } \varphi = f, g, \sigma, \tau, l, h. \end{cases}
$$

Notations: we will write $\varphi(x,\cdot,\alpha,\beta) \in C_{\text{per}}(\mathbb{R}^m)$ if φ is continuous and $\mathbb{Z}^m$-periodic with respect to y.

Next we define the admissible strategies and the values of the game following Fleming and Souganidis [**FS89**]. In the following we identify two admissible controls $\alpha_\cdot, \widetilde{\alpha}_\cdot \in \mathcal{A}(t)$ on $[0,s]$ if $P_t\,(\alpha_\cdot = \widetilde{\alpha}_\cdot$ a.e. in $[0,s]) = 1$, and the analogous identification holds in $\mathcal{B}(t)$. An *admissible strategy* α for the first player is a map $\alpha : \mathcal{B}(t) \to \mathcal{A}(t)$ such that for all admissible controls $b_\cdot, \widetilde{b}_\cdot \in \mathcal{B}(t)$ identical on $[0,s]$ the responses $\alpha[b_\cdot]_\cdot$ and $\alpha[\widetilde{b}_\cdot]_\cdot$ are identical on $[0,s]$. The admissible strategies β for the second player are defined in the obvious symmetric way and we denote with $\Gamma(t)$ and $\Delta(t)$, respectively, the sets of admissible strategies of the first and the second player.

The *lower value function* u^ε of the stochastic differential game with finite horizon $t > 0$ is defined as

$$
u^\varepsilon(t,x,y) := \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} J(t,x,y,\alpha[\beta],\beta), \tag{1.8}
$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and the *upper value function* $\widetilde{u}^\varepsilon$ is defined as

$$
\widetilde{u}^\varepsilon(t,x,y) := \sup_{\beta\in\Delta(t)} \inf_{\alpha\in\mathcal{A}(t)} J(t,x,y,\alpha,\beta[\alpha]).
$$

If the upper and the lower value coincide we say that the game has a value.

When $\sigma \equiv 0$ and $\tau \equiv 0$ the system is deterministic and the admissible controls in $\mathcal{A}(t)$ (resp., $\mathcal{B}(t)$) are simply all measurable functions $[0,t] \to A$ (resp., $[0,t] \to B$). Then we have a *deterministic (two-person zero-sum) differential game*. The admissible strategies in this case are usually called nonanticipating, or causal, or progressive, or Varaiya-Roxin-Elliott-Kalton strategies, see [**ES84, BCD97, FS06**] and the references therein.

Another important special case of the stochastic differential game described above is the *stochastic optimal control* over a finite horizon. This occurs if one of the two players is missing (i.e., its controls take values in a singleton). If we keep, for instance, only the first player, the system becomes

$$\begin{aligned} dx_s &= f(x_s, y_s, \alpha_s)\, ds + \sigma(x_s, y_s, \alpha_s)\, dW_s, \\ dy_s &= \tfrac{1}{\varepsilon} g(x_s, y_s, \alpha_s)\, ds + \tfrac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s, \alpha_s)\, dW_s, \\ x_0 &= x, \quad y_0 = y, \end{aligned} \tag{1.9}$$

the value functions coincide and are given by

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \mathcal{A}(t)} E_{(x,y)} \left[\int_0^t l(x_s, y_s, \alpha_s)\, ds + h(x_t, y_t) \right]. \tag{1.10}$$

If, instead, we keep only the second player, we end up with a maximization problem. Finally, if σ and τ are null and there is only one player we have a *deterministic optimal control problem.*

The goal of the singular perturbation problem is studying the limit as $\varepsilon \to 0+$ of u^ε and $\widetilde{u}^\varepsilon$. We expect that in this procedure the state variables y evolving on a faster time-scale are eliminated, and therefore the dimension of the problem is reduced from $n + m$ to n.

1.3. The Bellman-Isaacs equations

In this section we associate to the lower and upper value functions of the game a Cauchy problem for a fully nonlinear 2nd order partial differential equation.

We begin with some notations. We use the dot "·" to indicate the scalar product of vectors as well as the scalar product of matrices $P \in \mathbb{M}^{N,K}$ and $Q \in \mathbb{M}^{N,K}$, that is,

$$P \cdot Q := \operatorname{trace}(PQ^T) = \operatorname{trace}(Q^T P) = \sum_{i=1}^{N} \sum_{j=1}^{K} P_{ij} Q_{ij} = P_{ij} Q_{ij}.$$

We associate to the dispersion matrices σ and τ of the controlled system (1.7) the diffusion matrices

$$a := \sigma\sigma^T/2, \quad b := \tau\tau^T/2, \quad c := \tau\sigma^T/2,$$

where ^{T} denotes the transpose. For $\alpha \in A$, $b \in B$, $x, p \in \mathbb{R}^n$, $y, q \in \mathbb{R}^m$, $X \in \mathbb{S}^n$, $Y \in \mathbb{S}^m$, $Z \in \mathbb{M}^{n,m}$, where $\mathbb{S}^k$ denotes the space of $k \times k$ symmetric matrices, we define

$$\begin{aligned} L^{\alpha,\beta}(x, y, p, q, X, Y, Z) &:= -X \cdot a(x, y, \alpha, \beta) - Y \cdot b(x, y, \alpha, \beta) \\ &\quad - 2Z \cdot c(x, y, \alpha, \beta) - p \cdot f(x, y, \alpha, \beta) - q \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta). \end{aligned} \tag{1.11}$$

Note that if we formally replace p and q with D_x and D_y (i.e., respectively, the gradient with respect to the x and the y variables), X and Y with D_{xx} and D_{yy} (i.e., respectively, the Hessian of pure second derivatives with respect to x and y), and Z with D_{xy} (i.e., the $n \times m$ matrix of mixed second derivatives), then $L^{\alpha,\beta}$ becomes the infinitesimal generator of the diffusion process (1.7) with constant control functions $\alpha_s = \alpha$, $\beta_s = \beta$ for all s. Now we can define the 2nd order Bellman-Isaacs Hamiltonians

$$H(x, y, p, q, X, Y, Z) := \min_{\beta \in B} \max_{\alpha \in A} L^{\alpha,\beta}(x, y, p, q, X, Y, Z), \tag{1.12}$$

$$\widetilde{H}(x,y,p,q,X,Y,Z) := \max_{\alpha\in A}\min_{\beta\in B} L^{\alpha,\beta}(x,y,p,q,X,Y,Z).$$

The Cauchy problem associated to the lower value function $u^\varepsilon(t,x,y)$ is

$$(\mathrm{HJ}_\varepsilon)\qquad \begin{cases} u_t^\varepsilon + H\left(x,y,D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx}u^\varepsilon, \frac{D_{yy}u^\varepsilon}{\varepsilon}, \frac{D_{xy}u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad (0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^m, \\ u^\varepsilon(0,x,y) = h(x,y) \qquad \text{for all } (x,y)\in\mathbb{R}^n\times\mathbb{R}^m, \end{cases}$$

and the one for the upper value $\widetilde{u}^\varepsilon$ is the same with H replaced $\widetilde{H}$. Under the current assumptions H and $\widetilde{H}$ are continuous functions $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{S}^n\times\mathbb{S}^m\times\mathbb{M}^{n,m}\to\mathbb{R}$, and they are degenerate elliptic in the sense that

$$H(x,y,p,q,X,Y,Z) \le H(x,y,p,q,X',Y',Z'), \text{ if } \begin{pmatrix} X & Z \\ Z^T & Y \end{pmatrix} \ge \begin{pmatrix} X' & Z' \\ Z'^T & Y' \end{pmatrix}.$$

Therefore the Hamilton-Jacobi-Bellman-Isaacs (briefly, HJBI) equation in $(\mathrm{HJ}_\varepsilon)$ is degenerate parabolic. Moreover, the Hamiltonians satisfy a regularity property with respect to x, y that is usually called *structure condition* and implies the Comparison Principle between bounded viscosity sub- and supersolutions of $(\mathrm{HJ}_\varepsilon)$, see, e.g., [**CIL92**]. Therefore $(\mathrm{HJ}_\varepsilon)$ has at most one bounded and uniformly continuous viscosity solution. The next result, due to Fleming and Souganidis [**FS89**], says the lower value u^ε is indeed such solution. In the deterministic case it goes back to Evans and Souganidis [**ES84**], for a general presentation see the books [**BCD97, FS06**]. In the case of a single player the result is due to P.-L. Lions [**Lio83**], see also the monograph by Fleming & Soner [**FS06**].

THEOREM 1.1. *Under the standing assumptions the lower value function u^ε is the unique bounded and uniformly continuous viscosity solution of* $(\mathrm{HJ}_\varepsilon)$ *and the upper value function $\widetilde{u}^\varepsilon$ is the unique such solution of* $(\mathrm{HJ}_\varepsilon)$ *with H replaced by $\widetilde{H}$.*

PROOF We only show how the result can be deduced from Theorem 2.6 of [**FS89**]. Consider the stochastic system (1.7) for negative times $s\in[r,0]$, with initial condition $x_r = x$, $y_r = y$, and the functional

$$\widehat{J}(r,x,y,\alpha,\beta) := E_{(x,y)}\left[\int_r^0 l(x_s,y_s,\alpha_s,\beta_s)\,ds + h(x_t,y_t)\right],$$

for admissible controls on $[r,0]$ defined in the obvious way. Let $\widehat{\mathcal{A}}(r)$, $\widehat{\mathcal{B}}(r)$ denote the sets of admissible controls and $\widehat{\Gamma}(r)$, $\widehat{\Delta}(r)$ the sets of admissible strategies on $[r,0]$ defined in the natural way. Set

$$v^\varepsilon(r,x,y) := \inf_{\alpha\in\widehat{\Gamma}(r)}\sup_{\beta\in\widehat{\mathcal{B}}(r)} \widehat{J}(r,x,y,\alpha[\beta],\beta).$$

Since all data are f,σ,g,τ,l,h are independent of time, this game and the original game in $[0,t]$ with $t=-r$ are equivalent and the value functions satisfy

$$v^\varepsilon(-t,x,y) = u^\varepsilon(t,x,y), \qquad \forall t>0,\ \forall x,y.$$

It is proved in [**FS89**] that $v^\varepsilon(r,x,y)$ is a viscosity solution of

$$-v_r^\varepsilon + H\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}, D_{xx}v^\varepsilon, \frac{D_{yy}v^\varepsilon}{\varepsilon}, \frac{D_{xy}v^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \quad \text{in } (-\infty,0)\times\mathbb{R}^n\times\mathbb{R}^m.$$

Therefore u^ε is a viscosity solution of the Bellman-Isaacs PDE in $(\mathrm{HJ}_\varepsilon)$. □

Note that if $H = \widetilde{H}$ the lower and upper value coincide, $u^\varepsilon = \widetilde{u}^\varepsilon$, so the game has a value. The equality of the two Hamiltonians is called Isaacs condition, or solvability of the small game.

In the case of a single player the PDE in (HJ$_\varepsilon$) is the Hamilton-Jacobi-Bellman (briefly, HJB) equation of stochastic control. If, for instance, there is only the minimizing player, the Hamiltonian becomes

$$H(x, y, p, q, X, Y, Z) := \max_{\alpha \in A} L^\alpha(x, y, p, q, X, Y, Z). \tag{1.13}$$

If the system is deterministic, namely, $\sigma \equiv 0, \tau \equiv 0$, then the Bellman-Isaacs PDE is of first order and takes the form

$$u_t^\varepsilon + \min_{\beta \in B} \max_{\alpha \in A} \left\{ -D_x u^\varepsilon \cdot f(x, y, \alpha, \beta) - \frac{D_y u^\varepsilon}{\varepsilon} \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta) \right\} = 0. \tag{1.14}$$

The singular perturbation problem stated in the previous section is now translated into the PDE problem of letting $\varepsilon \to 0+$ in (HJ$_\varepsilon$) and finding a limit Cauchy problem in the reduced space dimension n.

CHAPTER 2

Abstract ergodicity, stabilization, and convergence

2.1. Ergodicity and the effective Hamiltonian

In this section we recall the three equivalent definitions of ergodicity of the operator H from [**AB03**] and we explain their meaning in terms of differential game problems. These interpretations allow to check the ergodicity on various examples and give formulas for the effective Hamiltonian. The second definition also shows the connection with classical ergodic theory and motivates the name. We also provide, at the end of the section, a PDE characterization of ergodicity that will be used throughout.

Fix $(\overline{x}, \overline{p}, \overline{X})$. The first definition is based on the *cell δ-problem*, for $\delta > 0$,

$$(\mathrm{CP}_\delta) \qquad \delta w_\delta + H(\overline{x}, y, \overline{p}, Dw_\delta, \overline{X}, D^2 w_\delta, 0) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic.}$$

By standard viscosity theory, under the current assumptions, it has a unique periodic viscosity solution that we denote with $w_\delta(y; \overline{x}, \overline{p}, \overline{X})$ so as to display its dependence on the frozen slow variables. The PDE in (CP_δ) is the stationary Bellman-Isaacs equation

$$\delta w_\delta + \min_{\beta\in B}\max_{\alpha\in A}\left\{-D^2 w_\delta \cdot b(\overline{x}, y, \alpha, \beta) - Dw_\delta \cdot g(\overline{x}, y, \alpha, \beta) - L(y, \alpha, \beta)\right\} = 0 \tag{2.1}$$

where

$$L(y, \alpha, \beta) = L(y, \alpha, \beta; \overline{x}, \overline{p}, \overline{X}) := \overline{X} \cdot a(\overline{x}, y, \alpha, \beta) + \overline{p} \cdot f(\overline{x}, y, \alpha, \beta) + l(\overline{x}, y, \alpha, \beta). \tag{2.2}$$

Then, by the results of Fleming and Souganidis [**FS89**] and Swiech [**Swi96**], w_δ can be represented as the lower value function of the differential game with infinite horizon discounted cost functional

$$w_\delta(y; \overline{x}, \overline{p}, \overline{X}) = \inf_{\alpha\in\Gamma}\sup_{\beta\in\mathcal{B}} E_y \int_0^{+\infty} L(y_s, \alpha[\beta]_s, \beta_s; \overline{x}, \overline{p}, \overline{X}) e^{-\delta s}\, ds, \tag{2.3}$$

where

$$\Gamma := \Gamma(+\infty), \quad \mathcal{B} := \mathcal{B}(+\infty)$$

and y_s denotes the path of the stochastic differential equation

$$dy_s = g(\overline{x}, y_s, \alpha[\beta]_s, \beta_s) ds + \tau(\overline{x}, y_s, \alpha[\beta]_s, \beta_s) dW_s, \quad y_0 = y. \tag{2.4}$$

Note that this m-dimensional controlled system can be obtained from the subsystem of the fast components $y_.$ in the full two-scale system (1.7) by freezing the slow components $x_.$ to $\overline{x}$ and ε to 1. We will call it the *fast subsystem*.

Definition 1. We say that the Hamiltonian H (or the operator) is (uniquely or uniformly) *ergodic* in the fast variable at $(\overline{x}, \overline{p}, \overline{X})$ if

$$\delta w_\delta(y; \overline{x}, \overline{p}, \overline{X}) \to \text{const} \quad \text{as } \delta \to 0, \text{ uniformly in } y.$$

We say that it is ergodic at $\overline{x}$ if it is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ for all $(\overline{p}, \overline{X})$, and that it is ergodic if it is ergodic at every $\overline{x} \in \mathbb{R}^n$.

It turns out that, except for trivial cases, assumptions that ensure ergodicity of the Hamiltonian will be made only on the dynamics. We therefore say that *the fast subsystem* (2.4) *is uniquely ergodic* if, for every bounded functions $L : \mathbb{R}^m \times A \times B \to \mathbb{R}$ such that

$$L(\cdot, \alpha, \beta) \in C_{\text{per}}(\mathbb{R}^m), \qquad \text{uniformly in } (\alpha, \beta), \tag{2.5}$$

we have that δw_δ converges uniformly to a constant, where w_δ is the value function given by (2.3). Here, (2.5) means that L is $\mathbb{Z}^m$-periodic in y and there are a constant C and a modulus of continuity ω such that

$$|L(y, \alpha, \beta)| \le C, \qquad |L(y', \alpha, \beta) - L(y, \alpha, \beta)| \le \omega(|y' - y|)$$

for all α, β, y and y'. When we speak of the dynamical system, we shall call the property *unique* ergodicity to avoid any confusion with the other more classical notions of ergodicity. The name is motivated by the characterization of the uncontrolled systems with this property as those system that possess a unique invariant measure, see Chapter 3.

DEFINITION 2. When the operator is ergodic at $(\overline{x}, \overline{p}, \overline{X})$, we set

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) := -\lim_{\delta \to 0} \delta w_\delta(y; \overline{x}, \overline{p}, \overline{X}).$$

The function $\overline{H}$ is called the *effective operator*, or *effective Hamiltonian.*

By (2.4) $\overline{H}$ has the following representation formula.

PROPOSITION 2.1. *If H is ergodic, then, for all initial positions y of* (2.4),

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = -\lim_{\delta \to 0+} \inf_{\alpha \in \Gamma} \sup_{\beta \in \mathcal{B}} E_y\, \delta \int_0^{+\infty} L(y_s, \alpha[\beta]_s, \beta_s; \overline{x}, \overline{p}, \overline{X}) e^{-\delta s}\, ds. \tag{2.6}$$

The right hand side of this formula is the value function of an asymptotic problem for a differential game in the fast variables $y \in \mathbb{R}^m$, for frozen slow variables. More precisely, writing the above formula as

$$\begin{aligned} \overline{H}(\overline{x}, \overline{p}, \overline{X}) = \lim_{\delta \to 0+} \sup_{\alpha \in \Gamma} \inf_{\beta \in \mathcal{B}} E_y\, \delta \int_0^{+\infty} \big(&-\overline{X} \cdot a(\overline{x}, y_s, \alpha[\beta]_s, \beta_s) \\ &- \overline{p} \cdot f(\overline{x}, y_s, \alpha[\beta]_s, \beta_s) - l(\overline{x}, y_s, \alpha[\beta]_s, \beta_s)\big) e^{-\delta s}\, ds, \end{aligned}$$

we see that the integrand is essentially the infinitesimal generator for the slow variable $(\overline{x}, \overline{p}, \overline{X})$, but the inf and sup operation are performed with respect to all the controlled fast trajectories and not only with respect to the control sets A and B.

Proposition 3 in [**AB03**] states that $\overline{H}$ is automatically continuous in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n$ and degenerate elliptic, i.e.,

$$\overline{H}(x, p, X) \le \overline{H}(x, p, X') \quad \text{if } X \ge X'.$$

It has moreover linear growth with respect to (p, X)

$$|\overline{H}(x, p, X)| \le C\big(1 + |p| + |X|\big).$$

The second definition of ergodicity is based on the *cell t-problem*, that is,

$$\text{(CP)} \qquad \begin{cases} w_t + H(\overline{x}, y, \overline{p}, D_y w, \overline{X}, D^2_{yy} w, 0) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^m, \\ w(0,y) = 0, \quad w \text{ periodic in } y. \end{cases}$$

This Cauchy problem has a unique viscosity solution $w(t, y; \overline{x}, \overline{p}, \overline{X})$ that can be written as the lower value function of the differential game with finite horizon cost functional

$$w(t, y; \overline{x}, \overline{p}, \overline{X}) = \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E_y \int_0^t L(y_s, \alpha[\beta]_s, \beta_s; \overline{x}, \overline{p}, \overline{X})\, ds, \tag{2.7}$$

where y_s and L are given by (2.4) and (2.2). The next result follows immediately from the Abelian-Tauberian Theorem 4 of [**AB03**].

Proposition 2.2. *The Hamiltonian H is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ if and only if*

$$\frac{w(t, y; \overline{x}, \overline{p}, \overline{X})}{t} \to \mathit{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y,$$

and when this occurs the constant is $-\overline{H}(\overline{x}, \overline{p}, \overline{X})$. Therefore, for all $y \in \mathbb{R}^m$,

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = -\lim_{t\to+\infty} \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E_y \frac{1}{t}\int_0^t L(y_s, \alpha[\beta]_s, \beta_s; \overline{x}, \overline{p}, \overline{X})\, ds. \tag{2.8}$$

If there are no controls, the formula shows the connection with the classical ergodic theory for diffusion processes (see, e.g., [**Has80**]) and for deterministic dynamical systems [**AA67, CFS82**]. This will be explained in detail in Chapter 3. In the case of a single player the existence of this limit and its independence on the initial position of the system is often called an ergodic control problem [**Ben88, AL98**]. The general case of two players has not been studied so far, to our knowledge. We call it an *ergodic differential game*.

The third characterization of the ergodicity of H is given in terms of the *true cell problem*

$$\lambda + H(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2\chi, 0) = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic} \tag{2.9}$$

for some constant λ. It has been shown that there is at most one $\lambda \in \mathbb{R}$ such that (2.9) has a continuous solution χ, and, if it exists, then $\lambda = -\overline{H}(\overline{x}, \overline{p}, \overline{X})$. The function χ (which is nonunique) is called a (first) *corrector*. This is the definition of the effective Hamiltonian in many papers, e.g., [**LPV86, Eva89, Eva92**]. However, in the current generality there may be no pair (λ, χ) with continuous χ solving (2.9), see Proposition 7.2 or [**AL98**]. Theorem 4 in [**AB03**] asserts also that H is ergodic if and only if

$$\begin{aligned} \lambda_1 &:= \sup\{\lambda \mid \exists \text{ a u.s.c. subsolution of (2.9)}\} \\ &= \lambda_2 := \inf\{\lambda \mid \exists \text{ a l.s.c. supersolution of (2.9)}\}, \end{aligned} \tag{2.10}$$

and in this case

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = -\lambda_1 = -\lambda_2.$$

A major objective of this paper is to determine sufficient conditions on the dynamics for unique ergodicity. The main tool is the following result that characterizes it in terms of the equicontinuity of some value functions and of the strong maximum principle for the homogeneous equation

$$H'(\overline{x}, y, D_y v, D^2_{yy} v) = 0 \quad \text{in } \mathbb{R}^m, \qquad v \text{ periodic,} \tag{2.11}$$

where H' is the following Hamiltonian, positively 1-homogeneous in (q, Y),

$$H'(\overline{x}, y, q, Y) := \min_{\beta\in B}\max_{\alpha\in A}\{-Y \cdot b(\overline{x}, y, \alpha, \beta) - q \cdot g(\overline{x}, y, \alpha, \beta)\}.$$

For further reference, we call H' the *recession function* or *homogeneous part* in the fast derivatives (q, Y) of the Hamiltonian H.

PROPOSITION 2.3. *The dynamical system* (2.4) *is uniquely ergodic if and only if the following holds.*

- Equicontinuity of the discounted value functions. *For every L satisfying* (2.5)*, the family $\{\delta w_\delta \mid 0 < \delta \le 1\}$ is equicontinuous, where w_δ is the value function given by* (2.3).
- Strong maximum principle (or Liouville property) for the stationary homogeneous problem. *The constants are the only viscosity solution of the homogeneous equation* (2.11).

PROOF We first assume the unique ergodicity of the dynamical system. Fix L satisfying (2.5) and denote by w_δ the associated value function. The function $w_\delta(y)$ is continuous in $(\delta, y) \in \mathbb{R}^*_+ \times \mathbb{R}^m$ because of the classical continuity in y for fixed δ and because of the elementary estimate $\|w_\delta - w_{\delta'}\|_\infty \le C\left|\delta^{-1} - (\delta')^{-1}\right|$. As δw_δ converges uniformly as $\delta \to 0$, we conclude that the family $\{\delta w_\delta \mid 0 < \delta \le 1\}$ is equicontinuous. To show the strong maximum principle for the stationary equation, we pick a viscosity solution v of (2.11). The value function

$$w_\delta(y) = \inf_{\alpha\in\Gamma}\sup_{\beta\in\mathcal{B}} E_y \int_0^{+\infty} v(y_s)e^{-\delta s}\, ds$$

is readily seen to be the function $\delta^{-1}v$ because the two functions solve the HJBI equation

$$\delta w_\delta + H'(\overline{x}, y, D_y w_\delta, D^2_{yy} w_\delta) = v \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic.}$$

But, δw_δ converges to a constant as $\delta \to 0$ by the assumed unique ergodicity of the dynamical system. Thus, v must be constant.

Conversely, let L satisfy (2.5) and denote by w_δ the associated value function. As $\{\delta w_\delta\}$ is equicontinuous and equibounded (by $\|L\|_\infty$), we can extract a subsequence converging uniformly to a certain function v. Multiplying (2.1) by δ and sending it to 0, we deduce from the stability properties of viscosity solutions that v must solve (2.11). By the strong maximum principle, we obtain that v is actually a constant, say λ. The proof that there is at most one constant that can be the limit of a converging subsequence of $\{\delta w_\delta\}$ follows from formula (2.10). This implies that the whole family $\{\delta w_\delta\}$ converges uniformly to λ as $\delta \to 0$. Since L is arbitrary, the dynamical system is uniquely ergodic. □

Note that, if the slow subsystem is an uncontrolled deterministic system, the condition called strong maximum principle in the last proposition becomes the classical characterization of ergodicity of a dynamical system via the first integral equation.

