Existence results for a class of evolution equations of mixed type

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Abstract

We give an existence result for the evolution equation $(\mathcal{R}u)' + \mathcal{A}u = f$ in the space $\mathcal{W} = \{u \in \mathcal{V} \mid (\mathcal{R}u)' \in \mathcal{V}'\}$ where $\mathcal{V}$ is a Banach space and $\mathcal{R}$ is a non-invertible operator (the equation may be partially elliptic and partially parabolic, both forward and backward) and we study the "Cauchy-Dirichlet" problem associated to this equation (indeed also for the inclusion $(\mathcal{R}u)' + \mathcal{A}u \ni f$). We also investigate continuous and compact embeddings of $\mathcal{W}$ and regularity in time of the solution. At the end we give some examples of different $\mathcal{R}$.

1 Introduction

We present an existence result for the solution of the abstract evolution equation

$$(\mathcal{R}u)' + \mathcal{A}u = f\,.$$

Given V Banach space and H Hilbert space, $V \subset H$, defined $\mathcal{V} = L^p(0,T;V)$, $\mathcal{R}$ will be a linear operator defined in $L^p(0,T;H)$ and the solution will be taken in the space $\mathcal{W} = \{u \in \mathcal{V} \mid (\mathcal{R}u)' \in \mathcal{V}'\}$. Notations and definitions for the abstract equation will be given in the second section, so for the moment we confine ourselves to explain a concrete and simple problem (see also examples in Section 4): suppose to have a bounded open set Ω of $\mathbf{R}^n$, $T > 0$, a function $r : \Omega \times [0,T] \to \mathbf{R}$, $r \in L^\infty(\Omega \times (0,T))$, define $\Omega_+(t) = \{x \in \Omega \mid r(x,t) > 0\}$ and $\Omega_-(t) = \{x \in \Omega \mid r(x,t) < 0\}$, consider $H = L^2(\Omega)$ and $V = H^1_0(\Omega)$, and define $\mathcal{R}u(x,t) = r(x,t)u(x,t)$. Consider

$$\begin{cases} \dfrac{\partial}{\partial t}(r(x,t)u(x,t)) - \Delta u(x,t) = f(x,t) & \text{on } \Omega \times (0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = \varphi(x) & x \in \Omega_+(0) \\ u(x,T) = \psi(x) & x \in \Omega_-(T) \end{cases} \tag{1}$$

where $f : \Omega \times (0,T) \to \mathbf{R}$, $\varphi : \Omega_+(0) \to \mathbf{R}$, $\psi : \Omega_-(T) \to \mathbf{R}$ are the data of the problem. Notice that for this kind of Cauchy-Dirichlet problem we prescribe an "initial datum" in the

part of Ω in which at time $t = 0$ the coefficient r is positive, a "final datum" in the part of Ω in which at time $t = T$ the coefficient r is negative and no datum where r is zero both at time $t = 0$ and time $t = T$. A simple situation in which we have existence and uniqueness of the solution is stated in the following theorem (but $\mathcal{R}$ can be much less regular, e. g. the case $r : \Omega \times (0, T) \to \{-1, 0, 1\}$ is also possible: see examples in Section 4).
Denote by r_+ and r_- respectively the positive and the negative part of r and, for a positive function λ, by $L^2(\omega, \lambda)$ the space L^2 with respect to the measure λdx on ω. Then the following result holds.

Theorem *For every* $f \in L^2(0, T; H^{-1}(\Omega))$, $\varphi \in L^2(\Omega_+(0), r_+(\cdot, 0))$, $\psi \in L^2(\Omega_-(T), r_-(\cdot, T))$ *problem* (1) *admits a unique solution in the space* $\{u \in L^2(0, T; H_0^1(\Omega)) \mid (ru)' \in L^2(0, T; H^{-1}(\Omega))\}$ *if* $r, \dfrac{\partial r}{\partial t} \in L^\infty(\Omega \times (0, T))$. *Moreover a lower bound on* $\frac{\partial r}{\partial t}$ *is assumed, precisely* $\|u\|^2_{H_0^1(\Omega)} + 1/2 \operatorname{ess\,inf} \frac{\partial r}{\partial t} \|u\|^2_{L^2(\Omega)} \geqslant \alpha \|u\|^2_{H_0^1(\Omega)}$ *for some positive* α *(see Theorem* (3.8)*).*

Many diffusion problems lead to mixed problems, partially elliptic and partially parabolic. For example in some composite materials there can be a zone in which the evolution is so quick to consider the problem stationary, and there the problem can be considered elliptic, and another zone in which the problem is parabolic. This situation corresponds to take $r \geqslant 0$ in the example (1) above. This problem is not new: perhaps the first papers about it are [3] and [13] (see also [14], Section III.3, [17], Chap. 5, and [12], [11] for homogenization and G-convergence of elliptic-parabolic operators). We also refer to [3], chap. 3, and the references therein for many applications.

Also the more general problem (r arbitrary) was already considered: for example, in [2] (for this see also Section 3.2.6 in [7]) the Authors considered the following equation

$$x \frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t) \qquad x \in \mathbf{R}\,. \tag{2}$$

An equation like that arises from kinetic theory (see [4]). Later in [9] and [8] the following problem is considered

$$\operatorname{sgn}(x) \frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) + k u(x, t) = f(x, t) \qquad x \in \mathbf{R}\,.$$

Probably the first general paper about it, but with $\mathcal{R}$ independent of time, is [10] (see moreover [1] for a recent work on homogenization for changing type evolution equations).

In the present paper we study the problem of existence and uniqueness of a solution to a more general problem (see (18) and (19) in Section 3).
If $\mathcal{R} = \mathrm{Id}$ the space $\mathcal{W}$ embeds continuously in $C([0, T]; H)$ and compactly in $L^p(0, T; H)$. In Section 2 we investigate these embeddings if $\mathcal{R} \neq \mathrm{Id}$ and study the density of regular functions in $\mathcal{W}$.
Moreover we study regularity in time of the solution (see Theorem 3.11, Corollary 3.13, Remark 3.14 and Example 3.15).
In the last section we give examples of some possible choices of the function r, but also for the operator $\mathcal{R}$, like $\mathcal{R}u(t) = \int_\Omega r(x, y, t) u(y, t) dy$.

2 Notations, hypotheses and preliminary results

Consider a triplet

$$V \subset H \subset V'$$

where V is a reflexive and separable Banach space, V' its dual space, H a Hilbert space and the embedding $V \subset H$ is continuous and dense. Fix $T > 0$, $p \in (1, +\infty)$ and define

$$\mathcal{V} = L^p(0, T; V)\,, \qquad \mathcal{V}' = L^{p'}(0, T; V')\,.$$

In the next section we want to study the problem of existence, and uniqueness, of a solution to an abstract evolution equation of the form

$$(\mathcal{R}u)' + \mathcal{A}u = f$$

where $\mathcal{A}$ may be a pseudomonotone, coercive and bounded operator from $\mathcal{V}$ to $\mathcal{V}'$ and $\mathcal{R}$ may be the sum of one non-negative and one non-positive operator (positive and negative in the sense we are going to explain). Therefore in this section we focalize our attention on hypotheses about the operator $\mathcal{R}$ and to define and study the space which the solution to the equation should belong to, leaving considerations about $\mathcal{A}$ to the next section where we discuss the existence results.
We consider a family of linear operators from H to H

$$R : [0, T] \longrightarrow \mathcal{L}(H) \tag{3}$$

and write, for every $t \in [0, T]$, $H = H_+(t) \oplus H_0(t) \oplus H_-(t)$ where $H_0(t)$ is the kernel of $R(t)$, $H_+(t)$ and $H_-(t)$ are defined respectively as the subspaces of H such that $(R(t)u, u)_H > 0$ for every $u \in H_+(t)$, $u \neq 0$, and $(R(t)u, u)_H < 0$ for every $u \in H_-(t)$, $u \neq 0$, and decompose $R(t)$ as $R_+(t) + R_0(t) - R_-(t)$ where

$$R_+(t) : H_+(t) \to H_+(t) \qquad \text{defined by} \qquad R_+(t)u = R(t)u$$
$$R_-(t) : H_-(t) \to H_-(t) \qquad \text{defined by} \qquad R_-(t)u = -R(t)u$$

and $R_0(t) : H_0(t) \to H_0(t)$ is the operator $R_0(t)u = 0$ for every $u \in H_0(t)$.
Before giving assumptions on R we need a definition.

Definition 2.1 *We say that $B : [0, T] \to \mathcal{L}(X, X')$, X Banach space, is differentiable if, for every $u, v \in X$ the function*

$$t \mapsto \big\langle B(t)u, v \big\rangle_{X' \times X}$$

is absolutely continuos on $[0, T]$ and there exists a function $b \in L^1(0, T)$ such that

$$\Big| \frac{d}{dt} \big\langle B(t)u, v \big\rangle_{X' \times X} \Big| \leqslant b(t) \|u\|_X \|v\|_X \quad \textit{for a. e. } t \in [0, T]\,.$$

Observe that $B'(t) : X \to X'$ is linear for almost every $t \in (0, T)$.

Remark 2.2 - If B is differentiable in the sense just defined the following fomula holds: if $u \in W^{1,p}(0, T; X)$ for any p, then

$$\big(B(t)u(t)\big)' = B'(t)u(t) + B(t)u'(t) \qquad \text{for a. e. } t \in (0, T)\,.$$

In the special case $B(t) = B$ for every t, taking $v \in L^p(0, T; X)$ and $u(t) = \displaystyle\int_0^t v(s)ds$, we also derive from the formula above that

$$B \int_0^t v(s)ds = \int_0^t Bv(s)ds\,. \tag{4}$$

About $R : [0,T] \longrightarrow \mathcal{L}(H)$ we require that for every $u, v \in V$

$$\left| \begin{array}{l} R(t) \quad \text{self-adjoint} \\ \sup_{t\in(0,T)} \|R(t)\|_{\mathcal{L}(H)} \leqslant C_1 \, , \\ t \mapsto \big(R(t)u, v\big)_H \text{ absolutely continuos on } [0,T] \, , \\ \big|\dfrac{d}{dt}\big(R(t)u, v\big)_H\big| \leqslant C_2 \; \|u\|_V \|v\|_V \quad \text{for a. e. } t \in [0,T] \, . \end{array} \right. \tag{5}$$

Then we define the operator $\mathcal{R}$ by

$$\mathcal{R} : L^p(0,T;H) \longrightarrow L^p(0,T;H) \qquad \text{by} \qquad \mathcal{R}u(t) := R(t)u(t) \tag{6}$$

which turns out to be linear and bounded by the constant C_1. Notice that we require an assumption which is weaker with respect to that in Definition 2.1, i. e. $|\frac{d}{dt}\big(R(t)u,v\big)_H| \leqslant C_2 \; \|u\|_V\|v\|_V$ and not $|\frac{d}{dt}\big(R(t)u,v\big)_H| \leqslant C_2 \; \|u\|_H\|v\|_H$. By (5) we can define a family of equibounded operators

$$R' : [0,T] \to \mathcal{L}(V,V') \qquad \text{by} \qquad \langle R'(t)u, v\rangle_{V'\times V} := \frac{d}{dt}\big(R(t)u,v\big)_H$$

and an operator

$$\mathcal{R}' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \langle \mathcal{R}'u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t)\rangle_{V'\times V} dt$$

which turns out to be linear and bounded by C_2.

