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CRACK GROWTH WITH NON-INTERPENETRATION: A SIMPLIFIED PROOF FOR THE PURE NEUMANN PROBLEM

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Abstract. We present a recent existence result concerning the quasistatic evolution of cracks in hyperelastic brittle materials, in the framework of finite elasticity with non-interpenetration. In particular, here we consider the problem where no Dirichlet conditions are imposed, the boundary is traction-free, and the body is subject only to time-dependent volume forces. This allows us to present the main ideas of the proof in a simpler way, avoiding some of the technicalities needed in the general case, studied in [9].

1. **Introduction.** In this paper we consider some problems of fracture mechanics for *brittle materials*: we assume a perfectly elastic behaviour out of the cracks and no force transmission through the cracks. We are interested in *quasistatic evolutions*, i.e., motions that are so slow that the system remains in equilibrium at each instant. The time scale of external loads is larger than the intrinsic time scale of the process, so in our analysis we do not "see" the oscillations of the system and we can ignore all dynamical effects.

The physical model relies on Griffith's principle [18] that the propagation of a crack is the result of the competition between the elastic energy released when the crack opens and the energy spent to produce new crack. We focus our attention on the variational model proposed in Francfort-Marigo [14]: it is based on time discretization and on the solution to some incremental minimum problems, involving the stored elastic energy and the energy dissipated by the crack. The continuous-time evolution is then obtained in the limit as the time step tends to zero. This is a standard scheme also in the treatment of other rate-independent processes (see Mielke [20] and the references therein). Notice that in this model the crack path is not prescribed a priori and is determined by the energy criterion.

The state of the system is described by a pair of variables (u, Γ), where Γ is the crack, an $(n-1)$-dimensional subset of the reference configuration $\Omega \subset \mathbb{R}^n$, and u is the deformation, defined in $\Omega \backslash \Gamma$. The total energy of the system at time $t \in [0, T]$, which will be minimized under suitable constraints in the incremental problems, is given by

$$\mathcal{E}(t)(u, \Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma) - \mathcal{G}(t)(u)\,, \tag{1}$$

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where $\mathcal{W}(u)$ is the bulk energy, $\mathcal{K}(\Gamma)$ is the energy dissipated to open the crack Γ, and $\mathcal{G}(t)(u)$ is the potential of volume forces acting at time t. Notice that the functional is highly non-convex, so we cannot expect uniqueness of minimizers.

In the first existence results for quasistatic evolutions, the dimension n of the space is 2, the crack Γ is supposed to be a one-dimensional connected closed set, the bulk energy is quadratic, and the deformation u is a Sobolev function on the domain $\Omega\backslash\Gamma$. These existence results are obtained in Dal Maso-Toader [10] for the antiplane case and in Chambolle [5] for the case of planar elasticity.

In the formulation of Francfort-Larsen [13], the function u is scalar and belongs to the space of special functions of bounded variation $SBV(\Omega)$, introduced by De Giorgi-Ambrosio [11], the bulk energy $\mathcal{W}(u)$ is still quadratic, while the crack Γ is a rectifiable set containing the jump set $S(u)$ of u. This allows them to treat the problem for an arbitrary space dimension n and to avoid the topological restrictions on the cracks, which have no mechanical justification.

These results are generalized in Dal Maso-Francfort-Toader [8], with a non-quadratic bulk energy term

$$\mathcal{W}(u) := \int_{\Omega\backslash\Gamma} W(\nabla u(x))\,\mathrm{d}x$$

and a vector-valued deformation $u\colon \Omega\backslash\Gamma \to \mathbb{R}^n$. The function W is only assumed to be quasiconvex and to satisfy the polynomial growth condition $c\,|A|^p \leq W(A) \leq C\,|A|^p$, with $c, C > 0$ and $p > 1$. This requires a slightly different functional setting and suitable coercivity hypotheses on the potential of external forces $\mathcal{G}(t)$.

The usual assumption in finite elasticity is that the determinant of the deformation gradient is positive on all deformations with finite energy, and that the strain energy diverges when this determinant tends to zero:

$$\mathcal{W}(u) = +\infty \text{ if } \det\nabla u \leq 0 \qquad \text{and} \qquad \mathcal{W}(u) \to +\infty \text{ if } \det\nabla u \to 0^+\,. \tag{2}$$

Unfortunately, (2) is incompatible with polynomial growth, which is a basic tool in the above mentioned articles [10, 5, 13, 8] for proving lower semicontinuity and controlling energy from above. In [9] we extend the previous results, adopting some general hypotheses compatible with finite elasticity, that were introduced in Ball [3], Francfort-Mielke [15], and Fusco-Leone-March-Verde [16].

In this paper we present the results of [9] under the simplifying assumption that the body is subject only to time-dependent volume forces, no Dirichlet conditions are imposed, and the boundary is traction-free, so that the natural Neumann conditions are satisfied. This allows us to present the main ideas of the proof in a simpler way, avoiding some technicalities.

The first step in this proof is the existence of solutions to the incremental minimum problems for (1) under our new assumptions (see Remark 5). Then, we show that the approximate solutions obtained by incremental minimization converge, as the time step tends to zero, to a function $t \mapsto (u(t), \Gamma(t))$ (up to a subsequence). We prove also that for every t the pair $(u(t), \Gamma(t))$ satisfies a suitable minimality condition involving the functional (1), called *global stability* (see (23)). Finally, we obtain the *energy-dissipation balance* (see (24)), which states that the time derivative of the internal energy $\mathcal{E}^{\mathrm{int}}(u(t), \Gamma(t)) := \mathcal{W}(u(t)) + \mathcal{K}(\Gamma(t))$ equals the power of the external forces. These results prove the existence (see Theorem 4.1) of a quasistatic evolution with prescribed initial conditions, according to Definition 3.1.