2.2. Stabilization and the effective terminal cost

In this section we describe the property of stabilization to a constant introduced in [**AB03**] and its meaning for game problems. Then we define the effective terminal cost $\overline{h}$.

Fix $\overline{x}$. Since $h(\overline{x},\cdot) \in C_{\text{per}}(\mathbb{R}^m)$ the *cell Cauchy problem* for the homogeneous part H' of the Hamiltonian H

$$\text{(CP}'\text{)} \qquad \begin{cases} w_t + H'(\overline{x}, y, D_y w, D^2_{yy} w) = 0 & \text{in } (0,+\infty) \times \mathbb{R}^m, \\ w(0,y) = h(\overline{x}, y), \quad w \text{ periodic}, \end{cases}$$

has a unique bounded viscosity solution $w(t, y; \overline{x})$. It is the lower value of a finite horizon differential game [**FS89**]

$$w(t,y;\overline{x}) = \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E_y h(\overline{x}, y_t), \tag{2.12}$$

where y_s is the trajectory of the stochastic differential equation (2.4) for the fast variables.

Definition 3. We say that the pair (H, h) is *stabilizing* (to a constant) at $\overline{x}$ if

$$w(t,y;\overline{x}) \to \text{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y, \tag{2.13}$$

and in this case the *effective initial data*, or *effective terminal cost*, is

$$\overline{h}(\overline{x}) := \lim_{t\to+\infty} w(t,y;\overline{x}). \tag{2.14}$$

We call the pair stabilizing if it is stabilizing at every $\overline{x} \in \mathbb{R}^n$, and we say that the Hamiltonian is stabilizing if the pair (H, h) is stabilizing for every initial data $h \in BUC(\mathbb{R}^{n+m})$. We say equivalently that the fast subsystem (2.4) is stabilizing, because the recession Hamiltonian H' only depends on the dynamics.

From the representation formula (2.12), $h \in BUC(\mathbb{R}^{n+m})$ and the comparison principle we obtain the following.

Proposition 2.4. *If (H, h) is stabilizing, then*

$$\overline{h}(\overline{x}) = \lim_{t\to+\infty} \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E_y h(\overline{x}, y_t), \tag{2.15}$$

where y_s solves (2.4). *Moreover, $\overline{h} \in BUC(\mathbb{R}^n)$.*

Note that this is a representation formula for the effective terminal cost as the asymptotic value of a differential game in the fast variables $y \in \mathbb{R}^m$, for frozen slow variables.

Remark If $h(\overline{x}, y) = h(\overline{x})$ is a constant with respect to y, then for any Hamiltonian H the pair (H, h) is stabilizing at $\overline{x}$ and $\overline{h}(\overline{x}) = h(\overline{x})$. This follows immediately from the formula (2.15).

The following PDE characterization of stabilization will be of constant use later.

Proposition 2.5. *The dynamical system is stabilizing if and only if the following holds*

- Equicontinuity of the value function. *For every $h \in C_{per}(\mathbb{R}^m)$, $\{w(t,\cdot) \mid t \geq 0\}$ is equicontinuous, where w the value function given by* (2.12).

– Strong maximum principle for the evolutionary problem. *The only viscosity solutions in $BUC(\mathbb{R}\times\mathbb{R}^m)$ of the homogeneous parabolic equation*

$$v_t + H'(\overline{x}, y, D_y v, D^2_{yy} v) = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^m, \qquad v \text{ periodic in } y, \tag{2.16}$$

that achieve an interior maximum are constants.

PROOF We first assume that the diffusion is stabilizing. We write $w(t,y) = S_t h(y)$ the value function associated to h so as to display its dependency with the initial condition. For every $h \in C_{\mathrm{per}}(\mathbb{R}^m)$, the functions $S_t h(y)$ are known to be continuous in (t,y). Since $S_t h$ converges uniformly as $t \to +\infty$, the family $\{S_t h\}$ must be equicontinuous. Now, consider a viscosity solution $v \in BUC(\mathbb{R}\times\mathbb{R}^m)$ of the evolutionary equation (2.16). By Ascoli theorem, we can extract a subsequence $v(-s_k, \cdot)$ that converges uniformly to a function $\widetilde{v}$ as $s_k \to +\infty$. Since the dynamical system is stabilizing, $S_r\widetilde{v}$ converges uniformly to a constant C as $r \to +\infty$. For every $t \in \mathbb{R}$ fixed and for all k large enough so that $t \geq -s_k$, we have the identity $v(t,\cdot) = S_{t+s_k} v(-s_k, \cdot)$, because v solves the parabolic equation (2.16). Therefore,

$$\begin{aligned} |v(t,\cdot) - C| = |S_{t+s_k} v(-s_k,\cdot) - C| &\leq |S_{t+s_k} v(-s_k,\cdot) - S_{t+s_k}\widetilde{v}| + |S_{t+s_k}\widetilde{v} - C| \\ &\leq \|v(-s_k,\cdot) - \widetilde{v}\|_\infty + |S_{t+s_k}\widetilde{v} - C|. \end{aligned}$$

In the last inequality we have used the fact that S_t is non-expansive for the uniform norm. Sending $s_k \to +\infty$, we deduce that $v(t,\cdot) = C$. Since t is arbitrary, we conclude that v is constant.

It is of interest to note that we have shown a Liouville property, as we have proved that the only viscosity solutions of (2.16) are constants without assuming a priori that the function achieves an interior maximum.

We now prove the converse. We therefore assume that, for every $h \in C_{\mathrm{per}}(\mathbb{R}^m)$, the family $\{S_t h \mid t \geq 0\}$ is equi-continuous and that the strong maximum principle holds for (2.16). The first step is to show that $w(t,y) = S_t h(y)$ is in $BUC([0,+\infty)\times\mathbb{R}^m)$. Boundedness follows from the obvious estimate $\|w\|_\infty \leq \|h\|_\infty$. To prove the uniform continuity of w, we first assume that h is smooth. By the comparison principle, we have the estimate $|w(t,y) - h(y)| \leq Ct$ on $[0,+\infty)\times\mathbb{R}^m$, for the constant $C := \sup_y |H'(\overline{x}, y, D_y h, D^2_{yy} h)|$. By applying again the comparison principle, we obtain $|w(t+s,y) - w(t,y)| \leq \sup_{y\in\mathbb{R}^m} |w(s,y) - h(y)| \leq Cs$ on $[0,+\infty)\times\mathbb{R}^m$ for all $s > 0$. Therefore, the function w is Lipschitz continuous in t uniformly in y. Since $\{w(t,\cdot) \mid t \geq 0\}$ is equicontinuous by assumption, we conclude that w is uniformly continuous. When h is merely continuous, we take a sequence of smooth functions (h_k) converging uniformly to h. The comparison principle implies that the associated sequence of solutions (w_k) converges uniformly to w on $[0,\infty)\times\mathbb{R}^m$. Therefore, w must be uniformly continuous.

The second step consists in showing that $w(t,\cdot)$ converges uniformly as $t \to +\infty$ to

$$M = \limsup_{t\to+\infty} \sup_{\mathbb{R}^m} w(t,\cdot).$$

Let (t_k, y_k) be a sequence so that $w(t_k, y_k) \to M$ with $t_k \to +\infty$. By extracting a subsequence, we can assume that y_k converge to some $\widetilde{y}$. The family $\{w(t_k + \cdot, \cdot)\}$ is equicontinuous and equibounded because $w \in BUC$. Along a subsequence, it therefore converges uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}^m$ to a function $\widetilde{w}$. By the stability results of viscosity solutions, it is a solution of

$$\partial_t\widetilde{w} + H'(\overline{x}, y, D_y\widetilde{w}, D^2_{yy}\widetilde{w}) = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^m, \qquad \widetilde{w} \text{ periodic.}$$

On the other hand, by the definition of M, one has $\widetilde{w} \leq M$ and $\widetilde{w}(0, \widetilde{y}) = M$. Thus $\widetilde{w}$ achieves an interior maximum. By the assumed strong maximum principle for (2.16), we deduce that $\widetilde{w} \equiv M$. The identity $\widetilde{w}(0, \cdot) \equiv M$ means that $\{w(t_k, \cdot)\}$ converges uniformly to M. By the comparison principle, we obtain

$$\sup_{[t_k,+\infty[\times\mathbb{R}^m} |w(t, y) - M| = \sup_{\mathbb{R}^m} |w(t_k, y) - M|.$$

As the right hand term converges to 0, we conclude that $\{w(t, \cdot)\}$ converges uniformly to M as $t \to +\infty$. □

2.3. The general convergence result

Most of this section is devoted to recalling the general convergence result of [**AB03**] for the family $\{u^\varepsilon\}$ of solutions of (HJ_ε). It roughly says that, whenever the Hamiltonian H is ergodic and stabilizing in the fast variable, u^ε will converge to the solution of the effective Hamilton-Jacobi equation ($\overline{\mathrm{HJ}}$). A subtle issue arises here. In most cases, one can show that the limit equation has a unique continuous solution and satisfies the comparison principle. This is roughly guaranteed if we can prove that the effective Hamiltonian is locally Lipschitz continuous with respect to the slow variable. Provided comparison holds, we shall show that u^ε converges uniformly on the compact sets of $(0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique solution of ($\overline{\mathrm{HJ}}$). In the general case, the effective Hamiltonian will be only continuous and the comparison principle will not hold. We shall construct an explicit elementary example in Chapter 8 for which the family u^ε does not converge uniformly (the effective Hamiltonian turns out to be merely Hölder continuous with respect to x). In full generality, uniform convergence has to be relaxed to bounds on the lower and upper semilimits of u^ε.

Theorem 2.9 and Theorem 2.10 give simple sufficient conditions for uniform convergence. Proposition 2.6 and Theorem 2.7 provide information on the semilimits in the general case. Theorem 2.7 is new, whereas the other results of this section are adapted from [**AB03**] for later use.

The family $\{u^\varepsilon\}$ is equibounded under the current assumptions. We can therefore define the upper semilimit $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ as follows

$$\overline{u}(t, x) := \limsup_{\varepsilon\to 0,\ (t',x')\to(t,x)} \sup_y u^\varepsilon(t', x', y) \quad \text{if } t > 0,$$

$$\overline{u}(0, x) := \limsup_{(t',x')\to(0,x),\ t'>0} \overline{u}(t', x') \quad \text{if } t = 0.$$

It is a bounded u.s.c. function. We define analogously the lower semilimit $\underline{u}$ by replacing limsup with liminf and sup with inf. The two-steps definition of the semilimit for $t = 0$ permits to sweep away an expected initial layer.

Our starting point is the main convergence result of [**AB03**] (Theorem 1).

Proposition 2.6. *Assume that the Hamiltonian defined by* (1.12) *is ergodic and the pair* (H, h) *is stabilizing. Then the semilimits* $\overline{u}$ *and* $\underline{u}$ *are, respectively, a subsolution and a supersolution of the effective Cauchy problem*

($\overline{\mathrm{HJ}}$)

$$u_t + \overline{H}(x, Du, D^2u) = 0 \quad \textit{in } (0, +\infty) \times \mathbb{R}^n, \qquad u(0, x) = \overline{h}(x) \quad \textit{for all } x \in \mathbb{R}^n.$$

We deduce from the Proposition some bounds on the semilimits which entail the pointwise convergence on a certain set.

THEOREM 2.7. *Assume H is ergodic and the pair (H,h) is stabilizing. Then for any $T>0$ there exist a maximal subsolution $\widetilde{u}\in BUSC([0,T)\times\mathbb{R}^n)$ and a minimal supersolution $\underset{\sim}{u}\in BLSC([0,T)\times\mathbb{R}^n)$ of the limit equation ($\overline{\mathrm{HJ}}$). Therefore, we have*

$$\underset{\sim}{u}\le\underline{u}\le\overline{u}\le\widetilde{u}$$

In particular,

$$\lim_{\varepsilon\to0}u^\varepsilon(x)=\widetilde{u}(x)\quad\text{for all } x \text{ such that } \widetilde{u}(x)=\underset{\sim}{u}(x).$$

The proof is an immediate consequence of the following auxiliary result and of the properties of the effective Hamiltonian recalled in Section 2.1.

LEMMA 2.8. *Assume that $\overline{H}\in C(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n)$ is a degenerate elliptic Hamiltonian such that*

$$|\overline{H}(x,p,X)|\le C\big(1+|p|+|X|\big)$$

and let $\overline{h}\in BUC(\mathbb{R}^n)$. Then, there exist a maximal subsolution $\widetilde{u}\in BUSC([0,T)\times\mathbb{R}^n)$ and a minimal supersolution $\underset{\sim}{u}\in BLSC([0,T)\times\mathbb{R}^n)$ of ($\overline{\mathrm{HJ}}$).

PROOF We construct a maximal subsolution by regularizing the Hamiltonian by inf-convolution. More explicitely, given $k>C$, consider the following Hamiltonian

$$\begin{aligned}\overline{H}_k(x,p,X)&=\inf\{\overline{H}(x',p',X')+k\big(|x-x'|+|p-p'|+|X-X'|\big)\}\\&=\inf\{\overline{H}(x-x',p-p',X-X')+k\big(|x'|+|p'|+|X'|\big)\}.\end{aligned}$$

It is well defined, Lipschitz continuous with respect to all the variables and degenerate elliptic. Moreover, $|\overline{H}_k(x,0,0)|\le C$. Since $\overline{H}_k$ is degenerate elliptic and Lipschitz continuous in x uniformly with respect to (p,X), it satisfies the structural condition of the User's Guide (condition (3.14) p. 18 in [**CIL92**]). Therefore, the Cauchy problem

(2.17)

$$u_t+\overline{H}_k(x,Du,D^2u)=0\quad\text{in } (0,T)\times\mathbb{R}^n,\qquad u(0,x)=\overline{h}(x)\quad\text{for all } x\in\mathbb{R}^n$$

has a unique solution $u_k\in BUC([0,T)\times\mathbb{R}^n)$.

It is clear that the sequence $(\overline{H}_k)$ is nondecreasing with respect to k and satisfies $\overline{H}_k\le\overline{H}$ for each k. Moreover, one obtains from the continuity of $\overline{H}$ that $\overline{H}_k\to\overline{H}$ uniformly on the compact sets. By the comparison principle, the sequence (u_k) is nonincreasing. We set

$$\widetilde{u}=\inf u_k.$$

As the sequence is nondecreasing, $\widetilde{u}(x)$ is the relaxed semi-limit $\limsup_{k\to\infty,\,y\to x}u_k(y)$. We deduce from the stability result for viscosity solutions that $\widetilde{u}$ is a subsolution of ($\overline{\mathrm{HJ}}$). Moreover, as every subsolution u of ($\overline{\mathrm{HJ}}$) is a subsolution of (2.17), we have $u\le u_k$. Passing to the limit, we obtain that $u\le\widetilde{u}$. Hence, $\widetilde{u}$ is the maximal subsolution.

The construction of the minimal supersolution is similar and it is left to the reader. □

Now we turn to the uniform convergence. If we know a priori that u^ε converges uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to some function u, and extend u at $t=0$ by setting $u(0,\cdot)=\overline{h}$, then u must be a viscosity solution of ($\overline{\mathrm{HJ}}$) by Proposition 2.6. This is in fact Corollary 1 in [**AB03**]. But, it is hard to use in practice because knowing a priori the local uniform convergence of u^ε (or a subsequence) requires proving its equicontinuity, a delicate question for singular perturbation problems. A simpler route is to prove that the comparison principle holds for the limit equation ($\overline{\text{HJ}}$), i.e.

$$\text{if } u \in BUSC \text{ is a subsolution of } (\overline{\text{HJ}}) \text{ and } v \in BLSC \text{ is a supersolution, then } u \le v \text{ on } [0,T) \times \mathbb{R}^n. \tag{2.18}$$

Indeed, in this case, the maximal subsolution and minimal supersolution of Theorem 2.7 are equal and so are the semilimits. This implies that u^ε converges locally uniformly to the function $\overline{u} = \underline{u}$ which is the unique continuous viscosity solution of ($\overline{\text{HJ}}$). We thus have proved the following result (Corollary 2 in [**AB03**]).

THEOREM 2.9. *Assume H is ergodic, the pair (H, h) is stabilizing, and $\overline{H}$ satisfies the comparison principle* (2.18). *Then u^ε converges uniformly on the compact subsets of* $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* ($\overline{\text{HJ}}$).

Some sufficient conditions for the comparison principle (2.18) were given in [**AB01**] for the HJB equations of optimal control with a single player. We give immediately a result under the rather simple (although restrictive) assumption that the fast dynamic does not depend on the slow variables. It will be substantially improved in the next Chapters 3-6 by a sharp ad hoc analysis of the regularity of the corrector and of its dependence with respect to the parameters (especially the slow variable). This theorem follows from Proposition 2 in [**AB03**].

THEOREM 2.10. *Assume that the Hamiltonian defined by* (1.12) *and* (1.11) *is ergodic, the pair (H, h) is stabilizing, and $g(y, \alpha, \beta)$, $\tau(y, \alpha, \beta)$ do not depend on x. Then u^ε converges uniformly on the compact subsets* $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* ($\overline{\text{HJ}}$).

REMARK In the case when the initial data h are independent of y, the pair (H, h) is automatically stabilizing and the effective initial data is of course h, as explained at the end of Section 2.2. One gets easily that the above convergence results are uniform on the compact subsets of $[0,T) \times \mathbb{R}^n \times \mathbb{R}^m$.

CHAPTER 3

Uncontrolled fast variables and averaging

In this chapter we consider the case when the fast dynamics

$$dy_s = g(\overline{x}, y_s)ds + \tau(\overline{x}, y_s)dW_s, \quad y_0 = y \tag{3.1}$$

is uncontrolled. Our purpose is to revisit the relationship between the ergodic properties of a diffusion and its invariant measures. This chapter generalizes to the stochastic setting and simplifies Appendix 6.1 in [**AB03**].

We recall the semigroup approach to the diffusion (3.1), which is well adapted to the description of its ergodic properties. The linear semigroup associated to the diffusion is

$$S_t\varphi(y) := E_y[\varphi(y_t)], \qquad \varphi \in C_{\mathrm{per}}(\mathbb{R}^m), \qquad y_\cdot \text{ solving (3.1)},$$

and its infinitesimal generator is the linear operator

$$\mathcal{L}_{\overline{x}}\varphi := D^2_{yy}\varphi \cdot b(\overline{x}, \cdot) + D_y\varphi \cdot g(\overline{x}, \cdot)$$

that is defined for the functions of class C^2. The function $w(t, y) = S_t\varphi(y)$ is therefore the unique viscosity solution of the Cauchy problem

$$w_t - \mathcal{L}_{\overline{x}}w = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \quad w(0, y) = \varphi(y), \quad w \text{ periodic}.$$

We denote by $M_{\mathrm{per}}(\mathbb{R}^m)$ the set of the periodic Radon measures on $\mathbb{R}^m$, which we identify with the set of the Radon measures on the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. By duality, we can define the adjoint semigroup to S_t acting on $\mu \in M_{\mathrm{per}}(\mathbb{R}^m)$ by

$$S_t^*\mu(\varphi) := \mu(S_t\varphi), \qquad \text{for all } \varphi \in C_{\mathrm{per}}(\mathbb{R}^m).$$

Here $\mu(\varphi) = \int_{[0,1[^m} \varphi(y)\, d\mu(y)$. Hence, when μ is a probability measure, $S_t^*\mu$ is the law of the diffusion process y_t with initial law μ.

A Radon measure μ is said to be *invariant* for the diffusion (3.1) (or stationary) if $S_t^*\mu = \mu$ for all $t \geq 0$, i.e., if

$$\mu(S_t\varphi) = \mu(\varphi) \qquad \text{for all } t \geq 0 \text{ and } \varphi \in C_{\mathrm{per}}(\mathbb{R}^m).$$

There is of course an infinitesimal characterization of invariance. For every $\mu \in M_{\mathrm{per}}(\mathbb{R}^m)$, define the distribution

$$\mathcal{L}^*_{\overline{x}}\mu := \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}\big(b_{ij}(\overline{x}, y)\mu\big) - \sum_i \frac{\partial}{\partial y_i}\big(g_i(\overline{x}, y)\mu\big).$$

As $\mathcal{L}^*_{\overline{x}}$ is the formal adjoint of $\mathcal{L}_{\overline{x}}$, it is the infinitesimal generator of the adjoint semi-group S_t^*. Therefore, μ is an invariant measure if and only if it solves the equation

$$-\mathcal{L}^*_{\overline{x}}\mu = 0 \quad \text{in } \mathbb{R}^m, \qquad \mu \text{ periodic}, \tag{3.2}$$

in the sense of distributions.

3.1. Ergodicity

The following proposition states the relationship between invariant measures and ergodicity. It is a classical result for discrete-time systems [**CFS82, Wal82**] that we adapt to diffusion processes.

PROPOSITION 3.1. *Under the standard assumptions*

(i) *the diffusion* (3.1) *has an invariant probability measure;*

(ii) *there is a unique invariant probability measure* μ *if and only if, for every* $\varphi \in C_{per}(\mathbb{R}^m)$,

$$\frac{1}{t}\int_0^t S_s\varphi(y)\,ds \to const \quad \text{uniformly in } y, \qquad \text{as } t \to +\infty; \tag{3.3}$$

(iii) *if this is the case the constant in* (3.3) *is* $\mu(\varphi)$.

PROOF We follow standard ideas in ergodic theory, see, e.g., Walters [**Wal82**]. We first show the existence of an invariant probability measure. Given an arbitrary probability measure ν, we define the occupation probability measure by

$$\mu_t(\varphi) = \frac{1}{t}\int_0^t \nu(S_s\varphi)\,ds, \qquad \varphi \in C_{\text{per}}(\mathbb{R}^m).$$

It gives the time average of the expected value of φ evaluated along the diffusion with initial law ν. Since $\mathbb{T}^m$ is compact, we can extract a subsequence μ_{t_k} that converges weakly-$*$ to a probability measure μ. For every periodic φ and every $t > 0$, we have

$$\begin{aligned}\mu_{t^k}(S_t\varphi) = \frac{1}{t_k}\int_0^{t_k} \nu\big(S_s(S_t\varphi)\big)\,ds = \frac{1}{t_k}\int_0^{t_k} \nu(S_{s+t}\varphi)\,ds = \frac{1}{t_k}\int_t^{t_k+t} \nu(S_s\varphi)\,ds \\ = \frac{1}{t_k}\int_0^{t_k} \nu(S_s\varphi)\,ds + O\big(\frac{t\|\varphi\|_\infty}{t_k}\big) = \mu_{t^k}(\varphi) + O\big(\frac{t\|\varphi\|_\infty}{t_k}\big).\end{aligned}$$

Sending $k \to +\infty$, we deduce that $\mu(S_t\varphi) = \mu(\varphi)$ for all φ and all $t > 0$. Hence, μ is an invariant probability.

We now suppose that the diffusion has a unique invariant probability measure μ. We consider the occupation probability measure starting from y

$$\mu_t^y(\varphi) = \frac{1}{t}\int_0^t S_s\varphi(y)\,ds, \qquad \varphi \in C_{\text{per}}(\mathbb{R}^m).$$

To prove (3.3) amounts to showing that $\mu_t^y(\varphi) \to \mu(\varphi)$ as $t \to +\infty$ uniformly in y for all $\varphi \in C_{\text{per}}(\mathbb{R}^m)$. If this were false, there would be a continuous periodic function φ, a real number $\varepsilon > 0$, and sequences $t_k \to +\infty$ and (y_k) so that $|\mu_{t_k}^{y_k}(\varphi) - \mu(\varphi)| \geq \varepsilon$. Extracting a subsequence, we can suppose that the probability measures $\mu_{t_k}^{y_k}$ converge weakly-$*$ to a probability measure μ'. By construction, we must have that $|\mu'(\varphi) - \mu(\varphi)| \geq \varepsilon$. But one checks, exactly as in the preceding paragraph, that the measure μ' must be invariant. By uniqueness, this implies that $\mu'(\varphi) = \mu(\varphi)$, which is a contradiction. Hence (3.3) holds.