Remark 2.3 - Notice that with these assumptions on R, i. e. (5), $R \in W^{1,\infty}(0,T;\mathcal{L}(V,V'))$. Indeed for every $\eta \in C^1_c((0,T);\mathbf{R})$, $u, v \in V$ (see, e. g., Proposition 23.9 in [16])

$$\int_0^T \langle R(t)u,v\rangle_{V'\times V}\eta'(t)dt = -\int_0^T \langle R'(t)u,v\rangle_{V'\times V}\eta(t)dt = -\Big\langle \Big[\int_0^T \eta(t)R'(t)dt\Big]u, v\Big\rangle_{V'\times V}$$

and

$$\int_0^T \langle R(t)u,v\rangle_{V'\times V}\eta'(t)dt = \Big\langle \Big[\int_0^T R(t)\eta'(t)dt\Big]u, v\Big\rangle_{V'\times V} .$$

Now we define the space

$$\mathcal{W} = \big\{ v \in \mathcal{V} \mid (\mathcal{R}v)' \in \mathcal{V}' \big\} \, , \qquad\qquad \|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} \tag{7}$$

where the derivative is to be intended in the sense of distributions. Notice that if R satisfies (5) we have, for $v \in \mathcal{W}$, that $(\mathcal{R}v)' \in H^1(0,T;V')$ and $\mathcal{R}'v \in L^2(0,T;V') = \mathcal{V}'$. For this reason it makes sense to evaluate for almost every $t \in (0,T)$

$$\begin{aligned} (\mathcal{R}v')(t) \quad &:= \quad (\mathcal{R}v)'(t) - \mathcal{R}'(t)v(t) \\ &= \quad \lim_{h\to 0} \frac{[R(t+h)v(t+h) - R(t)v(t)]}{h} - \lim_{h\to 0} \frac{[R(t+h)v(t) - R(t)v(t)]}{h} \\ &= \quad \lim_{h\to 0} R(t+h)\frac{[v(t+h)-v(t)]}{h} = R(t)\lim_{h\to 0}\frac{[v(t+h)-v(t)]}{h} \, . \end{aligned} \tag{8}$$

Proposition 2.4 *Suppose $\mathcal{R}$ regular, i. e. satisfying* (5). *Then the space $C^1([0,T];V)$ is dense in $\mathcal{W}$.*

Proof - Consider a family of mollifiers $(\rho_\epsilon)_\epsilon$ given by $\epsilon^{-1}\rho(t/\epsilon)$, $t \in \mathbf{R}$, $\rho \in C^1_c(\mathbf{R})$ an even function such that $\rho(0) = 1$ and $\mathrm{supp}(\rho) \subset [-1,1]$. For $w \in L^2(0,T;X)$, X Banach, consider

$$\overline{w}(t) = \begin{cases} w(t) & t \in [0,T] \\ 0 & t \in \mathbf{R} \setminus [0,T] \end{cases}$$

and define

$$w_\epsilon(t) = \int_{\mathbf{R}} \overline{w}(s)\rho_\epsilon(t-s)ds\,. \tag{9}$$

We consider the family $(u_\epsilon)_{\epsilon>0}$ which is contained in $C^\infty([0,T];V)$ and approximate u in $\{u \in \mathcal{V} \mid u' \in \mathcal{V}'\}$ (see [16], ex. 23.10) and in particular in $\mathcal{V}$.
In the same way $(\mathcal{R}u)_\epsilon = (\mathcal{R}u)*\rho_\epsilon$ converges to $\mathcal{R}u$ in $\{w \in L^p(0,T;H) \mid w' \in L^{p'}(0,T;V')\}$, and in particular

$$(\mathcal{R}u)'_\epsilon \to (\mathcal{R}u)' \quad \text{in } \mathcal{V}'\,.$$

To see that also $(\mathcal{R}u_\epsilon)'$ approximate $(\mathcal{R}u)'$ in $\mathcal{V}'$ first observe that, since $\mathcal{R}$ is regular then $\mathcal{R}u_\epsilon \in \{w \in L^p(0,T;H) \mid w' \in L^{p'}(0,T;V')\}$ and in particular that $(\mathcal{R}u_\epsilon)' \in \mathcal{V}'$. Since $R' : [0,T] \longrightarrow \mathcal{L}(V,V')$ we have that

$$\mathcal{R}'u_\epsilon \to \mathcal{R}'u \quad \text{in } \mathcal{V}'\,.$$

Then it is sufficient to show that $\mathcal{R}u'_\epsilon \to \mathcal{R}u' = (\mathcal{R}u)' - \mathcal{R}'u$ in $\mathcal{V}'$. By (4) we have

$$\begin{aligned} R(t)u'_\epsilon(t) &= \int_{\mathbf{R}} R(t)u(t-s)\rho'_\epsilon(s)ds = \\ &= \int_{\mathbf{R}} \big[R(t-s) + R(t) - R(t-s)\big]u(t-s)\rho'_\epsilon(s)ds = \\ &= (\mathcal{R}u)'_\epsilon(t) + \int_{\mathbf{R}} \Big[\frac{R(t)-R(t-s)}{s}\Big]u(t-s)s\rho'_\epsilon(s)ds\,. \end{aligned} \tag{10}$$

Now observe that for every $\delta > 0$ there exists $\epsilon > 0$ such that if $|s| < \epsilon$ we have that

$$\Big\| R'(t) - \frac{R(t)-R(t-s)}{s} \Big\|_{\mathcal{L}(V,V')} < \delta \qquad \text{since } R \in W^{1,\infty}(0,T;\mathcal{L}(V,V'))\,.$$

Finally observe that

$$s \mapsto -s\rho'_\epsilon(s) \qquad\qquad \text{are mollifiers}\,.$$

Then, for a fixed δ and for ϵ sufficiently small, we have that

$$\begin{aligned} &\int_0^T \Big\| \mathcal{R}'u_\epsilon(t) + \int_{\mathbf{R}} \Big[\frac{R(t)-R(t-s)}{s}\Big]u(t-s)s\rho'_\epsilon(s)ds \Big\|^2_{V'} dt = \\ &\int_0^T \Big\| \mathcal{R}'u_\epsilon(t) - \int_{\mathbf{R}} \Big[\frac{R(t)-R(t-s)}{s}\Big]u(t-s)[-s\rho'_\epsilon(s)]ds \Big\|^2_{V'} dt = \\ &\quad = \int_0^T \Big\| \int_{\mathbf{R}} \Big[R'(t) - \frac{R(t)-R(t-s)}{s}\Big]u(t-s)[-s\rho'_\epsilon(s)]ds \Big\|^2_{V'} dt \leqslant \\ &\quad \leqslant \int_0^T \Big[\int_{\mathbf{R}} \Big\| \Big[R'(t) - \frac{R(t)-R(t-s)}{s}\Big]u(t-s) \Big\|_{V'} [-s\rho'_\epsilon(s)]ds \Big]^2 dt \leqslant \\ &\quad \leqslant \int_0^T \Big[\int_{\mathbf{R}} \delta \|u(t-s)\|_{V'} [-s\rho'_\epsilon(s)]ds \Big]^2 dt = \\ &\quad = \delta^2 \int_0^T \big(\|u(t)\|_V\big)^2_\epsilon dt \leqslant c\,\delta^2\,. \end{aligned}$$

Therefore we obtain that the last term in the equality in (10) is arbitrarly near to

$$(\mathcal{R}u)'_\epsilon(t) - \mathcal{R}' u_\epsilon(t)$$

and this converge in $\mathcal{V}'$ to $(\mathcal{R}u)' - \mathcal{R}'u$ as we wanted. $\square$

Denote $\tilde{H}_0(t) = H_0(t) = \mathrm{Ker}\, R(t)$ and

$$\tilde{H}(t)\,, \tilde{H}_+(t)\,, \tilde{H}_-(t) = \text{the completion respectively of } H\,, H_+(t)\,, H_-(t) \tag{11}$$

with respect to the norm $\|w\|_{\tilde{H}(t)} = \||R(t)|^{1/2} w\|_H$, where by $|R(t)|$ we mean $R_+(t) + R_-(t)$. By density of regular functions we obtain the formula of integration by parts and the continuity result for the functions in $\mathcal{W}$ stated in Proposition 2.6 which extends the following classical result (see, e. g., [16]).

Proposition 2.5 *If $\mathcal{R} = Id$ the embedding $\mathcal{W} \subset C([0,T]; H)$ is continuos, i. e. there is $c > 0$ such that $\max_{0 \leqslant t \leqslant T} \|u(t)\|_H \leqslant c\, \|u\|_{\mathcal{W}}$.*

Proposition 2.6 *For every $u, v \in \mathcal{W}$ the following holds:*

$$\frac{d}{dt}(\mathcal{R}u(t), v(t))_H = \langle \mathcal{R}'u(t), v(t)\rangle_{V' \times V} + \langle \mathcal{R}u'(t), v(t)\rangle_{V' \times V} + \langle \mathcal{R}v'(t), u(t)\rangle_{V' \times V}\,, \tag{12}$$

so in particular we have

$$\int_s^t \big\langle \mathcal{R}'u(\tau), v(\tau)\big\rangle_{V' \times V} d\tau + \int_s^t \big\langle \mathcal{R}u'(\tau), v(\tau)\big\rangle_{V' \times V} d\tau + \int_s^t \langle \mathcal{R}v'(\tau), u(\tau)\rangle_{V' \times V} d\tau =$$
$$= (R(t)u(t), v(t))_H - (R(s)u(s), v(s))_H\,.$$

Moreover the function $t \mapsto (R(t)u(t), v(t))_H$ is continuous and there exists a constant c, which depends only on $\|\mathcal{R}\|_{\mathcal{L}(L^p(0,T;H))}$ and T, such that

$$\max_{[0,T]} |(R(t)u(t), v(t))_H| \leqslant c\, \Big[\|(\mathcal{R}u)'\|_{\mathcal{V}'} \|v\|_{\mathcal{V}} + \|(\mathcal{R}v)'\|_{\mathcal{V}'} \|u\|_{\mathcal{V}} + \|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}\Big]\,. \tag{13}$$

In particular if $u = v$ we have

$$2\int_s^t \big\langle (\mathcal{R}u)'(\tau), u(\tau)\big\rangle_{V' \times V} d\tau =$$
$$= (R(t)u(t), u(t))_H - (R(s)u(s), u(s))_H + \int_s^t \langle \mathcal{R}'u(\tau), u(\tau)\rangle_{V' \times V} d\tau$$

or equivalently

$$\int_s^t \big\langle \mathcal{R}u'(\tau), u(\tau)\big\rangle_{V' \times V} d\tau + \frac{1}{2}\int_s^t \langle \mathcal{R}'u(\tau), u(\tau)\rangle_{V' \times V} d\tau =$$
$$= \frac{1}{2}\Big[(R(t)u(t), u(t))_H - (R(s)u(s), u(s))_H\Big]$$

and

$$\max_{[0,T]} |(R(t)u(t), u(t))_H| \leqslant c\, \|u\|^2_{\mathcal{W}}\,.$$

Proof - One can follow the proof in [16] (see ex. 23.10) or the one contained in [15] (see Lemma 40.2) and use the previous density result to obtain (12) for every $u, v \in C^1([0,T];V)$ and from this derive also (13). $\square$

REMARK 2.7 - If $\mathcal{R} \geqslant 0$ or if $\mathcal{R} \leqslant 0$ from Proposition 2.6 we derive the embedding of $\mathcal{W}$ in $C([0,T];\tilde{H}(t))$, and moreover that this is continuous. But in general we cannot derive, at least from Proposition 2.6, neither an inclusion of $\mathcal{W}$ in $C([0,T];\tilde{H}(t))$, because by (13) we only have that

$$\text{for } \textit{every } t \in [0,T] \quad |(R_+(t)u(t),u(t))_H - (R_-(t)u(t),u(t))_H| \quad \text{is finite,}$$

but only (see also Lemma 2.13)

$$\text{for } \textit{almost every } t \in [0,T] \quad |(R_+(t)u(t),u(t))_H| \quad \text{and} \quad |(R_-(t)u(t),u(t))_H| \quad \text{are finite}\,.$$

Moreover, in any case, even if we had the inclusion of $\mathcal{W}$ in $C([0,T];\tilde{H}(t))$, this could not be continuos, as the example below shows, since by (13) the natural estimation should be

$$\max_{[0,T]} |(|R(t)|u(t),u(t))_H| \leqslant c\, \left[\|u\|_{\mathcal{V}} + \|(|\mathcal{R}|u)'\|_{\mathcal{V}'}\right]^2 . \tag{14}$$

and not

$$\max_{[0,T]} |(|R(t)|u(t),u(t))_H| \leqslant c\, \left[\|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'}\right]^2 .$$

The following example show this fact: consider, for instance, a function $\varphi \in H^1_0(0,1)$, the function $\psi_\alpha(t) = \sqrt{\alpha}e^{-\alpha t^2}$ with $\alpha > 0$ and define $u_\alpha(x,t) = \psi_\alpha(t)\varphi(x)$. Consider the function $r(x)$ which is 1 in $(0,1/2)$ and -1 in $(1/2,1)$ and, for $v \in H^1_0(0,1)$, define $R_h v(x) = r(hx)v(x)$ with $h \in \mathbf{N}$.
Then $u_\alpha \in \{u \in L^2(-1,1;H^1_0(0,1)) \mid R_h u' \in L^2(-1,1;H^{-1}(0,1))\}$ for every $h \in \mathbf{N}$. But $\|u_\alpha\|_{C([-1,1];L^2(0,1))} = \sqrt{\alpha}\|\varphi\|_{L^2}$, $\|u_\alpha\|_{L^2(-1,1;H^1_0(0,1))} \leqslant c$ where c is independent of α and $\|R_h u'\|_{L^2(-1,1;H^{-1}(0,1))} \to 0$.
Since $|R_h| = Id$ for every h it is sufficient to choose α big enough to see that the last estimation above is not true.