We impose on the solutions a strong *non-interpenetration* requirement, the so-called Ciarlet-Nečas condition [7] (see Definition 2.2). The deformations must be globally invertible and orientation-preserving. The latter property is ensured by the finiteness of the energy (see (2)). A stability theorem for the global invertibility condition was recently proven in GIACOMINI-PONSIGLIONE [17] in the SBV setting: this result is crucial for our treatment of the problem.

There are three main difficulties in passing from the polynomial growth condition to the context of finite elasticity:

- lower semicontinuity of the bulk energy,
- upper bound for the energy,
- jump transfer.

When one has to prove the lower semicontinuity of the bulk energy, all theorems for quasiconvex functions require a polynomial growth, forbidden by (2); on the other hand, the convexity assumption is not compatible with finite elasticity, as shown in [3]. We overcome this difficulty by assuming the intermediate property of polyconvexity: this allows us to apply a recent result [16], which requires only suitable bounds from below (see Theorem 2.4).

To control the energy from above, we replace polynomial growth by an upper bound compatible with (2): namely, we suppose that the *energy-momentum tensor* $P(A) := W(A)I - A^{\mathrm{T}}\mathrm{D}_A W(A)$ satisfies the inequality

$$|P(A)| \leq c^1_W (W(A) + c^0_W)\,, \tag{3}$$

where $c^0_W \geq 0$ and $c^1_W > 0$ are two constants The *multiplicative stress estimate* (3) is well known in mechanics [3] and holds, for instance, in the case of OGDEN materials [21, 22], a class of natural rubbers (see Section 2.2).

As we will see in Section 4.2, to prove the global stability condition for the limit pair $(u(t), \Gamma(t))$ we have to employ the Jump Transfer Lemma, first established in [13]. Since the original construction uses a reflection argument, which is forbidden by (2), we must modify the proof: reflections are substituted by suitable dilations and the corresponding energies are controlled thanks to (3).

When time-dependent Dirichlet boundary conditions are imposed, there are some further difficulties that require the use of the multiplicative decomposition introduced in [15]. In this paper we do not address this problem, which is completely solved in [9, 19].

2. **The mechanical assumptions.**

We present here a variational model for quasistatic crack growth in brittle hyperelastic materials with non-interpenetration, following FRANCFORT-MARIGO [14].

2.1. **Cracks and deformations.**

We consider an elastic body, whose *reference configuration* is a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. The physical cases are $n = 2$ and $n = 3$; however, the mathematical theory can be developed for every $n \geq 2$. The state of the system is determined by two variables: the crack set $\Gamma \subset \Omega$ (assumed to be $(n-1)$-dimensional) and the deformation $u\colon \Omega\backslash\Gamma \to \mathbb{R}^n$. We assume that the uncracked part of the material is hyperelastic.

For mathematical reasons it is convenient to consider u as a function defined almost everywhere (a.e.) on Ω, with a discontinuity set contained in Γ. A suitable functional framework, chosen, e.g., by FRANCFORT-LARSEN [13], is the space of special functions of bounded variation $SBV^p(\Omega;\mathbb{R}^n)$, introduced by DE GIORGI-AMBROSIO in the context of free-discontinuity problems [11]. The precise choice of the exponent $p > 1$ will be done later.

For every $u \in SBV^p(\Omega;\mathbb{R}^n)$ we can give a precise definition [2] of the jump set $S(u)$, which is $(\mathcal{H}^{n-1}, n-1)$-rectifiable [12], and of the approximate gradient $\nabla u \in L^p(\Omega;\mathbb{M}^{n\times n})$, where $\mathbb{M}^{n\times n}$ is the space of $n\times n$ real matrices. Notice that ∇u does not coincide with the distributional derivative; for instance, for piecewise smooth functions, ∇u is the gradient of u out of $S(u)$, but it does not include the contribution of the jump to the distributional derivative.

We fix a compact set with Lipschitz boundary K such that $\Omega \subset K$: it represents a *container* for all the deformed configurations, i.e., we suppose $u(x) \in K$ for a.e. $x \in \Omega$, for every deformation u of Ω. The space of functions $u \in SBV^p(\Omega;\mathbb{R}^n)$ that satisfy this condition is denoted by $SBV^p(\Omega;K)$. This confinement assumption ensures an a-priori bound on u in $L^\infty(\Omega;\mathbb{R}^n)$, and avoids some problems due to the non-coercivity of the energy functional, which led to a slightly different choice of the functional setting in [8].

In the approximation procedures, we will use the following notion of convergence.

Definition 2.1. A sequence u_k *converges to* u *weakly** *in* $SBV^p(\Omega;K)$ if

- $u_k, u \in SBV^p(\Omega;K)$;
- $u_k \to u$ in measure;
- $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega;\mathbb{M}^{n\times n})$;
- $\mathcal{H}^{n-1}(S(u_k))$ is bounded uniformly with respect to k.

Finally, we assume that every deformation u satisfies a property that reflects the non-interpenetration of matter: we suppose that u is globally injective (a.e.) and preserves orientation. This requirement was studied by Ciarlet-Nečas [7] in the case of Sobolev spaces; the generalization to SBV functions was done by Giacomini-Ponsiglione [17]. For a discussion about the physical meaning of this condition, we refer to [9, Appendix A].