Conversely, we assume that (3.3) holds and we denote by C_φ the right-hand constant. Then, for every invariant measure μ, we have

$$\mu\Big(\frac{1}{t}\int_0^t S_s\varphi\,ds\Big) = \frac{1}{t}\int_0^t \mu(S_s\varphi)\,ds = \mu(\varphi).$$

By the dominated convergence theorem, we know that the left-hand term converges to $C_\varphi \mu(1)$ as $t \to +\infty$. Therefore, the invariant measure is unique up to a multiplicative constant. In particular, there is at most one invariant probability measure. Moreover, such measure satisfies $\mu(1) = 1$, so $C_\varphi = \mu(\varphi)$, which proves (iii). □

From the last proposition we get a formula for the effective Hamiltonian $\overline{H}$ as an average of H with respect to the invariant measure. We define

$$H_1(\overline{x}, y, \overline{p}, \overline{X}) := \min_{\beta\in B} \max_{\alpha\in A} \left\{ -L(y, \alpha, \beta; \overline{x}, \overline{p}, \overline{X}) \right\} = H(x, y, p, 0, X, 0, 0) \tag{3.4}$$

$$= \min_{\beta\in B} \max_{\alpha\in A} \left\{ -\overline{X} \cdot a(\overline{x}, y, \alpha, \beta) - \overline{p} \cdot f(\overline{x}, y, \alpha, \beta) - l(\overline{x}, y, \alpha, \beta) \right\}, \tag{3.5}$$

and observe that for a fast subsystem of the form (3.1) the Hamiltonian becomes

$$H(\overline{x}, y, \overline{p}, q, \overline{X}, Y) = -Y \cdot b(\overline{x}, y) - p \cdot g(\overline{x}, y) + H_1(\overline{x}, y, \overline{p}, \overline{X}). \tag{3.6}$$

Corollary 3.2. *Assume* (A) *and, for some* $\overline{x} \in \mathbb{R}^n$, $g = g(\overline{x}, y)$ *and* $\tau = \tau(\overline{x}, y)$ *independent of* α *and* β. *Then the Hamiltonian* H *is ergodic at* $\overline{x}$ *for all data* f, σ, l *if and only if the diffusion* (3.1) *has a unique invariant probability measure* $\mu_{\overline{x}}$, *and the effective Hamiltonian is*

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \int_{[0,1)^m} H_1(\overline{x}, y, \overline{p}, \overline{X})\, d\mu_{\overline{x}}(y). \tag{3.7}$$

Proof For the current Hamiltonian the cell t-problem is the linear equation

$$w_t - D^2_{yy} w \cdot b(\overline{x}, y) - D_y w \cdot g(\overline{x}, y) + H_1(\overline{x}, y, \overline{p}, \overline{X}) = 0 \ \text{ in } (0, +\infty) \times \mathbb{R}^m, \ w(0, y) = 0,$$

whose unique viscosity solution is

$$w(t, y) = E_y \int_0^t [H_1(\overline{x}, y_s, \overline{p}, \overline{X})]\, ds = \int_0^t S_s H_1(y)\, ds$$

with $y_.$ solving (3.1). Then, by Proposition 2.2, H is ergodic at $\overline{x}$ for all data f, σ, l if and only if (3.3) holds for all $\varphi \in C_{\text{per}}(\mathbb{R}^m)$. Therefore Proposition 3.1 gives all the conclusions. □

3.2. Stabilization

We recall that the uncontrolled diffusion (3.1) is *stabilizing* if

$$S_t \varphi(y) \to \text{const} \quad \text{uniformly in } y, \qquad \text{as } t \to +\infty \tag{3.8}$$

for all $\varphi \in C_{\text{per}}(\mathbb{R}^m)$. There is not such a simple characterization of stabilization as for unique ergodicity in terms of the invariant measure. However, the following holds.

Proposition 3.3. *Under the standing assumptions a stabilizing diffusion is uniquely ergodic. Moreover, the constant in* (3.8) *is* $\mu(\varphi)$, *where* μ *is the unique invariant probability measure of* (3.1).

Proof Fix $\varphi \in C_{\text{per}}(\mathbb{R}^m)$ and assume that $S_t\varphi \to C_\varphi$ uniformly as $t \to +\infty$ for some constant C_φ. Then, for all $t \geq r \geq 0$, we have

$$\begin{aligned}\|\frac{1}{t}\int_0^t S_s\varphi\, ds - C_\varphi\|_\infty &= \|\frac{1}{t}\int_0^t \big(S_s\varphi - C_\varphi\big)\, ds\|_\infty \\ &\leq \|\frac{1}{t}\int_0^r \big(S_s\varphi - C_\varphi\big)\, ds\|_\infty + \|\frac{1}{t}\int_r^t \big(S_s\varphi - C_\varphi\big)\, ds\|_\infty \\ &\leq \frac{2r}{t}\|\varphi\|_\infty + \frac{(t-r)}{t}\sup_{s\geq r}\|S_s\varphi - C_\varphi\|_\infty.\end{aligned}$$

Sending $t \to +\infty$ and then $r \to +\infty$, we conclude that $\frac{1}{t}\int_0^t S_s\varphi\, ds \to C_\varphi$ uniformly. Therefore, the diffusion is uniquely ergodic. □

Remark It is easy to construct uniquely ergodic diffusions that are not stabilizing. For instance, there are uniquely ergodic deterministic systems. A classical example consists of the translations on the torus $y_t = y + \xi t$ with a vector $\xi \in \mathbb{R}^m$ whose coordinates are rationnaly independent (i.e. $\xi \cdot k \neq 0$ for all $k \in \mathbb{Z}^m \backslash \{0\}$) (see Chapter 7). However, no deterministic uncontrolled system

$$\dot{y}_s = g(y_s), \qquad y_0 = y,$$

can be stabilizing. Indeed, for all $\varphi \in C_{\text{per}}(\mathbb{R}^m)$, we have

$$\|S_t\varphi\|_\infty = \sup_{y\in\mathbb{T}^m} |\varphi(y_t)| = \|\varphi\|_\infty,$$

because, for all $t \geq 0$, the map $y \mapsto y_t$ is a bijection on the torus. Hence, for every constant C, we have $\|S_t\varphi - C\|_{L^\infty} = \|\varphi - C\|_{L^\infty}$. Therefore, $S_t\varphi$ cannot uniformly converge to a constant unless φ is constant.

Remark It is an open question whether stabilization implies unique ergodicity for controlled diffusions.

Corollary 3.4. *Assume* (A) *and, for some* $\overline{x}$, $g = g(\overline{x}, y)$, $\tau = \tau(\overline{x}, y)$ *independent of* α *and* β. *If the Hamiltonian* H *is stabilizing at* $\overline{x}$, *then, for any terminal cost* h,

$$\overline{h}(\overline{x}) = \int_{[0,1)^m} h(\overline{x}, y)\, d\mu_{\overline{x}}(y), \tag{3.9}$$

where $\mu_{\overline{x}}$ *is the unique invariant probability measure of* (3.1).

3.3. Uniform convergence

Here we limit ourselves to a simple convergence result that exploits the explicit formula for the effective Hamiltonian of Corollary 3.4. We refer to the next chapters for other kinds of assumptions.

Theorem 3.5. *Suppose that the differential game satisfies the standing assumptions* (A) *with a fast subsystem that does not depend on the controls* α, β. *Assume also that the subsystem* (3.1) *is stabilizing for all* $\overline{x}$ *and that the unique invariant* *probability measure* $\mu_{\overline{x}}$ *is independent of the slow variable* $\overline{x}$*. Then the value functions* $u^\varepsilon(t,x,y)$ *converge uniformly on the compact subsets of* $(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^m$ *as* $\varepsilon\to 0$ *to the unique viscosity solution* $u(t,x)$ *of*

$$\begin{cases} u_t + \overline{H}(x, D_x u, D^2_{xx}u) = 0, & in\ (0,+\infty)\times\mathbb{R}^n \\ u(x,0) = \int_{(0,1)^m} h(x,y)d\mu(y), \end{cases} \tag{3.10}$$

where

$$\overline{H}(x,p,X) = \int_{[0,1)^m} \min_{\beta\in B}\max_{\alpha\in A} \{-X\cdot a(x,y,\alpha,\beta) - p\cdot f(x,y,\alpha,\beta) - l(x,y,\alpha,\beta)\}\, d\mu(y).$$

PROOF Since the Hamiltonian is given by the average

$$\overline{H}(\overline{x},\overline{p},\overline{X}) = \int_{[0,1)^m} H_1(\overline{x},y,\overline{p},\overline{X})\, d\mu(y)$$

with respect to a measure independent of $\overline{x}$, it satisfies the comparison principle (2.18). It is indeed a routine excercise to verify that it fulfils the structure condition of the User's Guide [**CIL92**]. Convergence then follows from Theorem 2.9. □

Note that this theorem reduces the singular perturbation problem to an *averaging process* that determines the effective Hamiltonian and the effective initial data.

We end this section with some examples where the last theorem applies. The simplest sufficient condition for the invariant measure to be independent of x is that the fast subsystem be so, that is,

$$g = g(y), \qquad \tau = \tau(y).$$

In this case the uniform convergence follows also from Theorem 2.10. A different sufficient condition is given by the next result.

COROLLARY 3.6. *Assume* (A), $g = g(x,y)$, $\tau = \tau(x,y)$, (3.1) *stabilizing for all* $\overline{x}$*, and*

$$\sum_{i,j} \frac{\partial^2 b_{ij}}{\partial y_i \partial y_j} = \sum_i \frac{\partial g_i}{\partial y_i} \tag{3.11}$$

in the sense of distributions. Then the invariant probability measure of (3.1) *is the Lebesgue measure for all* $\overline{x}$ *and the conclusion of Theorem 3.5 hold with* $d\mu(y) = dy$*. This is the case, for instance, if* $g = g(x)$ *and* $\tau = \tau(x)$ *are independent of* y*, or if the infinitesimal generator* $\mathcal{L}$ *is in divergence form, i.e.,*

$$\mathcal{L}_x\varphi(y) = \sum_{i,j} \frac{\partial}{\partial y_j}\left(b_{ij}(x,y)\frac{\partial\varphi}{\partial y_i}(y)\right).$$

PROOF If (3.11) holds the constants are solutions of the adjoint equation (3.2). Therefore the Lebesgue measure on $[0,1)^m$ is an invariant probability measure of (3.1) for all $\overline{x}$ and we conclude by Theorem 3.5. For the last statement it is enough to observe that $\mathcal{L}$ is in divergence form when $\sum_j \frac{\partial b_{ij}}{\partial y_j} = g_i$ for all i. □

Explicit sufficient conditions for the ergodicity and stabilization of (3.1), implying the uniform convergence of u^ε by the results above, will be given in the next chapters. The uniformly nondegenerate case of Chapter 4 and the nonresonant case of Chapter 7 are treated directly in the general situation of controlled fast subsystem. Instead, the hypoelliptic case of Chapter 5 exploits the linear structure of the Hamiltonian occurring when the fast subsystem is uncontrolled.

3.4. An explicit formula for the limit control problem

Under some additional assumptions we can describe an explicit control problem whose value function is the limit of the value functions u^ε. We call it the *effective control problem* in the case of a single player and *effective differential game* in the general case. Consider the system in split form

$$
\begin{aligned}
&dx_s = [f_1(x_s,\alpha_s,\beta_s) + f_2(x_s,y_s)]\, ds + [\sigma_1(x_s,\alpha_s,\beta_s) + \sigma_2(x_s,y_s)]\, dW_s,\\
&dy_s = \tfrac{1}{\varepsilon} g(x_s,y_s)\, ds + \tfrac{1}{\sqrt{\varepsilon}}\tau(x_s,y_s)\, dW_s. \qquad (3.12)\\
&x_0 = x, \quad y_0 = y,
\end{aligned}
$$

with the cost functional

$$
J(t,x,y,\alpha,\beta) := E_{(x,y)}\left[\int_0^t \left(l_1(x_s,\alpha_s,\beta_s) + l_2(x_s,y_s)\right) ds + h(x_t,y_t)\right]. \qquad (3.13)
$$

Assume the fast subsystem is uniquely ergodic and the invariant measure μ_x is independent of x. Since the fast variables and the controls appear in different terms of the dynamics and of the running cost, the effective Hamiltonian (3.7) is

$$
\begin{aligned}
\overline{H}(x,p,X) = &\min_{\beta\in B}\max_{\alpha\in A}\{-X\cdot a_1(x,\alpha,\beta) - p\cdot f_1(x,\alpha,\beta) - l_1(x,\alpha,\beta)\}\\
&- X\cdot\int a_2(x,y)\, d\mu(y) - p\cdot\int f_2(x,y)\, d\mu(y) - \int l_2(x,y)\, d\mu(y)\\
&- \frac{1}{2}\int \min_{\beta\in B}\max_{\alpha\in A}\left(\sigma_1\sigma_2^T + \sigma_2\sigma_1^T\right)\cdot X\, d\mu(y),
\end{aligned}
$$

where $a_i := \sigma_i\sigma_i^T/2$, $i = 1,2$. To arrive at a simple explicit expression for the effective system we must assume that the last term vanishes

$$
\int_{[0,1)^m}\min_{\beta\in B}\max_{\alpha\in A}\left(\sigma_1(x,\alpha,\beta)\sigma_2^T(x,y) + \sigma_2(x,y)\sigma_1^T(x,\alpha,\beta)\right)\cdot X\, d\mu(y) = 0,\ \forall\, X\in\mathbb{S}^n,
$$

which is true, for instance, if $\sigma_1\sigma_2^T \equiv 0$. This assumptions describes a sort of uncorrelation of the two diffusion terms σ_1 and σ_2. Now suppose there is a Lipschitz continuous, $n\times k$ matrix valued function $\overline{\sigma}_2(x)$ such that

$$
\frac{\overline{\sigma}_2(x)\overline{\sigma}_2(x)^T}{2} = \int_{[0,1)^m} a_2(x,y)\, d\mu(y),
$$

and define

$$
\overline{f}_2(x) := \int_{[0,1)^m} f_2(x,y)\, d\mu(y), \qquad \overline{l}_2(x) := \int_{[0,1)^m} l_2(x,y)\, d\mu(y).
$$

Then the system associated to the effective Hamiltonian $\overline{H}$ is

(3.14)

$$
dx_s = \left[f_1(x_s,\alpha_s,\beta_s) + \overline{f}_2(x_s)\right] ds + [\sigma_1(x_s,\alpha_s,\beta_s) + \overline{\sigma}_2(x_s)]\, dW_s, \qquad x_0 = x,
$$

and the cost functional associated to $\overline{H}$ and to the effective terminal cost $\overline{h}(x) = \int h(x, y)\, d\mu(y)$ is

$$\overline{J}(t, x, \alpha, \beta) := E_x \left[\int_0^t \left(l_1(x_s, \alpha_s, \beta_s) + \bar{l}(x_s) \right) ds + \overline{h}(x_t) \right]. \tag{3.15}$$

The next result now follows immediately from Theorem 3.5 and from the characterization of the unique solution of the effective Cauchy problem as the value function of the corresponding control problem (if there is a single player) or differential game [**FS06, FS89**].

PROPOSITION 3.7. *Under the preceding assumptions, the value function* u^ε *of the differential game with system* (3.12) *and cost functional* (3.13) *converges uniformly on the compact subsets* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the (lower) value function of the differential game with the effective system* (3.14) *and the effective terminal cost* (3.15).

CHAPTER 4

Uniformly nondegenerate fast diffusion

We say that the fast subsystem

$$dy_s = g(x, y_s, \alpha[\beta]_s, \beta_s)ds + \tau(x, y_s, \alpha[\beta]_s, \beta_s)dW_s, \quad y_0 = y, \tag{4.1}$$

is a uniformly non-degenerate diffusion for a frozen $x = \overline{x}$ if

$$\text{for some } \nu > 0, \quad b(\overline{x}, y, \alpha, \beta) \geq \nu I_m \quad \forall\, y \in \mathbb{R}^m,\ \alpha \in A,\ \beta \in B, \tag{4.2}$$

where I_m denotes the m-dimensional identity matrix and $b = \tau\tau^T/2$. Under this condition and the boundedness of the data, the Hamiltonian is uniformly elliptic with respect to the fast variables, i.e., there is also a positive constant ν' depending only on $\overline{x},\ \overline{p},\ \overline{X}$ such that

$$\nu \operatorname{tr} W \leq H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) - H(\overline{x}, y, \overline{p}, q, \overline{X}, Y + W, 0) \leq \nu' \operatorname{tr} W, \tag{4.3}$$
$$\forall W \in \mathbb{S}^m,\ W \geq 0,\ \forall y, q, Y.$$

4.1. Ergodicity

The first result concerns ergodicity for uniformly non-degenerate diffusions. It generalizes Lemma 3.1 in Evans [**Eva92**]. The proof will serve as a reference for the study of ergodicity under alternative assumptions on the dynamics. We therefore show the ergodicity in details, adapting the demonstration by Arisawa, Lions [**AL98**] to the case of non convex Hamiltonians. Though we always assume that the Hamiltonian is of the form (1.12) and is therefore associated to stochastic games, the result is valid under the classical structural assumptions for fully non-linear uniformly elliptic operator to which the $C^{1,\alpha}$ regularity theory applies (see, e.g., Gilbarg, Trudinger [**GT83**], Trudinger [**Tru89**] and Cabré, Caffarelli [**CC95**]) and does not need the min-max form of Bellman-Isaacs Hamiltonians.

THEOREM 4.1. *Assume that the Hamiltonian is given by* (1.12) *with the data satisfying the standing assumptions* (A) *as well as the uniform non degeneracy assumption* (4.2).

Then the Hamiltonian is ergodic at $\overline{x}$. More precisely, for every $(\overline{p}, \overline{X})$, there is a unique constant λ and a corrector $\chi \in C_{per}(\mathbb{R}^m)$, that is unique up to an additive constant, for which the true cell problem (2.9) *has a solution. Moreover, there are constants $C > 0$ and $\gamma \in (0, 1]$ such that $\chi \in C^{1,\gamma}_{per}(\mathbb{R}^m)$ and we have the estimate*

$$\|\chi - \chi(0)\|_{C^{1,\gamma}_{per}(\mathbb{R}^m)} \leq C\big(1 + |\overline{p}| + |\overline{X}|\big).$$

PROOF Until the last paragraph, we fix the slow data $(\overline{x}, \overline{p}, \overline{X})$. We shall denote in the proof by C various constants that depend only on the Hamiltonian H, i.e. depending in the constants in the assumptions (A) and (4.2), and by K constants that may depend also on $(\overline{x}, \overline{p}, \overline{X})$.

The constant functions $\min_y$ and $\max_y$ of $H(\overline{x}, y, \overline{p}, 0, \overline{X}, 0, 0)/\delta$ are, respectively, a sub- and a supersolution of (CP$_\delta$), so we have the uniform bound

$$|\delta w_\delta(y)| \le K \qquad \text{for all } y. \tag{4.4}$$

Next we show that $\{w_\delta - w_\delta(0)\}$ is equibounded following [**AL98**]. Were it false, there would be a subsequence $\delta_k \to 0$ such that the sequence $\varepsilon_k = \|w_{\delta_k} - w_{\delta_k}(0)\|_{L^\infty}^{-1}$ converges to 0 as $k \to +\infty$. The function $\psi_k = \varepsilon_k\big(w_{\delta_k} - w_{\delta_k}(0)\big)$ satisfies

$$-K\varepsilon_k \le H'(\overline{x}, y, D\psi_k, D^2\psi_k) \le K\varepsilon_k \quad \text{in } \mathbb{R}^m \tag{4.5}$$

in the viscosity sense. Since H' is uniformly elliptic and $\|\psi_k\|_{L^\infty} = 1$, we can apply the regularity theory for viscosity solutions of fully nonlinear uniformly elliptic equations (as exposed in Trudinger [**Tru89**] and [**Tru88**]) and obtain that the family $\{\psi_k\}$ is equi Hölder continuous. (Note that the Hölder bound proved by Trudinger [**Tru88**, Th. 5.1] for solutions of uniformly elliptic equations

$$H'(\overline{x}, y, \overline{p}, D\psi, \overline{X}, D^2\psi, 0) = f(y)$$

with f bounded is still valid when this is relaxed to a pair of inequalities

$$-K \le H'(\overline{x}, y, \overline{p}, D\psi, \overline{X}, D^2\psi, 0) \le K.$$

Indeed, the Hölder estimate results from the two Harnack inequalities which hold for sub and supersolutions and not only for solutions.) Extracting a subsequence, we get that ψ_k converges uniformly to a function ψ. We must have $\|\psi\|_{L^\infty} = 1$ and $\psi(0) = 0$. On the other hand, as $\varepsilon_k \to 0$, the function ψ is a viscosity solution of $H'(\overline{x}, y, D\psi, D^2\psi) = 0$. Since H' is uniformly elliptic with $H'(\overline{x}, \cdot, 0, 0) \equiv 0$, we deduce from the strong maximum principle [**Tru88, BL99**] that ψ must be constant. We have reached a contradiction.

Going back to w_δ, we see that it satisfies

$$-K \le H(\overline{x}, y, \overline{p}, Dw_\delta, \overline{X}, D^2 w_\delta, 0) \le K \quad \text{in } \mathbb{R}^m \tag{4.6}$$

in the viscosity sense, where K is the uniform bound for δw_δ. In view of the equiboundedness of $\{w_\delta - w_\delta(0)\}$, we can apply again the uniformly elliptic regularity theory to obtain that the family $\{w_\delta - w_\delta(0)\}$ is uniformly bounded in $C^{0,\gamma_0}_{\text{per}}(\mathbb{R}^m)$, for some $\gamma_0 > 0$. By Ascoli theorem, we can extract a sequence $\delta_k \to 0$ such that $\delta_k w_{\delta_k}$ converges uniformly to a constant λ and $w_{\delta_k} - w_{\delta_k}(0)$ converges to $\chi \in C^{0,\gamma_0}_{\text{per}}(\mathbb{R}^m)$ uniformly. By the stability of viscosity solutions, λ and χ solve the true cell problem (2.9).

The proof that there is at most one constant λ for which the true cell problem is solvable follows from a standard argument based on the comparison principle which we omit here (see, e.g., the proof of Theorem 1 in [**Ari98**] or that of Theorem 4 in [**AB03**]). We deduce that the whole family δw_δ converges uniformly to λ.

The uniqueness of the corrector up to an additive constant results from the strong comparison principle for uniformly elliptic equations : if u is a subsolution and v is a supersolution, then $u - v$ cannot achieve its maximum unless it is constant.

Invoking once again the regularity theory for viscosity solutions of uniformly elliptic equations [**Tru89**], we get that the corrector χ is of class C^1 with Hölder continuous derivatives. To have a sharp estimate of $\|\chi - \chi(0)\|_{C^{1,\gamma}_{\text{per}}(\mathbb{R}^m)}$, we mimic the argument given in [**AB01**] for the optimal control problem. We first note that the constant K in (4.5) can be choosen of the form $C\big(1 + |\overline{p}| + |\overline{X}|\big)$. By a compactness argument, one can deduce that

$$\|\chi - \chi(0)\|_{L^\infty} \le C\big(1 + |\overline{p}| + |\overline{X}|\big).$$

We leave the verification of the claim to the reader as this estimate is a immediate adaptation of the similar one proved in detail in Proposition 12 of [**AB01**]. On the other hand, the constant K in (4.4), which is the same as the one in (4.6), can also be taken of the form $K = C\big(1 + |\overline{p}| + |\overline{X}|\big)$. Following carefully the proof of the $C^{1,\gamma}$ bound of Trudinger [**Tru89**, Th. 2.1] and especially the dependency of the various constants with respect to the data, we note that the estimate depends linearly in the bound for the non-homogeneous part of the operator (the constant μ_0 with the notations of Trudinger). In our problem, this constant is the only one that is not uniform in $(\overline{x}, \overline{p}, \overline{X})$ but grows at most linearly in $(\overline{p}, \overline{X})$. Consequently, there are constants $C > 0$ and $\gamma \in (0, 1]$ that are independent of $(\overline{x}, \overline{p}, \overline{X})$ for which $\chi \in C^{1,\gamma}_{\mathrm{per}}(\mathbb{R}^m)$ with the bound

$$\|\chi - \chi(0)\|_{C^{1,\gamma}_{\mathrm{per}}(\mathbb{R}^m)} \le C\big(1 + |\overline{p}| + |\overline{X}| + \|\chi - \chi(0)\|_{L^\infty}\big).$$

Combining these two bounds gives the claimed estimate. □

Remark Assume that only the second player may act on the diffusion of the fast variable, i.e. $b = b(\overline{x}, y, \beta)$. Then, the Hamiltonian is concave with respect to the matrix Y. In this context, the regularity theory for uniformly elliptic equations improves the regularity of the corrector to $C^{2,\gamma}$ if the running cost l is Hölder continuous in y uniformly in the other entries, see in particular Safonov [**Saf89**]. Moreover, we have the a priori bound

$$\|\chi - \chi(0)\|_{C^{2,\gamma}_{\mathrm{per}}(\mathbb{R}^m)} \le C\big(1 + |\overline{p}| + |\overline{X}|\big).$$

The proof of this claim is exactly the same as the one of [**AB01**, Prop. 12, case (I)], because the only feature of the Hamiltonian that is used is its concavity with respect to the Hessian matrix.

4.2. Stabilization

The next result concerns stabilization for uniformly non-degenerate diffusions. The assumption (4.2) entails the uniform ellipticity of the recession function H'

$$\nu \operatorname{tr} W \le H'(\overline{x}, y, q, Y) - H'(\overline{x}, y, q, Y + W) \le \nu' \operatorname{tr} W, \qquad \forall\, W \in \mathbb{S}^m,\ W \ge 0,\ \forall\, y, q, Y. \tag{4.7}$$

As for ergodicity, the same proof holds under structural assumptions on the Hamiltonian. For consistency with the rest of the paper we state it for the Bellman-Isaacs operator under the standing assumptions on the controlled system. Our result is related to Theorem II.2 for HJB equations in Arisawa, Lions [**AL98**] and follows most of its argument. Once again, we provide a detailed proof as this will serve as a reference for the other stabilization results in the paper.

Theorem 4.2. *Assume that the Hamiltonian is given by* (1.12) *with the data satisfying the standing assumptions* (A) *as well as the uniform non degeneracy assumption* (4.2).

