SOME APPLICATIONS OF THE LEAST SQUARES METHOD TO DIFFERENTIAL EQUATIONS AND RELATED PROBLEMS

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Abstract. The first part of this paper contains an overview of the least squares method applied to various problems in ordinary and partial differential equations. In particular, we discuss various applications to the homogenization of transport equations, to the characterization of entropy solutions to scalar conservation laws and to the asymptotic behaviour of the action functional obtained through the reaction-diffusion approximation of mean curvature flow. In the last part of the paper we introduce and discuss the related problem of the quasi-potential for scalar conservation laws.

1. Introduction

Let $T > 0$ and consider the ordinary differential equation

$$\dot{x} = b(x)\,, \tag{1.1}$$

on some bounded interval $I = [0, T]$, where b is a bounded Lipschitz function in $\mathbb{R}$. Then (1.1) admits a global C^1 solution, which is unique once we fix a condition $x(t_0) = x_0$, for some $t_0 \in I$. Let us now consider the functional

$$F(x) := \frac{1}{2} \int_I [\dot{x} - b(x)]^2 \, dt\,, \tag{1.2}$$

defined for instance for $x \in W^{1,2}(I)$ and extended to $+\infty$ in $L^2(I) \setminus W^{1,2}(I)$. If necessary, we can require x to satisfy the initial condition, by adding to F an indicator term of the form $\chi_{x_0}(x)$, which vanishes if $x(t_0) = x_0$ and is equal to $+\infty$ elsewhere.

A function x in the domain of F satisfies $F(x) = 0$ if and only if x is an almost everywhere solution to (1.1) which, under our assumptions, is equivalent to say that x is of class C^1 and satisfies (1.1) everywhere. It follows that x is a minimum point of F if and only if x is a C^1 solution to (1.1). This observation translates the problem of finding a solution to (1.1) into the problem of finding the zero-level set of the functional F, or equivalently of finding the global minimizers of F. This latter viewpoint falls within the framework of Calculus of Variations; in particular, for applying the direct methods, the lower semicontinuity and the coercivity of F, for instance with respect to the $L^2(I)$-topology, become of interest. Note that the problem of minimizing F makes sense also if coupled with boundary conditions which are not usual for (1.1), such as for instance two point boundary value problems. We address generically the method of finding a solution to (1.1) through the minimization of the functional F as the *least squares method*.

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The situation is more complicated if we weaken the regularity assumptions on b, for instance in such a way that (1.1) may not have a C^1 solution; just to fix ideas, we can think of the case when b is bounded but discontinuous. Then one can look for suitable weak solutions of (1.1) (see, for instance, [19, 21, 13, 2]). Almost everywhere solutions in $W^{1,2}(I)$ could also be considered: in general, depending on the choice of the function b and on the initial condition, it may happen that (1.1) does not admit an almost everywhere solution, or even that it admits infinitely many almost everywhere solutions. This means that $\{F = 0\}$ may be empty, or may be rather large (even under the constraint given by the initial condition). In addition, it may also happen that the functional F fails to be lower semicontinuous.

In a variational framework, it is natural to work with functionals which are lower semicontinuous; if F does not share this property, one is led to consider the $L^2(I)$-lower semicontinuous envelope $\overline{F}$ of F (the so-called $L^2(I)$-relaxed functional of F). Observe that $0 \leq \overline{F} \leq F$, and hence

$$\{F = 0\} \subseteq \{\overline{F} = 0\}\,. \tag{1.3}$$

In this viewpoint, one reasonable definition of weak solution to (1.1) is the one that we could call variational weak solution, namely a function x in the domain of $\overline{F}$ for which $\overline{F}(x) = 0$ (or, if $\overline{F}(x) = 0$ is empty, such that $\overline{F}(x)$ is minimum, provided the minimum exists). We recall that $\overline{F}(x) = 0$ if there exists a sequence (x^h) of functions in the domain of F converging to x in $L^2(I)$ and such that $F(x^h)$ converges to zero as $h \to +\infty$. This definition does not in general solve the problem of devising a unique solution to (1.1) for various reasons. Firstly, it does not give any further information when $F = \overline{F}$; secondly, in view of the inclusion (1.3), in case of nonuniqueness of almost everywhere solutions, non-uniqueness of variational weak solutions is a fortiori expected. On the other hand, when $F \neq \overline{F}$, the advantage of considering the zero-level set of $\overline{F}$ is that such a set is closed. Inside this closed set one can think of selecting a particular solution: this may be considered as a separate problem, to be faced using other arguments. We give in Section 2 some more details on variational weak solutions for ODE's.