Definition 2.2. A function $u \in SBV^p(\Omega;K)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if the following properties hold:

- u *preserves orientation*, i.e.,
$$\det \nabla u(x) > 0 \quad \text{for a.e. } x \in \Omega\,; \tag{4}$$
- u is *a.e.-injective*, i.e.,
$$\text{there exists } N \subset \Omega \text{ such that } \mathcal{L}^n(N) = 0 \text{ and } u \text{ is injective on } \Omega\backslash N\,. \tag{5}$$

In the approximation procedures, we will employ a stability result of the Ciarlet-Nečas condition under weak* convergence in $SBV^p(\Omega;K)$, proven in [17, Theorem 3.4].

Theorem 2.3 (Stability of the Ciarlet-Nečas condition)**.** *Let u_k be a sequence converging to u weakly* in $SBV^p(\Omega;K)$. Suppose that every u_k satisfies (4) and (5), u satisfies (4), and $\det \nabla u_k \rightharpoonup \det \nabla u$ weakly in $L^1(\Omega)$. Then u satisfies (5).*

Now we can define the set of *admissible cracks* of Ω as
$$\mathcal{R} := \big\{\Gamma\colon (\mathcal{H}^{n-1}, n-1)\text{-rectifiable}, \, \Gamma \subset \Omega\,,\, \mathcal{H}^{n-1}(\Gamma) < +\infty\big\}\,, \tag{6}$$
where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and the set of *admissible deformations* with crack $\Gamma \in \mathcal{R}$ as
$$AD(\Gamma) := \Big\{u \in SBV^p(\Omega;K)\colon u \text{ satisfies (4), (5), and } S(u) \subset \Gamma\Big\}\,. \tag{7}$$

According to Griffith's theory, when the material is homogeneous and isotropic, the work done to produce a crack Γ is given by $\kappa\,\mathcal{H}^{n-1}(\Gamma)$, where the constant $\kappa > 0$ depends on the material and is called *fracture toughness*; for the sake of simplicity, we assume $\kappa = 1$. More general crack energies can be considered, in order to describe non-homogeneous or non-isotropic materials [9, 19].

We study the evolution problem in a time interval $[0, T]$. A time-dependent volume force is acting on the body and the whole boundary is traction-free. Since the uncracked part is hyperelastic, at each time $t \in [0, T]$ the *elastic energy* of a deformation u is given by

$$\mathcal{E}^{\mathrm{el}}(t)(u) := \mathcal{W}(u) - \mathcal{G}(t)(u)\,, \tag{8}$$

where

$$\mathcal{W}\colon SBV^p(\Omega; K) \to [0, +\infty]$$

is the *bulk energy* and

$$\mathcal{G}(t)\colon L^\infty(\Omega; K) \to \mathbb{R}$$

is the *potential* of the *volume force* acting at time t. Finally, given $\Gamma \in \mathcal{R}$ and $u \in AD(\Gamma)$,

$$\mathcal{E}(t)(u, \Gamma) := \mathcal{E}^{\mathrm{el}}(t)(u) + \mathcal{H}^{n-1}(\Gamma) \tag{9}$$

is the *total energy* of the system.

2.2. **Bulk energy.** Given a crack $\Gamma \in \mathcal{R}$, we suppose that the uncracked part $\Omega\backslash\Gamma$ is hyperelastic and that the *bulk energy* on $\Omega\backslash\Gamma$ of any deformation $u \in SBV^p(\Omega; K)$ with $S(u) \subset \Gamma$ can be written as

$$\mathcal{W}(u) := \int_{\Omega\backslash\Gamma} W(\nabla u(x))\,\mathrm{d}x = \int_{\Omega} W(\nabla u(x))\,\mathrm{d}x\,, \tag{10}$$

where $W\colon \mathbb{M}^{n\times n} \to [0, +\infty]$ is a continuous function, independent of Γ. We assume the standard property of *frame indifference*:

$$W(QA) = W(A) \text{ for every } A \in \mathbb{M}^{n\times n} \text{ and every } Q \in SO_n\,. \tag{11}$$

We suppose that

$$W(A) < +\infty \quad \Longleftrightarrow \quad \det A > 0 \tag{12}$$

and that $A \mapsto W(A)$ is C^1 on $\{A \in \mathbb{M}^{n\times n}\colon \det A > 0\}$. This is a standard hypothesis in *finite elasticity*, which ensures that every deformation with finite energy preserves orientation, as in (4).

In the approximation procedures, we will need a lower semicontinuity property for $\mathcal{W}$. To this aim, the usual assumption is the *quasiconvexity* of W; however, all known semicontinuity results for quasiconvex functions require some conditions of polynomial growth from above (as in Ambrosio [1]), which are incompatible with (12). On the other side, convexity is too strong, so we employ the intermediate notion of *polyconvexity*, i.e., we suppose that there exists a continuous and convex function $\widetilde{W}\colon \mathbb{R}^\tau \to [0, +\infty]$ such that

$$W(A) = \widetilde{W}(M(A)) \quad \text{for every } A \in \mathbb{M}^{n\times n}\,,$$

where $M(A) := (\mathrm{adj}_1 A, \dots, \mathrm{adj}_n A)$, $\mathrm{adj}_j A$ is the vector composed of all minors of A of order j, and τ is the dimension of $M(A)$.