In spite of the previous remark in some cases $\mathcal{W}$ is continuously embedded in $C([0,T];\tilde{H}(t))$, as in the example contained in [2] and as we state in the following proposition which generalizes Theorem 1 in [2].

Proposition 2.8 *If $\mathcal{R} : \mathcal{V} \to \mathcal{V}$ and moreover $\mathcal{R}$ is bounded from $\mathcal{V}$ to $\mathcal{V}$ the space $\mathcal{W}$ is included in $C([0,T];\tilde{H}(t))$ and the embedding is continuous.*

EXAMPLE 2.9 - $\mathcal{R}$ satisfies assumptions of the previous proposition if, for example, $V = H^1_0(\Omega)$, $H = L^2(\Omega)$ and $\mathcal{R} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is defined by

$$\mathcal{R}u(t)(x) = r(x,t)u(x,t)\,, \qquad \text{with } r \in L^\infty(0,T;W^{1,\infty}(\Omega))\,.$$

Proof - Suppose that $\mathcal{R}u \in \mathcal{V}$ for every $u \in \mathcal{V}$. Consider the space $\mathcal{Z} = \{u \in \mathcal{V} \mid u' \in \mathcal{V}'\}$. Notice that $\mathcal{Z} \subset \mathcal{W}$: indeed $\mathcal{R}'v \in \mathcal{V}'$ and moreover it is possible to define $\mathcal{R}v' : \mathcal{V} \to \mathcal{V}'$ as

$$\langle \mathcal{R}v', u\rangle_{\mathcal{V}'\times\mathcal{V}} := \langle v', \mathcal{R}u\rangle_{\mathcal{V}'\times\mathcal{V}}\,.$$

As done to obtain (13) one deduces that

$$\begin{aligned}
&\max_{[0,T]} |(R(t)u(t), v(t))_H| \leqslant \\
&\qquad \leqslant c \int_0^T \big[\langle \mathcal{R}'u(t), v(t)\rangle_{V'\times V} + \langle \mathcal{R}u'(t), v(t)\rangle_{V'\times V} + \langle \mathcal{R}v'(t), u(t)\rangle_{V'\times V}\big] dt \\
&\qquad \leqslant c \left[\|(\mathcal{R}u)'\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}\|\mathcal{R}u\|_{\mathcal{V}}\right].
\end{aligned}$$

Now we choose a particular function v in the space $\mathcal{Z}$. Fix an arbitrary $\bar{t} \in [0,T]$ and the function $\eta = 1$ on $H_+(\bar{t})$ and 0 on $H_0(\bar{t}) \oplus H_-(\bar{t})$. Then for $f \in \mathcal{V}'$ consider v_1 the solution to the problem

$$\begin{cases} -v' + \mathcal{A}v = f \\ v(\bar{t}) = (R_+(\bar{t}))^{-1/2}\eta\,, \end{cases}$$

on the interval $[0,\bar{t}]$ and v_2 the solution to the problem

$$\begin{cases} v' + \mathcal{A}v = f \\ v(\bar{t}) = (R_+(\bar{t}))^{-1/2}\eta\,, \end{cases}$$

on the interval $[\bar{t},T]$. Finally the function $v = v_1$ on $[0,\bar{t}]$ and $v = v_2$ on $[\bar{t},T]$. Clearly $v \in \mathcal{Z} \subset \mathcal{W}$. Applying Proposition 2.6 we in particular obtain

$$\begin{aligned}
|(R(\bar{t})u(\bar{t}), v(\bar{t}))_H| = \|R_+(\bar{t})^{1/2}u(\bar{t})\|_H \; &\leqslant c\left[\|(\mathcal{R}u)'\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}\|\mathcal{R}u\|_{\mathcal{V}}\right] \\
&\leqslant c\left[\|(\mathcal{R}u)'\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}\|\mathcal{R}\|_{\mathcal{L}(\mathcal{V},\mathcal{V})}\|u\|_{\mathcal{V}}\right] \\
&\leqslant c\,\|u\|_{\mathcal{W}}\,.
\end{aligned}$$

This clearly implies, since by Proposition 2.6 $|(R(\bar{t})u(\bar{t}), u(\bar{t}))_H|$ is controlled by $\|u\|_{\mathcal{W}}$, that also $\|R_-(\bar{t})^{1/2}u(\bar{t})\|_H$ is finite and bounded by the norm of u. Since $\bar{t}$ is arbitrary one conclude. $\square$

Remark 2.10 - More in general we have the inclusion if from $u \in \mathcal{W}$ we can derive $(\mathcal{R}_+u)'$, $(\mathcal{R}_-u)' \in \mathcal{V}'$ (see, for instance, example 9) in the last section).

In [7] it is stated the following lemma, by which one can deduce the theorem that follows: below we generalize these results to our situation.
Consider two reflexive Banach spaces B_0, B_1, a Hilbert space B such that $B_0 \subset B \subset B_1$ with continuous embeddings.

Lemma 2.11 *Suppose the embedding $B_0 \subset B$ is compact. For every $\eta > 0$ there exists $c_\eta > 0$ such that for every $v \in B_0$*

$$\|v\|_B \leqslant \eta\|v\|_{B_0} + c_\eta\|v\|_{B_1}\,.$$

Theorem 2.12 *Fix two numbers $p_0, p_1 \in (1,\infty)$ and $T > 0$. The space $\{v \in L^{p_0}(0,T;B_0) \mid v' \in L^{p_1}(0,T;B_1)\}$ compactly embeds in $L^{p_0}(0,T;B)$.*

Lemma 2.13 *Suppose the embedding $B_0 \subset B$ is compact. Let $R : B \to B$ linear, bounded, self-adjoint (then $|R|$ is linear, bounded, self-adjoint). Then for every $\eta > 0$ there exists $c_\eta > 0$ such that for every $v \in V$*

$$\||R|^{1/2}v\|_B \leqslant \eta\|v\|_{B_0} + c_\eta\|Rv\|_{B_1}\,.$$

Proof - If $R = 0$ it is trivial. Otherwise by contradiction suppose that for a fixed $\overline{\eta}$ the thesis is false. Then for every $h \in \mathbf{N}$ we can find $u_h \in B_0$, $\|u_h\|_{B_0} = 1$, such that

$$\||R|^{1/2}u_h\|_B \geqslant \overline{\eta}\|u_h\|_{B_0} + h\|Ru_h\|_{B_1} .$$

Then, since Ker $|R|^{1/2} =$ Ker R, adapting the proof of Lemma 5.1 in [7] one can conclude. □

Theorem 2.14 *Fix two numbers $p_0, p_1 \in (1, \infty)$ and $T > 0$. Consider a family $R : [0,t] \to \mathcal{L}(B)$ of linear, equibounded, self-adjoint operators which defines an operator $\mathcal{R} : L^{p_0}(0,T;B) \to L^{p_0}(0,T;B)$ such that $\mathcal{R} \neq 0$. Suppose the embedding $B_0 \subset B$ is compact and moreover the embedding $B \subset B_1$ is compact. Denote $\tilde{B}(t)$ the analogous to $\tilde{H}(t)$ in* (11).
Then the space $\mathcal{W} = \{v \in L^{p_0}(0,T;B_0) \mid (\mathcal{R}v)' \in L^{p_1}(0,T;B_1)\}$ compactly embeds in $L^{p_0}(0,T;\tilde{B}(t))$.

REMARK 2.15 - If $\mathcal{R}|_{L^{p_0}(0,T;B_0)} : L^{p_0}(0,T;B_0) \to L^{p_0}(0,T;B_0)$ it is not necessary to require that the embedding $B \subset B_1$ is compact.

Proof - We give a sketch of the proof, since even in this case one can follow the proof of Theorem 5.1 in [7]. Consider a sequence $(u_h)_h$ such that $\|u_h\|_{\mathcal{W}} \leqslant c$. In particular, up to a subsequence, $u_h \to u$ weakly in $L^{p_0}(0,T;B_0)$ and suppose $u = 0$. Then by the previous lemma

$$\||\mathcal{R}|^{1/2}u_h\|_{L^{p_0}(0,T;B)} \leqslant \eta\|u_h\|_{L^{p_0}(0,T;B_0)} + c_\eta\|\mathcal{R}u_h\|_{L^{p_1}(0,T;B_1)}$$

and defining $v_h = u_h/\|u_h\|_{L^{p_0}(0,T;B_0)}$ we have

$$\||\mathcal{R}|^{1/2}v_h\|_{L^{p_0}(0,T;B)} \leqslant \eta + c_\eta\|\mathcal{R}v_h\|_{L^{p_1}(0,T;B_1)} .$$

Since $\mathcal{R}v_h \in L^{p_0}(0,T;B)$ and $(\mathcal{R}v_h)' \in L^{p_1}(0,T;B_1)$ we have that $\mathcal{R}v_h \in C([0,T];B_1)$, so it is sufficient to prove that $\mathcal{R}v_h(t) \to 0$ strongly in B_1 for every $t \in [0,T]$. To prove this fact it is sufficient to follow the proof in [7]. The only thing to stress is that since we consider $\mathcal{R}v_h(t)$ which belong to B (while if $\mathcal{R} = Id$ one has $v_h(t) \in B_0$) in the proof we will need $B \subset B_1$ compact (while in the classical case one needs the embedding $B_0 \subset B_1$ compact which is free from hypothesis of Lemma 2.13). □

The constant c_η in Lemma 2.13 is in fact depending also on the operator R. Indeed if we consider a sequence of equibounded operators we need more assumptions and one can generalize Lemma 2.13 as follows.

Lemma 2.16 *Suppose $B_0 \subset B$ is compact. Let $R_h : B \to B$ a sequence of linear, bounded, self-adjoint operators such that one of the two following hypotheses is satisfied:*

i) $R_h u \to Ru$ in B for every $u \in B$,

ii) $R_h u \to Ru$ in B_1 for every $u \in B$ and R injective.

Then for every $\eta > 0$ there exists $c_\eta > 0$ such that for every $v \in V$

$$\||R_h|^{1/2}v\|_B \leqslant \eta\|v\|_{B_0} + c_\eta\|R_h v\|_{B_1} .$$

EXAMPLE 2.17 - The following example can clarify the assumptions of the lemma above: suppose $V = H^1_0(0,1)$ and $H = L^2(0,1)$. Define $r(x) = 1$ on $(0,a)$ and -1 on $(a,1)$ for $a \in (0,1)$ and $R_h u(x) = r_h(x)u(x) = r(hx)u(x)$. Notice that in this case $|R_h|u = u$ for every h. If $a = 1/2$ $R_h u \to 0$ in B_1 and we cannot hope that the lemma above is true (while for $a \neq 1/2$ condition ii) of Lemma 2.16 is satisfied).