As suggested by De Giorgi in [17], it is possible to introduce suitable quadratic functionals F also in the context of partial differential equations (with various boundary conditions). One may then define the corresponding variational weak solutions as the functions u satisfying $\overline{F}(u) = 0$ (where $\overline{F}$ is the relaxed functional of F in a suitable topology). The topology in which one relaxes the functional plays an essential role. Indeed, if τ_1 and τ_2 are different topologies with τ_1 weaker than τ_2, then $0 \leq \overline{F}_{\tau_1} \leq \overline{F}_{\tau_2}$ and hence

$$\{\overline{F}_{\tau_2} = 0\} \subseteq \{\overline{F}_{\tau_1} = 0\}.$$

If we exceed in weakening the topology we may get a useless notion of solution since the zero-set may become too large. On the other hand, a weaker topology makes it easier to satisfy coerciveness (which in this approach is a delicate point). As usual, these two opposite requirements suggest a balanced choice of the topology.

The ideas illustrated so far may be likewise helpful when dealing with perturbed ordinary or partial differential equations. A first example is given by the homogenization theory. Consider for instance the ε-dependent system of ordinary differential equations

$$\dot{x} = \mathrm{f}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \tag{1.4}$$

on some interval I, and the associated ε-dependent linear transport equation

$$u_t + \mathrm{f}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \cdot \nabla u = 0\,, \tag{1.5}$$

under suitable periodicity assumptions on f. The homogenization problem for (1.4) (resp. (1.5)) consists in investigating the convergence of the solutions x^ε (resp. u^ε) as $\varepsilon \to 0$, in determining the limit function and eventually in finding also a limit equation satisfied by such a limit. If we introduce the quadratic functionals associated with (1.4) and (1.5),

$$G_\varepsilon(x) := \int_I \left| \dot{x} - \mathrm{f}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \right|^2 dt\,, \tag{1.6}$$

and

$$F_\varepsilon(u) := \int_\Omega \left[u_t + \mathrm{f}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \cdot \nabla u \right]^2 dx\, dt\,, \tag{1.7}$$

it is natural to study their Γ-limits. Note that the integrand in (1.7) is a *non-coercive* quadratic form. From what has been said so far, it is reasonable to define the solutions x^0 and u^0 of the limit problems, as the zeroes of the Γ-limits (if any) of the functionals G_ε and F_ε, respectively. Using Γ-convergence results we may try to characterize the zeroes without knowing the explicit representation of the Γ-limit. On the other hand, if we succeed in the full identification of the Γ-limit we may also get a limit equation. A further discussion on these issues is reported in Section 3.

Motivated by large deviations results for some stochastic models in [37, 22], the idea of the least squares method has been recently applied also in the context of nonlinear one-dimensional scalar conservation laws [6], which do not admit, in general, smooth global solutions, no matter how regular the initial datum is. This leads to a definition of variational weak solution for the equation

$$u_t + f(u)_x = 0\,, \tag{1.8}$$

which turns out to coincide with the Kruzkhov solution (here the flux f is assumed to be smooth, Lipschitz and nowhere affine, and the function u takes values in the interval $[-1, 1]$). More precisely, for $\varepsilon > 0$ one considers the viscous approximation to (1.8), that is the parabolic equation

$$u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x = 0\,, \tag{1.9}$$

where $t \in [0,T]$ for some $T > 0$ and x runs on a one-dimensional torus $\mathbb{T}$. (1.9) is associated with the functional

$$I_\varepsilon(u) := \frac{1}{2}\Big\|u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x\Big\|^2_{\mathcal{D}^{-1}}\,, \tag{1.10}$$