Furthermore, we require a *bound from below* in terms of the minors of A: for every $A \in \mathbb{M}^{n\times n}$

$$W(A) \geq \sum_{j=1}^{n} \beta_W^j \left|\mathrm{adj}_j A\right|^{p_j} \,, \tag{13}$$

where $\beta_W^1, \dots, \beta_W^n > 0$ and

$$p_1 \geq 2\,, \quad p_j \geq p_1' := \frac{p_1}{p_1 - 1} \text{ for } j = 2, \dots, n-1\,, \quad p_n > 1\,. \tag{14}$$

Henceforth, we will take $p := p_1$.

Under these hypotheses, the lower semicontinuity of $\mathcal{W}$ under weak* convergence in $SBV^p(\Omega; K)$ is guaranteed by a result of Fusco-Leone-March-Verde [16, Theorem 3.5] (see also [9, Theorem 3.1]).

Theorem 2.4 (Semicontinuity)**.** *Let $u_k \rightharpoonup u_\infty$ weakly* in $SBV^p(\Omega; K)$. Then*

$$\mathcal{W}(u_\infty) \leq \liminf_{k\to\infty} \mathcal{W}(u_k)\,. \tag{15}$$

The importance of this theorem is that it does not require any bound from above on the energy density. Nevertheless, for the energy estimates we will need some growth condition, which must be compatible with (12). Therefore, we assume the following *multiplicative stress estimate*, which is usual in elasticity (see, e.g., Ball [3]): for every $A \in \mathbb{M}^{n\times n}$ the *energy-momentum tensor*

$$P(A) := W(A) I - A^{\mathrm{T}} \mathrm{D}_A W(A) \tag{16}$$

satisfies the inequality

$$|P(A)| \leq c_W^1 (W(A) + c_W^0)\,, \tag{17}$$

where $c_W^0 \geq 0$ and $c_W^1 > 0$ are two constants. In these formulas, $\mathrm{D}_A W(A)$ denotes the matrix whose entries are the partial derivatives of W with respect to the corresponding entries of A, while $|\cdot|$ is the Euclidean norm on $\mathbb{M}^{n\times n}$.

Example (Ogden Materials)**.** An important class of hyperelastic isotropic materials in dimension $n = 3$ was studied by Ogden in 1972 [21, 22] to describe the behaviour of natural rubbers. These materials provide a classical example in finite elasticity [6, Section 4.10]. Consider the strain-energy defined for $A \in \mathbb{M}^{3\times 3}$ by

$$W(A) := \begin{cases} \beta_W^1 |A|^{p_1} + \beta_W^2 |\mathrm{cof} A|^{p_2} + \beta_W^3 |\det A|^{p_3} + \gamma |\det A|^{-q} & \text{if } \det A > 0\,, \\ +\infty & \text{otherwise,} \end{cases}$$

where $p_1 = p \geq 2$, $p_2 \geq p'$, $p_3 > 1$, and $\beta_W^j > 0$, as in (13), $q > 0$, $\gamma > 0$, and $\mathrm{cof} A := (\det A)\ A^{-\mathrm{T}}$ is the cofactor matrix of A. It is possible to prove that W satisfies all the hypotheses listed above; we refer to [9, Example 1.8] for the details.

2.3. **Forces.** In our model, the body is subjected to a conservative volume force, depending on time. This means that there exists a function $G\colon [0,T] \times \Omega \times K \to \mathbb{R}$ such that the force density at x per unit volume in the reference configuration is given by $\mathrm{D}_y G(t, x, u(x))$, where $\mathrm{D}_y G(t, x, y)$ is the partial gradient of G with respect to y and u is the deformation of the body. So, the potential of the body force is given, up to an additive constant, by

$$\mathcal{G}(t)(u) := \int_\Omega G(t, x, u(x))\, \mathrm{d}x\,. \tag{18}$$

We make the following assumptions on $G\colon [0,T] \times \Omega \times K \to \mathbb{R}$:

- $x \mapsto G(t, x, y)$ is $\mathcal{L}^n$-measurable on Ω for every $(t, y) \in [0,T] \times K$;

- $(t, y) \mapsto G(t, x, y)$ is C^1 on $[0, T] \times K$ for every $x \in \Omega$;
- there is a constant $a_G > 0$ such that

$$|G(t, x, y)| + |\mathrm{D}_t G(t, x, y)| + |\mathrm{D}_y G(t, x, y)| \leq a_G\,, \tag{19}$$

  for every $(t, x, y) \in [0, T] \times \Omega \times K$.

Under these hypotheses, for any $u \in SBV^p(\Omega; K)$ the function $t \mapsto \mathcal{G}(t)(u)$ is C^1 on $[0, T]$ and its derivative $\dot{\mathcal{G}}(t)(u)$ is given by

$$\dot{\mathcal{G}}(t)(u) = \int_\Omega \mathrm{D}_t G(t, x, u(x))\,\mathrm{d}x\,. \tag{20}$$

Remark 1. The presence of the confinement hypothesis $u(x) \in K$ allows us to avoid the growth conditions with respect to y, required in [8]. Here we do not focus on the minimal hypotheses on the time dependence, which are studied in [19].

Remark 2. The assumption that the density of the body force per unit volume in the reference configuration can be expressed as $\mathrm{D}_y G(t, x, u(x))$ is a very strong physical requirement, which is satisfied when the force is generated by a conservative field acting on a charge distribution which is deformed with the body; this excludes, for instance, hydrostatic pressure forces. For further comments, we refer to [8, Remark 3.5].