Then, for every continuous h, the pair (H,h) is stabilizing. In other words, for every fixed $\overline{x}$, the solution $w(t,\cdot)$ of (CP′) *converges uniformly to a constant as $t\to+\infty$.*

PROOF Since H' is homogeneous, any constant solves the PDE; consequently, we have the bounds $\min_y h(\overline{x},y)\le w\le \max_y h(\overline{x},y)$ by the comparison principle.

We begin with the case of smooth h. The first step is to show that w is uniformly continuous. This could follow simply from the regularity theory of viscosity solutions for uniformly parabolic equations. We prove it here in a more deviate manner, by applying the theory of uniformly elliptic equations for t fixed. This approach will reveal more convenient later (in the hypoelleptic case notably), as the regularity theory for stationary equation is much more developed. From the comparison principle, we get the estimate $|w(t,y)-h(y)|\le Ct$ on $[0,+\infty)\times\mathbb{R}^m$, for the constant $C:=\sup_y|H'(\overline{x},y,D_yh,D^2_{yy}h)|$. By applying again the comparison principle we obtain $|w(t+s,y)-w(t,y)|\le \sup_{y\in\mathbb{R}^m}|w(s,y)-h(y)|\le Cs$ on $[0,+\infty)\times\mathbb{R}^m$ for all $s>0$. In particular, we have $|\partial_t w|\le C$ in the viscosity sense. From this, it is easy to deduce that, for all $t>0$, the partial function $w(t,\cdot)$ satisfies

$$-C\le H'(\overline{x},y,D_yw,D^2_{yy}w)\le C \quad \text{in } \mathbb{R}^m \tag{4.8}$$

in the viscosity sense (see, for instance, [**BCD97**, Lemma 5.17]). In view of the uniform ellipticity of H' (4.7), we can apply the regularity theory for viscosity solutions of uniformly elliptic equations. As the family $\{w(t,\cdot)\}$ is uniformly bounded, it is uniformly bounded in $C^{0,\gamma}(\mathbb{R}^m)$ for some $\gamma>0$. Since w is Lipschitz continuous in t, we conclude that it is uniformly continuous in $[0,+\infty)\times\mathbb{R}^m$.

The second step is to prove that the function $\overline{w}(y)=\limsup_{t\to+\infty} w(t,y)$ is constant. To do so, we note that the rescaled function $w_\eta(t,y)=w(t/\eta,y)$ solves the Hamilton-Jacobi equation

$$\eta\,\frac{\partial w_\eta}{\partial t}+H'(\overline{x},y,D_yw_\eta,D^2_{yy}w_\eta)=0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m.$$

By the stability results for viscosity solutions, we deduce that

$$\overline{\overline{w}}(t,y)=\limsup_{\eta\to0,\ t'\to t,\ y'\to y} w_\eta(t',y')$$

is a subsolution of

$$H'(\overline{x},y,D_y\overline{\overline{w}},D^2_{yy}\overline{\overline{w}})\le 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m.$$

But, for every $t>0$, we have that

$$\overline{\overline{w}}(t,y)=\limsup_{t'\to+\infty,\ y'\to y} w(t',y')$$

The right-hand side is $\overline{w}(y)$ because the family $\{w(t,\cdot)\}$ is equicontinuous. We conclude that $\overline{w}$ is a subsolution of

$$H'(\overline{x},y,D_y\overline{w},D^2_{yy}\overline{w})\le 0 \quad \text{in } \mathbb{R}^m.$$

By the strong maximum principle [**BL99**], we obtain that $\overline{w}$ is constant.

The last step is to prove the uniform convergence of $w(t,\cdot)$ to $\overline{w}$ as $t\to+\infty$. Let (t_n) be a sequence converging to $+\infty$ so that $w(t_n,0)\to\overline{w}$. The family $\{w(t_n+\cdot,\cdot)\}$ is equicontinuous and equibounded. Along a subsequence, it therefore converges uniformly on the compact subsets of $\mathbb{R} \times \mathbb{R}^m$ to a function $\widetilde{w}$. By the stability results of viscosity solutions, it is a solution of

$$\partial_t \widetilde{w} + H'(\overline{x}, y, D_y \widetilde{w}, D^2_{yy} \widetilde{w}) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^m.$$

On the other hand, by the definition of $\overline{w}$, one has $\widetilde{w} \leq \overline{w}$ and $\widetilde{w}(0,0) = \overline{w}$. Thus $\widetilde{w}$ achieves an interior maximum at $t = 0$. By the strong maximum principle for uniformly parabolic equations (see [**DaL04**] or [**CKS00**]), we deduce that $\widetilde{w} = \overline{w}$ on $]-\infty, 0] \times \mathbb{R}^m$. The identity $\widetilde{w}(0, \cdot) \equiv \overline{w}$ means that $\{w(t_n, \cdot)\}$ converges uniformly to $\overline{w}$. By the comparison principle, we obtain

$$\sup_{[t_n, +\infty[\times\mathbb{R}^m} |w(t,y) - \overline{w}| = \sup_{\mathbb{R}^m} |w(t_n, y) - \overline{w}|.$$

As the right hand term converges to 0, we conclude that $\{w(t,\cdot)\}$ converges uniformly to $\overline{w}$ as $t \to +\infty$.

When h is merely continuous, we take a sequence of smooth functions (h_k) converging uniformly to h. The comparison principle implies that the associated sequence of solutions (w_k) converges uniformly to w on $[0, \infty) \times \mathbb{R}^m$. Moreover, for each fixed k, $w_k(t, \cdot)$ converges uniformly to a constant $\overline{w}_k$ as $t \to +\infty$. Since the sequence (h_k) is uniformly bounded, so are the sequences (w_k) and $(\overline{w}_k)$, thus a subsequence of $\overline{w}_k$ converges to some $\overline{w} \in \mathbb{R}$. Now it is easy to see that $w(t, \cdot)$ converges uniformly to $\overline{w}$ as $t \to +\infty$ by first choosing k large and then t large. □

4.3. Uniform convergence

In this section, we apply the ergodicty and stabilization properties of uniformly non-degenerate diffusions to the study of the singular perturbation problem. We shall use the general convergence results stated in Section 2.3. Under the sole assumption (4.2), Proposition 2.6 and Theorem 2.7 guarantee the convergence in a weak sense of the value functions u^ε by providing a priori bounds to the semilimits $\overline{u}$ and $\underline{u}$. Under some additional mild assumptions that guarantee the validity of the comparison principle for the limit equation ($\overline{\text{HJ}}$), the semilimits coincide and one can actually apply Theorem 2.9 to get the uniform convergence of u^ε. The purpose of this section is to list such assumptions.

The first result is a simple restatement of Theorem 2.10 that concerns the case when the fast dynamics is independent of the slow variable.

Corollary 4.3. *Assume that the differential game satisfies the usual assumptions* (A) *with uniformly non-degenerate fast diffusions* (4.2). *Assume also that the fast dynamics $g(y, \alpha, \beta)$ and $\tau(y, \alpha, \beta)$ do not depend on x.*

Then $\overline{H}$ and $\overline{h}$ exist and the value functions u^ε converge uniformly on the compact subsets of $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as $\varepsilon \to 0$ to the unique viscosity solution of* ($\overline{\text{HJ}}$).

The second result takes advantage of the regularity of the corrector. It allows the fast dynamics to depend on the slow variables but requires that the dynamics for the slow variables is either deterministic

$$a(x, y, \alpha, \beta) = 0 \qquad \text{for all } (x, y, \alpha, \beta) \tag{4.9}$$

or uniformly non-degenerate

$$\text{for some } \mu > 0, \quad a(x, y, \alpha, \beta) \geq \mu I_n \qquad \text{for all } (x, y, \alpha, \beta). \tag{4.10}$$

We give two results in this sense.

THEOREM 4.4. *Assume that the differential game satisfies the usual assumptions* (A) *with uniformly non-degenerate fast diffusions, i.e., for some* $\nu > 0$ (4.2) *holds for all* $\overline{x}$. *Assume also that*
(i) *the slow dynamics is either deterministic* (4.9) *or uniformly non-degenerate* (4.10)*;*
(ii) *the fast diffusion* $\tau(y, \alpha, \beta)$ *does not depend on* x.

Then $\overline{H}$ *and* $\overline{h}$ *exist and the value functions* u^ε *converge uniformly on the compact subsets* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of* $(\overline{\mathrm{HJ}})$.

PROOF The proof follows the argument introduced in [**AB01**, Prop. 12]. We have to establish some regularity for the effective Hamiltonian that guarantees the comparison principle.

The first estimate on the effective Hamiltonian is its Lipschitz regularity in the derivatives (p, X)

$$|\overline{H}(x, p', X') - \overline{H}(x, p, X)| \le C(|p' - p| + |X' - X|)$$

for all (x, p, p', X, X'). This is deduced from the uniform estimate

$$|H(x, y, p', q, X', Y, 0) - H(x, y, p, q, X, Y, 0)| \le C(|p' - p| + |X' - X|)$$

and a kind of comparison principle with respect to λ for the true cell problem (see below or Proposition 12 in [**AB01**] for a detailed proof).

When the slow dynamics is deterministic, the Hamiltonian H is of first order with respect to x, i.e. is independent of X. This implies that the effective Hamiltonian is of first order. When the slow diffusion is uniformly non-degenerate, the Hamiltonian H is uniformly elliptic with respect to X. This implies that the effective Hamiltonian is uniformly elliptic with the same ellipticity constants (same argument as for the Lipschitz regularity in (p, X), see [**AB01**] for the details).

The second estimate we need is the regularity of the effective Hamiltonian with respect to the state variable

$$|\overline{H}(x', p, X) - \overline{H}(x, p, X)| \le C|x' - x|(1 + |p| + |X|) + \omega(|x' - x|)$$

for some constant C and modulus ω, and for all x, x', p, X. Since the diffusion for the fast variable is independent of x, the Hamiltonian H satisfies

$$H(x', y, p, q, X, Y, 0) \le H(x, y, p, q, X, Y, 0) + C|x' - x|(1 + |p| + |q| + |X|) + \omega(|x' - x|),$$

where ω is the modulus of continuity of l. Therefore, any corrector χ at the point (x, p, X) will be a subsolution of

$$\begin{aligned} &H(x', y, p, D_y\chi, X, D^2_{yy}\chi, 0) \\ &\quad \le H(x, y, p, D_y\chi, X, D^2_{yy}\chi, 0) + C|x' - x|(1 + |p| + |D_y\chi| + |X|) + \omega(|x' - x|) \\ &\quad = \overline{H}(x, p, X) + C|x' - x|(1 + |p| + |D_y\chi| + |X|) + \omega(|x' - x|) \\ &\quad \le \overline{H}(x, p, X) + C|x' - x|(1 + |p| + |X|) + \omega(|x' - x|), \end{aligned}$$

where, for the last inequality, we have used the regularity of the corrector and the estimate of Theorem 4.1 to bound from above $|D_y\chi|$ by $C(1 + |p| + |X|)$. Thus, the corrector χ at x is a subsolution for the true cell problem at x' for the constant given above. By a standard argument (see e.g. [**AB03**, Th. 4]) this implies

$$\overline{H}(x', p, X) \le \overline{H}(x, p, X) + C|x' - x|(1 + |p| + |X|) + \omega(|x' - x|).$$

We get the claimed Lipschitz bound for $\overline{H}$ after exchanging x and x'.

The above regularity of the effective Hamiltonian ensures the validity of the comparison principle for the effective Hamilton-Jacobi equation ($\overline{\mathrm{HJ}}$) as it is either of first order or uniformly elliptic (see e.g. [**IL90**] for a proof of the comparison principle under these assumptions). □

Whenever the fast diffusion is independent of the first player's control, we can use the Remark after Theorem 4.1 to get a smoother corrector. This allows the fast diffusion to depend on the slow variable. The proof of the next theorem is omitted as it follows from the demonstration of [**AB01**, Prop. 12]. It can be easily reconstructed by modifying the proof of Theorem 4.4.

THEOREM 4.5. *Assume that the differential game satisfies the usual assumptions* (A) *with uniformly non-degenerate fast diffusions* (4.2). *Assume also that*
(i) *the slow dynamics is either deterministic* (4.9) *or uniformly non-degenerate* (4.10)*;*
(ii) *the fast diffusion* $\tau(x, y, \beta)$ *does not depend on the first player's control and* l *is Hölder continuous in* y *uniformly in* x, α, β.
Then $\overline{H}$ *and* $\overline{h}$ *exist and the value functions* u^ε *converge uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of* ($\overline{\mathrm{HJ}}$).

REMARK If the Isaacs condition holds, namely if, in the definition (1.12) of H we have $\min_\beta \max_\alpha = \max_\alpha \min_\beta$, then the conclusion of Theorem 4.5 holds also if the fast diffusion depends on the first player's control α, but not on the second player's control β.

By combining the assumptions of the Theorems 4.4 and 4.5 and of the remark we can give many examples of singularly perturbed differential games whose value functions converge uniformly. We only show two simple cases.

Example 1: deterministic games with small noise on the fast variables. Consider the system

$$\dot{x}_s = f(x_s, y_s, \alpha_s, \beta_s), \qquad \varepsilon dy_s = g(x_s, y_s, \alpha_s, \beta_s)ds + \sqrt{\varepsilon} dW_s.$$

The lower value $u^\varepsilon(t, x, y)$ of the game satisfies the PDE

$$u_t^\varepsilon + \min_{\beta \in B} \max_{\alpha \in A} \left\{ -D_x u^\varepsilon \cdot f(x, y, \alpha, \beta) - \frac{D_y u^\varepsilon}{\varepsilon} \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta) \right\} = \frac{\Delta_y u^\varepsilon}{\varepsilon},$$

where $\Delta_y := \mathrm{trace} D_{yy}$ denotes the Laplacian with respect to the y variables. By Theorem 4.4 there are continuous $\overline{H}$ and $\overline{h}$ such that u^ε converges locally uniformly to the unique solution $u(t, x)$ of the first-order problem

$$u_t + \overline{H}(x, Du) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^n, \qquad u(0, x) = \overline{h}(x) \quad \text{in } \mathbb{R}^n.$$

Example 2: systems with fully nondegenerate noise. Suppose that the whole $n + m$-dimensional system (1.7) is affected by a uniformly nondegenerate noise, that is, both (4.2) and (4.10) hold. Then the value function u^ε converges locally uniformly to the unique solution of the effective Cauchy problem ($\overline{\mathrm{HJ}}$) if the dispersion matrix of the fast variables $\tau(x, y, \alpha, \beta)$ depends on at most two among the three entries x, α, and β.

CHAPTER 5

Hypoelliptic diffusion of the fast variables

In this chapter we assume the fast subsystem (4.1) is uncontrolled for a fixed $x = \overline{x}$, i.e., $g = g(\overline{x}, y)$, $\tau = \tau(\overline{x}, y)$. We denote with $\tau^i(\overline{x}, y)$, $i = 1, \ldots, r$, the columns of the covariance matrix τ, and we assume they are C^∞ vector fields in $\mathbb{R}^m$. Next we suppose

$$(5.1)\qquad \begin{aligned} &g(\overline{x}, y) = \widetilde{g}(\overline{x}, y) + \eta(\overline{x}, y),\\ &\widetilde{g}(\overline{x}, y) := \zeta_i(\overline{x}, y)\tau^i(\overline{x}, y), \quad \eta(\overline{x}, y) := \frac{1}{2} D\tau^i(\overline{x}, y)\tau^i(\overline{x}, y), \end{aligned}$$

where we are adopting the summation convention, $\zeta_i(\overline{x}, \cdot) : \mathbb{R}^m \to \mathbb{R}$ are C^∞ functions, and $D\tau^i$ denotes the Jacobian matrix of τ^i with respect to the y variables; thus

$$\eta_j := \frac{1}{2} \frac{\partial \tau_{ji}}{\partial y_k} \tau_{ki}, \quad j = 1, \ldots, m.$$

Note that η is the correction term appearing in the drift when one writes a Stratonovich integral in Ito form, so our uncontrolled fast subsystem is equivalent to

$$(5.2)\qquad dy_s = \widetilde{g}(\overline{x}, y_s)dt + \tau(\overline{x}, y_s) \circ dW_s,$$

where $\circ$ indicates that we are using the Stratonovich calculus. We will also denote with $X_i := \tau^i \cdot \nabla$ the operator associated to the vector field τ^i, $i = 1, \ldots, r$. Then the cell δ-problem (CP$_\delta$) can be written as

$$(5.3)\qquad \delta w_\delta - \frac{1}{2} \sum_{i=1}^r X_i^2 w_\delta - \sum_{i=1}^r \zeta_i X_i w_\delta + H_1 = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic},$$

where H_1 is defined by (3.4), and the true cell problem (2.9) is

$$(5.4)\qquad \lambda - \frac{1}{2} \sum_{i=1}^r X_i^2 \chi - \sum_{i=1}^r \zeta_i X_i \chi + H_1 = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic}.$$

Note that the operator is a sum of squares of vector fields plus lower order terms. Our last assumption is

$$(5.5)\qquad \begin{cases} X_1, \ldots, X_r \text{ and their commutators} \\ \text{up to a certain fixed order } \overline{r} \\ \text{span } \mathbb{R}^m \text{ at each point of } \mathbb{R}^m. \end{cases}$$

By a classical theorem of Hörmander [**Hor68**] the operator in (5.3) and (5.4) is hypoelliptic under this assumption.

5.1. Ergodicity and stabilization

THEOREM 5.1. *Assume* (A), (5.1), *and* (5.5). *Then*
(i) H *is ergodic at* $\overline{x}$, *and for all* $\overline{p}, \overline{X}$ *there exists a Hölder continuous corrector* χ *solving* (5.4) *with* $\lambda = \overline{H}(\overline{x}, \overline{p}, \overline{X})$;
(ii) for every continuous h *the pair* (H, h) *is stabilizing at* $\overline{x}$;
(iii) there is a unique invariant measure $\mu_{\overline{x}}$ *with density* $\varphi(\overline{x}, \cdot) \in C^\infty$ *such that*

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \int_{[0,1)^m} H_1(\overline{x}, y, \overline{p}, \overline{X})\, \varphi(\overline{x}, y)\, dy, \quad \overline{h}(\overline{x}) = \int_{[0,1)^m} h(\overline{x}, y)\, \varphi(\overline{x}, y)\, dy,$$

with H_1 *defined by* (3.4).

PROOF *(i)* The structure of the proof is the same as that of Proposition 4.1, so we only explain the changes. In order to prove the uniform Hölder continuity of $\{w_\delta - w_\delta(0)\}$ we first mollify H_1. The estimate will depend on the L^∞ norm of H_1 but not on its modulus of continuity. So, by the stability of viscosity solutions, this estimate for smooth H_1 carries through to the general case. Then we will assume H_1 smooth in the sequel. Next we observe that the viscosity solution of (5.3) is also a distribution solution: it is enough to add a small viscosity term to the equation, so that the solution is smooth, use the uniform L^∞ bound on the solutions to take the vanishing viscosity limit, and observe it is a solution in both senses and coincides with w_δ by the uniqueness for (5.3). Now we can use the hypoellipticity of the linear equation (5.3) to conclude that $w_\delta \in C^\infty(\mathbb{R}^m)$. We proceed as in the proof of Proposition 4.1 to consider $\psi_k := \varepsilon_k(w_{\delta_k} - w_{\delta_k}(0))$. We denote with X_i^* the formal adjoint operator of X_i and we set

$$\mathcal{L}\psi := \frac{1}{2}\sum_{i=1}^r X_i^2\psi + \sum_{i=1}^r \zeta_i X_i \psi = -\frac{1}{2}X_i^* X_i \psi - \left(\frac{1}{2}\frac{\partial \tau_{ji}}{\partial y_j} - \zeta_i\right) X_i \psi.$$

Then ψ_k is a classical solution of

$$\mathcal{L}\psi_k = \varphi_k \quad \text{in } \mathbb{R}^m, \quad \varphi_k \in C^\infty(\mathbb{R}^m), \quad \|\varphi_k\|_{L^\infty} \le C\varepsilon_k \le C.$$

The subellittic regularity theory, e.g., Theorem 17 on p.167 of [**Xu90**], gives the existence of C' and $\overline{\delta} > 0$ independent of k such that

$$\|\psi_k\|_{C^{\overline{\delta}}} \le C'(\|\psi_k\|_{L^\infty} + 1) \le 2C'.$$

This equi-Hölder estimate allows to get the equiboundedness of $\{w_\delta - w_\delta(0)\}$ as in in the proof of Proposition 4.1. Next we use that

$$\mathcal{L}(w_\delta - w_\delta(0)) = \varphi_\delta, \quad \text{in } \mathbb{R}^m, \quad \varphi_\delta \in C^\infty(\mathbb{R}^m), \quad \|\varphi_\delta\|_{L^\infty} \le C$$

and Theorem 17 of [**Xu90**] again to deduce that $\{w_\delta - w_\delta(0)\}$ is uniformly bounded in $C^{\overline{\delta}}(\mathbb{R}^m)$. From now on the proof is exactly the same as in Proposition 4.1: the strong maximum principles for subsolutions of $\mathcal{L}\overline{w} = 0$ is in Bony's seminal paper [**Bon69**] for classical solutions and in [**BL01**] for viscosity subsolutions.

(ii) We follow closely the proof of Theorem 4.2 and only indicate the changes. By the assumption (5.5) the equation

$$w_t - \frac{1}{2}\sum_{i=1}^r X_i^2 w - \sum_{i=1}^r \zeta_i X_i w = 0$$

is hypoelliptic in $\mathbb{R}^{m+1}$. Since w is also a distribution solution of this equation we get that $w \in C^\infty((0, +\infty) \times \mathbb{R}^m)$. Arguing as in the proof of Theorem 4.2 we obtain, for h smooth, $|w_t| \le C$. Then, for each fixed $t > 0$, $w(t, \cdot)$ satisfies $\mathcal{L}w = w_t$ whose right hand side is C^∞ and bounded. Thus the subelliptic regularity Theorem 17 of [**Xu90**] shows that the family $\{w(t, \cdot)\}$ is uniformly bounded in $C^{\overline{\delta}}(\mathbb{R}^m)$ for a $\overline{\delta} > 0$ independent of t. The rest of the proof follows closely that of Theorem 4.2. We only note that the strong maximum principles for subsolutions of $\partial_t \widetilde{w} - \mathcal{L}\widetilde{w} = 0$ are in [**Bon69**] for classical solutions and in [**DaL04**] for viscosity subsolutions.

(iii) The existence and uniqueness of an invariant measure μ follows from Proposition 3.1. Moreover, as recalled in Chapter 3, μ solves $\mathcal{L}^*\mu = 0$ in the sense of distributions. Therefore, by Hörmander hypoellipticity theorem, it has a C^∞ density. □

Remark The ergodicity of H stated in this theorem remains true for more general drifts $\widetilde{g}$ that are not linear combinations of the vector fields τ^i. This follows from the probabilistic arguments of Ichihara and Kunita, Proposition 6.1 in [**IK74**], and by the characterization of ergodicity via the uniqueness of the invariant measure, Proposition 3.1. Our analytic proof works if one has the uniform Hölder estimates, and it provides also a Hölder corrector.

On the other hand, it is interesting to note that the notion of ergodicity of this paper is false for a general hypoelliptic operator, that is, for any $\widetilde{g}$ of class C^∞ such that $Y := \widetilde{g} \cdot \nabla$, $X_1, \dots, X_r$, and their commutators up to a certain fixed order $\overline{r}$ span $\mathbb{R}^m$ at each point of $\mathbb{R}^m$, if (5.5) does not hold. The following counterexample is adapted from [**IK74**]. In $\mathbb{R}$ we take $\tau(y) = \sin 4\pi y$ and $\widetilde{g}(y) = \cos 4\pi y$. Their Lie bracket is the constant 4π, so the operator $\frac{1}{2}X^2 + Y$ is hypoelliptic in $\mathbb{R}$. However, both intervals $[0, 1/4]$ and $[1/2, 3/4]$ are invariant for the diffusion $y_s = gds + \tau dW_s$, because τ vanishes at their extrema and $\widetilde{g}(0) = \widetilde{g}(1/2) = 1$, $\widetilde{g}(1/4) = \widetilde{g}(3/4) = -1$. Then, if we take a running cost $L \equiv 0$ in $[0, 1/4]$ and $L \equiv 1$ in $[1/2, 3/4]$, we get $w_\delta \equiv 0$ in $[0, 1/4]$ and $w_\delta \equiv 1/\delta$ in $[1/2, 3/4]$, so the limit of δw_δ is not a constant in $[0, 1]$.

Further references on the ergodicity of hypoelliptic diffusions are [**IK77**] and [**AK87**].

Remark A simple example of degenerate operator satisfying the assumptions of Theorem 5.1 is obtained by taking $m = 2$, $\tau^1 = (0, 1)$, $\tau^2 = (\cos 2\pi y_2, \sin 2\pi y_2)$. In fact, the matrix $b = \tau\tau^T/2$ degenerates at $y_2 = \pi/4, 3\pi/4$, but the Lie bracket

$$[\tau^1, \tau^2] = 2\pi(-\sin 2\pi y_2, \cos 2\pi y_2)$$

has nonvanishing first component at such points.