Proof - As in the proof of Lemma 2.13 we find, by contradiction, a sequence v_h such that $\|v_h\|_{B_1} = 1$, such that

$$\||R|^{1/2} v_h\|_B \geqslant \overline{\eta} + h\|Rv_h\|_{B_1}$$

and then $\|Rv_h\|_{B_1} \to 0$ since $\||R|^{1/2} v_h\|_B \leqslant C\|v_h\|_B \leqslant C'\|v_h\|_{B_1} = C'$. From these we have, up to a subsequence,

$$v_h \to v \text{ in } B, \quad R_h v_h \to Rv \text{ weakly in } B, \quad R_h v_h \to 0 \text{ in } B_1 .$$

From this we obtain that $Rv = 0$ and then $|R|^{1/2} v = 0$. We want to conclude that $\||R|^{1/2} v_h\|_B \to 0$. If $R_h \to R$ strongly ($R_h u \to Ru$ in B for every $u \in B$) then $|R_h|^{1/2} \to |R|^{1/2}$ strongly from which $\||R|^{1/2} v_h\|_B \to 0$. Otherwise if R is injective we can conclude that $v = 0$, i. e. $\|v_h\|_B \to 0$ and from $\||R|^{1/2} v_h\|_B \leqslant C\|v_h\|_B$ we can conclude that $\||R|^{1/2} v_h\|_B \to 0$. In both cases we obtain a contradiction

$$0 > \overline{\eta} + \liminf h\|Rv_h\|_{B_1} \geqslant 0 . \qquad \square$$

In the same way the compactness result reads as follows.

Theorem 2.18 *Fix two numbers $p_0, p_1 \in (1, \infty)$ and $T > 0$. Consider a sequence $R_h : [0, T] \to \mathcal{L}(B)$ of linear, equibounded, self-adjoint which define operators $\mathcal{R}_h : L^{p_0}(0, T; B) \to L^{p_0}(0, T; B)$. Suppose $B_0 \subset B$ is compact and moreover $B \subset B_1$ is compact. Suppose $\mathcal{R}_h \to \mathcal{R}$ (see below) and denote $\tilde{B}(t)$ the analogous to $\tilde{H}(t)$ in (11) corresponding to the limit operator $\mathcal{R}$. Consider a sequence $(u_h)_h \subset \mathcal{V}$ such that $(\mathcal{R}_h u_h)' \in \mathcal{V}'$ and $\|u_h\|_{\mathcal{V}} + \|(\mathcal{R}_h u_h)'\|_{\mathcal{V}'} \leqslant c$ for any positive constant c. Then $(u_h)_h$ is relatively compact*

i) in $L^{p_0}(0, T; \tilde{B}(t))$ if $\mathcal{R}_h \to \mathcal{R}$ pointwise in $L^{p_0}(0, T; B)$

ii) in $L^{p_0}(0, T; B)$ if $\mathcal{R}_h \to \mathcal{R}$ pointwise in $L^{p_0}(0, T; B_1)$ and $\mathcal{R}$ is injective.

Example 2.19 - Boundedness $\|u_h\|_{\mathcal{V}} + \|(\mathcal{R}_h u_h)'\|_{\mathcal{V}'} \leqslant c$ it is not sufficient to have compactness. For example consider $H = L^2(0, 1)$ and $V = H^1_0(0, 1)$, $u_h(x, t) = \eta(x) \sin ht$ with $\eta \in C^1_c(0, 1)$ and $\mathcal{R}_h$ the operators which multiply by $\sin hx$. This sequence is bounded in the sense above, but does not converge in $L^2((0, 1) \times (0, T))$.

3 The existence result

Consider a family of operators

$$A(t) : V \longrightarrow V' \quad \text{with} \quad t \mapsto \langle A(t)u, v \rangle_{V' \times V} \quad \text{measurable on } [0, T]$$

such that if we define the abstract operator

$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}' , \qquad \mathcal{A}u(t) = A(t)u(t) \qquad 0 \leqslant t \leqslant T . \tag{15}$$

this turns out to be

$$\mathcal{A} \quad \text{pseudomonotone, coercive, bounded} \quad \text{(see Definition 3.4)} .$$

We denote

$$\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{V}', \qquad (\mathcal{P}u)(t) = (\mathcal{R}u)'(t) + \mathcal{A}u(t) \qquad 0 \leqslant t \leqslant T .$$

REMARK 3.1 - **The initial conditions** - By Proposition 2.6 we can evaluate the quantity $(R(t)u(t), u(t))_H$ for every $t \in [0, T]$ (and every $u \in \mathcal{W}$), but we are not sure $(R_+(t)u(t), u(t))_H$ and $(R_-(t)u(t), u(t))_H$ are finite for every t, at least not in every case (see also Remark 2.7). Since, for every $u \in \mathcal{W}$, $\mathcal{R}u \in L^p(0,T;H)$ and $(\mathcal{R}u)' \in L^{p'}(0,T;V')$ we have that the function $t \mapsto R(t)u(t)$ is continuous valued in V'. Define $V_+(0) := V \cap H_+(0)$, which is a subspace of V. Then

$$w \mapsto (R(0)u(0), w)_H = (R_+(0)u(0), w)_H \, , \quad w \in V_+(0)$$

is a linear and continuos form. Then one can give a pointwise meaning to $R_+(0)u(0)$ in the dual space of $V_+(0)$ (and more generally to $R_+(t)u(t)$ in $V_+(t) := V \cap H_+(t)$). In particular it makes sense to consider (see (11))

$$\varphi \in \tilde{H}_+(0) \qquad \text{and ask that} \quad R_+(0)u(0) = R_+(0)\varphi \, .$$

In this way if we denote the orthogonal projections

$$P_+(0) : \tilde{H}(0) \to \tilde{H}_+(0) \qquad \text{and} \qquad P_-(T) : \tilde{H}(T) \to \tilde{H}_-(T) \tag{16}$$

(and indeed one can define $P_+(t) : \tilde{H}(t) \to \tilde{H}_+(t)$, $P_-(t) : \tilde{H}(t) \to \tilde{H}_-(t)$, $P_0(t) : \tilde{H}(t) \to \tilde{H}_0(t)$ for every $t \in [0,T]$) and since $R_+(0)u(0) = R_+(0)P_+(0)u(0) = R_+(0)\varphi$ it makes sense to consider

$$P_+(0)u(0) = \varphi \qquad \text{in } \tilde{H}_+(0) \, .$$

In a similar way, for every $\psi \in \tilde{H}_-(T)$, one can give a meaning to

$$P_-(T)u(T) = \psi \qquad \text{in } \tilde{H}_-(T) \, .$$

Obviously since $(R(t)u(t), u(t))_H$ is finite for every t, if $P_+(0)u(0) = \varphi$ we obtain that

$$(R(0)u(0), u(0))_H = (R_+(0)\varphi, \varphi)_H - (R_-(0)u(0), u(0))_H$$

and since $\varphi \in \tilde{H}_+(0)$, $(R_+(0)\varphi, \varphi)_H$ is finite and necessarily also $(R_-(0)u(0), u(0))_H$ is finite. The same holds for $(R_+(T)u(T), u(T))_H$ if $P_-(T)u(T) = \psi$.

Thanks to the previous remark we can define

$$\mathcal{W}^0 = \{u \in \mathcal{W} \mid P_+(0)u(0) = 0 \, , P_-(T)u(T) = 0\} \, ,$$

$$\mathcal{W}^{\mathrm{per}} = \{u \in \mathcal{W} \mid (P_+ + P_-)u(0) = (P_+ + P_-)u(T)\}$$

(in the periodic case we consider $\mathcal{R}$ independent of t).
Consider the operators

$$\begin{aligned} &\mathcal{L}_i u = \mathcal{R}u' + \tfrac{1}{2}\mathcal{R}'u \, , \quad i = 1,2 \, , && D(\mathcal{L}_1) = \mathcal{W}^0 \, , \quad D(\mathcal{L}_2) = \mathcal{W}^{\mathrm{per}} \, , \\ &\mathcal{L}_i u = (\mathcal{R}u)' \, , \quad i = 3,4 \, , && D(\mathcal{L}_3) = \mathcal{W}^0 \, , \quad D(\mathcal{L}_4) = \mathcal{W}^{\mathrm{per}} \, . \end{aligned}$$

Proposition 3.2 *The operators $\mathcal{L}_i : D(\mathcal{L}_i) \subset \mathcal{V} \to \mathcal{V}'$, $i = 1, \ldots 4$, are maximal monotone if*

$$\langle \mathcal{R}'u, u \rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant 0 \qquad \textit{for every } u \in \mathcal{V} \, . \tag{17}$$

Assumption (17) *can be dropped if $i = 1, 2$.*

Proof - Denote for simplicity $\mathcal{L} = \mathcal{L}_i$ for some i till it is not ambiguous. The first step

$$\langle \mathcal{L}u, u\rangle \geqslant 0$$

follows from Proposition 2.6 since for $i = 1, 2$

$$\langle \mathcal{L}u, u\rangle = \frac{1}{2}\Big[(R(T)u(T), u(T))_H - (R(0)u(0), u(0))_H\Big] \geqslant 0$$

and indeed $\langle \mathcal{L}u, u\rangle = 0$ if $D(\mathcal{L}) = \mathcal{W}^{\mathrm{per}}$, while for $i = 3, 4$

$$\langle \mathcal{L}u, u\rangle = \frac{1}{2}\Big[(R(T)u(T), u(T))_H - (R(0)u(0), u(0))_H\Big] + \frac{1}{2}\langle \mathcal{R}'u, u\rangle$$

To see that it is maximal monotone we confine ourselves to $i = 1, 2$, being the proof for $i = 3, 4$ similar. Fix $w \in \mathcal{V}'$ and $v \in \mathcal{V}$ and suppose

$$\langle w - \mathcal{L}u, v - u\rangle \geqslant 0$$

for every $u \in D(\mathcal{L})$. We want to show that $v \in D(\mathcal{L})$ and $w = \mathcal{L}v$. If we define $z = v - u$ we obtain that

$$\langle w, z\rangle - \langle \mathcal{L}(v - z), z\rangle \geqslant 0\,.$$

Fix z and consider λz with $\lambda \in \mathbf{R}$. Since $\mathcal{L}$ is linear for every positive λ we obtain

$$\lambda\langle \mathcal{L}z, z\rangle \geqslant \langle \mathcal{L}v - w, z\rangle$$

and for every negative λ

$$\lambda\langle \mathcal{L}z, z\rangle \leqslant \langle \mathcal{L}v - w, z\rangle$$

for every $z \in v + D(\mathcal{L})$. Letting λ go to zero we obtain that $\langle w - \mathcal{L}v, z\rangle = 0$, i. e.

$$\langle w - \mathcal{L}v, v\rangle = \langle w - \mathcal{L}v, u\rangle$$

for every $u \in D(\mathcal{L})$ which implies $\langle w - \mathcal{L}v, u\rangle = 0$ for every $u \in D(\mathcal{L})$. If we consider $u(t) = \varphi(t)y$ with $y \in V$ and $\varphi \in C^1([0, T])$ we obtain

$$\int_0^T \big\langle w(t) - \mathcal{L}v(t), y\big\rangle \varphi(t)dt = 0$$

for every $\varphi \in C^1([0, T])$ and $y \in V$. Then $w(t) = \mathcal{L}v(t)$ in V' for almost every $t \in [0, T]$ and then $w = \mathcal{L}v$. Then $\mathcal{L}v \in \mathcal{W}$: since $\mathcal{L}v = \mathcal{R}v' + \frac{1}{2}\mathcal{R}'v = (\mathcal{R}v)' - \frac{1}{2}\mathcal{R}'v$ and $\mathcal{R}'v \in \mathcal{V}'$ we have that $(\mathcal{R}v)' \in \mathcal{V}'$ and then $v \in \mathcal{W}$. We have to see now that $v \in D(\mathcal{L})$.
Consider first $\mathcal{L}_1$: for every $u \in D(\mathcal{L}_1)$ we have that

$$\begin{aligned} 2\langle \mathcal{L}(v - u), v - u\rangle_{\mathcal{V}'\times\mathcal{V}} \;\; = \;\; & \big(R_+(T)(v(T) - u(T), (v(T) - u(T)\big)_H - \big(R_-(T)v(T), v(T)\big)_H + \\ & -\big(R_+(0)v(0), v(0)\big)_H + \big(R_-(0)(v(0) - u(0), (v(0) - u(0)\big)_H\,. \end{aligned}$$