3. **Quasistatic evolution.** In the previous section we have introduced a mechanical framework where the strain-energy is compatible with non-interpenetration conditions. Now, we are interested in finding time evolutions $t \mapsto (u(t), \Gamma(t))$ along the stable configurations of the system.

According to Griffith's principle [18], an equilibrium of the system is an admissible configuration $(u(t), \Gamma(t))$ which is stationary for the functional $\mathcal{E}(t)(v, \Gamma)$ on the set of configurations (v, Γ) with $\Gamma \in \mathcal{R}$, $\Gamma(t) \subset \Gamma$, and $v \in AD(\Gamma)$. While the notion of stationarity can be made precise in the case of a prescribed crack path, up to now no consistent notion of critical point has been proposed for the general case, where the crack path is not prescribed.

For this reason, we focus our attention on *minimum energy configurations*, as proposed by Francfort-Marigo [14]: these are the admissible pairs $(u(t), \Gamma(t))$ such that

- $\Gamma(t) \in \mathcal{R}$,
- $u(t) \in AD(\Gamma(t))$,
- if $\Gamma \in \mathcal{R}$, $\Gamma(t) \subset \Gamma$, and $v \in AD(\Gamma)$, then $\mathcal{E}(t)(u(t), \Gamma(t)) \leq \mathcal{E}(t)(v, \Gamma)$.

As in Griffith, we consider a *unilateral minimum problem*, where the competitors for $(u(t), \Gamma(t))$ are only the configurations with crack bigger than $\Gamma(t)$. This restriction is due to the requirement of *irreversibility* of the fracture process: the function $t \mapsto \Gamma(t)$ must be monotone nondecreasing. The main difference from Griffith's theory is that we consider global minimizers rather than critical points.

Finally, the time evolution is governed by the *energy-dissipation balance*, which is expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{E}(t)(u(t), \Gamma(t))\big) = -\dot{\mathcal{G}}(t)(u(t))\,. \tag{21}$$

When $t \mapsto u(t)$ is regular enough, this is equivalent to the usual equality between the time derivative of the internal energy and the power of the external force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{E}^{\mathrm{int}}(u(t), \Gamma(t))\big) = -\int_\Omega \mathrm{D}_y G(t, x, u(t)(x)) \cdot \dot{u}(t)(x)\,\mathrm{d}x\,, \tag{22}$$

where

$$\mathcal{E}^{\rm int}(u,\Gamma) := \mathcal{W}(u) + \mathcal{H}^{n-1}(\Gamma)\ .$$

These properties define the notion of *quasistatic evolution.*

Definition 3.1. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,T]$ in $SBV^p(\Omega;K) \times \mathcal{R}$ is a quasistatic evolution of minimum energy configurations if the following hold:

- *Irreversibility:* for every $s < t$ we have $\Gamma(s) \subset \Gamma(t)$;
- *Unilateral global stability:* for every $t \in [0,T]$ we have $u(t) \in AD(\Gamma(t))$ and
$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(v,\Gamma) \tag{23}$$
for every $\Gamma \in \mathcal{R}$ with $\Gamma(t) \subset \Gamma$ and every $v \in AD(\Gamma)$;
- *Energy-dissipation balance:* $t \mapsto \mathcal{E}(t)(u(t),\Gamma(t))$ is absolutely continuous and
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{E}(t)(u(t),\Gamma(t))\big) = -\dot{\mathcal{G}}(t)(u(t)) \quad \text{for a.e. } t \in [0,T]\,. \tag{24}$$

Remark 3. In this definition, we require no regularity for $t \mapsto u(t)$. Indeed, in some examples the evolutions exhibit a discontinuity time. This is the reason why we prefer (21) rather than (22), which requires some regularity on $t \mapsto u(t)$. Actually, even the measurability of the solutions with respect to time is a nontrivial issue. In [9, Section 6] we consider the problem of finding quasistatic evolutions such that $t \mapsto u(t)$ is measurable, with respect to a suitable Banach space structure on $SBV^p(\Omega;\mathbb{R}^n)$.

Remark 4. Global stability and energy balance have been used to define a notion of variational solution for a wide class of *rate-independent problems* (see Mielke [20]). Rate-independence means the following property: if $u(t)$ is a solution for the problem with datum $\mathcal{G}(t)$, then $u \circ \alpha(t)$ is a solution for the problem with datum $\mathcal{G} \circ \alpha(t)$ for every increasing reparametrization α of the time interval.

4. **Existence result.** Our main result is the existence of quasistatic evolutions starting from a fixed initial condition. Let $\Gamma_0 \in \mathcal{R}$ and $u_0 \in AD(\Gamma_0)$ satisfy the unilateral minimality condition at time 0:

$$\mathcal{E}(0)(u_0,\Gamma_0) \le \mathcal{E}(0)(u,\Gamma) \tag{25}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma_0 \subset \Gamma$ and every $u \in AD(\Gamma)$. To simplify the exposition, we assume that there exists a function $v_0 \in SBV^p(\Omega;\mathbb{R}^n)$ such that $S(v_0) = \Gamma_0$; this happens, for instance, when Γ_0 is an $(n-1)$-dimensional C^1-manifold with boundary. However, all results remain valid without this assumption, with a slightly different proof.

Theorem 4.1 (Existence of quasistatic evolutions). *If* (25) *holds, then there exists a quasistatic evolution* $t \mapsto (u(t),\Gamma(t))$ *with* $(u(0),\Gamma(0)) = (u_0,\Gamma_0)$.