The most classical hypoelliptic example is the Heisenberg operator, whose coefficients are not 1-periodic. Homogenization problems for this operator were studied on a suitable periodic pavage associated to the Heisenberg group, see [**BMT96**]. We believe our methods can be adapted to that context.

5.2. Uniform convergence

The first result follows immediately from Theorem 2.10 and the special form of the effective Hamiltonian in the case of uncontrolled fast variables, see Section 3.3.

COROLLARY 5.2. *Assume that the differential game satisfies the usual assumptions* (A) *with* $g = g(y)$ *and* $\tau = \tau(y)$ *independent of* x *and of the controls, of class* C^∞*, and satisfying* (5.1) *and* (5.5).

Then the conclusions of Theorem 3.5 hold with $d\mu(y) = \varphi(y)dy$*, where* $\varphi \in C^\infty(\mathbb{R}^m)$ *is the density of the invariant measure* μ *of the hypoelliptic generator of the process* (5.2).

The second result corresponds to the case when the invariant measure μ_x of the fast subsystem is the Lebesgue measure for all x. The special form of the effective Hamiltonian derived in Section 3.3. allows to check immediately that it satisfies the structure condition of the comparison principle and therefore to apply Theorem 2.9.

COROLLARY 5.3. *Assume that the differential game satisfies the usual assumptions* (A) *with uncontrolled* $g = g(x, y)$ *and* $\tau = \tau(x, y)$ *of class* C^∞ *in* y *and satisfying* (5.1) *and* (5.5). *Suppose also that*

$$\frac{\partial}{\partial y_k}\left[\tau_{ki}\left(\zeta_i - \frac{1}{2}\frac{\partial \tau_{ji}}{\partial y_j}\right)\right] = 0 \qquad \forall\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^m.$$

Then the value functions u^ε *converge uniformly on the compact subsets* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of*

$$\begin{cases} u_t + \overline{H}(x, D_x u, D^2_{xx} u) = 0, & \text{in } (0, +\infty) \times \mathbb{R}^n \\ u(x, 0) = \int_{(0,1)^m} h(x, y)\, dy, \end{cases} \tag{5.6}$$

where

$$\overline{H}(x, p, X) = \int_{[0,1)^m} \min_{\beta \in B} \max_{\alpha \in A} \{-X \cdot a(x, y, \alpha, \beta) - p \cdot f(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\}\, dy.$$

CHAPTER 6

Controllable fast variables

6.1. Bounded-time controllability and ergodicity

In this section we show that a sufficient condition for ergodicity is a property that we call *bounded-time (complete) controllability*, where the controllability refers to the first player α for all possible behaviors of the opponent β, and it means that he can drive the system to any given state. It extends to games and to controlled diffusions on the flat torus the classical complete controllability of deterministic systems with a single player, see [**CK00**] and the references therein. The bounded-time complete controllability is also known in the literature with different names, such as *uniform exact controllability* [**Ari98**] and *total controllability* [**AG00**].

We begin with the deterministic system

$$\dot{y}_s = g(x, y_s, \alpha[\beta]_s, \beta_s) \quad y_0 = y. \tag{6.1}$$

We say that the system (6.1) is *bounded-time controllable for* $x = \overline{x}$ (by the first player) if, for some $S > 0$ depending only on $\overline{x}$, and for all $y, \widetilde{y} \in \mathbb{R}^m$, there is a strategy $\widetilde{\alpha} \in \Gamma$ such that for all control functions $\beta \in \mathcal{B}$

$$\exists\, t^{\#} = t^{\#}(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \beta) \le S \text{ such that } y_{t^{\#}} - \widetilde{y} \in \mathbb{Z}^m, \tag{6.2}$$

where $y_.$ is the trajectory of (6.1) with $x = \overline{x}$ and $\alpha = \widetilde{\alpha}$.

THEOREM 6.1. *If the system* (6.1) *is bounded-time controllable for* $x = \overline{x}$*, then the Isaacs Hamiltonian*

$$H(x, y, p, q) = \min_{\beta \in B} \max_{\alpha \in A} \{-q \cdot g(x, y, \alpha, \beta) - p \cdot f(x, y, \alpha, \beta) - L(x, y, \alpha, \beta)\}, \tag{6.3}$$

is ergodic at $\overline{x}$*.*

PROOF For fixed $y, \widetilde{y}$ we define $t^{\#} = t^{\#}(\overline{x}, y, \widetilde{y}, \alpha, \beta)$ as the minimum t such that $y_t - \widetilde{y} \in \mathbb{Z}^m$, if this ever occurs, $+\infty$ otherwise. Since it is a nonanticipating functional we can use the Dynamic Programming Principle Theorem 3.1 in [**EI84**] to get, for all $T > 0$,

$$w_\delta(y) = \inf_{\alpha \in \Gamma} \sup_{\beta \in \mathcal{B}} \left\{ \int_0^{t^{\#} \wedge T} L(y_s, \alpha[\beta]_s, \beta_s) e^{-\delta s}\, ds + e^{-\delta(t^{\#} \wedge T)} w_\delta(y_{t^{\#} \wedge T}) \right\}.$$

By (6.2) there is $\widetilde{\alpha} \in \Gamma$ such that, for $t^{\#} = t^{\#}(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \beta)$

$$w_\delta(y) \le \sup_{\beta \in \mathcal{B}} \left\{ \int_0^{t^{\#}} L(y_s, \widetilde{\alpha}[\beta]_s, \beta_s) e^{-\delta s}\, ds + e^{-\delta t^{\#}} w_\delta(\widetilde{y}) \right\},$$

where $y_.$ is the trajectory of (6.1) with $\alpha = \widetilde{\alpha}$. Since L and δw_δ are uniformly bounded there is a constant C such that

$$\delta w_\delta(y) - \delta w_\delta(\widetilde{y}) \le C(1 - e^{-\delta S}). \tag{6.4}$$

Now we exchange the roles of y and $\widetilde{y}$ to get

$$\lim_{\delta\to 0+} |\delta w_\delta(y) - \delta w_\delta(\widetilde{y})| = 0 \quad \text{uniformly in } y, \widetilde{y} \in \mathbb{R}^m.$$

If for fixed $\widetilde{y}$ we choose a sequence $\delta_k \to 0$ such that $\delta_k w_{\delta_k}(\widetilde{y}) \to \mu$, we obtain the uniform convergence of $\delta_k w_{\delta_k}$ to μ.

We claim that μ is independent of the sequence δ_k. This implies the uniform convergence of the whole net δw_δ to μ, as desired. To prove the claim we recall the true cell problem (2.9), which in the current case is

$$\lambda + H(\overline{x}, y, \overline{p}, D\chi) = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic,} \tag{6.5}$$

where λ is a constant and H is given by (6.3). We use the inequality

$$\lambda_1 := \sup\{\lambda \mid \exists \text{ a u.s.c. subsolution of (6.5)}\}$$
$$\leq \lambda_2 := \inf\{\lambda \mid \exists \text{ a l.s.c. supersolution of (6.5)}\},$$

which follows from a standard argument based on the comparison principle (see, e.g., the proof of Theorem 1 in [**Ari98**] or that of Theorem 4 in [**AB03**]). From the equation (CP$_\delta$) satisfied by w_δ, i.e.,

$$\delta w_\delta + H(\overline{x}, y, \overline{p}, Dw_\delta) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic,}$$

we see that, for $\lambda < \mu$, $v = w_{\delta_k}$ is a subsolution of (6.5) for k large enough. Then $\mu \leq \lambda_1$. The same argument gives $\lambda_2 \leq \mu$. Therefore $\mu = \lambda_1 = \lambda_2$, which proves the claim. □

Under a stronger controllability assumption we obtain also the existence of a corrector, i.e., a continuous solution χ of the true cell problem (2.9), which in this case is the first order equation in $\mathbb{R}^m$

$$\lambda + \min_{\beta\in B}\max_{\alpha\in A} \{-D\chi \cdot g(\overline{x}, y, \alpha, \beta) - \overline{p}\cdot f(\overline{x}, y, \alpha, \beta) - l(\overline{x}, y, \alpha, \beta)\} = 0, \ \chi \text{ periodic.} \tag{6.6}$$

We say that the system (6.1) is *small-time controllable for* $x = \overline{x}$ (by the first player) if it is bounded-time controllable and there exists a modulus ω and a constant $\gamma > 0$ such that the time $t^\#$ defined in (6.2) verifies

$$t^\#(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \beta) \leq \omega(|y - \widetilde{y}|) \qquad \text{for all } |y - \widetilde{y}| \leq \gamma \text{ and all } \beta \in \mathcal{B}. \tag{6.7}$$

Note that, if there is no second player (B singleton), (6.7) reduces to the classical notion of *small-time local controllability* at every point of the state space for deterministic control systems.

Proposition 6.2. *Assume the system* (6.1) *is small-time controllable for* $x = \overline{x}$. *Then for all* $\overline{p} \in \mathbb{R}^n$ *there exists a corrector* $\chi \in C(\mathbb{R}^m)$ *solving* (6.6) *with* $\lambda = -\overline{H}(\overline{x}, \overline{p})$, *and for a constant* C

$$|\chi(y) - \chi(\widetilde{y})| \leq C\omega(|y - \widetilde{y}|) \qquad \forall y, \widetilde{y},$$

where ω *is the modulus appearing in* (6.7).

Proof We follow the proof of Theorem 6.1 and we use (6.7) to improve (6.4) to

$$w_\delta(y) - w_\delta(\widetilde{y}) \leq C\frac{1 - e^{-\delta t^\#}}{\delta} \leq Ct^\# \leq C\omega(|y - \widetilde{y}|).$$

Then the net $\{w_\delta - w_\delta(0)\}$ is equicontinuous and, by the compactness of the flat torus, also equibounded. We extract a sequence $\delta_k \to 0$ such that $w_{\delta_k} - w_{\delta_k}(0) \to$

χ uniformly. The equation (CP_δ) satisfied by w_δ and the stability of viscosity solutions imply that χ satisfies the true cell problem (6.6) with $\lambda = \overline{H}(\overline{x}, \overline{p})$. □

Next we give some examples of controllable systems.

Example 1: first order controllability and coercivity of H. Assume that for some $\nu > 0$

$$B(0, \nu) \subset \overline{\mathrm{conv}}\{g(\overline{x}, y, \alpha, \beta) \mid \alpha \in A\}, \quad \forall y \in \mathbb{R}^m, \beta \in B, \tag{6.8}$$

where $B(0, \nu) \subset \mathbb{R}^m$ denotes the open ball of radius ν centered at the origin. From the theory of deterministic differential games (see, for instance, Corollary 3.7 in [**Sor93**]) it is known that the system is small-time controllable and the time necessary to reach a point $\widetilde{y}$ from y satisfies an estimate of the form

$$t^{\#}(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \beta) \leq \frac{C}{\nu}|y - \widetilde{y}|.$$

Therefore Theorem 6.1 and Proposition 6.2 imply the ergodicity of H and the existence of a Lipschitz continuous corrector.

Note that the assumption (6.8) is equivalent to the *coercivity of the Hamiltonian* given by (6.3) with respect to $D_y u$, that is, the inequality

$$H(\overline{x}, y, \overline{p}, q) \geq \nu|q| - C(1 + |\overline{p}|), \quad \forall\, y, q, \tag{6.9}$$

for some constant C. The ergodicity of H under this coercivity assumption goes back to the pioneering papers on the subject [**LPV86, Eva92**].

Example 2: higher order controllability. Here we assume that the system (6.1) for $x = \overline{x}$ is also independent of the second player and symmetric, that is, it takes the form

$$\dot{y}_s = g(\overline{x}, y_s, \alpha_s) = \sum_{i=1}^{k} \alpha_s^i g^i(y_s), \tag{6.10}$$

where the control $\alpha = (\alpha^1, \ldots, \alpha^k)$ varies in a neighborhood of the origin $A \subset \mathbb{R}^k$, and each g^i is a C^∞ vector field in $\mathbb{R}^m$. Moreover, we suppose that

$$\begin{cases} \text{the vector fields } g^1, \ldots, g^k \\ \text{and their commutators of any order} \\ \text{span } \mathbb{R}^m \text{ at each point of } \mathbb{R}^m. \end{cases} \tag{6.11}$$

By the classical Chow's theorem of geometric control theory the system (6.10) is small-time locally controllable at all points of the state space. It also known that that the modulus ω in (6.7) is $\omega(s) = Cs^{1/(\overline{r}+1)}$ if $\overline{r}$ is a uniform bound to the number of bracket operations necessary to generate the whole space (see, e.g., [**BCD97**] and the references therein). Moreover, for any small $t > 0$ the reachable set from y in time t is a neighborhood of y, and the same holds for the reachable set backward in time. From this, using the compactness of the flat torus, it is easy to see that the whole state space is an invariant control set in the terminology of [**CK00**]. Then the global bounded-time controllability follows from Lemma 3.2.21 in [**CK00**]. In conclusion, H is ergodic with a Hölder continuous corrector.

The same properties hold if the system depends on the second player via an additional term that can be killed by the first, that is, g of the form

$$g(\overline{x}, y, \alpha, \beta) = \sum_{i=1}^{k} \alpha^i g^i(y) + g^{k+1}(y, \alpha, \beta)$$

with $g^1, \dots, g^k$ as above, $\alpha = (\alpha^1, \dots, \alpha^k, \alpha^{k+1}, \dots, \alpha^{k+j})$ with $(\alpha^1, \dots, \alpha^k)$ varying in a neighborhood of the origin in $\mathbb{R}^k$, and

$$\forall\, y \in \mathbb{R}^m,\ \beta \in B,\ \alpha^1, ..., \alpha^k,\ \exists\, \alpha_*^{k+1}, ..., \alpha_*^{k+j} \text{ such that } g^{k+1}(y, \alpha^1, ..., \alpha^k, \alpha_*^{k+1}, ..., \alpha_*^{k+j}, \beta) = 0.$$

We refer to Grüne [**Gru98**] and the Ph.D. thesis of Arisawa (see [**Ari97, Ari98**]) for further results on vanishing discount limits for deterministic systems with a single controller (g independent of β) in connection with controllability properties.

Next we suppose there is a diffusion term but no second player β. So we have the stochastic system with a single controller

$$dy_s = g(x, y_s, \alpha_s)ds + \tau(x, y_s, \alpha_s)dW_s, \quad y_0 = y. \tag{6.12}$$

For given $y, \widetilde{y} \in \mathbb{R}^m$ we call $t^\#(x, y, \widetilde{y}, \alpha, \omega)$ the minimum t such that $y_t - \widetilde{y} \in \mathbb{Z}^m$, if this ever occurs, $+\infty$ otherwise. We say that (6.12) is *bounded time controllable for $x = \overline{x}$* if for some $S > 0$ depending only on $\overline{x}$, and all $y, \widetilde{y} \in \mathbb{R}^m$, there is a control function $\widetilde{\alpha} \in \mathcal{A}$ whose trajectory satisfies $\widetilde{t} := t^\#(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \omega) \le S$ almost surely.

PROPOSITION 6.3. *If the system* (6.12) *is bounded time controllable for $x = \overline{x}$, then the HJB Hamiltonian* (1.13) *is ergodic at $\overline{x}$. Assume, in addition, there exist a modulus μ and $\gamma > 0$ such that for all $y, \widetilde{y} \in \mathbb{R}^m$, $|y - \widetilde{y}| \le \gamma$, there is $\widetilde{\alpha} \in \mathcal{A}$ satisfying*

$$t^\#(\overline{x}, y, \widetilde{y}, \widetilde{\alpha}, \omega) \le \mu(|y - \widetilde{y}|) \quad \textit{for almost every } \omega. \tag{6.13}$$

Then, for all $\overline{p} \in \mathbb{R}^n, \overline{X} \in \mathbb{S}^n$, there exists a periodic corrector $\chi \in C(\mathbb{R}^m)$ solving

$$\lambda + \max_{\alpha \in A} \left\{ -D^2\chi \cdot \frac{\tau\tau^T}{2}(\overline{x}, y, \alpha) - D\chi \cdot g(\overline{x}, y, \alpha) - L(y, \alpha) \right\} = 0, \quad \textit{in } \mathbb{R}^m, \tag{6.14}$$

where

$$\lambda = -\overline{H}(\overline{x}, \overline{p}.\overline{X}), \qquad L(y, \alpha) := \overline{X} \cdot a(\overline{x}, y, \alpha) + \overline{p} \cdot f(\overline{x}, y, \alpha) + l(\overline{x}, y, \alpha),$$

and χ satisfies $|\chi(y) - \chi(\widetilde{y})| \le C\mu(|y - \widetilde{y}|)$ for some C and all $y, \widetilde{y}$.

PROOF We fix y and $\widetilde{y}$. Since $t^\#$ is a stopping time we can use the Dynamic Programming Principle (see, e.g., [**FS06**]) to get, for all $T > 0$,

$$w_\delta(y) = \inf_{\alpha \in \mathcal{A}} E_y \left[\int_0^{t^\# \wedge T} L(y_s, \alpha_s)e^{-\delta s}\, ds + e^{-\delta(t^\# \wedge T)} w_\delta(y_{t^\# \wedge T}) \right].$$

By the periodicity of w_δ and the controllability assumption

$$w_\delta(y) - w_\delta(\widetilde{y}) \le E_y \left[\int_0^{\widetilde{t}} L(y_s, \widetilde{\alpha}_s)e^{-\delta s}\, ds + (e^{-\delta\widetilde{t}} - 1)w_\delta(\widetilde{y}) \right],$$

where $\widetilde{t} = t^{\#}(\widetilde{\alpha})$. The uniform boundedness of $\widetilde{t}$, L, and δw_δ gives

$$\delta w_\delta(y) - \delta w_\delta(\widetilde{y}) \leq C(1 - e^{-\delta S}),$$

and from now on the proof of the ergodicity is exactly the same as that of Theorem 6.1, with the Hamiltonian

$$H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) = \max_{\alpha \in A} \left\{ -Y \cdot \frac{\tau\tau^T}{2}(\overline{x}, y, \alpha) - q \cdot g(\overline{x}, y, \alpha) - L(y, \alpha) \right\}$$

in the true cell problem (2.9). The proof of the existence of a continuous corrector under the assumption (6.13) is the same as the proof of Proposition 6.2. □

Example 3: controllability of a deterministic subsystem. Suppose that the controlled diffusion (6.12) has a deterministic subsystem, i.e.,

$$\exists\, A' \subset A \text{ such that } \tau(\overline{x}, y, \alpha) = 0 \quad \forall\ \alpha \in A',\ y \in \mathbb{R}^m,$$

and that the subsystem

$$\dot{y}_s = g(\overline{x}, y_s, \alpha_s) \quad y_0 = y, \quad \alpha_s \in A'\ \forall s,$$

is bounded time controllable. Then the stochastic system (6.12) is bounded time controllable and the associated Hamiltonian is ergodic. If, for instance, this subsystem can be written in the form (6.10) with α taking values in a neighborhood A' of the origin in $\mathbb{R}^k$, then Proposition 6.3 applies and there exists a Hölder continuous corrector.

The last result of this section is about general second order non-convex Hamiltonians satisfying a *coercivity* condition with respect to $D_y u$, namely,

$$H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) \geq \nu|q| - C(1 + |\overline{p}| + |\overline{X}|), \quad \forall\ y, q, Y. \tag{6.15}$$

for some constants $\nu > 0$ and C. The proof uses only PDE methods and it is a natural extension of results in [**LPV86, AL98**]; the result was also announced in Proposition 9 of [**AB03**].

PROPOSITION 6.4. *If the Hamiltonian* (1.12) *satisfies* (6.15)*, then it is ergodic at* $(\overline{x}, \overline{p}, \overline{X})$*, and there exists a Lipschitz corrector* χ *solving the true cell problem* (2.9) *with* $\lambda = -\overline{H}(\overline{x}, \overline{p}, \overline{X})$*. Moreover, there is a constant* K *such that*

$$|D_y \chi| \leq K(1 + |\overline{p}| + |\overline{X}|), \quad \textit{for a.e. } y. \tag{6.16}$$

PROOF It is enough to modify the proof of Theorem 4.1 at the points where the regularity theory is used. Observe that if H satisfies (6.15) also its recession function H' is coercive, that is, for some $C > 0$

$$H'(\overline{x}, y, q, Y) \geq \nu|q| - C, \quad \forall\ y, q, Y.$$

We follow the proof of Theorem 4.1 until (4.5). Then the coercivity of H' implies that $|D\psi_k| \leq K'$ for all k in viscosity sense, thus the family $\{\psi_k\}$ is equi-Lipschitz continuous. This allows to continue and arrive at (4.6). Here we use (6.15) to get

$$\nu|D_y w_\delta| \leq C'(1 + |\overline{p}| + |\overline{X}|), \tag{6.17}$$

so the family $\{w_\delta\}$ is uniformly Lipschitz continuous in $\mathbb{R}^m$. Then we can extract a sequence $\delta_k \to 0$ such that $\delta_k w_{\delta_k}$ converges uniformly to a constant λ and $w_{\delta_k} - w_{\delta_k}(0)$ converges to $\chi \in C^{0,1}_{\text{per}}(\mathbb{R}^m)$ uniformly. By the stability of viscosity solutions λ and χ solve the true cell problem (2.9). The estimate (6.16) follows from (6.17). □

Example 4: stochastic games with coercive H. The coercivity assumption (6.15) on H correspond to stochastic differential games having a deterministic subsystem first-order controllable by the minimizing player. The precise condition is

$$\begin{cases} \exists\, A' \subset A \text{ such that } \tau(\overline{x}, y, \alpha, \beta) = 0 \;\forall \alpha \in A' \text{ and} \\ B(0,\nu) \subset \overline{\text{conv}}\{g(\overline{x}, y, \alpha, \beta) \mid \alpha \in A'\} \;\forall y \in \mathbb{R}^m, \beta \in B. \end{cases} \tag{6.18}$$

Remark The definition of bounded-time controllability can be extended from deterministic to stochastic games by requiring that (6.2) is satisfied almost surely. The proof that it implies the ergodicity relies on a suitable Dynamic Programming Principle due to Swiech [**Swi96**], see [**AB07**]. Note that in the case of a single player this condition is stronger than the one we gave for (6.12). However, it is satisfied if there is a deterministic subsystem independent of the second player and bounded-time controllable (e.g., small-time controllable as in Example 2).

6.2. Stabilization and a formula for the effective initial data

In this section we give a sufficient condition for the property of stabilization of the pair (H, h) that is a slight strengthening of the bounded-time controllability of the fast subsystem (2.4). We call it *stoppability* by the first player, and it also implies a simple explicit formula for the effective terminal cost $\overline{h}$.

The deterministic system (6.1) is called *stoppable* for $x = \overline{x}$ (by the first player) if

$$\forall\, \widetilde{y} \in \mathbb{R}^m,\; \beta \in B,\; \exists\, \alpha_{\beta,\widetilde{y}} \in A \text{ such that } g(\overline{x}, \widetilde{y}, \alpha_{\beta,\widetilde{y}}, \beta) = 0, \tag{6.19}$$

which implies the property $H'(\overline{x}, y, q) \geq 0$ for all y and q, where H' is the homogeneous part of the Hamiltonian H. Note that both Examples 1 and 2 of Section 6.1 satisfy these properties.

Similarly, the controlled diffusion (6.12) is called *stoppable* for $x = \overline{x}$ if

$$\forall\, \widetilde{y} \in \mathbb{R}^m\; \exists\, \alpha_{\widetilde{y}} \in A \text{ such that } g(\overline{x}, \widetilde{y}, \alpha_{\widetilde{y}}) = 0 \text{ and } \tau(\overline{x}, \widetilde{y}, \alpha_{\widetilde{y}}) = 0, \tag{6.20}$$

so that the homogeneous part of H satisfies

$$H'(\overline{x}, y, q, Y) \geq 0, \quad \forall\, y, q, Y.$$

Example 3 of Section 6.1 satisfies this property if the controllable deterministic subsystem is of the form of the Examples 1 or 2.

In the general case of the stochastic system with two players (2.4), we will assume the coercivity of the Hamiltonian (6.15). No extra condition is needed in this case. In fact, the proof requires only the coercivity of the homogeneous part of H, namely

$$H'(\overline{x}, y, q, Y) \geq \nu |q|, \quad \forall\, y, q, Y, \tag{6.21}$$

which is easily seen to follow from (6.15) or from (6.18).