Choose a sequence (u_n) bounded in $\mathcal{W}$ such that $u_n \to v$ in $\mathcal{V}$ and such that $P_+(T)u_n(T) = P_+(t)v(T)$ and $P_-(0)u_n(0) = P_-(0)v(0)$. Taking the limit we obtain

$$-\big(R_-(T)v(T), v(T)\big)_H - \big(R_+(0)v(0), v(0)\big)_H \geqslant 0$$

which implies that $\big(R_-(T)v(T), v(T)\big)_H = \big(R_+(0)v(0), v(0)\big)_H = 0$.
Consider now $\mathcal{L}_2$: for every $u \in D(\mathcal{L}_2)$ ($\mathcal{R}$ is independent of time in this case) we have that

$$\begin{aligned}
0 \leqslant 2\langle(\mathcal{R}(v-u))', v-u\rangle_{\mathcal{V}'\times\mathcal{V}} = \\
&= \big(R(v(T)-u(T), (v(T)-u(T)\big)_H - \big(R(v(0)-u(0), v(0)-u(0)\big)_H = \\
&= \big(Rv(T), v(T)\big)_H - \big(Rv(0), v(0)\big)_H + 2\big(R(v(0)-v(T)), u(0)\big)_H
\end{aligned}$$

since $\mathcal{R}$ is independent of t and $(Ru(0), u(0))_H = (Ru(T), u(T))_H$. This equality is true for every choice of $u(0) \in V$ and then $R(v(0)-v(T)) = 0$ in H, i. e. $v \in D(\mathcal{L}_2)$. □

Definition 3.3 *We say solution of the problem (see* (6) *for the definition of* $\mathcal{R}$ *and* (16) *for* $P_+(0)$ *and* $P_-(T)$*)*

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi\,, \end{cases}$$

$f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0), \psi \in \tilde{H}_-(T)$, *if*

$$(\mathcal{R}u)'(t) + \mathcal{A}u(t) = f(t) \qquad \text{for a. e. } t \in [0,T]$$

where we can consider $\varphi \in \tilde{H}_+(0), \psi \in \tilde{H}_-(T)$ *(see Remark* 3.1*).*

Under the assumption of the previous section we consider the following elliptic-parabolic problems

$$\text{(I)} \quad \begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = 0 \\ P_-(T)u(T) = 0 \end{cases} \qquad \text{(II)} \quad \begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+u(0) = P_+u(T) \\ P_-u(0) = P_-u(T) \end{cases} \tag{18}$$

In the periodic problem, the second one, we consider $\mathcal{R}$ independent of time. Notice that the periodic conditions mean $u(0) = u(T)$ but in the kernel of $R : H \to H$.

Before the main theorem we recall some definitions.

Definition 3.4 *We say that an operator* $B : X \to X'$ *is coercive if*

$$\lim_{\|x\|\to+\infty} \frac{\langle Bx, x\rangle}{\|x\|} \to +\infty\,,$$

is bounded if it maps a bounded set in a bounded set, is pseudomonotone if

$$x_n \to x \text{ in } X\text{-weak} \quad \text{and} \quad \limsup_n \langle Bx_n, x_n - x\rangle \leqslant 0$$

implies that

$$\langle Bx, x-y\rangle \leqslant \liminf_n \langle Bx_n, x_n - y\rangle \qquad \text{for every } y \in X\,.$$

We recall now a classical result (see, for instance, Section 32.4 in [16]).

Theorem 3.5 *Let $B : X \to X'$ (X' the dual space of X, X Banach space) be pseudomonotone, bounded and coercive. Suppose $L : X \to 2^{X'}$ to be maximal monotone. Then for every $f \in X'$ the following equation has a solution*

$$Lu + Bu \ni f$$

and in particular if L, B are single-valued the equation $Lu + Bu = f$ has a solution.

The idea now is to use the previous theorem for the equation $(\mathcal{R}u)' + \mathcal{A}u = f$. Notice that

$$(\mathcal{R}u)' + \mathcal{A}u = \mathcal{R}u' + \frac{1}{2}\mathcal{R}'u + \mathcal{A}u + \frac{1}{2}\mathcal{R}'u\,.$$

Theorem 3.6 *Suppose $\mathcal{R}$ satisfies assumptions* (5). *Suppose true one of the following*

i) $\mathcal{B} = \mathcal{A} + \frac{1}{2}\mathcal{R}'$ *pseudomonotone, corcive, bounded.*

ii) $\mathcal{B} = \mathcal{A}$ *pseudomonotone, corcive, bounded and* $\langle \mathcal{R}'u, u\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant 0$ *for every* $u \in \mathcal{V}$.

Then problems (18) *admit a solution for every* $f \in \mathcal{V}' = L^{p'}(0,T;V')$. *If moreover $\mathcal{B}$ is strictly monotone the solution is unique.*

Remark 3.7 - In fact we obtain an existence result also for the Cauchy problem

$$(\mathcal{R}u)' + \mathcal{A}u \ni f\,, \qquad u \in \mathcal{W}^0 \quad (\text{or } u \in \mathcal{W}^{\mathrm{per}})\,.$$

Moreover the solution is unique if the operator $\mathcal{B}$ of Theorem 3.6 is stricly monotone (see [14], chap. 4, and [6] for a more recent result with $\mathcal{R} \geqslant 0$)

Proof - By Theorem 3.5 (but see for more details Theorem 32.A, Corollary 32.25, Corollary 32.26 and also Proposition 27.6 in [16]) and Proposition 3.2 we obtain existence in $\mathcal{W}^0$ and $\mathcal{W}^{\mathrm{per}}$, i. e. for problems (18)-(I) and (18)-(II).
As regards uniqueness it is sufficient to observe that if u, v are two solutions we have

$$0 = \langle (\mathcal{R}(u-v))' + \mathcal{A}u - \mathcal{A}v, u - v\rangle \geqslant \langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geqslant 0.$$

Since $\mathcal{B}$ is stricly monotone we conclude that $u = v$. □

Now consider the Cauchy-Dirichlet problem with nonzero "initial" data

$$(\mathrm{III}) \begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi \end{cases} \tag{19}$$

and add some assumptions on $\mathcal{A}$. Before we assume also the following: define $V_+(0) = \{w \in V \mid [P_+(0)+P_0(0)]w \in V\} = V \cap (\tilde{H}_+(0) \oplus \tilde{H}_0(0))$ and $V_-(T) = \{w \in V \mid [P_-(T)+P_0(T)]w \in V\} = V \cap (\tilde{H}_-(T) \oplus \tilde{H}_0(T))$ (see (11) for the definition of $\tilde{H}_-, \tilde{H}_0, \tilde{H}_+$). Then we suppose

$$V_+(0) \neq \emptyset\,, V_-(T) \neq \emptyset\,, \qquad V_+(0) \text{ dense in } \tilde{H}_+(0)\,, \quad V_-(T) \text{ dense in } \tilde{H}_-(T)\,. \tag{20}$$

Then the following theorem holds.

Theorem 3.8 *Suppose* (20) *holds. Define an operator $\mathcal{P} : \mathcal{W} \to \mathcal{V}'$ by $\mathcal{P}u = (\mathcal{R}u)' + \mathcal{A}u$ where $\mathcal{R}$ satisfies* (5) *and $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ is continuous. Suppose that there exist two constants $\alpha, \beta > 0$ such that*

$$\begin{aligned} &\big\langle \mathcal{A}u - \mathcal{A}v + \tfrac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u - v \big\rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant \alpha \|u - v\|^2_{\mathcal{V}} \, , \\ &\|\mathcal{A}u + \tfrac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leqslant \beta \|u\|_{\mathcal{V}} \end{aligned} \tag{21}$$

or for some $p \in (1, +\infty)$, $p \neq 2$,

$$\begin{aligned} &\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant \alpha \|u - v\|^p_{\mathcal{V}} \, , \qquad \|\mathcal{A}u\|_{\mathcal{V}'} \leqslant \beta \|u\|^{p-1}_{\mathcal{V}} \\ &\langle \mathcal{R}'u, u \rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant 0 \end{aligned} \tag{22}$$

for every $u, v \in \mathcal{V}$. Then the following estimates hold:

(*i*) *there is a constant $c = c(\alpha, \beta, p)$ (depending only on α, β and p) such that for every $u \in \mathcal{W}^0$ and $u \in \mathcal{W}^{\mathrm{per}}$*

$$\|u\|_{\mathcal{W}} \leqslant c \; [\|\mathcal{P}u\|_{\mathcal{V}'} + \|\mathcal{P}u\|^{1/(p-1)}_{\mathcal{V}'}] \, ;$$

(*ii*) *there is a constant $c = c(\alpha, \beta, p)$ (depending only on α, β and p) such that for every $u \in \mathcal{W}$*

$$\|u\|_{\mathcal{W}} \leqslant c \, \Big[\|\mathcal{P}u\|_{\mathcal{V}'} + \|\mathcal{P}u\|^{1/(p-1)}_{\mathcal{V}'} + \|R^{1/2}_{-}(T)u(T)\|^{2/p}_{H_{-}(T)} + \|R^{1/2}_{+}(0)u(0)\|^{2/p}_{H_{+}(0)} \Big] \, .$$

Moreover for every $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_{+}(0)$, $\psi \in \tilde{H}_{-}(T)$ problem (19) *has a unique solution.*

Corollary 3.9 *If $\mathcal{R}$ does not depend on time the same holds requiring only that for some $p \in (1, +\infty)$, $\alpha, \beta > 0$ we have for every $u, v \in \mathcal{V}$*

$$\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant \alpha \|u - v\|^p_{\mathcal{V}} \, , \qquad \|\mathcal{A}u\|_{\mathcal{V}'} \leqslant \beta \|u\|^{p-1}_{\mathcal{V}} \, .$$

Proof of Theorem 3.8 - We prove the theorem assuming (22), being the proof assuming (21) similar, and indeed easier.
Since $u \mapsto (\mathcal{R}u)'$ is linear and monotone on $\mathcal{W}^0$ and $\mathcal{W}^{\mathrm{per}}$ and by (22) we have that

$$\alpha \|u\|^p_{\mathcal{V}} \leqslant \langle \mathcal{A}u, u \rangle \leqslant \langle \mathcal{P}u, u \rangle$$

by which we estimate $\|u\| \leqslant (1/\alpha \|\mathcal{P}u\|_{\mathcal{V}'})^{1/(p-1)}$. Since $(\mathcal{R}u)' = \mathcal{P}u - \mathcal{A}u$ we obtain that $\|(\mathcal{R}u)'\|_{\mathcal{V}'} \leqslant \|\mathcal{P}u\|_{\mathcal{V}'} + \beta \|u\|^{p-1}_{\mathcal{V}} \leqslant \|\mathcal{P}u\|_{\mathcal{V}'} + \beta/\alpha \|\mathcal{P}u\|_{\mathcal{V}'}$ so we have

$$\begin{aligned} &\|\mathcal{A}u\|_{\mathcal{V}'} \leqslant \frac{\beta}{\alpha} \|\mathcal{P}u\|_{\mathcal{V}'} \qquad \text{and} \qquad \|(\mathcal{R}u)'\|_{\mathcal{V}'} \leqslant \Big(1 + \frac{\beta}{\alpha}\Big) \|\mathcal{P}u\|_{\mathcal{V}'} \\ &\|u\|_{\mathcal{W}} \leqslant \Big[\frac{1}{\alpha} \|\mathcal{P}u\|_{\mathcal{V}'} \Big]^{1/(p-1)} + \Big(1 + \frac{\beta}{\alpha}\Big) \|\mathcal{P}u\|_{\mathcal{V}'} \, . \end{aligned} \tag{23}$$