The previous theorem is proven in [9] following a scheme which is usual in the variational theory of fracture mechanics and in other rate-independent problems [14, 10, 5, 13, 8, 4, 15, 20].

1. First one discretizes time and defines an approximate solution by solving a sequence of *incremental minimum problems.*
2. One proves that the approximate solutions satisfy a *discrete energy inequality,* which provides the bounds needed for the compactness arguments.
3. Passing to the limit as the time step vanishes, one finds an *incrementally-approximable evolution* which satisfies the *irreversibility* condition.

4. One proves that the property of *global stability*, valid for each approximant, passes to the limit.
5. Finally, one proves the *energy balance*, showing that the energy inequality passes to the limit, while the reverse inequality is a consequence of stability.

Now we sketch the steps of the proof, referring to [9] for the details.

4.1. **The discrete-time problems.** We consider a *time discretization*, i.e., a sequence of subdivisions $\{t_k^i\}_{0\le i\le k}$ of the interval $[0,T]$, such that

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = 1 \quad \text{and} \quad \lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0\,.$$

For every time discretization, we define a corresponding sequence of approximate solutions.

Given $k \in \mathbb{N}$, we set $(u_k^0, \Gamma_k^0) := (u_0, \Gamma_0)$. For $i = 1, \dots, k$, we define inductively (u_k^i, Γ_k^i) to be a solution to the incremental problem

$$\min\left\{\mathcal{E}(t_k^i)(u,\Gamma)\colon \Gamma\in\mathcal{R},\ \Gamma_k^{i-1}\subset\Gamma,\ u\in AD(\Gamma)\right\}. \tag{26}$$

Since $\mathcal{W}(I) < +\infty$, by minimality we find a uniform bound on $\mathcal{E}(t_k^i)(u_k^i, \Gamma_k^i)$; hence $u_k^i \in SBV^p(\Omega; K)$ and $\left\|\nabla u_k^i\right\|_{L^p(\Omega;\mathbb{M}^{n\times n})}$ is uniformly bounded, by coercivity.

Remark 5. The existence of a minimum configuration for problem (26) is obtained using the Direct Method of the Calculus of Variations. Notice that there is no uniqueness, since the functional is non-convex. In particular, for what concerns the deformations, we use

- the compactness of "bounded" subsets of $SBV^p(\Omega; K)$ [2, Theorem 4.8];
- the lower semicontinuity property of the bulk energy (Theorem 2.4);
- the stability of non-interpenetration (Theorem 2.3).

We now consider the piecewise constant interpolation, setting

$$u_k(t) := u_k^i \quad \text{and} \quad \Gamma_k(t) = \Gamma_k^i \quad \text{if } t\in\left[t_k^i, t_k^{i+1}\right)$$

for every $i = 0, \dots, k-1$. Let us show that each approximate solution satisfies an a-priori bound, which will allow us to apply some compactness results. Taking $(u, \Gamma) = (u_k^{i-1}, \Gamma_k^{i-1})$ in (26) we get

$$\begin{aligned}\mathcal{E}(t_k^i)(u_k^i,\Gamma_k^i) &\le \mathcal{E}(t_k^i)(u_k^{i-1},\Gamma_k^{i-1}) = \mathcal{W}(u_k^{i-1}) + \mathcal{H}^{n-1}\left(\Gamma_k^{i-1}\right) - \mathcal{G}(t_k^i)(u_k^{i-1})\\ &= \mathcal{W}(u_k^{i-1}) + \mathcal{H}^{n-1}\left(\Gamma_k^{i-1}\right) - \mathcal{G}(t_k^{i-1})(u_k^{i-1}) - \int_{t_k^{i-1}}^{t_k^i} \dot{\mathcal{G}}(s)(u_k(s))\,\mathrm{d}s\\ &= \mathcal{E}(t_k^{i-1})(u_k^{i-1},\Gamma_k^{i-1}) - \int_{t_k^{i-1}}^{t_k^i} \dot{\mathcal{G}}(s)(u_k(s))\,\mathrm{d}s\,.\end{aligned}$$

Summing up from 1 to i, we obtain the discrete energy inequality

$$\mathcal{E}(t)(u_k(t),\Gamma_k(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) - \int_0^t \dot{\mathcal{G}}(s)(u_k(s))\,\mathrm{d}s + o(1)\,. \tag{27}$$

Since the right-hand side is bounded, we find a bound on $\|\nabla u_k(t)\|_{L^p(\Omega;\mathbb{M}^{n\times n})}$ and $\mathcal{H}^{n-1}(\Gamma_k(t))$, uniform in k and t.

Now we would like to pass to the limit in the approximate solutions as $k\to\infty$. Thanks to the uniform bounds on $\|\nabla u_k(t)\|_{L^p(\Omega;\mathbb{M}^{n\times n})}$ and $\mathcal{H}^{n-1}(S(u_k(t)))$, a subsequence of $u_k(t)$ has a weak*-limit in $SBV^p(\Omega; K)$ by [2, Theorem 4.8].

As for the cracks, the usual notions of convergence (like the Hausdorff convergence) are not useful because they do not ensure the lower semicontinuity of the relevant energy terms. Hence, we consider a set convergence, called *σ^p-convergence* and introduced in [8], which has all properties of semicontinuity and compactness needed for our purposes.

The σ^p-convergence is related to the weak* convergence in $SBV^p(\Omega;K)$: indeed, the main property is that, if $v_k \rightharpoonup v$ weakly* in $SBV^p(\Omega;K)$, $S(v_k) \subset \Gamma_k$, and Γ_k σ^p-converges to Γ, then $S(v) \subset \Gamma$. Actually, the σ^p-limit is the minimal set Γ satisfying this property. We refer to [8] for the details.