Proposition 6.5. *Assume that, for $x = \overline{x}$, at least one of the following condition holds*

- *the system is deterministic, i.e., of the form* (6.1)*, bounded-time controllable, and stoppable;*
- *the system has only one player, i.e., of the form* (6.12)*, bounded-time controllable, and stoppable;*

– *the system* (2.4) *satisfies* (6.18) *, i.e., the Hamiltonian is coercive and* (6.21) *holds.*

Then, for any $h \in BUC(\mathbb{R}^{n+m})$*, the pair* (H, h) *is stabilizing at* $\overline{x}$ *and*

$$\overline{h}(\overline{x}) = \min_y h(\overline{x}, y).$$

PROOF We begin with the deterministic case. The solution w of (CP′) is

$$w(t, y; \overline{x}) = \inf_{\alpha\in\Gamma} \sup_{\beta\in\mathcal{B}} h(\overline{x}, y_t) \geq \min_y h(\overline{x}, y),$$

where y_t is the trajectory of (6.1). We are going to prove that the equality holds for t large enough, so we will get the conclusions from the definitions (2.13) and (2.14). We fix $\widetilde{y} \in \mathrm{argmin}_y h(\overline{x}, y)$ and y and use the bounded-time controllability to choose a strategy $\widetilde{\alpha}$ steering the system to $\widetilde{y}$ for any disturbance β in a finite time $t^{\#}$. Since $t^{\#}$ is a nonanticipating functional, we can construct a nonanticipating strategy α^* by setting

$$\alpha^*[b](s) = \widetilde{\alpha}[b](s) \text{ if } s \leq t^{\#}, \quad \alpha^*[b](s) = \alpha_{b(s),\widetilde{y}} \text{ if } s > t^{\#},$$

where $\alpha_{b,\widetilde{y}}$ is given by the stoppability condition (6.19). Since $t^{\#} \leq S$ we obtain, for the trajectories $y^*_\cdot$ corresponding to α^*,

$$w(t, y; \overline{x}) \leq \sup_{\beta\in\mathcal{B}} h(\overline{x}, y^*_t) = h(\overline{x}, \widetilde{y}) = \min_y h(\overline{x}) \quad \text{for all } t \geq S,$$

which completes the proof for this case.

In the stochastic single-player case the proof is similar. The solution of (CP′) is

$$w(t, y; \overline{x}) = \inf_{\alpha\in\mathcal{A}} E_y h(\overline{x}, y_t) \geq \min_y h(\overline{x}, y),$$

where y_t is the trajectory of (6.12). Since $t^{\#}$ is a stopping time we can construct a control function $\alpha^* \in \mathcal{A}$ by taking

$$\alpha^*(s) = \widetilde{\alpha}(s) \text{ if } s \leq t^{\#}, \quad \alpha^*(s) = \alpha_{\widetilde{y}} \text{ if } s > t^{\#}.$$

The trajectory $y^*_\cdot$ corresponding to α^* satisfies $y^*_t = \widetilde{y}$ for all $t > t^{\#}$ a.s., so

$$w(t, y; \overline{x}) \leq E_y h(\overline{x}, y^*_t) = h(\overline{x}, \widetilde{y}) = \min_y h(\overline{x}) \quad \text{for all } t \geq S,$$

which completes the proof for this case.

The proof in the coercive case uses purely PDE methods and is given in [**AB03**], Proposition 10. □

6.3. An explicit formula for the effective Hamiltonian and the limit differential game

In this section we derive a formula for $\overline{H}$ under an additional assumption on the Hamiltonian. We also derive from it an *effective differential game* describing the limit of the singular perturbation problem. Define, as in Chapter 3,

$$\begin{aligned} H_1(x, y, p, X) &:= H(x, y, p, 0, X, 0, 0) \\ &= \min_{\beta\in B} \max_{\alpha\in A} \{-X \cdot a(x, y, \alpha, \beta) - p \cdot f(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\} \end{aligned}$$

and

$$H_2(x, y, p, q, X, Y) := H(x, y, p, q, X, Y, 0) - H_1(x, y, p, X).$$

PROPOSITION 6.6. *Assume the HJBI Hamiltonian* (1.12) *is ergodic at* $\overline{x}$. *Then*

$$\min_y H_1(\overline{x}, y, \overline{p}, \overline{X}) \le \overline{H}(\overline{x}, \overline{p}, \overline{X}) \le \max_y H_1(\overline{x}, y, \overline{p}, \overline{X}) \quad \forall\ \overline{p}, \overline{X}. \tag{6.22}$$

If, moreover,

$$H_2(\overline{x}, y, \overline{p}, q, \overline{X}, Y) \ge 0 \qquad \text{for all } y, q, Y, \tag{6.23}$$

then

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \max_y H_1(\overline{x}, y, \overline{p}, \overline{X}) \quad \text{for all } \overline{x}, \overline{p}, \overline{X}. \tag{6.24}$$

PROOF We prove the inequalities (6.22) by means of the formula (2.8), where w satisfies the cell t-problem

$$w_t + H(\overline{x}, y, \overline{p}, D_y w, \overline{X}, D^2_{yy} w, 0) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \quad w(0, y) = 0, \quad w \text{ periodic.}$$

We omit the frozen entries of H_1 and write $H_1(y) := H(\overline{x}, y, \overline{p}, 0, \overline{X}, 0, 0)$. Observe that $-t \max_y H_1$ and $-t \min_y H_1$ are, respectively, a sub- and a supersolution of this Cauchy problem. Therefore the comparison principle gives

$$-t \max_y H_1 \le w(t, y) \le -t \min_y H_1.$$

We divide by t and let $t \to +\infty$. Since $w(t, y)/t \to -\overline{H}$ we obtain (6.22).

To prove the second statement we omit the frozen arguments $\overline{x}, \overline{p}, \overline{X}$ also in $\overline{H}$ and H_2 and assume by contradiction that $\overline{H} < H_1(y)$ in a neighborhood of a maximum point of H_1. By the cell δ-problem (CP$_\delta$) and the uniform convergence of δw_δ to $-\overline{H}$ we get

$$H_2(y, Dw_\delta, D^2 w_\delta) = \overline{H} - H_1(y) + o(1) < 0 \qquad \text{as } \delta \to 0$$

in an open set. This is a contradiction with the assumption $H_2 \ge 0$. □

If the slow subsystem and the running cost do not depend on the second player, the effective Hamiltonian $\overline{H}$ given by (6.24) takes the form

$$\overline{H}(x, p, X) = \max_{y \in [0,1]^n} \max_{\alpha \in A} \left\{ -X \cdot \frac{\sigma \sigma^T}{2}(x, y, \alpha) - p \cdot f(x, y, \alpha) - l(x, y, \alpha) \right\}.$$

This is the HJB Hamiltonian of a new control problem, where the dynamics is given by the slow subsystem with controls (y, α) taking values in $[0, 1]^n \times A$, and the cost functional is

$$\overline{J}(t, x, y_\cdot, \alpha_\cdot) := E_x \left[\int_0^t l(x_s, y_s, \alpha_s)\, ds + \overline{h}(x_t) \right].$$

Therefore, in this case, provided H is ergodic and stabilizing, we have an explicit optimal control problem as the limit of the singular perturbation problem, that we call the *effective control problem.* It has a very simple interpretation: in the limit the fast state variables become controls of the minimizing player.

In order to give a similar interpretation of the limit when the slow subsystem or the running cost depend on the second player β, we assume the Isaacs-type condition

$$\min_{\beta \in B} \max_{\alpha \in A} \{-X \cdot a - p \cdot f - l\} = \max_{\alpha \in A} \min_{\beta \in B} \{-X \cdot a - p \cdot f - l\}, \ \forall x, y, p, X. \tag{6.25}$$

Then the effective Hamiltonian $\overline{H}$ given by (6.24) takes the form

$$\overline{H}(x,p,X) = \max_{y\in[0,1]^n} \max_{\alpha\in A} \min_{\beta\in B} \left\{ -X \cdot \frac{\sigma\sigma^T}{2}(x,y,\alpha,\beta) - p \cdot f(x,y,\alpha,\beta) - l(x,y,\alpha,\beta) \right\},$$

which is a HJBI Hamiltonian. It corresponds to the upper value of a stochastic differential game with dynamics

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \quad x_o = x, \tag{6.26}$$

where $(y_., \alpha_.)$ are the controls of the first player taking values in $[0,1]^n \times A$, and the cost functional is

$$\overline{J}(t,x,y_.,\alpha_.,\beta_.) := E_x \left[\int_0^t l(x_s,y_s,\alpha_s,\beta_s)\, ds + \overline{h}(x_t) \right], \qquad \overline{h}(x) := \min_{y\in\mathbb{R}^n} h(x,y).$$

We call this the *effective differential game*. The main result of this section states that its (upper) value is in fact the limit of the values u^ε. We denote with $\mathcal{Y}(t)$ the set of admissible controls $[0,t] \to [0,1]^n$ and with $\Delta_e(t)$ the set of nonanticipating strategies $\beta : \mathcal{Y}(t) \times \mathcal{A}(t) \to \mathcal{B}(t)$ of the second player.

Theorem 6.7. *Assume that* (6.23) *and* (6.25) *hold, and the Hamiltonian H is ergodic and stabilizing. Then the lower value functions u^ε converge uniformly on the compact subsets of* $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as $\varepsilon \to 0$ to the upper value function of the game with cost functional $\overline{J}$ for the system* (6.26)*, that is,*

$$\widetilde{u}(t,x) := \sup_{\beta\in\Delta_e(t)} \inf_{(y_.,\alpha_.)\in\mathcal{Y}(t)\times\mathcal{A}(t)} \overline{J}(x,y_.,\alpha_.,\beta[y_.,\alpha_.]).$$

Proof Since $\overline{H}$ is a HJBI Hamiltonian with σ, f, l Lipschitz in x uniformly in y, α, β, the effective Cauchy problem ($\overline{\text{HJ}}$) satisfies the comparison principle (2.18) . Then Theorem 2.9 gives the local uniform convergence of u^ε to the unique solution of ($\overline{\text{HJ}}$). By the results of [**FS89**], this solution is the upper value function of the game described above. □

We end this section with a brief discussion of the assumption (6.23), that we rewrite explicitely

$$\min_{\beta\in B}\max_{\alpha\in A} \{-X\cdot a - Y\cdot b - p\cdot f - q\cdot g - l\} \geq \min_{\beta\in B}\max_{\alpha\in A} \{-X\cdot a - p\cdot f - l\}, \qquad \forall x,y,p,q,X,Y.$$

As the other conditions in the previous Sections 6.1 and 6.2, it concerns the directions the first player can choose for the fast subsystem. However, different from them, it involves also the slow subsystem. The simplest case for a comparison is when the slow subsystem and the running cost l are independent of the controls α and β. Then $H_2 \geq 0$ if and only if $H' \geq 0$, and this is related to the stoppability conditions of Section 6.2 and is weaker that the coercivity of the Hamiltonian (6.15). More general cases are discussed in the next examples.

Example 5: Separated controls. This is an extension to games of an example in [**BB98, AB01**] We assume that the first player uses different components of his control for the slow and for the fast variables. More precisely the controls of the first player are of the form $\alpha = (\alpha^S, \alpha^F) \in A^S \times A^F$, and $f = f(\overline{x}, y, \alpha^S, \beta)$,

$\sigma = \sigma(\overline{x}, y, \alpha^S, \beta)$, whereas $g = g(\overline{x}, y, \alpha^F, \beta)$ and $\tau = \tau(\overline{x}, y, \alpha^F, \beta)$. We also assume that $l = l(\overline{x}, y, \alpha^S, \beta)$ and

$$\min_{\beta\in B} \max_{\alpha^F\in A^F} \left\{-Y \cdot b(\overline{x}, y, \alpha^F, \beta) - q \cdot g(\overline{x}, y, \alpha^F, \beta)\right\} \geq 0 \quad \text{for all } q, Y. \tag{6.27}$$

Note that this is satisfied under the stoppability condition

$\forall\, y \in \mathbb{R}^m$, $\beta \in B$, $\exists\, \alpha_* \in A^F$ such that $g(\overline{x}, y, \alpha_*, \beta) = 0$, and $\tau(\overline{x}, y, \alpha_*, \beta) = 0$.

Under these assumptions $H_2 \geq 0$. In fact

$$H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) = \min_{\beta\in B} \left(\max_{\alpha^S\in A^S} \left\{-\overline{X} \cdot a - \overline{p} \cdot f - l\right\} + \max_{\alpha^F\in A^F} \{-Y \cdot b - q \cdot g\} \right) \geq H_1 + \min_{\beta\in B} \max_{\alpha^F\in A^F} \{-Y \cdot b - q \cdot g\} \geq H_1$$

for all y, q, Y, where $H_1 := H_1(\overline{x}, y, \overline{p}, \overline{X})$. Therefore, if H is ergodic, from Proposition 6.6 we get the formula

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \max_{y\in\mathbb{R}^m} \min_{\beta\in B} \max_{\alpha^S\in A^S} \left\{-\overline{X} \cdot a(\overline{x}, y, \alpha^S, \beta) - \overline{p} \cdot f(\overline{x}, y, \alpha^S, \beta) - l(\overline{x}, y, \alpha^S, \beta)\right\}. \tag{6.28}$$

Example 6: Almost separated controls. We can generalize the preceding example by allowing the fast subsystem to depend also on α^S

$$g = g(\overline{x}, y, \alpha^S, \alpha^F, \beta), \qquad \tau = \tau(\overline{x}, y, \alpha^S, \alpha^F, \beta),$$

provided that α^F is stronger than α^S in the following sense:

$$\forall q,\ Y,\ \beta,\ ,\alpha^S,\ \exists\alpha^F\ :\ -Y \cdot b(\overline{x}, y, \alpha^S, \alpha^F, \beta) - q \cdot g(\overline{x}, y, \alpha^S, \alpha^F, \beta) \geq 0.$$

Then

$$\max_{\alpha^S\in A^S} \left\{-\overline{X} \cdot a - \overline{p} \cdot f - l + \max_{\alpha^F\in A^F} [-Y \cdot b - q \cdot g]\right\} \geq \max_{\alpha^S\in A^S} \left\{-\overline{X} \cdot a - \overline{p} \cdot f - l\right\}.$$

We take the min over $\beta \in B$ to obtain $H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) \geq H_1(\overline{x}, y, \overline{p}, \overline{X})$, so (6.23) is satisfied. If H is ergodic the formula for $\overline{H}$ is again (6.28), as in the preceding example.

6.4. Uniform convergence

The results of the Sections 6.1-6.3 give some controllability-type conditions that allow to apply the general convergence theorems of Section 2.3 for the singular perturbation problem. Here we list some groups of assumptions ensuring that the convergence is uniform. The first result is a simple combination of Theorem 2.10 with Theorem 6.1 and Propositions 6.3 and 6.5.

Corollary 6.8. *Assume that the differential game satisfies the usual assumptions* (A) *and that the fast dynamics $g(y, \alpha, \beta)$ and $\tau(y, \alpha, \beta)$ do not depend on x. Suppose also that the fast subsystem is either deterministic, i.e., of the form $\dot{y} = g(y, \alpha, \beta)$, or independent of the second player, i.e., of the form $dy_s = g(y_s, \alpha_s)ds + \tau(y_s, \alpha_s)dW_s$, and in both cases assume it is bounded-time controllable and stoppable. Then $\overline{H}$ exists and the value functions u^ε converge uniformly on the compact subsets of $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ as $\varepsilon \to 0$ to the unique viscosity solution of the effective Cauchy problem* ($\overline{\text{HJ}}$) *with $\overline{h}(x) = \min_y h(x, y)$.*

The second result is obtained by combining the proof of Theorem 4.4 with the Propositions 6.4 and 6.5. We will say that H is coercive in q if for some constant C (6.15) holds for all $\overline{x}, \overline{p}, \overline{X}$. This is equivalent to the existence of a deterministic subsystem first-order-controllable by the minimizing player, that is, condition (6.18) with A' and ν independent of x.

Corollary 6.9. *Assume that the differential game satisfies the usual assumptions* (A)*, H is coercive in q, and τ is independent of x. Suppose also that either the diffusion in the slow variables is nondegenerate, i.e., for some $\nu > 0$*

$$\frac{\sigma\sigma^T}{2}(x,y,\alpha,\beta) \geq \nu I_n, \quad \textit{for all } x,y,\alpha,\beta,$$

or the slow subsystem is deterministic, i.e., $\sigma \equiv 0$. Then $\overline{H}$ exists and the value functions u^ε converge uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ as $\varepsilon \to 0$ to the unique viscosity solution of the effective Cauchy problem ($\overline{\text{HJ}}$) *with $\overline{h}(x) = \min_y h(x,y)$.*

The last results exploits the representation formula for $\overline{H}$ of Proposition 6.6.

Corollary 6.10. *Assume that the differential game satisfies the usual assumptions* (A) *and* (6.23)*, that is,*

$$H(x,y,p,q,X,Y,0) \geq H(x,y,p,0,X,0,0)$$

Suppose also that either H is coercive in q, or the fast subsystem is deterministic, bounded-time controllable and stoppable, or the fast subsystem is independent of the second player, bounded-time controllable and stoppable. Then the value functions u^ε converge uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ as $\varepsilon\to 0$ to the unique viscosity solution of

$$\begin{cases} u_t + \max_y H(x,y,Du,0,D^2u,0,0) = 0 \ \ \textit{in } (0,T)\times\mathbb{R}^n, \\ u(0,x) = \min_y h(x,y) \ \ \textit{for all } x\in\mathbb{R}^n. \end{cases}$$

Proof The Hamiltonian is ergodic by Theorem 6.1 and Propositions 6.3 and 6.4, it is stabilizing by Proposition 6.5. By the assumption (6.23) we can use Proposition 6.6 to get the explicit formula

$$\overline{H}(x,p,X) = \\ \max_y \min_{\beta\in B}\max_{\alpha\in A}\left\{-X\cdot\frac{\sigma\sigma^T}{2}(x,y,\alpha,\beta) - p\cdot f(x,y,\alpha,\beta) - l(x,y,\alpha,\beta)\right\}.$$

By the periodicity of the data the $\max_y$ is taken on $[0,1]^n$. Since σ, f, l are Lipschitz in x uniformly in y,α,β, by a standard argument $\overline{H}$ satisfies the conditions for the comparison principle. Therefore we conclude by Theorem 2.9. □

Example 7. We test the assumptions of the three corollaries of this section on systems with general slow subsystem

$$dx_s = f(x_s,y_s,\alpha_s,\beta_s)\,ds + \sigma(x_s,y_s,\alpha_s,\beta_s)\,dW_s$$

and a fast subsystem either deterministic or independent of the second player, and of a somewhat special form. We start with the deterministic subsystem

$$\dot{y}_s = \frac{1}{\varepsilon}g(x_s,y_s,\alpha_s,\beta_s)$$

where g is

$$g(x,y,\alpha,\beta)=\sum_{i=1}^{k}\alpha^i g^i(x,y)+\alpha^{k+1}g^{k+1}(x,y,\alpha,\beta),$$
$$\alpha=(\alpha^1,\dots,\alpha^{k+j})\in A\subseteq\mathbb{R}^{k+j},\ j\geq 1,$$

with A such that $(\alpha^1,\dots,\alpha^k)$ takes values in a neighborhood of the origin in $\mathbb{R}^k$ and α^{k+1} can be 0. The value functions u^ε of the singular perturbation problem converge uniformly for any running cost l to the solution of ($\overline{\text{HJ}}$) if one of the following groups of conditions hold:

- the fields g^i are independent of x, for all $i=1,...,k+1$, and $g^1,\dots,g^k$ satisfy the Lie algebra condition (6.11) (by Corollary 6.8);
- for all x,y, $\text{span}\{g^1,\dots,g^k\}=\mathbb{R}^m$, and either $\sigma\sigma^T/2\geq\nu I_n$ or $\sigma\equiv 0$ (by Corollary 6.9);
- for all x fixed $g^1,\dots,g^k$ satisfy the Lie algebra condition (6.11), and $f$and σ depend only on $\alpha^{k+2},\dots,\alpha^{k+j}$ (by Corollary 6.10 and Example 6 of Section 6.3).

Next we consider a fast subsystem independent of the second player of the form

$$dy_s=\frac{1}{\varepsilon}\sum_{i=1}^{k}\alpha^i g^i(x_s,y_s,\alpha_s)\,ds+\frac{1}{\sqrt{\varepsilon}}\alpha^{k+1}\tau(x_s,y_s,\alpha_s)\,dW_s$$

with A as above. The value functions u^ε converge uniformly for any running cost l to the solution of ($\overline{\text{HJ}}$) under any of the three previous groups of conditions, where g^{k+1} is replaced by τ in the first group.

6.5. The reduction order formula for the effective control problem

In this section we consider a fast dynamics depending on the second player but not on the first. In search of a formula for the effective Hamiltonian $\overline{H}$ we try to follow the classical Levinson-Tychonov approach to the singular perturbations of ODEs and set formally $\varepsilon=0$ in the controlled system (1.7). This leads to the differential-algebraic system

$$dx_s=f(x_s,y_s,\alpha_s,\beta_s)\,ds+\sigma(x_s,y_s,\alpha_s,\beta_s)\,dW_s,$$
$$g(x_s,y_s,\beta_s)=0,\quad \tau(x_s,y_s,\beta_s)=0,$$

and the Hamiltonian for the differential game associated to this system is

$$H_0(x,p,X):=\min_{(y,\beta)\in Z(x)}\max_{\alpha\in A}-L(y,\alpha,\beta;x,p,X),$$

where

$$Z(x):=\{(y,\beta)\in\mathbb{R}^m\times B\,:\,g(x,y,\beta)=0,\ \tau(x,y,\beta)=0\}$$

and

$$L(y,\alpha,\beta)=L(y,\alpha,\beta;x,p,X):=X\cdot a(x,y,\alpha,\beta)+p\cdot f(x,y,\alpha,\beta)+l(x,y,\alpha,\beta).$$

In the next result we compare H_0 with $\overline{H}$ by means of the formula (2.8) in Section 2.1. The equality holds if H_0 satisfies an Isaacs-type condition and there exists an asymptotically stable optimal trajectory for the ergodic control problem of the system

$$dy_s=g(x,y_s,\beta_s)ds+\tau(x,y_s,\beta_s)dW_s,\quad y_0=y. \tag{6.29}$$

PROPOSITION 6.11. *Besides the standing assumptions* (A) *suppose that g and τ do not depend on α and $Z(x) \neq \emptyset$. Then*
(i) $H_0(x,p,X) \geq \overline{H}(x,p,X)$ for all x,p,X;
(ii) $H_0(x,p,X) = \overline{H}(x,p,X)$ under the following additional conditions:

$$H_0(x,p,X) = \max_{\alpha\in A} \min_{(y,\beta)\in Z(x)} -L(y,\alpha,\beta;x,p,X), \tag{6.30}$$

for all $\overline{\alpha} \in A$ there exist sequences $\beta^n \in \mathcal{B}$ and $t_n \to +\infty$ such that

$$E_y \int_0^{t_n} -L\left(y_s^{\beta^n}, \overline{\alpha}, \beta_s^n\right) ds = \inf_{\beta\in\mathcal{B}(t_n)} E_y \int_0^{t_n} -L\left(y_s, \overline{\alpha}, \beta_s\right) ds + o(t_n), \tag{6.31}$$

where y_s is the trajectory of (6.29), *and for some $(y^*, \beta^*) \in Z(x)$*

$$\lim_n E_y \frac{1}{t_n} \int_0^{t_n} \left(\left| y_s^{\beta^n} - y^* \right| + |\beta_s^n - \beta^*| \right) ds = 0. \tag{6.32}$$

PROOF *(i)* We fix $(\overline{y}, \overline{\beta}) \in Z(x)$ and note that $y_s \equiv \overline{y}$ if $\beta_s \equiv \overline{\beta}$. For any strategy $\alpha \in \Gamma(t)$

$$E_y \frac{1}{t} \int_0^t -L\left(\overline{y}, \alpha[\overline{\beta}]_s, \overline{\beta}\right) ds \geq \inf_{\beta\in\mathcal{B}(t)} E_y \frac{1}{t} \int_0^t -L\left(y_s, \alpha[\beta]_s, \beta_s\right) ds. \tag{6.33}$$

Now observe that

$$\sup_{\alpha\in\Gamma(t)} E_y \frac{1}{t} \int_0^t -L\left(\overline{y}, \alpha[\overline{\beta}]_s, \overline{\beta}\right) ds = \max_{\alpha\in A} -L\left(\overline{y}, \alpha, \overline{\beta}\right).$$

Then, taking $\sup_{\alpha\in\Gamma(t)}$ and then $\lim_{t\to+\infty}$ in (6.33), by the formula (2.8) for the effective Hamiltonian we get

$$\max_{\alpha\in A} -L\left(\overline{y}, \alpha, \overline{\beta}\right) \geq \overline{H}.$$

By the arbitrariness of $(\overline{y}, \overline{\beta}) \in Z(x)$ we have proved that $H_0 \geq \overline{H}$.

(ii) The assumption (6.31) gives, for a fixed $\overline{\alpha} \in A$,

$$\sup_{\alpha\in\Gamma(t_n)} \inf_{\beta\in\mathcal{B}(t_n)} E_y \frac{1}{t_n} \int_0^{t_n} -L\left(y_s, \alpha[\beta]_s, \beta_s\right) ds + o(1) \geq E_y \frac{1}{t_n} \int_0^{t_n} -L\left(y_s^{\beta^n}, \overline{\alpha}, \beta_s^n\right) ds,$$

and the left hand side tends to $\overline{H}$ as $n \to \infty$ by (2.8). On the other hand, the right hand side tends to $-L(y^*, \overline{\alpha}, \beta^*)$. In fact, if we denote with ω_L the modulus of continuity of L with respect to y and β and use its concavity, by Jensen's inequality we get

$$\left| E_y \frac{1}{t_n} \int_0^{t_n} -L\left(y_s^{\beta^n}, \overline{\alpha}, \beta_s^n\right) ds + L(y^*, \overline{\alpha}, \beta^*) \right| \leq \omega_L \left(E_y \frac{1}{t_n} \int_0^{t_n} \left(\left| y_s^{\beta^n} - y^* \right| + |\beta_s^n - \beta^*| \right) ds \right),$$

and the right hand side tends to 0 by (6.32). Therefore, $(y^*, \beta^*) \in Z(x)$ gives

$$\overline{H} \geq \min_{(y,\beta)\in Z(x)} -L(y, \overline{\alpha}, \beta).$$

Now the arbitrariness of $\overline{\alpha}$ and the Isaacs-type condition (6.30) imply $\overline{H} \geq H_0$. □

CHAPTER 7

Nonresonant fast variables

In this chapter, we assume that the fast dynamical system is independent of the fast variable y and of the strategy of the first player α. The fast dynamics is therefore

$$dy_s = g(\overline{x}, \beta_s)ds + \tau(\overline{x}, \beta_s)dW_s, \quad y_0 = y, \tag{7.1}$$

The remaining data (the slow dynamics, the running cost and the initial cost) can depend as usual on all the variables.