To see point ii) we estimate ($q = p/(p-1)$), for every $\epsilon > 0$,

$$\|\mathcal{P}u\|_{\mathcal{V}'} \|u\|_{\mathcal{V}} \leqslant \frac{\epsilon^p}{p} \|u\|^p + \frac{1}{q} \Big(\frac{1}{\epsilon}\Big)^{q/p^2} \|\mathcal{P}u\|^q \tag{24}$$

and, by Proposition 2.6,

$$\begin{aligned} 2\langle(\mathcal{R}u)',u\rangle &= (R(T)u(T),u(T))_H - (R(0)u(0),u(0))_H + \langle \mathcal{R}'u,u\rangle \geqslant \\ &\geqslant (R_-(T)u(T),u(T))_H - (R_+(0)u(0),u(0))_H\,. \end{aligned}$$

Then, since $\langle \mathcal{A}u,u\rangle = \langle \mathcal{P}u,u\rangle - \langle(\mathcal{R}u)',u\rangle$, we have

$$\alpha\|u\|^p \leqslant \|\mathcal{P}u\|\|u\| + \frac{1}{2}\big[-(R_-(T)u(T),u(T))_H + (R_+(0)u(0),u(0))_H\big]$$

and using (24) we infer

$$\|u\|_{\mathcal{V}} \leqslant c(\alpha,p,\epsilon)\|\mathcal{P}u\|_{\mathcal{V}'}^{1/(p-1)} + \frac{1}{2^{1/p}}\Big[-(R_-(T)u(T),u(T))_H + (R_+(0)u(0),u(0))_H\Big]^{1/p}.$$

We conclude estimating $\|(\mathcal{R}u)'\|$ as in (23).
For problem (19) consider first $\Phi,\Psi\in V$ with $P_+(0)\Phi=\varphi$, $P_-(T)\Psi=\psi$ and define $\eta=\varphi+\psi$. The problem

$$\begin{cases} (\mathcal{R}v)' + \mathcal{A}(v+\eta) = f - \mathcal{R}'\eta \\ P_+(0)v(0) = 0 \\ P_-(T)v(T) = 0 \end{cases}$$

has a unique solution v. Indeed the operator $\tilde{\mathcal{A}}(v) := \mathcal{A}(v+\eta)$ is bounded and pseudomonotone. Moreover if we suppose (22) we see that $\tilde{\mathcal{A}}$ is also coercive:

$$\langle \tilde{\mathcal{A}}v,v\rangle = \langle \mathcal{A}(v+\eta)-\mathcal{A}\eta, v+\eta-\eta\rangle + \langle \mathcal{A}\eta,v\rangle\,.$$

Dividing by $\|v\|$ we obtain for the second term that

$$\Big|\frac{\langle \mathcal{A}\eta,v\rangle}{\|v\|}\Big| \leqslant \|\mathcal{A}\eta\|$$

and for the first

$$\frac{\langle A(v+\eta)-\mathcal{A}\eta, v+\eta-\eta\rangle}{\|v\|} \geqslant \alpha\|v\|^{p-1}\,.$$

Then the function $u(t)=v(t)+\eta$ satisfies (19).
In general we can consider, thanks to assumption (20), a sequence $\eta_h=\Phi_h+\Psi_h$ with $\Phi_h,\Psi_h\in V$ and $P_+(0)\Phi_h=\varphi_h\to\varphi$ in $\tilde{H}_+(0)$, $P_-(T)\Psi_h=\psi_h\to\psi$ in $\tilde{H}_-(T)$ and denote by u_h the solution corresponding to the data f,φ_h,ψ_h. As done for point ii) we can obtain

$$\begin{aligned} \alpha\|u-v\|_{\mathcal{V}}^p \leqslant \|\mathcal{P}u-\mathcal{P}v\|_{\mathcal{V}'}\|u-v\|_{\mathcal{V}} + \frac{1}{2}\big[-(R_-(T)(u(T)-v(T)),u(T)-v(T))_H + \\ +(R_+(0)u(0)-v(0),u(0)-v(0))_H\big] \end{aligned}$$

for every $u,v\in\mathcal{W}$. In particular for the solutions u_h and u_k we have

$$\alpha\|u_h-u_k\|^p \leqslant c\,\big[-(R_-(T)(\psi_h-\psi_k),\psi_h-\psi_k)_{H_-(T)} + (R_+(0)(\varphi_h-\varphi_k),\varphi_h-\varphi_k)_{H_+(0)}\big]\,.$$

Since $(\mathcal{R}u_h)'-(\mathcal{R}u_k)' = \mathcal{A}u_h-\mathcal{A}u_k$ we can estimate this difference in an analogous way to obtain that $u_h\to u\in\mathcal{W}$ and we define u to be the solution of the limit problem. □

Remark 3.10 - If $\mathcal{R} \geqslant 0$ ($\mathcal{R} \leqslant 0$) and $\mathcal{A}$ is linear we also have the corresponding existence results for the problem

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u + \lambda \mathcal{R}u = f \\ P_+(0)u(0) = \varphi \ \ (P_-(T)u(T) = \psi) \end{cases}$$

for every $\lambda \in \mathbf{R}$. It is sufficient indeed to consider the change of variable

$$v(t) = e^{\lambda t}u(t) \qquad (v(t) = e^{\lambda(t-T)}u(t))$$

to obtain

$$\begin{cases} (\mathcal{R}v)' + \mathcal{A}v = \tilde{f} = fe^{\lambda t} \ \ (fe^{\lambda(t-T)}) \\ P_+(0)u(0) = \varphi \ \ (P_-(T)u(T) = \psi) \end{cases}$$

which has a unique solution v. Then $u(t) = v(t)e^{-\lambda t}$ $(u(t) = v(t)e^{-\lambda(t-T)})$ solves the original problem.

Regularity - We conclude with a discussion about time regularity of the solution, already considered in [12]. We will see the result for

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \end{cases} \tag{25}$$

with $\mathcal{A}$ **linear (here** $p = 2$**)** and $\mathcal{R} \geqslant 0$ (obviously the analogous holds if $\mathcal{R} \leqslant 0$).
About the operator $\mathcal{A}$ we will make the following assumptions: that the family

$$A : [0, T] \to \mathcal{L}(V, V') \qquad \text{with} \qquad \mathcal{A}u(t) = A(t)u(t)$$

is such that A is regular in the sense of Definition 2.1 for a. e. $t \in (0, T)$ and

$$\left| \begin{array}{ll} t \mapsto \big\langle A(t)u, v \big\rangle_{V' \times V} & \text{absolutely continuos on } [0, T] \\ \alpha \|u\|_V^2 \leqslant \big\langle A(t)u, u \big\rangle_{V' \times V} \leqslant \beta \|u\|_V^2 & \\ |\big\langle A(t)u, v \big\rangle_{V' \times V}| \leqslant M \big\langle A(t)u, u \big\rangle_{V' \times V}^{1/2} \big\langle A(t)v, v \big\rangle_{V' \times V}^{1/2} & \\ |\dfrac{d}{dt} \big\langle A(t)u, v \big\rangle_{V' \times V}| = |\big\langle A'(t)u, v \big\rangle_{V' \times V}| \leqslant M' \|u\|_V \|v\|_V & \end{array} \right. \tag{26}$$

with $M' > 0$, for almost every $t \in (0, T)$ and for every $u, v \in V$. In this way we can define an operator $\mathcal{A}'$ as follows

$$A' : [0, T] \to \mathcal{L}(V, V') \qquad \text{with} \qquad \mathcal{A}'u(t) = A'(t)u(t)$$

bounded by M'. About the operator $\mathcal{R}$ we will assume (5) and the same for $\mathcal{R}'$, i. e. we define $\mathcal{R}'u(t) := R'(t)u(t)$ where R' is such that

$$\left| \begin{array}{l} R' : [0, T] \to \mathcal{L}(H)\,, \quad R'(t) \ \text{self-adjoint} \\ \sup_{t \in (0,T)} \|R'(t)\|_{\mathcal{L}(H)} \leqslant K_1\,, \\ t \mapsto \big(R'(t)u, v\big)_H \ \text{absolutely continuos on } [0, T]\,, \\ \dfrac{d}{dt}\big(R'(t)u, v\big)_H \leqslant K_2 \ \|u\|_V \|v\|_V \quad \text{for a. e. } t \in [0, T]\,. \end{array} \right. \tag{27}$$

As done for $\mathcal{R}$ we can define

$$\langle R''(t)u, v\rangle_{V'\times V} := \frac{d}{dt}\big(R'(t)u, v\big)_H$$

and an operator $\mathcal{R}''$ by

$$\mathcal{R}'' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \langle \mathcal{R}''u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \langle R''(t)u(t), v(t)\rangle_{V'\times V}\, dt$$

which turns out to be linear and bounded by K_2.

Theorem 3.11 *Consider the problem* (25). *Assume the operator* A *satisfies* (26). *We suppose* $\mathcal{R} \geqslant 0$ *and that it satisfies* (5) *and* (27). *Moreover we will suppose* $f \in H^1(0,T;V')$ *and the existence of* $u_0 \in V$ *such that* $P_+(0)u_0 = \varphi$ *in such a way that* $f(0) - A(0)u_0 - R'(0)u_0 \in \mathrm{Im}\ R(0)$. *Then the solution* u *satisfies*

$$u \in H^1(0,T;V)\,.$$

Remark 3.12 - Condition about φ is not so restrictive, on the contrary it is quite natural. Indeed one can choose, e. g., u_0 to be the solution of the problem

$$\begin{cases} \big[A(0) + R'(0)\big]w = f(0) \\ w \in V\,. \end{cases}$$

and consider $\varphi = P_+(0)u_0$.

Proof - Let u be the solution to (25) and consider

$$\begin{cases} (\mathcal{R}v)' + \mathcal{A}v = f' - \mathcal{A}'u - (\mathcal{R}'u)' \\ P_+(0)v(0) = R_+^{-1}(0)\Big[P_+(0)\big(f(0) - A(0)u_0 - R'(0)u_0\big)\Big] \end{cases} \tag{28}$$

This problem has a unique solution by Theorem 3.8 since $\mathcal{A} + \frac{1}{2}\mathcal{R}'$ is bounded and strictly coercive. Denote by v the solution to (28) and define

$$w(t) = u_0 + \int_0^t v(s)ds\,.$$

Integrating the equation in (28) one obtains

$$\mathcal{R}v(t) = -A(0)u_0 - R'(0)u_0 - \int_0^t \mathcal{A}v(s)ds - \int_0^t \mathcal{R}'v(s)ds + f(t) - \int_0^t \mathcal{A}'u(s)ds - \int_0^t \mathcal{R}''u(s)ds\,. \tag{29}$$

In this expression we can write

$$\int_0^t \mathcal{A}v(s)ds = \int_0^t \mathcal{A}w'(s)ds = \mathcal{A}w(t) - A(0)w(0) - \int_0^t \mathcal{A}'w(s)ds$$
$$\int_0^t \mathcal{R}'v(s)ds = \int_0^t \mathcal{R}'w'(s)ds = \mathcal{R}'w(t) - R'(0)w(0) - \int_0^t \mathcal{R}''w(s)ds$$

and finally, since

$$(\mathcal{R}w)'(t) = \mathcal{R}'w(t) + \mathcal{R}w'(t) = \mathcal{R}'w(t) + \mathcal{R}v(t)\,,$$

using (29) we compute

$$(\mathcal{R}w)'(t) - (\mathcal{R}u)'(t) + \mathcal{A}w(t) - \mathcal{A}u(t)\,.$$

We obtain

$$(\mathcal{R}w)'(t) - (\mathcal{R}u)'(t) + \mathcal{A}w(t) - \mathcal{A}u(t) = \int_0^t (\mathcal{A}'w(s) - \mathcal{A}'u(s))ds$$

therefore the function $w - u$ solves the problem

$$\begin{cases} (\mathcal{R}y)' + \mathcal{A}y = h \\ P_+ y(0) = 0 \end{cases}$$

where $h(t) = \int_0^t (\mathcal{A}'w(s) - \mathcal{A}'u(s))ds$. By hypotheses on $\mathcal{A}'$ we have that

$$\|h(t)\|_{V'} \leqslant \int_0^t \|\mathcal{A}'(w-u)(s)\|_{V'} ds \leqslant \int_0^t M'(s)\|w(s) - u(s)\|_V ds\,.$$