where D, assumed smooth and uniformly positive, represents the diffusion coefficient and $\mathcal{D}^{-1}$ is a suitable H^{-1}-like norm, depending on an arbitrary function σ of u itself. Then it results (exploiting a Young measures' setting) that points in the zero level set of the Γ-limit of I_ε are the measure-valued solutions to (1.8), which is a much larger class of solutions than the Kruzkhov solutions. These latter may be identified with the zeroes of the Γ-limit of the rescaled functionals $H_\varepsilon := \varepsilon^{-1} I_\varepsilon$. In [6] a candidate Γ-limit H of the sequence (H_ε) is introduced. An interesting role in the analysis is played by the so-called *entropy measure* solutions to (1.8), namely those functions forming the domain of H. The Γ-convergence proof is partially established, since the Γ-limsup inequality has been proved only for a special class of entropy measure solutions (containing piecewise smooth solutions), called entropy-splittable solutions. Various problems related to these functions remain open, such as their regularity (in general they are not of bounded variation) and the density (in topology and in energy) of the entropy splittable functions. If we consider a perturbed version of (1.9) of the form

$$u_t + f(u)_x - \frac{\varepsilon}{2}\big(D(u)u_x\big)_x = -(\sigma(u)E_\varepsilon)_x\,, \tag{1.11}$$

then the least squares method clarifies when solutions to (1.11) still converge to an entropic solution or to an entropy measure solution of (1.8) as $\varepsilon \to 0$. Here $\sigma : [-1,1] \ni v \mapsto \sigma(v) \geq 0$ is an arbitrary smooth function such that $\sigma(v) > 0$ for $v \in (-1,1)$. Typical examples are $\sigma \equiv 1$ and $\sigma(v) = v(1-v)$. If σ is uniformly positive, convergence to an entropic solution takes place if $\lim_{\varepsilon\to 0} \varepsilon^{1/2}\|E_\varepsilon\|_2 = 0$, while convergence to an entropy-measure solution happens if $\varepsilon^{1/2}\|E_\varepsilon\|_2$ is uniformly bounded. Note that the least squares method can be applied to the related Hamilton-Jacobi equation

$$w_t + f(w_x) = 0\,, \tag{1.12}$$

the analog of the functional I_ε being, for $D \equiv \sigma \equiv 1$

$$J_\varepsilon(w) = \frac{1}{2}\int_{[0,T]\times\mathbb{R}} [w_t + f(w_x) - \varepsilon w_{xx}]^2\, dxdt\,. \tag{1.13}$$

In this case viscosity solutions to (1.12) can be characterized as the zero set of the Γ-limit of the rescaled functionals $K_\varepsilon := \varepsilon^{-1} J_\varepsilon$. In this case it may be of interest to consider the functions in the domain of the Γ-limit, that are the counterparts of the entropy measure solutions. Further details on these issues are given in Section 4.

Entropic solutions to (1.8) correspond to the zero set of H; so that H represents a *cost* associated with non-entropic solutions. In particular, as proven in [6], for $u \in BV$, $H(u)$ quantifies the violation of the entropy condition along non-entropic shocks of u. Then, given two space profiles $u_i(x)$ (the initial one) and $u_f(x)$ (the final one), a question that naturally arises is the following: what is the cheapest cost one needs to pay in terms of the violation of the entropy when moving from u_i to u_f? In mathematical terms this issues translates in the study of the quantity

$$V(u_i,u_f):=\inf\big\{H_T(u)\,:\,T>0,\,u\,:\,u(0)=u_i,\,u(T)=u_f\big\}. \tag{1.14}$$

Note that constant solutions $u(t,x)\equiv m\in[-1,1]$ are attractors of entropic solutions to (1.8), namely the entropic solution of (1.8) corresponding to an initial datum u_0 converges to the mean m of u_0, $m:=\int_{\mathbb{T}}u_0(x)dx$. Since the functional H also appears as candidate large deviations functional for suitable sequences of stochastic processes in [22, 37, 26], one is particularly interested in finding an explicit expression for $V(m,u_f)$. Indeed, the quasi-potential usually provides information for the underlying processes. To motivate this issue, let us recall a few results for finite dimensional equations, for which we refer to [20].