Now there is a technical problem: the subsequence of $u_k(t)$ used to define $u(t)$ may depend on t. To overcome this difficulty, we use the monotonicity of $\Gamma_k(t)$ with respect to t and apply a Helly-type theorem [8, Theorem 4.8], which ensures that a subsequence (independent of t and still denoted by $\Gamma_k(t)$) σ^p-converges to a set function $\Gamma(t)$. Since $\Gamma(t)$ turns out to be monotone nondecreasing and $\Gamma(0) = \Gamma_0$, the irreversibility and the initial conditions are satisfied.

As for the deformations, we have to choose carefully the subsequence of $u_k(t)$ which defines the limit $u(t)$. We employ a trick introduced in [8] and select a subsequence k_j, depending on t, such that

$$\lim_{j\to\infty} \theta_{k_j}(t) = \limsup_{k\to\infty} \theta_k(t)\,, \tag{28}$$

where

$$\theta_k(t) := -\dot{\mathcal{G}}(t)(u_k(t))\,. \tag{29}$$

The compactness result in $SBV^p(\Omega;K)$ [2, Theorem 4.8] allows us to find a further subsequence (not relabelled) converging weakly* to a function $u(t)$.

We point out the properties of $(u(t),\Gamma(t))$:

- $\Gamma(t) \in \mathcal{R}$, $u \in SBV^p(\Omega;K)$, and $S(u) \subset \Gamma(t)$;
- $\Gamma_k(t)$ σ^p-converges to $\Gamma(t)$;
- there is a subsequence $u_{k_j}(t)$, depending on t, such that (28) holds and $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega;K)$.

We are left to prove that $(u(t),\Gamma(t))$ satisfies global stability and energy balance.

4.2. **Stability of minimizers.** We now give a rough idea of the proof of the unilateral global stability: given $t \in [0,T]$, we should see that $u(t) \in AD(\Gamma(t))$ and

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(v,\Gamma) \tag{30}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma(t) \subset \Gamma$ and every $v \in AD(\Gamma)$. To highlight the difficulties, we consider here only the particular case $\Gamma = \Gamma(t)$, so (30) reads

$$\mathcal{W}(u(t)) - \mathcal{G}(t)(u(t)) \le \mathcal{W}(v) - \mathcal{G}(t)(v) \quad \text{for every } v \in AD(\Gamma(t))\,. \tag{31}$$

We want to obtain (31) passing to the limit in the inequality

$$\mathcal{W}(u_{k_j}(t)) - \mathcal{G}(t_j)(u_{k_j}(t)) \le \mathcal{W}(v_j) - \mathcal{G}(t_j)(v_j) \quad \text{for every } v_j \in AD(\Gamma_{k_j}(t))\,,$$

which is a consequence of (26); here, t_j stands for $t^i_{k_j}$, where i is the unique index such that $t \in [t^i_{k_j}, t^{i+1}_{k_j})$.

Therefore, given $v \in AD(\Gamma(t))$, we look for some functions $v_j \in AD(\Gamma_{k_j}(t))$, converging to v in measure, such that $\mathcal{W}(v_j) \to \mathcal{W}(v)$. Once we have found these competitors, by semicontinuity we get

$$\begin{aligned}\mathcal{W}(u(t)) - \mathcal{G}(t)(u(t)) &\le \liminf_{j\to\infty} \left[\mathcal{W}(u_{k_j}(t)) - \mathcal{G}(t_j)(u_{k_j}(t))\right] \\ &\le \lim_{j\to\infty} \left[\mathcal{W}(v_j) - \mathcal{G}(t_j)(v_j)\right] = \mathcal{W}(v) - \mathcal{G}(t)(v)\,,\end{aligned}$$

so (31) follows.

Given $v \in AD(\Gamma(t))$, in order to construct this sequence $v_j \in AD(\Gamma_{k_j}(t))$ we have to "transfer" the jumps of v from $\Gamma(t)$ to $\Gamma_{k_j}(t)$, without increasing the bulk energy too much: this is done using the Jump Transfer Lemma, originally proven in Francfort-Larsen [13, Theorem 2.1]. First, we find a suitable sequence of closed sets C_j such that $\mathcal{L}^n(C_j) \to 0$ and $\mathcal{H}^{n-1}(\Gamma(t)\backslash C_j) \to 0$, and define $v_j := v$ in $\Omega\backslash C_j$. In the region C_j, we transform v so that its jumps lying in $\Gamma(t)$ are transferred to $\Gamma_{k_j}(t)$.

In the original proof by Francfort and Larsen, under the polynomial growth assumptions on the bulk energy, v was modified in C_j through some reflections about hyperplanes. Unfortunately, under the hypotheses of finite elasticity (12) such transformations are forbidden, because they change the sign of the determinant of the deformation gradient.

Hence, we modify the proof of the lemma using, instead of reflections, a stretching argument, which allows us to control the increase in energy. Indeed, in C_j we will have $\nabla v_j = \nabla v\, \Lambda_j$, where $\Lambda_j(x)$ is a matrix depending on the dilations used in C_j. Then the energy of v_j is bounded through the estimate

$$|\Lambda_j - I| < \gamma \quad\Longrightarrow\quad W(\nabla v\, \Lambda_j) + c_W^0 \le \frac{n}{n-1}\left(W(\nabla v) + c_W^0\right)\,,$$

where $\gamma \in (0,1)$ is a suitable constant: this is a consequence of the multiplicative stress estimate (17) (see [3, Section 2.4]). We conclude by choosing Λ_j uniformly close to the identity. For the details of this technical proof, we refer to [9, Lemma 4.1].