Let T' be the greatest value less or equal to T such that $w(t) = u(t)$ for a. e. $t \in [0, T']$. Then $h(t) = 0$ for $t \in [0, T']$. Then by Theorem 3.8, (i), there is constant $c = c(\alpha, \beta, p)$ such that

$$\begin{aligned}
\int_0^T \|w(t) - u(t)\|_V^2 dt &= \int_{T'}^T \|w(t) - u(t)\|_V^2 dt \\
&\leqslant c \int_{T'}^T \|h(t)\|_{V'}^2 dt \leqslant c \int_{T'}^T \Big(\int_0^t M'(s)\|w(s) - u(s)\|_V ds\Big)^2 dt \\
&= c \int_{T'}^T \Big[\int_0^t |M'(s)|^2 ds \int_0^t \|w(s) - u(s)\|_V^2 ds \Big] dt \\
&\leqslant c\,(T - T')\|M'\|_{L^2(0,T)}^2 \|w - u\|_{L^2(T',T;V)}^2
\end{aligned}$$

that is $1 \leqslant c\,(T - T')\|M'\|_{L^2(0,T)}^2$. Since every estimate is independent of T, letting T go to T' we obtain a contradiction and consequently $T' = T$. □

Corollary 3.13 *Under the same hypotheses as in Theorem* 3.11*, suppose for simplicity that* $f(0) - A(0)\varphi - R'(0)\varphi \in \mathrm{Im}\ R(0)$*, the solution* u *of* (25) *satisfies*

$$(\mathcal{R}u')' \in L^2(0, T; V')$$

and moreover there exists a positive constant c depending (only) on $\|\mathcal{A}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$, $\|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$, $\|\mathcal{R}''\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$, α, β *such that*

$$\begin{aligned}
&\|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}} + \|(\mathcal{R}u')'\|_{\mathcal{V}'} + \|u\|_{C([0,T];V)} \leqslant \\
&\leqslant c\,\Big[\|f\|_{\mathcal{V}'} + \|f'\|_{\mathcal{V}'} + \|R_+^{1/2}(0)\varphi\|_{H_+(0)} + \|R_+^{-1/2}(0)P_+(0)\big(f(0) - A(0)\varphi - R'(0)\varphi\big)\|_{H_+(0)} \Big].
\end{aligned}$$

Proof - The thesis follows from the fact that the function v by which the function w in the proof of the preceeding theorem is defined is the solution to (28), from the estimates of Theorem 3.8 and since the embedding $H^1(0, T; X) \subset C([0, T]; X)$ is continuous. □

REMARK 3.14 - In the general case, i. e. for a problem

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi \end{cases}$$

with $\mathcal{R}$ both negative and positive, we are not able to prove a regularity result as Theorem 3.11. May be the solution is regular, but surely obtaining an estimation as in Corollary 3.13 is impossible, as the following example shows.

EXAMPLE 3.15 - Consider the sequence of functions $(r_h)_h$ of Example 2.17, with $a \neq 1/2$, where their mean value is

$$\mu = \int_0^1 r_h(x)dx = a - 1/2 \neq 0\,,$$

consider $T > 0$, two functions φ, ψ defined in $(0,1)$, f in $(0,1) \times (0,T)$ and the problems

$$\begin{cases} r_h(x)\dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 0 & t \in (0,T) \\ P_{+,h}u(x,0) = \varphi(x) & x \in (0,1) \\ P_{-,h}u(x,T) = \psi(x) & x \in (0,1)\,, \end{cases}$$

i. e. we restrict φ as initial datum at time 0 where r_h is positive and ψ as final datum at time T where r_h is negative. Notice that $|R_h| = Id$ and $R'_h = |R_h|' = 0$.
Even if the solutions, with regular data, were in $H^1(0,T;H^1_0(0,1))$ they surely would be unbounded in the same space. In fact, since $\mu \neq 0$, by Theorem 2.18 we have that the sequence of solutions (u_h) is compact in $L^2(0,T;L^2(0,1))$, so we have a limit $u \in L^2(0,T;L^2(0,1))$. Moreover we have $\|u_h\|_{L^2(0,T;H^1_0(0,1))}$ and $\|R_h u'_h\|_{L^2(0,T;H^{-1}(0,1))}$ bounded, so it easy to see that, up to subsequences,

$$u_h \to u \in L^2(0,T;H^1_0(0,1)) - w\,, \qquad R_h u'_h \to ru' \in L^2(0,T;H^{-1}(0,1)) - w$$

and u satisfies the limit equation

$$\mu\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t)\,.$$

If $(u_h)_h$ were bounded in $H^1(0,T;H^1_0(0,1))$ we would also have $(u_h)_h$ compact in $C([0,T];L^2(0,1))$ and in particular

$$u_h(x,0) \to u(x,0) \quad u_h(x,T) \to u(x,T) \qquad \text{in } L^2(0,1)\,.$$

If we denote $\chi_h(x)$ the function which is 1 where $r_h = 1$ and 0 where $r_h = -1$ we have

$$u_h(x,0) = \chi_h(x)u_h(x,0) + (1-\chi_h(x))u_h(x,0) = \chi_h(x)\varphi(x) + (1-\chi_h(x))u_h(x,0)$$

and taking the limit

$$u(x,0) = a\varphi(x) + (1-a)u(x,0)$$

and then we deduce $u(x,0) = \varphi(x)$. In the same way we would obtain $u(x,T) = \psi(x)$ and so we would have a function u satisfying

$$\begin{cases} r\dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & (0,1)\times(0,T) \\ u(0,t) = u(1,t) = 0 & t \in (0,T) \\ u(x,0) = \varphi(x) & x \in (0,1) \\ u(x,T) = \psi(x) & x \in (0,1)\,, \end{cases}$$

which in general is impossible.

4 Examples

In this section we present some example of possible choices of $\mathcal{R}$. In what follows we consider $T > 0$, Ω open bounded subset of $\mathbf{R}^n$ with Lipschitz boundary. In the first examples we consider

$$r : \Omega \times [0,T] \to \mathbf{R} \qquad (\text{denoting by } r(t) : \Omega \to \mathbf{R} \text{ the function } r(t)(x) = r(x,t))\,.$$

For every $t \in [0,T]$ we denote by $\Omega_+(t)$, $\Omega_0(t)$, $\Omega_-(t)$ respectively the subsets of Ω in which $r(t)$ is positive, null, negative and $r_+(t) = r(t)|_{\Omega_+(t)}$, $r_-(t) = -r(t)|_{\Omega_-(t)}$. The examples we are going to show will be different situations of the following problem (we mean that our interest will be mainly directed on different choices of the function r)

$$\begin{cases} \dfrac{\partial}{\partial t}(ru) + \mathcal{A}u = f(x,t) & \text{on } \Omega\times(0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega\times(0,T) \\ u(x,0) = \varphi(x) & x \in \Omega_+(0) \\ u(x,T) = \psi(x) & x \in \Omega_-(T)\,. \end{cases} \tag{30}$$

In the former examples we confine ourselves to the following simple situation: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and then $\mathcal{V} = L^2(0,T;H_0^1(\Omega))$, $\mathcal{V}' = L^2(0,T;H^{-1}(\Omega))$ and $\mathcal{H}$ will be the space $L^2(0,T;L^2(\Omega))$. Therefore $f \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi \in L^2(\Omega_+(0), r_+(\cdot,0)), \psi \in L^2(\Omega_-(T), r_-(\cdot,T))$ where

r_+ is the positive part of r

$L^2(\Omega_+(0), r_+(\cdot,0))$ is the completion of $C_c(\Omega_+(0))$

with respect to the norm $\displaystyle \|w\|^2 = \int_{\Omega_+(0)} w^2(x) r_+(\cdot,0)dx\,.$

Analogously we define $L^2(\Omega_-(T), r_-(\cdot,T))$. In this examples we will suppose $\mathcal{A}$ to be a linear operator from $\mathcal{V}$ to $\mathcal{V}'$ defined by a family $A : [0,T] \to \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ verifying

$$\lambda_0 \|u\|^2_{H_0^1(\Omega)} \leqslant \langle A(t)u, u\rangle_{V'\times V} \leqslant \Lambda_0 \|u\|^2_{H_0^1(\Omega)}\,,$$

for every $u, v \in L^2(0,T;H_0^1(\Omega))$, where $\Lambda_0 \geqslant \lambda_0 > 0$. We define $\mathcal{R}$ by the family

$$R : [0,T] \longrightarrow \mathcal{L}(H)\,, \qquad (R(t)w)(x) = r(x,t)w(x)\,.$$

If $r = r(x)$ then Corollary 3.9 holds with $\alpha = \lambda_0$ and $\beta = \Lambda_0$.

1) If $r \equiv 1$, i. e. $R(t) = \mathrm{Id}$ for every t, then $\Omega_+(t) = \Omega$ for every $t \in [0,T]$ and $\Omega_0(t) = \Omega_-(t) = \emptyset$ for every $t \in [0,T]$. Then problem (30) is the classical Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}u = f(x,t) & \text{on } \Omega \times (0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = \varphi(x) & x \in \Omega\,. \end{cases}$$

If $R(t) = -\mathrm{Id}$ we have the inverse problem with the datum ψ at time T.

2) If $r \equiv 0$, i. e. $R(t) = 0$ for every t, then $\Omega_-(t) = \Omega_+(t) = \emptyset$ for every t and therefore we have the following problem

$$\begin{cases} \mathcal{A}u = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}$$

or equivalently a family (the parameter is t) of elliptic problems

$$\begin{cases} \mathcal{A}u(t) = A(t)u(t) = f(t) & \text{on } \Omega \quad \text{for a. e. } t \in [0,T] \\ u(t) = 0 & \text{on } \partial\Omega \quad \text{for a. e. } t \in [0,T]\,. \end{cases}$$

3) Suppose $r = r(x)$. Suppose first $r \geqslant 0$. Then, in order to have (5) satisfied, we only need $r \in L^\infty(\Omega)$ (being constant with respect to t the operator $\mathcal{R}$ turns out to be regular). Notice that no regularity on r is required, i. e. r could be also a characteristic function

$$r(x) = \begin{cases} 1 & \text{on } \Omega_+ \\ 0 & \text{on } \Omega_0\,. \end{cases}$$

Then **every non-negative** $r \in L^\infty(\Omega)$ is admitted and in particular Ω_+ can be also a *Cantor set*.
Indeed $H^1_0(\Omega)$ is dense in $L^2(\Omega, r)$ and assumption (20) is verified. Obviously if the measure of Ω_+ is zero, problem (30) is to be intended as explained in Definition 3.3, i. e. pointwise, for almost every $t \in [0,T]$, in $H^{-1}(\Omega)$, and then if $r(x) = 0$ for a. e. $x \in \Omega$ the problem is to be considered as a family of elliptic problems in the parameter t and the initial datum has no meaning. But if Ω_+ is a Cantor set of positive measure one can consider a problem which is "parabolic" on a Cantor set and "elliptic" on the complementary.

Now consider $r \in L^\infty(\Omega)$ not necessarily non-negative. Every function for which Ω_+ and Ω_- are open and Ω_+, Ω_- and Ω_0 have Lipchitz boundary is admitted. It is also adimitted a situation in which, for instance,

$$r(x) = \begin{cases} 1 & \text{on } \Omega_+ \\ 0 & \text{on } \Omega_0 \\ -1 & \text{on } \Omega_-\,, \end{cases}$$

with Ω_+ and Ω_- closed sets such that there are two open sets A_1, A_2 with

$$A_1 \cap A_2 = \emptyset\,, \qquad \Omega_+ \subset A_1\,, \qquad \Omega_- \subset A_2$$

in order to have (20) satisfied (possibly Ω_+ and Ω_- Cantor sets).