Let us consider the stochastic differential problem in the unknown $x:[0,T]\to\mathbb{R}$

$$\begin{aligned}&\dot{x}=b(x)+\varepsilon\dot{W}\,,\\&x(0)=x_0\end{aligned} \tag{1.15}$$

where W is a brownian motion. Here we assume that b is smooth with good coercivity properties (i.e. $b(x)\,x\le -c|x|^2$ for $|x|$ large enough), and that the origin O is a global attractor for the equation (1.1). Then the functional $F+\chi_{x_0}$ as defined in (1.2) is indeed the *large deviations rate functional* for the sequence of the probability laws associated with the solution to (1.15), as $\varepsilon\to 0$. Let the quasi-potential $v:\mathbb{R}\to[0,+\infty]$ be defined as

$$v(y):=\inf\big\{\frac{1}{2}\int_0^T[\dot{x}-b(x)]^2dt,\,T>0,\,x\in W^{1,2}([0,T])\,:\,x(0)=0,x(T)=y\big\}.$$

v provides useful information about the law of the solution of (1.15). It is the large deviations rate functional of the invariant measures of(1.15) as $\varepsilon\to 0$, it provides sharp estimates for the average time the process takes to exit from a given domain containing x_0, it characterizes the path the process will follow (up to small fluctuations vanishing as $\varepsilon\to 0$) when exiting from such a domain. Roughly speaking, $v(y)$ represents the probabilistic cost of moving the process from 0 to y.

In the infinite dimensional setting above, it is however more difficult to get information concerning the stochastic processes from the quasi-potential. The details for the explicit formula of $V(m,u_f)$, as well as the generalization to the case of general uniformly convex smooth flux, will appear in a forthcoming paper. It will be also shown there that the quasi-potential (1.14) (with $u_i\equiv m$) of H coincides with the quasi-potential of H_ε at any $\varepsilon>0$, thus providing an heuristic argument to support the guess that H is indeed the Γ-limit of H_ε. We also remark that in [5] the quasi-potential problem for the functional H is considered in the case of Dirichlet boundary conditions, which however requires different techniques to be addressed.

Eventually, another application of the least squares method for perturbed partial differential equations has been considered in [24, 23, 33, 35, 30, 8], and is related to the paper [17]. Let us consider the solutions u^ε to the Allen-Cahn equation, that is the singularly perturbed parabolic partial differential equation

$$\varepsilon u_t - \varepsilon\Delta u + \frac{1}{\varepsilon}W'(u) = 0\,, \tag{1.16}$$

where $W(s) = \frac{1}{4}(1-s^2)^2$, under appropriate boundary conditions on the boundary of a domain $\Omega \subseteq \mathbb{R}^n$, having an initial datum $u(0) = u_0^\varepsilon$ which approximates the characteristic function of a smooth bounded open set $E \subset\subset \Omega$ and having $\sup_{\varepsilon\in(0,1)} M_\varepsilon(u_0^\varepsilon) < +\infty$, where $M_\varepsilon(v) := \int_\Omega \varepsilon|\nabla v|^2 + \frac{1}{\varepsilon}W(v)\ dx$. It is known that (u^ε) converges as $\varepsilon \to 0$ to the motion by mean curvature of E, at least for all times before the appearance of singularities in the geometric flow.

Let now $T > 0$ and consider the action functional

$$A_\varepsilon(u) := \int_{[0,T]\times\mathbb{R}^n} \left(\sqrt{\varepsilon}u_t - \frac{1}{\sqrt{\varepsilon}}(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u))\right)^2 dxdt \tag{1.17}$$

defined for $u \in C^2(\mathbb{R}^n \times [0,T])$. We assume periodicity of u with respect to the unit cube in space. The functions in the domain of A_ε can be coupled with the two boundary points condition $u(0) = u_0^\varepsilon$, $u(T) = u_1^\varepsilon$, where we assume that $\sup_\varepsilon M_\varepsilon(u_0^\varepsilon) < +\infty$ and $\sup_\varepsilon M_\varepsilon(u_1^\varepsilon) < +\infty$. The question is then to understand whether there is a meaningful Γ-limit of the functionals A_ε and, in positive case, how such a limit is related to the mean curvature flow. We give some references for this problem in Section 6.