With this dynamics, the Hamiltonian for the cell problem (CP) has the simplified form

$$\begin{aligned}\min_{\beta\in B}\max_{\alpha\in A}&\left\{-Y\cdot b(\overline{x},\beta) - q\cdot g(\overline{x},\beta) - L(y,\alpha,\beta;\overline{x},\overline{p},\overline{X})\right\}\\ &= \min_{\beta\in B}\left\{-Y\cdot b(\overline{x},\beta) - q\cdot g(\overline{x},\beta) - \min_{\alpha\in A} L(y,\alpha,\beta;\overline{x},\overline{p},\overline{X})\right\},\end{aligned}$$

for

$$L(y,\alpha,\beta;\overline{x},\overline{p},\overline{X}) = \overline{X}\cdot a(\overline{x},y,\alpha,\beta) + \overline{p}\cdot f(\overline{x},y,\alpha,\beta) + l(\overline{x},y,\alpha,\beta).$$

The solution of the cell problem (CP) is therefore the value function of the optimal control problem for the second player

$$w(t,y;\overline{x},\overline{p},\overline{X}) = \sup_{\beta\in\mathcal{B}} E_y \int_0^t \min_{\alpha\in A} L(y_s,\alpha,\beta_s;\overline{x},\overline{p},\overline{X})\, ds,$$

where the state process y_s is given by (7.1). In addition, the solution of the homogeneous cell Cauchy problem (CP′) only depends on the second player and is given by

$$w'(t,y;\overline{x},\overline{p},\overline{X}) = \sup_{\beta\in\mathcal{B}} E_y h(\overline{x},y_s).$$

From the analytic point of view, the issue of ergodicity and stabilization under these assumptions is therefore the same as for an optimal control problem, where one seeks to maximize a certain gain.

7.1. Ergodicity

For optimal control problems with periodic fast dynamics that are independent of the state variable, Arisawa, Lions [**AL98**] provided a necessary and sufficient condition for ergodicty called the *non-resonance condition*

$$\text{for every } k\in\mathbb{Z}^m\backslash\{0\}, \text{ there is } \beta\in B \text{ such that} \quad b(x,\beta)k\neq 0 \ \text{ or } \ g(x,\beta)\cdot k\neq 0. \tag{7.2}$$

The assumption is the suitable extension to controlled diffusions of the classical characterization by Jacobi of the ergodic translations on the torus. The nonresonance condition is dramatically weaker than the uniform non-degeneracy assumption. For instance, it will hold if the second player can choose a direction for the state process that has rationally independent coordinates, i.e. if there is a vector ξ such that $\xi \cdot k \neq 0$ for all $k \in \mathbb{Z}^m \backslash \{0\}$ and a strategy β such that ξ is either $g(x, \beta)$ or a column vector of $\tau(x, \beta)$.

The next result is shown in Arisawa, Lions [**AL98**].

THEOREM 7.1. *Assume that the Hamiltonian is given by* (1.12) *with the data satisfying the standing assumptions* (A). *Assume that the fast dynamics $\tau(x, \beta)$ and $g(x, \beta)$ are independent of the fast variables and of the first player's controls and that it satisfies the non-resonance condition* (7.2). *Then the Hamiltonian is ergodic at x.*

REMARK The non-resonance condition is necessary for ergodicity. More precisely, if (7.2) does not hold we can exhibit a simple running cost $l \in C_{\text{per}}(\mathbb{R}^m)$ for which the corresponding Hamiltonian is not ergodic. Take $k \in \mathbb{Z}^m \backslash \{0\}$ such that $b(x, y, \beta)k = 0$ and $g(x, y, \beta) \cdot k = 0$ for every $\beta \in B, y \in \mathbb{R}^m$, and the running cost $l(y) = \cos(2\pi k \cdot y)$. Then the solution to the stationary cell problem

$$\delta w_\delta + \min_{\beta \in B} \big\{ -D^2 w_\delta \cdot b(\overline{x}, \beta) - D w_\delta \cdot g(\overline{x}, \beta) \big\} - l(y) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic}$$

is $w_\delta = \delta^{-1} \cos(2\pi k \cdot y)$, because $\cos(2\pi k \cdot y)$ is in the kernel of the differential operator. Therefore, δw_δ will not converge to a constant as $\delta \to 0$.

Example: the uncontrolled case. If the system is not controlled the non-resonance condition reads

$$b(x)k \neq 0 \text{ or } g(x) \cdot k \neq 0 \quad \forall k \in \mathbb{Z}^m \backslash \{0\}.$$

Then the invariant measure is the Lebesgue measure by Corollary 3.6 and

$$\overline{H}(x, p, X) = \int_{[0,1)^m} H(x, y, p, 0, X, 0, 0)\, dy.$$

The non-resonant case provides an elementary example of an ergodic Hamiltonian for which the true cell problem has no continuous solution.

PROPOSITION 7.2. *Let $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$ with rationally independent coordinates. Then there exists $f \in C_{per}(\mathbb{R}^2)$ such that the Hamiltonian*

$$H(y, Y) = -Y \cdot (\sigma \otimes \sigma) - f(y)$$

is ergodic and, for all λ, the associated cell problem

$$-D^2 \chi \cdot (\sigma \otimes \sigma) = f(y) - \lambda \quad \textit{in } \mathbb{R}^2, \qquad \chi \textit{ periodic}$$

has no viscosity solution.

PROOF We can assume without loss of generality that $\int f\, dy = 0$. The Hamiltonian H is ergodic as the non-resonance condition is satisfied. Moreover, the only λ for which the cell problem may have a solution is $\int f\, dy$, i.e. 0, by the example above.

Before constructing f, we note that if χ is a continuous solution of the cell problem, then f and χ admit a Fourier expansion

$$f(y) = \sum_{k\in\mathbb{Z}^2} \widehat{f}(k)e^{i2\pi k\cdot y}, \quad \chi(y) = \sum_{k\in\mathbb{Z}^2} \widehat{\chi}(k)e^{i2\pi k\cdot y}$$

with $\widehat{f}$, $\widehat{\chi}$ in $\ell^2(\mathbb{Z}^2)$ and their Fourier coefficients must be related through the formula

$$4\pi^2|\sigma\cdot k|^2\widehat{\chi}(k) = \widehat{f}(k).$$

This follows simply from the observation that it is equivalent for a continuous solution χ of an equation with smooth coefficients to solve the equation in the sense of distributions or in the viscosity sense.

Because σ_1 and σ_2 are rationally independent, the additive subgroup $\sigma_1\mathbb{Z}+\sigma_2\mathbb{Z}$ is dense in $\mathbb{R}$. Therefore, there is a sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{Z}^2$ with distinct non-zero terms so that

$$|\sigma\cdot k_j| \le e^{-j}.$$

The function

$$f(y) = \sum_{j\in\mathbb{N}} 4\pi^2|\sigma\cdot k_j|^2 e^{i2\pi k_j\cdot y}$$

is continuous. On the other hand, the Fourier coefficients of χ must be given for $k\neq 0$ by

$$\widehat{\chi}(k) = 1 \quad \text{if } k = k_j \text{ for some } j, \qquad \widehat{\chi}(k) = 0 \quad \text{otherwise}.$$

So, $\widehat{\chi}$ is not in $\ell^2(\mathbb{Z}^2)$. This is impossible. □

7.2. Stabilization

The next result, which is new, provides a necessary and sufficient condition for the stabilization problem related to the dynamics (7.1). The non-resonance assumption (7.2) has to be slightly strenghtened to

(7.3) for every $k\in\mathbb{Z}^m\backslash\{0\}$, there are $\beta,\beta'\in B$ such that

$$b(\overline{x},\beta)k\neq 0 \;\text{ or }\; g(\overline{x},\beta)\cdot k\neq g(\overline{x},\beta')\cdot k.$$

The extra condition on the drift is natural as it excludes the case of uncontrolled deterministic processes for which uniform stabilization cannot hold.

THEOREM 7.3. *Assume that the Hamiltonian is given by* (1.12) *with the data satisfying the standing assumptions* (A). *Assume that the fast dynamics* $\tau(x,\beta)$ *and* $g(x,\beta)$ *are independent of the fast variables and of the first player's control, and that they satisfy the non-resonance condition* (7.3). *Then the Hamiltonian is stabilizing.*

REMARK Condition (7.3) is necessary for the Hamiltonian to be stabilizing, as the following variant of the example for the ergodicity illustrates. Assume that the condition fails, so that there is $k\in\mathbb{Z}^m\backslash\{0\}$ and a constant c such that $b(\overline{x},\beta)k=0$ and $g(\overline{x},\beta)\cdot k = c$ for every $\beta\in B$. Then, the solution of (CP′) with initial data $h(y)=\cos\big(2\pi\, k\cdot y\big)$ is $w(t,y)=\cos\big(2\pi\, k\cdot y+2\pi ct\big)$. It does not converge to a constant as $t\to+\infty$.

Proof The proof is an adaptation of the proof for the uniformly elliptic case, Theorem 4.2. We shall keep the notations and only mention the main differences. One difference is that we shall use the minimum principle for supersolutions instead of the maximum principle for subsolutions, because the minimum principle provides much more information for optimal control problem with Hamiltonian of the $\min_{\beta \in B}$ form. All the inequalities in the argument are thus to be reversed.

We first observe that we can assume without loss of generality that

$$\text{there is } \beta_0 \in B \text{ so that} \quad g(\overline{x}, \beta_0) = 0. \tag{7.4}$$

Indeed, if this is not the case, we fix β_0 arbitrary and set $\widehat{b}(\overline{x}, \beta) = b(\overline{x}, \beta)$ and $\widehat{g}(\overline{x}, \beta) = g(\overline{x}, \beta) - g(\overline{x}, \beta_0)$. Of course, $\widehat{g}(\overline{x}, \beta_0) = 0$. Moreover, w is a solution of (CP$'$) if and only if $\widehat{w}(t, y) = w(t, y - tg(\overline{x}, \beta_0))$ is a solution of

$$\begin{cases} \partial_t \widehat{w} + \min_{\beta \in B}\{-\widehat{b}(x, \beta) \cdot D^2_{yy}\widehat{w} - D_y\widehat{w} \cdot \widehat{g}(x, \beta)\} = 0 & \text{in } (0, +\infty) \times \mathbb{R}^m, \\ \widehat{w}(0, y) = h(\overline{x}, y) \text{ on } \mathbb{R}^m. \end{cases}$$

Thus, if $\widehat{w}(t, \cdot)$ is proved to converge uniformly to a constant as $t \to +\infty$, $w(t, \cdot)$ will converge uniformly as $t \to +\infty$ to the same constant. Note that, under (7.4), the non-resonance condition (7.3) is exactly the one needed for ergodicity, namely (7.2).

As for the uniformly elliptic case, we can assume without loss of generality that the initial data is smooth with respect to y. This implies that w is Lipschitz continuous in t. Moreover, since h is Lipschitz continuous in y and since H' is independent of y, the comparison principle shows that $w(t, \cdot)$ is Lipschitz continuous in y with a Lipschitz constant that is not larger than that of h. Therefore, w is globally Lipschitz continuous in $[0, +\infty) \times \mathbb{R}^m$.

Step 2 is unchanged. Indeed, the non-resonance condition (7.2) implies the validity of the strong minimum principle for the periodic supersolutions of the stationary problem

$$H'(\overline{x}, D_y\underline{w}, D^2_{yy}\underline{w}) \geq 0 \qquad \text{in} \quad \mathbb{R}^m$$

(see Arisawa, Lions [**AL98**]; this is also a special case of the strong minimum principle for parabolic equations we prove below). Therefore, we conclude that $\underline{w}(y) = \liminf_{t \to +\infty} w(t, y)$ must be constant.

We now follow step 3. To obtain the uniform convergence of $w(t, \cdot)$ to the constant $\underline{w}$ as $t \to +\infty$, it is enough to show the following version of the parabolic minimum principle : if $\widetilde{w}$ is a bounded uniformly continuous function in (t, y), periodic in y, that solves

$$\partial_t \widetilde{w} + H'(\overline{x}, D_y\widetilde{w}, D^2_{yy}\widetilde{w}) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^m$$

and if it achieves its minimum $\underline{w}$ at $(0, 0)$, then $\widetilde{w}$ must be constant.

To see this, we use the description of the propagation set for the minimum points of solutions to Hamilton-Jacobi-Bellman equations (see [**BL01**]) to deduce that $\widetilde{w} = \underline{w}$ along the trajectories of the control problem

$$\begin{cases} \dot{t} = -V(s), \\ \dot{y} = b(\overline{x}, \beta(s))U(s) + g(\overline{x}, \beta(s))V(s) \quad \text{for } s > 0, \\ t(0) = 0, \quad y(0) = 0 \end{cases}$$

with $\beta \in B$, $U \in [-1,1]^m$ and $V \in [0,1]$. By choosing piecewise constant controls, we get that

$$\widetilde{w}(-s,y) = \underline{w} \quad \text{for all } s \geq 0 \text{ and}$$
$$y \in \operatorname{span}\{b(\overline{x},\beta)\xi \mid \beta \in B,\ \xi \in \mathbb{R}^m\} + s\operatorname{conv}\{g(\overline{x},\beta) \mid \beta \in B\},$$

where $\operatorname{conv} X$ denotes the convex hull of the set X. As the family $\{\widetilde{w}(t,\cdot)\}$ is equibounded and equi-Lipschitz, we can construct a subsequence $t_p \to -\infty$ so that $\widetilde{w}(t_p,\cdot)$ converges uniformly to some periodic Lipschitz function v as $p \to +\infty$. Taking the limit in the preceding identity, we deduce that

$$v \equiv \underline{w} \qquad \text{on} \quad C$$

for the set

$$C := \operatorname{span}\{b(\overline{x},\beta)\xi \mid \beta \in B,\ \xi \in \mathbb{R}^m\} + \operatorname{cone}\{g(\overline{x},\beta) \mid \beta \in B\},$$

where $\operatorname{cone} X := \{s\,c \mid s \geq 0,\ c \in \operatorname{conv} X\}$. We have used here the fact that every point in $\operatorname{cone}\{g(\overline{x},\beta) \mid \beta \in B\}$, which is of the form $s_0\,c$ for some $s_0 > 0$ and $c \in \operatorname{conv}\{g(\overline{x},\beta) \mid \beta \in B\}$, is actually in $s\operatorname{conv}\{g(\overline{x},\beta) \mid \beta \in B\}$ for all $s \geq s_0$ because $0 \in \operatorname{conv}\{g(\overline{x},\beta) \mid \beta \in B\}$ by (7.4).

To complete the argument, we need a lemma whose proof is deferred after the end of the proof of the theorem; it extends the characterization by Jacobi of the translations on the torus whose orbits are dense.

Lemma 7.4. *Let C be a convex cone. Then the set $C + \mathbb{Z}^m$ is dense in $\mathbb{R}^m$ if and only if the following condition holds*

$$\textit{for every } k \in \mathbb{Z}^m\backslash\{0\}, \textit{ there is } c \in C \textit{ such that} \quad c \cdot k \neq 0.$$

Elementary algebra reveals that the non-resonance condition (7.2) is exactly the condition of the lemma for the set C above. Therefore, the set $C + \mathbb{Z}^m$ is dense. But, v is continuous and periodic. As $v = \underline{w}$ on C, we conclude that $v \equiv \underline{w}$. This means that $\widetilde{w}(t_p,\cdot) \to \underline{w}$ uniformly as $p \to +\infty$. By the comparison principle, we know that $\sup\limits_{[t_p,+\infty)\times\mathbb{R}^m} |\widetilde{w}(t,\cdot) - \underline{w}| = \sup\limits_{\mathbb{R}^m} |\widetilde{w}(t_p,\cdot) - \underline{w}|$. Sending $p \to +\infty$, we conclude that $\widetilde{w} \equiv \underline{w}$. Recalling the definition of $\widetilde{w}$, we argue as in the proof of Theorem 4.2 to conclude that $w(t,\cdot) \to \underline{w}$ uniformly as $t \to +\infty$. □

Proof of Lemma 7.4 The condition is necessary. Indeed, assume on the contrary that there is $k \in \mathbb{Z}^m\backslash\{0\}$, such that $c \cdot k = 0$ for every $c \in C$. Then, the function $e^{2\pi i\, k\cdot y}$ is continuous and periodic, equals 1 on C, but is not identically 1. Therefore, $C + \mathbb{Z}^m$ is not dense in $\mathbb{R}^m$.

Conversely, assume that the condition holds and set $u(y) = d(y, C + \mathbb{Z}^m)$. To prove that the set $C + \mathbb{Z}^m$ is dense, we have to show that $u \equiv 0$. The function u is continuous and periodic. Moreover, for every $y \in \mathbb{R}^m$, $c \in C$ and $t > 0$, it satisfies

$$u(y - tc) = \inf\{|y - tc - d - k| \mid d \in C,\ k \in \mathbb{Z}^m\} \geq u(y),$$

because C is a convex cone. But, by periodicity, we must have $\int_{[0,1]^m} u(y - tc)\,dy = \int_{[0,1]^m} u(y)\,dy$. Therefore, we actually have $u(y - tc) = u(y)$ for every $y \in \mathbb{R}^m$, $c \in C$ and $t > 0$. Expanding u in Fourier series, $u(y) = \sum_{k\in\mathbb{Z}^m} a_k e^{2\pi i\, k\cdot y}$, we deduce that $a_k e^{-2\pi i t\, k\cdot c} = a_k$ for every $k \in \mathbb{Z}^m$, $c \in C$ and $t > 0$. The condition implies that $a_k = 0$ for every $k \in \mathbb{Z}^m\backslash\{0\}$. Thus u constant. But $u = 0$ on C, so $u \equiv 0$. □

7.3. Uniform convergence

As we are not ensured of the existence of the corrector under the non-resonance condition, there is no hope to take advantage of its regularity as it was the case in the previous Chapters. We can obtain the uniform convergence only in two cases. The first is the case of non-resonant uncontrolled $b(x)$ and $g(x)$ of the last example, because there is an explicit formula for $\overline{H}$. However, the stronger condition (7.3) cannot be satisfied by an uncontrolled drift, so we must assume $h = h(x)$ is independent of the fast variables. Then the convergence is uniform up to time $t = 0$.

The second case is when the fast dynamics is independent of the slow variable.

COROLLARY 7.5. *Assume that the Hamiltonian is given by* (1.12) *with the data satisfying the standing assumptions* (A)*, and that the fast dynamics* $\tau(\beta)$ *and* $g(\beta)$ *only depend on the second player's controls.*
(i) If $b = \tau\tau^T/2$ *and* g *satisfy the non-resonance condition* (7.3)*, then the value functions* u^ε *converge uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of* $(\overline{\text{HJ}})$*;*
(ii) if, instead, they satisfy only condition (7.2) *but* $h = h(x)$ *is independent of the fast variable* y*, then the same convergence occurs on the compact subsets of* $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$*.*

CHAPTER 8

A counterexample to uniform convergence

The purpose of this chapter is to construct a singular perturbation problem that is ergodic and stabilizing in the fast variables but whose value function u^ε does not converge uniformly on the compact sets. Since the problem is ergodic and stabilizing there are an effective Hamiltonian $\overline{H}$ and an effective initial cost $\overline{h}$. However, in our example $\overline{H}$ is not regular enough, so the comparison principle for the effective equation ($\overline{\text{HJ}}$) does not hold and we cannot apply Theorem 2.9 to get the uniform convergence. In the precise example we give, we shall see that the convergence is locally uniform in the complement of a hyperplane, and the limit is discontinuous on the hyperplane. This will follow from the explicit construction of the minimal and maximal solutions to the problem, from the determination of the points where they coincide, and from the application of Theorem 2.7.

We consider the value function for the deterministic singular perturbations problem

$$u^\varepsilon(t,x,y) := \inf\{h(x_t) \mid \dot{x}_s = \cos y_s + 1,\ \varepsilon\dot{y}_s = x_s + \alpha_s,\ |\alpha_s| \le 1,\ x_0 = x,\ y_0 = y\}$$

in $[0,+\infty) \times \mathbb{R} \times \mathbb{R}$, where $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function. The corresponding HJB equation is

$$\begin{cases} u^\varepsilon_t - (\cos y + 1)u^\varepsilon_x + \frac{1}{\varepsilon}\big(|u^\varepsilon_y| - xu^\varepsilon_y\big) = 0 & \text{in } (0,+\infty) \times \mathbb{R} \times \mathbb{R}, \\ u^\varepsilon(0,x,y) = h(x) & \text{on } \mathbb{R} \times \mathbb{R}. \end{cases}$$

The ergodicity in the fast variable is guaranteed either by Theorem 6.1, because the fast dynamics is bounded-time controllable on the torus $\mathbb{R}/(2\pi\mathbb{Z})$, or by Theorem 7.1, because there is only one player and because the fast dynamics is independent of the fast variable and satisfies the non-resonance condition (7.2). The pair (H, h) is trivially stabilizing since the initial cost is independent of the fast variable. We can therefore define an effective Hamilontian $\overline{H}$ and the effective initial cost $\overline{h} = h$. However, we cannot apply the uniform convergence Corollaries 4.3 and 7.5 because the fast dynamics depends on the slow variable, nor can we use Corollary 6.9 because the Hamiltonian is not coercive with respect to u^ε_y, nor can we use Corollary 6.10 because the quantity H_2 in the crucial condition (6.23) is $|q| - xq$ that is negative for some q if $|x| > 1$. The main result of this Chapter is the following.

PROPOSITION 8.1. *The family $\{u^\varepsilon\}$ converges uniformly on the compact subsets of $[0,+\infty) \times \mathbb{R}\backslash\{1\} \times \mathbb{R}$ but not on the compact subsets of $[0,+\infty) \times \mathbb{R} \times \mathbb{R}$.*

Before proving the result, let us explain in an informal way why the singularly perturbed control problem changes nature in a neighbourhood of $x = 1$, a change which is reflected in the lack of uniform convergence. The slow variable is always nondecreasing. Since the cost increases with x_t, the controller seeks to keep the slow variable at its initial value. To do so, he must choose a control that keeps the fast variable at $\pi \mod (2\pi)$. When $x \in]-1,1[$, the controller can always drive the fast variable to π and keep it stationary at π in a amount of time of order ε. But when $x > 1$, the fast variable will be forced to turn on the circle with positive speed $\dot{y} \geq \varepsilon^{-1}(x-1) > 0$. It turns out that, in the limit, this change of behavior in the fast variable will force the slow variable to increase, uniformly in the control and *uniformly* in the initial slow position $x > 1$.

The proof of the Proposition splits into two parts. In a first lemma, we get an explicit formula for the effective Hamiltonian. It corresponds to a control problem where the drift is Hölder continuous but not Lipschitz continuous in the state variable. Therefore, for a given control, there is no uniqueness of the trajectories. In a second lemma, we verify that the semilimits of $\{u^\varepsilon\}$ must differ.

LEMMA 8.2. *The effective Hamiltonian is given by*

$$\overline{H}(x,p) = \sup\{-pv \mid v \in [\overline{f}(x), 2 - \overline{f}(x)]\} \tag{8.1}$$

for the effective drift $\overline{f}$ *given by*

$$\overline{f}(x) = 1 - \cos\theta, \tag{8.2}$$

where $\theta \in [0, \pi/2[$ *is the unique solution of*

$$\tan\theta - \theta = \frac{\pi}{2}(|x|-1)^+.$$

In particular, the function $\overline{f}$ *is even with values in* $[0,1[$ *and is* 0 *in* $[-1,1]$. *It is of class* C^∞ *when* $|x| \neq 1$. *In a neighborhood of* 1 *and* -1, *it is Hölder continuous with exponent* $2/3$. *More precisely, we have the expansion*

$$\overline{f}(x) = c(|x|-1)^{2/3} + o\big((|x|-1)^{2/3}\big) \qquad \text{when } |x| > 1$$

for the constant $c = (3\pi)^{2/3}/2^{5/3}$.