With similar arguments, we can see that (30) holds in the general case. In order to show that $u(t) \in AD(\Gamma(t))$, we have just to check the non-interpenetration condition. Now, (4) is a consequence of (12) and (30); (5) comes from Theorem 2.3, using the convergence of the Jacobian determinants given by [16, Theorem 3.4].

4.3. **Energy balance.** In order to prove the energy balance, we first pass to the limit in (27) and get the so called energy inequality. Then we obtain the opposite inequality via a standard method based on stability. For quasistatic evolution problems in fracture mechanics, this procedure was introduced in [10, 13, 8]. The same scheme was developed independently in the variational study of other rate-independent processes [20].

Let us consider the function

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t)\,, \tag{32}$$

where $\theta_k(t)$ was defined in (29). By the Fatou Lemma, we have $\theta_\infty \in L^1([0,T])$ and

$$\limsup_{k\to\infty} \int_0^t \theta_k(s)\,\mathrm{d}s \le \int_0^t \theta_\infty(s)\,\mathrm{d}s\,. \tag{33}$$

For every $s \in [0,T]$, we recall the property of the subsequence $u_{k_j}(s)$ (see (28)):

$$\theta_\infty(s) = \lim_{j\to\infty} \theta_{k_j}(s) = -\dot{\mathcal{G}}(s)(u(s))\,, \tag{34}$$

where the last equality follows from the fact that $u_{k_j}(s)$ converges in measure to $u(s)$.

By [8, Theorem 4.3], the Hausdorff measure is lower semicontinuous under σ^p-convergence; hence, using the lower semicontinuity of $\mathcal{W}$ we get

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \liminf_{j\to\infty} \mathcal{E}(t)(u_{k_j}(t),\Gamma_{k_j}(t)) \le \limsup_{k\to\infty} \mathcal{E}(t)(u_k(t),\Gamma_k(t))\,.$$

From (27), (32), (33), and (34) we deduce that

$$\limsup_{k\to\infty} \mathcal{E}(t)(u_k(t),\Gamma_k(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) - \int_0^t \dot{\mathcal{G}}(s)(u(s))\,\mathrm{d}s\,.$$

This leads to the energy inequality

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(0)(u_0,\Gamma_0) + \int_0^t \dot{\mathcal{G}}(s)(u(s))\,\mathrm{d}s\,. \tag{35}$$

The opposite inequality is obtained again by time discretization: consider a subdivion such that

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{k-1} < s_k^k = t \quad \text{and} \quad \lim_{k\to\infty} \max_{1\le i\le k} (s_k^i - s_k^{i-1}) = 0\,.$$

By global stability at time s_k^{i-1} we find

$$\begin{aligned}
&\mathcal{E}(s_k^{i-1})(u(s_k^{i-1}),\Gamma(s_k^{i-1})) \le \mathcal{E}(s_k^{i-1})(u(s_k^i),\Gamma(s_k^i))\\
&= \mathcal{W}(u(s_k^i)) + \mathcal{H}^{n-1}\big(\Gamma(s_k^i)\big) - \mathcal{G}(s_k^{i-1})(u(s_k^i))\\
&= \mathcal{W}(u(s_k^i)) + \mathcal{H}^{n-1}\big(\Gamma(s_k^i)\big) - \mathcal{G}(s_k^i)(u(s_k^i)) + \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{G}}(s)(u(s_k^i))\,\mathrm{d}s\\
&= \mathcal{E}(s_k^i)(u(s_k^i),\Gamma(s_k^i)) + \int_{s_k^{i-1}}^{s_k^i} \dot{\mathcal{G}}(s)(\widetilde{u}_k(s))\,\mathrm{d}s\,,
\end{aligned}$$

where $\widetilde{u}_k(s) := u(s_k^i)$ for $s \in (s_k^{i-1}, s_k^i]$. Summing in i we get

$$\mathcal{E}(0)(u_0,\Gamma_0) \le \mathcal{E}(t)(u(t),\Gamma(t)) + \int_0^t \dot{\mathcal{G}}(s)(\widetilde{u}_k(s))\,\mathrm{d}s\,.$$

It is possible to pass to the limit in the last integral (see [9, Section 5.2]), so that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \ge \mathcal{E}(0)(u_0,\Gamma_0) - \int_0^t \dot{\mathcal{G}}(s)(u(s))\,\mathrm{d}s\,.$$

Recalling (35), we obtain

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(0)(u_0,\Gamma_0) - \int_0^t \dot{\mathcal{G}}(s)(u(s))\,\mathrm{d}s\,. \tag{36}$$

which proves the energy balance in integral form. This concludes the proof of Theorem 4.1.

Remark 6. Under suitable hypotheses, it is possible to treat the problem with time-dependent boundary conditions on a part $\partial_D\Omega$ of $\partial\Omega$ and time-dependent surface forces on a part $\partial_S\Omega$ of $\partial\Omega$. In particular, we have to assume a further continuity condition on the Kirchhoff stress tensor $\mathrm{D}_A W(A)A^{\mathrm{T}}$. For the proof we refer to [9, 19], where we exploit the method of "multiplicative splitting" introduced by Francfort-Mielke [15].

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