2. Least squares method for ODEs

In this section we continue the discussion on the properties of the functional (1.2) and on variational weak solutions for an ordinary differential equation. We refer to [12, 18, 1, 13] for further details. Let $T > 0$ and

$$F(x) := \frac{1}{2}\int_I [\dot{x} - b(x)]^2\, dt\,,$$

where $I = [0,T]$ is a bounded interval, b is a bounded Borel function, $x \in W^{1,2}(I)$, and $F := +\infty$ in $L^2(I) \setminus W^{1,2}(I)$. As already said in the Introduction, one can add to F the function χ_{x_0} forcing the condition $x(t_0) = x_0$. The functional F can be rewritten as

$$F(x) = \frac{1}{2}\int_I \dot{x}^2 + [b(x)]^2 dt - \int_I b(x)\dot{x}dt =: Q(x) - \int_I b(x)\dot{x}dt,$$

where we set

$$Q(x) := \frac{1}{2}\int_I \mathfrak{f}(x,\dot{x})dt,$$

and $\mathfrak{f}(s,\xi) := \xi^2 + [b(s)]^2$. It is possible to see that F is $L^2(I)$-coercive if and only if Q is $L^2(I)$-coercive, and if $\overline{F}$ (resp. $\overline{Q}$) denotes the $L^2(I)$-lower semicontinuous envelope of F (resp. of Q) and if $x \in W^{1,2}(I)$, then

$$\overline{F}(x) = \overline{Q}(x) - \int_I b(x)\dot{x}dt.$$

Moreover, if the map $s \mapsto [b(s)]^2$ is lower semicontinuous, then Q is $L^2(I)$-lower semicontinuous on $W^{1,2}(I)$.

Example 2.1. *Let $b : \mathbb{R} \to \mathbb{R}$ be the function defined as follows: $b(s) := 1$ if $s < 0$ and $b(s) := -1$ if $s \geq 0$. Then*

- *equation (1.1) coupled with the condition $x(0) = 0$ has no almost everywhere $W^{1,2}(I)$-solutions, and therefore $\{F + \chi_0 = 0\}$ is empty;*
- *$F = \overline{F}$ on $W^{1,2}(I)$ and hence $\{\overline{F} + \chi_0 = 0\}$ is empty;*
- *F is $L^2(I)$-coercive, hence the problem*

$$\min\{F(x) + \chi_0(x) : x \in W^{1,2}(I)\} =: m$$

has a solution. If $F(x_{\min}) = m$ it is possible to prove that $x_{\min} \equiv 0$, and therefore $F(x_{\min}) = \frac{T}{2} > 0$. Indeed,

$$F(x) = \frac{1}{2}\int_I \dot{x}^2 dt + \frac{T}{2} - \int_0^{x(T)} b(s)ds = \frac{1}{2}\int_I \dot{x}^2 dt + \frac{a}{2} + |x(T)|\,,$$

Despite $\{F+\chi_0 = 0\}$ is empty, we can consider $x_{\min}$ as the variational weak solution of (1.1) with $x(0) = 0$.

Example 2.2. *Let $b : \mathbb{R} \to \mathbb{R}$ be as: $b(s) := 1$ if $s \in \mathbb{R} \setminus \mathbb{Q}$, $b(s) := 0$ if $s \in \mathbb{Q}$, see [16]. Note that $b = b^2$. Then equation (1.1) has a lot of almost everywhere solutions, such as: $x(t) \equiv q$ for any $q \in \mathbb{Q}$; $x(t) = t$; $x(t)$ a Lipschitz continuous with graph consisting of linear segments with slope one and horizontal segments in correspondence of rational values of x. As observed in [16] and proved in [12], Q is not $L^2(I)$-lower semicontinuous. It turns out that*

$$\overline{Q}(x) = \frac{1}{2}\int_I \psi(\dot{x})dt, \qquad x \in W^{1,2}(I), \tag{2.1}$$

where $\psi(\xi) = |\xi|^2 + 1$ if $|\xi| \geq 1$, and $\psi(\xi) = 2|\xi|$ if $|\xi| < 1$. In particular variational weak solutions to (1.1) with $x(0) = 0$ are those Lipschitz functions x in I such that $x'(t) \in [0, 1]$ for almost every $t \in I$.