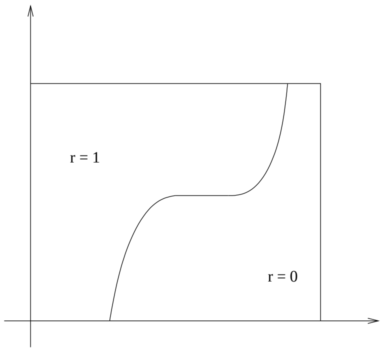

Figure 1: example of a **not** admitted situation.

4) Suppose $r = r(t)$. In this case we need some regularity. Assumptions in (5) are satisfied if $r \in W^{1,\infty}(0,T)$. If $r \geqslant c > 0$ the problem (30) is a standard forward parabolic problem with initial datum φ while if $r \leqslant c < 0$ we have a standard backward parabolic problem with final datum ψ. But suppose $r(0) \leqslant 0$ and $r(T) \geqslant 0$ or $r(0) \geqslant 0$ and $r(T) \leqslant 0$. We can admitt $r \in W^{1,\infty}(0,T)$ such that

$$\mu_0 = \operatorname*{ess\,inf}_{(0,T)} r'(t) \leqslant 0 \qquad \text{with} \qquad \lambda_0 + \frac{1}{2}\mu_0 c > 0\,,$$

where c is such that $\|u\|_{L^2} \leqslant c\,\|u\|_{H^1_0}$, so that (21) is satisfied. Then if $r'(t) > 0$ and $r(0) \leqslant 0$ and $r(T) \geqslant 0$ problem (30) is

$$\begin{cases} r(t)\dfrac{\partial u}{\partial t} + r'(t)u + \mathcal{A}u = f(x,t) & \text{on } \Omega \times (0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \end{cases}$$

without any initial and final data. A situation as that in Figure 1 is not admitted because γ' is not bounded. If $r'(t) < 0$ and $r(0) \geqslant 0$ and $r(T) \leqslant 0$ problem (30) is

$$\begin{cases} r(t)\dfrac{\partial u}{\partial t} + r'(t)u + \mathcal{A}u = f(x,t) & \text{on } \Omega \times (0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = \varphi(x) & x \in \Omega \\ u(x,T) = \psi(x) & x \in \Omega\,. \end{cases}$$

with initial and final data in the whole Ω.

5) Suppose now $r = r(x,t)$ and $r, \dfrac{\partial r}{\partial t} \in L^\infty(\Omega \times (0,T))$. We have that

$$\begin{aligned} (R(t)u,v)_H &= \int_\Omega u(x)v(x)r(x,t)dx \\ |(R'(t)u,v)_H| &= \Big|\int_\Omega u(x)v(x)\frac{\partial r}{\partial t}(x,t)dx\Big| \leqslant \\ &\leqslant \|\frac{\partial r}{\partial t}\|_\infty \|u\|_{L^2}\|v\|_{L^2} \leqslant \|\frac{\partial r}{\partial t}\|_\infty \|u\|_{H^1_0}\|v\|_{H^1_0} \end{aligned}$$

We can admit function r such that ($C_1, C_2 \geqslant 0$ arbitrary)

$$-C_1 \leqslant r \leqslant C_1\,, \qquad \mu_0 \leqslant \frac{\partial r}{\partial t} \leqslant C_2$$

with

$$\mu_0 \leqslant 0 \qquad \text{and} \qquad \lambda_0 + \frac{1}{2}\mu_0 c > 0$$

where c is the constant of example 4).

6) Suppose now $r = r(x,t)$, $r \in L^\infty(\Omega \times (0,T))$, but $\dfrac{\partial r}{\partial t} \notin L^\infty(\Omega \times (0,T))$. For simplicity consider the case we are given A and B subsets of $\Omega \times (0,T)$ and

$$r(x,t) = \chi_A(x,t) - \chi_B(x,t) \qquad \text{i. e.} \qquad r : \Omega \times (0,T) \to \{-1,0,1\}$$

where $\chi_E(y) = 1$ if $y \in E$ and 0 if $y \notin E$. We have that

$$R(t) = \mathrm{Id}_{H_+(t)} - \mathrm{Id}_{H_-(t)} \tag{31}$$

that is

$$(R(t)u,v)_{L^2(\Omega)} = \int_{\Omega_+(t)} u(x)v(x)dx - \int_{\Omega_-(t)} u(x)v(x)dx\,.$$

In this case to have conditions (5) satisfied we need for every $u, v \in H_0^1(\Omega)$

$$\begin{aligned}
&t \mapsto \int_{\Omega_+(t)} u(x)v(x)dx - \int_{\Omega_-(t)} u(x)v(x)dx \qquad \text{differentiable} \\
&\Big|\frac{d}{dt}\Big(\int_{\Omega_+(t)} u(x)v(x)dx - \int_{\Omega_-(t)} u(x)v(x)dx\Big)\Big| \leqslant C_2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\,.
\end{aligned} \tag{32}$$

Just to consider a simple but meaningful situation fix $n = 1$, suppose $A \cup B = \Omega \times (0,T)$, i. e. $r : \Omega \times (0,T) \to \{-1,1\}$, $\Omega = (0,L)$, $\Omega_+(t) = (0,\gamma(t))$, $\Omega_-(t) = (\gamma(t),L)$ ($\gamma : [0,T] \to [0,L]$). Then

$$\big(R(t)u,v\big)_{L^2(0,L)} = \int_0^{\gamma(t)} u(x)v(x)dx - \int_{\gamma(t)}^L u(x)v(x)dx$$

and its derivative is defined by

$$\langle R'(t)u,v\rangle_{H^{-1}(0,L)\times H_0^1(0,L)} = 2\gamma'(t)u(\gamma(t))v(\gamma(t))\,.$$

Then conditions (5) and (21) are satisfied if γ is differentiable, $\gamma' \in L^\infty$ and

$$\langle A(t)u,u\rangle_{H^{-1}\times H_0^1} + \gamma'(t)u^2(\gamma(t)) \geqslant \alpha\|u\|^2_{H_0^1}$$

for some positive constant α, that is if, denoted by c the constant such that $\|u\|_{C^0([0,T])} \leqslant c\|u\|_{H_0^1(0,L)}$ we can admit γ such that

$$\gamma \in W^{1,\infty}(0,L)\,, \qquad \operatorname*{ess\,inf}_{[0,T]} \gamma'(t) > -\frac{\lambda_0}{c^2}\,.$$

If $\mathcal{A}$ were non linear and $p \neq 2$, to satisfy (22) we can ask

$$\gamma \in W^{1,\infty}(0,L)\,, \qquad \gamma'(t) \geqslant 0\,.$$

If, on the contrary, $\Omega_-(t) = (0, \gamma(t))$, $\Omega_+(t) = (\gamma(t), L)$ we would have $\langle R'(t)u, v\rangle = -2\gamma'(t)u(\gamma(t))v(\gamma(t))$ and then we would admit γ such that

$$\gamma \in W^{1,\infty}(0,L)\,, \qquad \operatorname*{ess\,sup}_{[0,T]} \gamma'(t) < \frac{\lambda_0}{c^2}$$

and $\gamma'(t) \leqslant 0$ if $\mathcal{A}$ non linear with $p \neq 2$.
If the dimension $n \geqslant 2$ and $r : \Omega \times (0,T) \to \{-1, 0, 1\}$, we need to differentiate with respect to t the function

$$F(t) = \int_{\Omega_+(t)} u(x)v(x)dx - \int_{\Omega_-(t)} u(x)v(x)dx\,.$$

This is differentiable if $\Omega_+(t)$ and $\Omega_-(t)$ are open and the interfaces separating $\Omega_+(t)$, $\Omega_0(t)$, $\Omega_-(t)$ are Lipschitz continuous (see, e. g., Proposition 3, section 3.4.4, in [5]). Moreover, since $u, v \in H^1_0(\Omega)$, it makes sense to consider the trace on the interfaces (see, e. g., Theorem 1, section 4.3, in [5]).

Now we show some other examples: first we show that r may be also an unbounded function in L^1, then an example where $\mathcal{R}$ is an integral operator. Finally an example in which the embedding $\mathcal{W} \subset C([0,T]; \tilde{H}(t))$ is continuous.

7) It is also possible to consider *unbounded* coefficients (see [12]). Consider a function $\mu \in L^1(\Omega \times (0,T))$. In this case we consider, for a. e. $t \in (0,T)$, $H(t) = L^2(\Omega, |\mu(\cdot,t)|)$ and the equibounded operators $R(t)$ will be

$$R(t) = \mathrm{Id}_{H_+(t)} - \mathrm{Id}_{H_-(t)}$$

and observe that that is

$$\begin{aligned}
(R(t)u,v)_{H(t)} &= (R(t)u,v)_{L^2(\Omega,|\mu(\cdot,t)|)} = \\
&= \int_{\Omega_+(t)} u(x)v(x)|\mu(x,t)|dx - \int_{\Omega_-(t)} u(x)v(x)|\mu(x,t)|dx \\
&= \int_{\Omega} u(x)v(x)\mu(x,t)dx\,.
\end{aligned}$$

8) Let $H = L^2(\Omega)$ and $V = H^1_0(\Omega)$. Consider a function $r \in L^\infty(\Omega \times \Omega \times (0,T))$ and define $R(t) : H \to H$ as

$$\Big[R(t)u\Big](x) := \int_\Omega r(x,y,t)u(y)dy\,.$$

If this operator satisfies (5) under assumptions required on Theorem 3.8 we obtain an existence result for the problem

$$\begin{cases}
\dfrac{\partial}{\partial t}\Big[\displaystyle\int_\Omega r(x,y,t)u(y,t)dy\Big] + \mathcal{A}u(x,t) = f(x,t) & \text{on } \Omega \times (0,T) \\
u = 0 & \text{on } \partial\Omega \times (0,T) \\
u(x,0) = \varphi(x) & \text{on } \Omega_+(0) \\
u(x,T) = \psi(x) & \text{on } \Omega_-(T)
\end{cases}$$

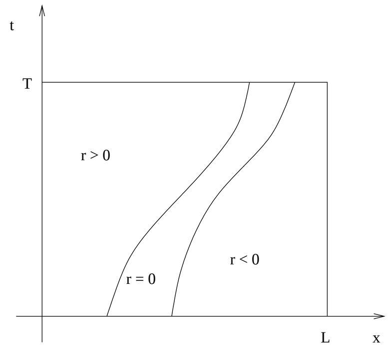

Figure 2:

9) We conclude with some examples in which inclusion $\mathcal{W} \subset C([0,T];\tilde{H}(t))$ holds. Consider as before $H = L^2(\Omega)$ and $V = H^1_0(\Omega)$.
A first situation is that stated in Proposition 2.8 and Example 2.9, i. e. if $r \in L^\infty(0,T;W^{1,\infty}(\Omega))$ so that

$$D_i(ru) = rD_iu + uD_ir \in L^2(\Omega)\,, \quad \text{i. e. } r(\cdot,t)u(\cdot,t) \in H^1_0(\Omega) \quad \text{for a. e. } t\,.$$

Another situation in which this inclusion is true is that $(\mathcal{R}_+u)', (\mathcal{R}_-u)' \in \mathcal{V}'$ (see Remark 2.10). This is true, for example, if (see Figure 2)

$$\text{dist }(\Omega_+,\Omega_-) > 0\,. \tag{33}$$

In this situation for every $u \in \mathcal{W}$ we can consider $\phi_1, \phi_2 \in C^1(\overline{\Omega})$ such that

$$\phi_1 = 1 \text{ in } \Omega_+\,, \qquad \phi_1 = 0 \text{ in } \Omega_-\,, \qquad\qquad \phi_2 = 0 \text{ in } \Omega_+\,, \qquad \phi_2 = 1 \text{ in } \Omega_-\,.$$

Then also $\phi_1u, \phi_2u \in \mathcal{W}$ and applying Proposition 2.6 by (13) we in particular infer that

$$|(R\phi_1u(0),\phi_1u(0))_H| = (R_+u(0),u(0))_H = \int_\Omega u^2(x,0)r_+(x)dx < +\infty$$
$$|(R\phi_2u(T),\phi_2u(T))_H| = (R_-u(T),u(T))_H = \int_\Omega u^2(x,0)r_-(x)dx < +\infty$$

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