PROOF The effective Hamiltonian is given by the formula

$$\overline{H}(x,p) = \lim_{T\to\infty} \sup\{\frac{1}{T}\int_0^T -p(\cos y_s + 1)\, ds \mid \dot{y}_s = x + \alpha_s,\ |\alpha_s| \leq 1,\ y_0 = y\}.$$

We set

$$\overline{f}(x) = \lim_{T\to\infty} \inf\{\frac{1}{T}\int_0^T (\cos y_s + 1)\, ds \mid \dot{y}_s = x + \alpha_s,\ |\alpha_s| \leq 1,\ y_0 = y\}.$$

Then, we have

$$\overline{H}(x,p) = -\overline{f}(x)p \quad \text{if } p \geq 0, \qquad \overline{H}(x,p) = -(2 - \overline{f}(x))p \quad \text{if } p \leq 0.$$

Indeed, the case $p \geq 0$ is trivial. When $p \leq 0$, we have

$$\begin{aligned}
\overline{H}(x,p) &= -p + p \lim_{T\to\infty} \inf\{-\frac{1}{T}\int_0^T \cos y_s\, ds \mid \dot{y}_s = x + \alpha_s,\ |\alpha_s| \leq 1,\ y_0 = y\}. \\
&= -p + p \lim_{T\to\infty} \inf\{\frac{1}{T}\int_0^T \cos y'_s\, ds \mid \dot{y}'_s = x + \alpha_s,\ |\alpha_s| \leq 1,\ y'_0 = y + \pi\}. \\
&= -p + p(\overline{f}(x) - 1).
\end{aligned}$$

The fact that $\overline{f}$ is even is obvious. Moreover, the function $\overline{f}$ is nonnegative. It is also ≤ 1 because the functional to minimize is 1 when the control α is a constant $\neq -x$. The above formula for $\overline{H}$ is then clearly equivalent to (8.1).

We now compute a candidate $\widetilde{f}$ for the function $\overline{f}$. We assume that $x \geq 0$. When $x \leq 1$, we get $\overline{f}(x) = 0$ by choosing the control $\alpha_s \equiv -x$ and $y = \pi$. From now on, we assume that $x > 1$. We begin by an informal computation that motivates our formula for $\overline{f}$. As the process y_s cannot be stationary, one is willing to consider a control that is $+1$ when $\cos y$ is large and -1 when it is small, in order to spend most time in this later case. The optimal control should therefore be of the form

$$\alpha = +1 \quad \text{when } y \in [-\pi + \sigma, \pi - \sigma] \pmod{2\pi},$$
$$\alpha = -1 \quad \text{when } y \in]\pi - \sigma, \pi + \sigma[\pmod{2\pi},$$

for some $\sigma \in [0, \pi]$. The associated trajectory y_s is periodic with period $T(\sigma) = \dfrac{4\sigma + 2\pi(x-1)}{x^2 - 1}$; the associated long run average cost is $\overline{f}(x, \sigma) = 1 - \dfrac{\sin \sigma}{\sigma + \frac{\pi}{2}(x-1)}$. The optimal average cost corresponds to the angle that minimizes $\overline{f}(x, \cdot)$. An immediate computation reveals that it is the unique solution in $[0, \pi/2[$ of $\tan\theta - \theta = \frac{\pi}{2}(x-1)$. The optimal average cost is then $\overline{f}(x, \theta) = 1 - \cos\theta$. This defines the function in (8.2) which we denote $\widetilde{f}$.

Let us now prove that $\overline{f} = \widetilde{f}$ by solving the true cell problem for $p = 1$

$$\text{(8.3)} \quad \sup_{|\alpha| \leq 1} \{-(x + \alpha)\chi_y - \cos y - 1\} + \widetilde{f}(x) = |\chi_y| - x\chi_y - \cos y - 1 + \widetilde{f}(x) = 0.$$

Take the following function

$$\chi(y) = \frac{1}{x+1}\big((\pi - \theta - y)(1 - \widetilde{f}(x)) + \sin\theta - \sin y\big) \quad \text{if } y \in [-\pi + \theta, \pi - \theta],$$
$$\chi(y) = \chi(-\pi + \theta) + \frac{1}{x-1}\big((\pi + \theta - y)(1 - \widetilde{f}(x)) - \sin\theta - \sin y\big)$$
$$\text{if } y \in]\pi - \theta, \pi + \theta[,$$

and extend it periodically. A tedious computation reveals that the function χ is of class C^1 and that it is a solution to the cell problem (8.3). This implies that $\overline{H}(x, 1) = -\widetilde{f}(x)$, hence $\widetilde{f}(x) = \overline{f}(x)$.

The C^∞ regularity of $\overline{f}$ in $\mathbb{R}\backslash\{-1, 1\}$ follows from the inverse mapping theorem. The behavior of $\overline{f}$ in a neighborhood of 1 and -1 when $|x| > 1$ is proved by an elementary Taylor expansion. □

Let us now give a complete description of the solutions of the effective equation

$$\text{(8.4)} \qquad u_t + \overline{H}(x, u_x) = 0 \quad (0, +\infty) \times \mathbb{R}, \qquad u(0, x) = h(x) \quad \text{on } \mathbb{R}.$$

From the formula for the effective Hamiltonian, a natural solution should be the value function

$$\overline{v}(t, x) := \inf\{h(x_t) \mid \dot{x}_s \in [\overline{f}(x_s), 2 - \overline{f}(x_s)],\ x_0 = x\}$$
$$= \inf\{h(x_t) \mid \dot{x}_s = \overline{f}(x_s),\ x_0 = x\}.$$

(The second identity follows from the fact that h is increasing.) Using the qualitative properties of the effective drift $\overline{f}$ given in Lemma 8.2, one can easily realize that the effective dynamical system

$$\dot{x}_t = \overline{f}(x_t), \qquad x_0 = x$$

has exactly one solution if and only if $x \neq 1$. In this case, we denote by $g_t x$ the associated flow. When $x = 1$, the dynamical system has infinitely many solutions, because the drift is not Lipschitz at $x = 1$. One can show easily that the smallest solution is the constant

$$x_t^- = 1,$$

while the largest one is defined by

$$x_0^+ = 1, \qquad \int_1^{x_t^+} \frac{du}{\overline{f}(u)} = t \quad \text{for } t > 0.$$

The integral is converging at 1 because $\overline{f}(u)$ behaves like $(u-1)^{2/3}$ near 1.

When we interpret this description of the effective flow in terms of the limit equation (8.4), we get the following result. It says in particular that the value function $\overline{v}$ is the minimal supersolution of (8.4).

Lemma 8.3. *The minimal supersolution of* (8.4) *and the maximal subsolution are given respectively by*

$$\underset{\sim}{u}(t,x) = h(g_t x) \quad \textit{if } x \neq 1, \qquad \underset{\sim}{u}(t,1) = h(x_t^-),$$

$$\widetilde{u}(t,x) = h(g_t x) \quad \textit{if } x \neq 1, \qquad \widetilde{u}(t,1) = h(x_t^+).$$

Proof Let $\overline{f}_\eta$ and $\overline{f}^\eta$ be Lipschitz continuous functions that converge uniformly to $\overline{f}$ as $\eta \to 0$ and satisfy $0 \leq \overline{f}_\eta \leq \overline{f} \leq \overline{f}^\eta \leq 1$ for all η. Define

$$u_\eta(t,x) = h(x_t) \text{ for } \dot{x}_s = \overline{f}_\eta(x_s) \text{ with } x_0 = x,$$

$$u^\eta(t,x) = h(x_t) \text{ for } \dot{x}_s = \overline{f}^\eta(x_s) \text{ with } x_0 = x.$$

By the construction of the approximated drifts and the definition of $\underset{\sim}{u}$ and $\widetilde{u}$, we get easily that $u_\eta \uparrow \underset{\sim}{u}$ and $u^\eta \downarrow \widetilde{u}$. One can check that u_η (resp. u^η) is the unique viscosity solution of

$$\text{(8.5)} \qquad u_t + H(x, u_x) = 0 \quad (0,+\infty) \times \mathbb{R}, \qquad u(0,x) = h(x) \quad \text{on } \mathbb{R}$$

for the Hamiltonian $H = H_\eta(x,p) := \sup\{-pv \mid v \in [\overline{f}_\eta(x), 2 - \overline{f}_\eta(x)]\}$ (resp. $H = H^\eta(x,p) := \sup\{-pv \mid v \in [\overline{f}^\eta(x), 2 - \overline{f}^\eta(x)]\}$).

Since H_η converges to $\overline{H}$ uniformly on the compact sets, we deduce from the stability properties of viscosity solutions that $\underset{\sim}{u}$ is a supersolution of (8.4). Since $H_\eta \geq \overline{H}$, any supersolution w of the effective equation (8.4) is a supersolution of (8.5) with $H = H_\eta$. By the comparison principle (note that H_η has enough regularity for this as it is Lipschitz continuous in x for p bounded), we get that $w \geq u_\eta$, whence $w \geq \underset{\sim}{u}$ after sending $\eta \to 0$. This proves that $\underset{\sim}{u}$ is a minimal supersolution. One proves analogously that $\widetilde{u}$ is the maximal subsolution of (8.4). □

Proof of Proposition 8.1 By Theorem 2.7, we know that the semilimits of $\{u^\varepsilon\}$ will satisfy the inequalities

$$\underset{\sim}{u} \leq \underline{u} \leq \overline{u} \leq \widetilde{u} \quad \text{on } [0,+\infty) \times \mathbb{R} \times \mathbb{R}.$$

Using the explicit formula for $\underset{\sim}{u}$ and $\widetilde{u}$, we observe that $(\underset{\sim}{u})^* = \widetilde{u}$ and $(\widetilde{u})_* = \underset{\sim}{u}$. Since $\underline{u}$ and $\underset{\sim}{u}$ are l.s.c. and $\overline{u}$ and $\widetilde{u}$ are u.s.c., this implies

$$\underline{u} = \underset{\sim}{u}, \qquad \overline{u} = \widetilde{u}.$$

On $[0, +\infty) \times \mathbb{R}\backslash\{1\} \times \mathbb{R}$, the minimal and maximal solutions are equal. So, the semilimits are equal and this implies that the family u^ε converges uniformly on the compact subsets of $[0, +\infty) \times \mathbb{R}\backslash\{1\} \times \mathbb{R}$. On $(0, +\infty) \times \{1\} \times \mathbb{R}$, the minimal and maximal solutions differ, so we must have $\underline{u} < \overline{u}$. By the well-known properties of the semilimits, this implies that u^ε cannot converge uniformly on a compact neighbourhood of any point in $(0, +\infty) \times \{1\} \times \mathbb{R}$. In particular, it cannot converge uniformly on the compact sets of $[0, +\infty) \times \mathbb{R} \times \mathbb{R}$. □

CHAPTER 9

Applications to homogenization

9.1. Periodic homogenization of 1st order H-J equations

Consider the Hamilton-Jacobi equation with oscillating Hamiltonian and initial data

$$\begin{cases} v^\varepsilon_t + G\left(x, \frac{x}{\varepsilon}, Dv^\varepsilon\right) = 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}\right) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{9.1}$$

If we look for a solution of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$, we see that $u^\varepsilon(t,x,y)$ solves the Cauchy problem

$$u^\varepsilon_t + G\left(x, y, D_x u^\varepsilon + \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^n,$$
$$u^\varepsilon(0,x,y) = h(x,y) \qquad \text{for } (x,y) \in \mathbb{R}^n\times\mathbb{R}^n,$$

which is a special case of our singular perturbation problem with $H(x,y,p,q) = G(x,y,p+q)$. This approach to homogenization problems was introduced in our papers [**AB01, AB03**] and it is a counterpart for fully nonlinear PDEs of the two-scale convergence by Allaire and Nguetseng [**All92**] for variational problems.

If the Hamiltonian G has the Bellman-Isaacs form

$$G(x,y,p) = \min_{\beta\in B}\max_{\alpha\in A}\{-p\cdot f(y,\alpha,\beta) - l(x,y,\alpha,\beta)\}, \tag{9.2}$$

the solution to (9.1) is the (lower) value function

$$v^\varepsilon(t,x) = \inf_{\alpha\in\Gamma(t)}\sup_{\beta\in\mathcal{B}(t)}\left[\int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)\right],$$

for the control system

$$\dot{x}_s = f\left(\frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right), \qquad x_0 = x.$$

The limit as $\varepsilon \to 0$ of v^ε gives informations on the homogenization of this (deterministic) differential game in a highly oscillating medium. The introduction of the fast variable $y = \frac{x}{\varepsilon}$ transforms the system into the singularly perturbed one

$$\begin{aligned} \dot{x}_s &= f(y_s,\alpha_s,\beta_s), \\ \dot{y}_s &= \tfrac{1}{\varepsilon} f(y_s,\alpha_s,\beta_s), \\ x_0 &= x, \quad y_0 = y, \end{aligned} \tag{9.3}$$

and u^ε is the value function

$$u^\varepsilon(t,x,y) = \inf_{\alpha\in\Gamma(t)}\sup_{\beta\in\mathcal{B}(t)}\left[\int_0^t l\left(x_s, y_s, \alpha_s, \beta_s\right) ds + h\left(x_t, y_t\right)\right].$$

Therefore we can use the theory of singular perturbations developed in the previous chapters: the local uniform convergence of $u^\varepsilon(t,x,y)$ to $u(t,x)$ clearly implies the local uniform convergence of $v^\varepsilon(t,x)$ to the same function u (by the periodicity in y of u^ε). Next we give two homogenization theorems that follow immediately from the results of Chapters 6 and 7, respectively. The first extends the classical result for Hamiltonians coercive in p [**LPV86, Eva92, AB01**] to the Bellman-Isaacs Hamiltonians associated to a bounded-time controllable system.

COROLLARY 9.1. *Assume the dynamics $f(y,\alpha,\beta)$ and the costs $l(x,y,\alpha,\beta)$, $h(x,y)$ satisfy the usual assumptions* (A). *Suppose also that the system*

$$\dot y = f(y,\alpha,\beta)$$

is bounded-time controllable and stoppable by the first player. Then there exists a continuous Hamiltonian $\overline{H}$ such that the value functions v^ε converge uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n$ as $\varepsilon\to 0$ to the unique viscosity solution of the effective Cauchy problem

$$u_t + \overline{H}(x,Du) = 0 \ \ \textit{in } (0,+\infty)\times\mathbb{R}^n, \qquad u(0,x) = \min_y h(x,y).$$

Example 1: the sub-riemannian eikonal equation. The PDE

$$u_t + \sum_{i=1}^{k} \left| g^i\left(\frac{x}{\varepsilon}\right)\cdot Du\right| = l\left(x,\frac{x}{\varepsilon}\right),$$

where the vector fields $g^1,\dots,g^k$ are C^∞ and generate a Lie algebra of full rank n at each point of $\mathbb{R}^n$, satisfies the assumptions of the Corollary, see Example 2 in Chapter 6. Here the Hamiltonian is not coercive in $p = Du$, although it is coercive with respect to the horizontal gradient associated to the family of vector fields.

Example 2. The PDE

$$u_t + \sum_{i=1}^{n} (u_{x_i})^+ = l\left(x,\frac{x}{\varepsilon}\right),$$

where r^+ denotes the positive part, satisfies the assumptions of the Corollary, in view of the periodicity of the state space, and the Hamiltonian is not coercive in $p = Du$.

The next corollary concerns the case of non-resonant systems and extend a result in [**AB01**] to the case of oscillating initial data.

COROLLARY 9.2. *Assume that the dynamics and the costs satisfy the usual assumptions* (A). *Suppose also that $f = f(\beta)$ depends only on the second player's control and satisfies the non-resonance condition*

$$\textit{for all } k\in\mathbb{Z}^m\backslash\{0\} \textit{ there exist } \beta,\beta'\in B \textit{ such that } f(\beta)\cdot k \neq f(\beta')\cdot k.$$

Then there exists a continuous Hamiltonian $\overline{H}$ and a continuous terminal cost $\overline{h}$ such that the value functions v^ε converge uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n$ as $\varepsilon\to 0$ to the unique viscosity solution of the effective Cauchy problem

$$u_t + \overline{H}(x,Du) = 0 \ \ \textit{in } (0,+\infty)\times\mathbb{R}^n, \qquad u(0,x) = \overline{h}(x).$$

Example 3. Consider the following system in $\mathbb{R}^2$ with two controls, say 0 and 1:

$$f(0) = (0,0), \qquad f(1) = (1,\pi).$$

It is not bounded-time controllable by either player, but it satisfies the non--resonance condition. Therefore the last Corollary applies and allows to homogenize the Cauchy problem

$$u_t - (u_{x_1} + \pi u_{x_2})^+ = l\left(x, \frac{x}{\varepsilon}\right), \qquad u(0,x) = h\left(x, \frac{x}{\varepsilon}\right).$$

9.2. Periodic homogenization of 2nd order equations

Consider the parabolic equation with oscillating coefficients and initial data

$$\text{(9.4)} \qquad \begin{cases} v_t^\varepsilon + F\left(x, \frac{x}{\varepsilon}, D^2 v^\varepsilon\right) = 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}\right) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

As in the previous section, we look again for a solution of the form

$$v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon}).$$

Now $u^\varepsilon(t,x,y)$ solves the Cauchy problem

$$u_t^\varepsilon + F\left(x, y, D_{xx}u^\varepsilon + \frac{D_{yy}u^\varepsilon}{\varepsilon^2} + \frac{D_{xy}u^\varepsilon}{\varepsilon} + \frac{(D_{xy}u^\varepsilon)^T}{\varepsilon}\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^n,$$
$$u^\varepsilon(0,x,y) = h(x,y) \qquad \text{for } (x,y)\in\mathbb{R}^n\times\mathbb{R}^n,$$

which is again a special case of our singular perturbation problem, now with $H(x,y,X,Y,Z) = F(x,y,X+Y+Z+Z^T)$.

If the operator F has the Bellman-Isaacs form

$$\text{(9.5)} \qquad F(x,y,X) = \min_{\beta\in B}\max_{\alpha\in A}\{-X\cdot a(x,y,\alpha,\beta) - l(x,y,\alpha,\beta)\}, \quad a = \sigma\sigma^T/2,$$

the solution to (9.4) is the (lower) value function

$$v^\varepsilon(t,x) = \inf_{\alpha\in\Gamma(t)}\sup_{\beta\in\mathcal{B}(t)} E_x\left[\int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right)ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)\right],$$

for the controlled diffusion

$$dx_s = \sigma\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right)dW_s, \qquad x_0 = x.$$

The limit as $\varepsilon\to 0$ of v^ε gives informations on the homogenization of this stochastic differential game in a highly oscillating medium. The introduction of the fast variable $y = \frac{x}{\varepsilon}$ transforms the system into the singularly perturbed one

$$\text{(9.6)} \qquad \begin{aligned} dx_s &= \sigma(x_s, y_s, \alpha_s, \beta_s)dW_s, \\ dy_s &= \tfrac{1}{\varepsilon}\sigma(x_s, y_s, \alpha_s, \beta_s)dW_s, \\ x_0 &= x, \quad y_0 = y, \end{aligned}$$

and u^ε is the value function

$$u^\varepsilon(t,x,y) = \inf_{\alpha\in\Gamma(t)}\sup_{\beta\in\mathcal{B}(t)} E_{(x,y)}\left[\int_0^t l(x_s, y_s, \alpha_s, \beta_s)\,ds + h(x_t, y_t)\right].$$

We can use again the theory of singular perturbations developed in the previous chapters and we give three homogenization theorems that follow immediately from the results of Chapters 4, 5, and 6, respectively. The first holds for uniformly non-degenerate diffusions, namely,

$$\text{for some } \nu > 0, \quad a(x, y, \alpha, \beta) \geq \nu I_n \quad \forall\, x, y \in \mathbb{R}^n,\ \alpha \in A,\ \beta \in B, \tag{9.7}$$

and it follows immediately from Theorem 4.4 and Theorem 4.5.

Corollary 9.3. *Assume that σ, l, and h satisfy the usual assumptions* (A) *and* (9.7)*. Suppose also that either $\sigma = \sigma(y, \alpha, \beta)$ is independent of x, or $\sigma = \sigma(x, y, \beta)$ is independent of the first player's control and l is Hölder continuous in y uniformly in x, α, β. Then there exist a continuous degenerate elliptic $\overline{H}$ and a continuous $\overline{h}$ such that the value functions v^ε converge uniformly on the compact subsets of $(0, T) \times \mathbb{R}^n$ as $\varepsilon \to 0$ to the unique viscosity solution of the effective Cauchy problem*

$$u_t + \overline{H}(x, D^2 u) = 0 \ \text{ in } (0, +\infty) \times \mathbb{R}^n, \qquad u(0, x) = \overline{h}(x). \tag{9.8}$$

The second result is about uncontrolled hypoelliptic diffusions with coefficients independent of x. Now v^ε solves

$$\begin{cases} v^\varepsilon_t - \frac{\sigma\sigma^T}{2}\left(\frac{x}{\varepsilon}\right) \cdot D^2 v^\varepsilon = l\left(x, \frac{x}{\varepsilon}\right) & \text{in } (0, T) \times \mathbb{R}^n, \\ v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{9.9}$$

the columns σ^i of σ are C^∞, and the associated vector fields $X_i = \sigma^i \cdot \nabla$ satisfy the Hörmander condition (5.5) (with $m = n$). Corollary 5.3 gives the following.

Corollary 9.4. *Assume that σ, l, and h satisfy the usual assumptions* (A) *and the diffusion $dy_s = \sigma(y_s) dW_s$ is hypoelliptic, as recalled above. Then there exist $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(y)dy$ is the invariant measure associated to the diffusion, and the solution $v^\varepsilon(t, x)$ of* (9.9) *converges uniformly on the compact subsets of $(0, T) \times \mathbb{R}^n$ as $\varepsilon \to 0$ to the unique viscosity solution of*

$$u_t + \int_{(0,1)^m} \left[-\frac{\sigma\sigma^T}{2}(y) \cdot D^2_{xx} u - l(x, y)\right] \varphi(y) dy = 0, \quad \text{in } (0, T) \times \mathbb{R}^n,$$

$$u(x, 0) = \int_{(0,1)^m} h(x, y) \varphi(y) dy \qquad \text{for } x \in \mathbb{R}^n.$$

The third and last result treats the homogenization for the Bellman equation

$$\begin{cases} v^\varepsilon_t + \min_{\beta \in B} \left\{-\frac{\sigma\sigma^T}{2}(\beta) \cdot D^2 v^\varepsilon - l\left(x, \frac{x}{\varepsilon}, \beta\right)\right\} = 0 & \text{in } (0, T) \times \mathbb{R}^n, \\ v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{9.10}$$

under the non-resonance condition. It follows from Corollary 7.5.

Corollary 9.5. *Assume the data satisfy the standing assumptions* (A)*. Suppose also that $\sigma = \sigma(\beta)$ depends only on the second player's control and satisfies the non-resonance condition*

$$\text{for all } k \in \mathbb{Z}^m \backslash \{0\} \text{ there exist } \beta \in B \text{ such that } \sigma(\beta)^T k \neq 0.$$

Then there exists a continuous degenerate elliptic Hamiltonian $\overline{H}$ and a continuous terminal cost $\overline{h}$ such that the solution v^ε of (9.10) *converges uniformly on the compact subsets of $(0, T) \times \mathbb{R}^n$ as $\varepsilon \to 0$ to the unique viscosity solution of the effective Cauchy problem* (9.8)*.*

Remark The homogenization of parabolic equations with first order terms, that is,

$$v_t^\varepsilon + F\left(x, \frac{x}{\varepsilon}, Dv^\varepsilon, D^2 v^\varepsilon\right) = 0$$

can be treated by a variant of these methods. Once we perform the usual ansatz $v^\varepsilon(t,x) = u^\varepsilon(t, x, \frac{x}{\varepsilon})$, we get for $u^\varepsilon(t,x,y)$ the PDE

$$u_t^\varepsilon + F\left(x, y, D_x u^\varepsilon + \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx}u^\varepsilon + \frac{D_{yy}u^\varepsilon}{\varepsilon^2} + \frac{D_{xy}u^\varepsilon}{\varepsilon} + \frac{(D_{xy}u^\varepsilon)^T}{\varepsilon}\right) = 0. \tag{9.11}$$

This singular perturbation problem does not have the same scaling as our main problem (HJ_ε) because the second order terms dominate. The natural guess is that the first order terms do not play a role in determining the effective Hamiltonian and terminal cost, which is in fact the same as that for

$$H(x, y, p, 0, X, Y, Z) = F(x, y, p, X + Y + Z + Z^T).$$

In terms of the control problem, the system associated to u^ε by setting $y = \frac{x}{\varepsilon}$ is

$$\begin{aligned} dx_s &= f(x_s, y_s, \alpha_s, \beta_s) + \sigma(x_s, y_s, \alpha_s, \beta_s)dW_s, \\ dy_s &= \tfrac{1}{\varepsilon} f(x_s, y_s, \alpha_s, \beta_s) + \tfrac{1}{\varepsilon}\sigma(x_s, y_s, \alpha_s, \beta_s)dW_s, \\ x_0 &= x, \quad y_0 = y, \end{aligned}$$

which explains why the drift f does not play a role in the calculation of the limit problem.

The proofs of these statements are obtained by an additional argument exploiting the fact that (9.11) can be treated as a regular perturbation of a singular perturbation problem of the form (HJ_ε). This is done in our paper [**AB01**] in some particular cases and in the companion paper with Marchi [**ABM07**] in full generality. The last paper also extend the singular perturbation and homogenization theory presented here from two scales to an arbitrary number of scales.

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