Let us show (2.1). Since ψ is convex the functional on the right hand side of (2.1) is $L^2(I)$-lower semicontinuous on $W^{1,2}(I)$, therefore

$$\overline{Q}(x) \geq \frac{1}{2}\int_I \psi(\dot{x})dt\,, \qquad x \in W^{1,2}(I).$$

We thus need to prove

$$\overline{Q}(x) \leq \frac{1}{2}\int_I \psi(\dot{x})dt\,, \qquad x \in W^{1,2}(I). \tag{2.2}$$

Let $\sigma < \tau$, $\sigma, \tau \in I$, and let $\alpha, \beta \in \mathbb{R}$ be given. Let us consider the minimum problem

$$\min\left\{\frac{1}{2}\int_\sigma^\tau (\dot{x}^2 + g(x))dt \,:\, x \in W^{1,2}([\sigma, \tau]),\, x(\sigma) = \alpha,\, x(\tau) = \beta\right\}.$$

Let $x \in W^{1,2}(I)$. Setting $J := \{t \in [\sigma,\tau] \,:\, x(t) \in \mathbb{R}\setminus\mathbb{Q}\}$, we have

$$\begin{aligned} Q(x,(\sigma,\tau)) &:= \frac{1}{2}\int_\sigma^\tau (\dot{x}^2 + g(x))dt = \frac{1}{2}\int_J (\dot{x}^2+1)dt + \frac{1}{2}\int_{[\sigma,\tau]\setminus J} \dot{x}^2 dt \\ &\geq \frac{1}{2}\int_J (\dot{x}^2+1)dt \geq \frac{\left(\int_J |\dot{x}|dt\right)^2}{2|J|} + \frac{|J|}{2}, \end{aligned}$$

where we use Schwarz-Hölder inequality and $|J|$ is the Lebesgue measure of J. We deduce that

$$Q(x,(\sigma,\tau)) \geq \frac{(\beta-\alpha)^2}{2|J|} + \frac{|J|}{2} \geq \min_{\ell\in[0,\tau-\sigma]} \left\{ \frac{(\beta-\alpha)^2}{2\ell} + \frac{\ell}{2} \right\}.$$

A direct computation gives

$$\begin{aligned} \min_{\ell\in[0,\tau-\sigma]} \left\{ \frac{(\beta-\alpha)^2}{2\ell} + \frac{\ell}{2} \right\} &= \begin{cases} |\beta-\alpha| & \text{if } \tau-\sigma \geq |\beta-\alpha| \\ \frac{(\beta-\alpha)^2}{2(\tau-\sigma)} + \frac{\tau-\sigma}{2} & \text{if } \tau-\sigma < |\beta-\alpha| \end{cases} \\ &= \min\left\{ \frac{1}{2}\int_\sigma^\tau \psi(\dot{y})dt \,:\, y \in W^{1,2}(I),\, y(\sigma)=\alpha,\, y(\tau)=\beta \right\}. \end{aligned}$$

It follows that, given $x \in W^{1,2}(I)$ with $x(\sigma)=\alpha$ and $x(\tau)=\beta$,

$$Q(x,(\sigma,\tau)) \geq \min\left\{ \frac{1}{2}\int_\sigma^\tau \psi(\dot{y})dt \,:\, y \in W^{1,2}(I),\, y(\sigma)=\alpha,\, y(\tau)=\beta \right\}.$$

Furthermore

$$\begin{aligned} &\min\left\{Q(x,(\sigma,\tau)) \,:\, x\in W^{1,2}(I),\, x(\sigma)=\alpha,\, x(\tau)=\beta\right\} \\ &= \min\left\{ \frac{1}{2}\int_\sigma^\tau \psi(\dot{y})dt \,:\, y \in W^{1,2}(I),\, y(\sigma)=\alpha,\, y(\tau)=\beta \right\}. \end{aligned} \tag{2.3}$$

Indeed, if $\tau-\sigma < |\beta-\alpha|$, the solution of the minimum problem on the left hand side of (2.3) *is given by the linear function $t \mapsto \dfrac{\beta-\alpha}{\tau-\sigma}(t-1)+\alpha$ while, if $\tau-\sigma \geq |\beta-\alpha|$, a solution is given by any nondecreasing piecewise linear function which matches the boundary conditions, with slopes in $\{0,1\}$, regions with slope* 1 *corresponding to a rational value.*

Given $x \in W^{1,2}(I)$ and $n \in \mathbb{N}$, let us pick points $t_1,\dots,t_{n+1}$ with $\inf I = t_1 < \dots < t_{n+1} = \sup I$, *and consider a function x_n which solves, for any $i \in \{1,\dots,n\}$,*

$$\min\left\{ \frac{1}{2}\int_{t_i}^{t_{i+1}} \psi(\dot{y})\,dt \,:\, y \in W^{1,2}(t_i,t_{i+1}),\, y(t_i)=x(t_i),\, y(t_{i+1})=x(t_{i+1}) \right\} =: m_i\,.$$

From (2.3) *it follows*

$$m_i = Q(x_n,(t_i,t_{i+1}))\,.$$

Moreover, since the sequence (x_n) *converges to* x *in* $L^2(I)$ *as* $n \to +\infty$, *we have*

$$\begin{aligned}\overline{Q}(x) &\leq \liminf_{n\to\infty} Q(x_n) = \liminf_{n\to\infty} \sum_{i=1}^{n} Q(x_n,(t_i,t_{i+1})) \\ &= \liminf_{n\to\infty} \sum_{i=1}^{n} m_i \leq \frac{1}{2}\int_I \psi(\dot{x})dt\end{aligned}$$

and (2.2) *is proved.*

A general result concerning the characterization of variational weak solutions to autonomous scalar ordinary differential equations can be found in [1, Theorem 4.2]. It would be interesting to characterize variational weak solutions to systems of ordinary differential equations and possibly to compare the results with the other notions of weak solutions in the literature.

3. Least squares method and homogenization of transport equations

In this section we apply the least squares method to the homogenization of the transport equation (1.5) and to the associated system (1.4). We refer to [27, 10, 17, 3] for all details. We initially consider the functionals (1.7) for $u \in C^1(\Omega)$, and (1.6) for $x \in C^1(I;\mathbb{R}^n)$, with Ω an open bounded subset of $\mathbb{R}^{n+1}$ and $I = [0,1]$. We assume that

$$\mathrm{f} \in C^1_{\mathrm{per}}(\mathbb{R}^{n+1};\mathbb{R}^n),$$

where $C^1_{\mathrm{per}}(\mathbb{R}^{n+1};\mathbb{R}^n)$ stands for the set of C^1 functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ which are 1-periodic in all variables.

The general expressions of the Γ-limits of the functionals (1.6) and (1.7) in the strong topology of $L^2(I)$ and of $L^2(\Omega)$, respectively, are given by the two following results (see [10] and [11]).

Theorem 3.1. *There exists* $G^s = \Gamma(L^2(I)) - \lim_{\varepsilon\to 0} G_\varepsilon$ *and*

$$G^s(x) = \int_I \psi(\dot{x})dt, \qquad x \in C^1(I;\mathbb{R}^n)\,, \tag{3.1}$$

where the convex map $\psi : \mathbb{R}^n \to [0,+\infty]$ *is given by*

$$\psi(\xi) = \lim_{T\to+\infty} \inf\left\{\frac{1}{T}\int_0^T |\dot{x} - \mathrm{f}(x,t)|^2 : x \in C^1([0,T];\mathbb{R}^n), x(0) = 0, x(T) = \xi T\right\}.$$

Theorem 3.2. *There exists* $F^s = \Gamma(L^2(\Omega)) - \lim_{\varepsilon\to 0} F_\varepsilon$ *and*

$$F^s(u) = \int_\Omega A\begin{pmatrix}\nabla u\\ u_t\end{pmatrix}\cdot\begin{pmatrix}\nabla u\\ u_t\end{pmatrix} dxdt, \qquad u \in C^1(\Omega), \tag{3.2}$$

where A is a constant, symmetric, positive semi-definite $(n+1)\times(n+1)$ matrix. Moreover for every $\xi \in \mathbb{R}^{n+1}$

$$A\xi\cdot\xi = \inf\Bigg\{ \int_{I^{n+1}} (u_t + \mathrm{f}(x,t)\cdot\nabla u)^2\,dx\,dt \ : \\ u(\xi) = \xi\cdot z + \varphi(z),\ z=(x,t),\ \varphi\in C^1_{\rm per}(\mathbb{R}^{n+1})\Bigg\}.$$

In [27], theorems 3.1 and 3.2 are applied to investigate the rotation set of the Poincaré map. Let $T_t^{(\varepsilon)}(x_0)$ be the value at time $t \geq 0$ of the solution to (1.4) with initial condition

$$x(0) = x_0. \tag{3.3}$$

Then $x \mapsto T_t^{(\varepsilon)}(x)$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. The mapping $T_1^{(1)}$ is called the Poincaré map associated with $\dot{x} = \mathrm{f}(x,t)$ and will be denoted by T. We also write $T^m = T\circ T^{m-1}$ for each $m \in \mathbb{N}$, where $T^0 := \mathrm{Id}$.

Definition 3.3. *Let $p \in \mathbb{R}^n$. The family of systems* (1.4)*,* (3.3) *G-converges to the "G-limit system"*

$$\dot{x} = p\,, \tag{3.4}$$

if for every $x_0 \in \mathbb{R}^n$ the solutions $T_t^{(\varepsilon)}(x_0)$ converge to the solution of problem (3.4)*,* (3.3)*, as $\varepsilon \to 0$ and uniformly with respect to t on compact intervals of $\mathbb{R}$. If the convergence is uniform with respect to the initial value x_0 in the compact subsets of $\mathbb{R}^n$, we say that* (1.4) *strongly G-converges to* (3.4)*.*

It has been proved that G-convergence is equivalent to strong G-convergence [31].

Remark 3.4 (See [14],[32])**.** *If $n = 1$ the G-limit equation exists and for every $x_0 \in \mathbb{R}$ we have*

$$p = \lim_{t\to\infty}\frac{T_t(x_0)}{t} = \lim_{m\to\infty}\frac{T^m(x_0)}{m}\,.$$

The limit $\displaystyle\lim_{m\to\infty}\frac{T^m(x_0)}{m}$ is called the rotation number *of T and measures the averaged advance of a solution of $\dot{x} = \mathrm{f}(x,t)$ starting at x_0 as t changes by a unit. The rotation number is independent of x_0 and it is rational if and only if some power of T has a fixed point.*

In dimension $n > 1$, the natural generalization of the rotation number is the rotation vector [28].

Definition 3.5. *A vector $p \in \mathbb{R}^n$ is called rotation vector for the Poincaré map T if there exist a sequence $\{p_h\} \subset \mathbb{R}^n$ and a subsequence m_h of positive integers such that*

$$p = \lim_{h\to\infty}\frac{T^{m_h}(p_h) - p_h}{m_h}.$$

The set of all rotation vectors, denoted by $\rho(T)$, is a nonempty compact connected subset of $\mathbb{R}^n$ and is related to the G-convergence as follows [27].

Proposition 3.6. *The family of equations* (1.4) *strongly G-converges to* (3.4) *if and only if $\rho(T) = \{p\}$.*

We are now in a position to state the main results proved in [27] concerning the form of the strong Γ-limits (3.1) and (3.2). Let us start with the case $n = 1$.

Theorem 3.7. *Let $n = 1$. Then the function ψ in* (3.1) *has just one zero and this zero is the rotation number of T.*

Remark 3.8. *Let $n = 1$, and x^ε and x^0 be the solutions to* (1.4), (3.3) *and to* (3.4) , (3.3) *respectively. We know that $x^\varepsilon \to x^0$ uniformly on compact sets and $\rho(T) = \{p\}$. Since $G_\varepsilon(x^\varepsilon) = 0$, we have $G^s(x^0) = 0$ and thus $\psi(p) = 0$. In addition, every zero of G^s is the G-limit of solutions of* (1.4).

Theorem 3.9. *Let $n = 1$. Then there exists a constant $k \in [0, 1]$ such that the sequence (F_ε) $\Gamma(L^2(\Omega))$-converges to the functional*

$$F^s(u) = k \int_\Omega [u_t(x,t) + pu_x(x,t)]^2 \, dx \, dt, \qquad u \in C^1(\Omega)\,,$$

where p is the rotation number of T.

Theorem 3.10. *The constant k in Theorem 3.9 is strictly positive if and only if T admits an invariant measure, which is absolutely continuous with respect to the Lebesgue measure with density in $L^2_{\mathrm{loc}}(\mathbb{R})$.*

If $n > 1$ G-convergence is not always ensured. Moreover, concerning the Γ-limit G^s, even if the sequence (1.4) G-converges to (3.4), we have $p \in \{\psi = 0\}$.

As shown in [27], if $n = 2$, a case in which G-convergence fails is when

$$\mathrm{f}(x, y, t) = (a(y), 0), \qquad x, y \in \mathbb{R},\ t \geq 0, \tag{3.5}$$

with a a non constant function in $C^1_{\mathrm{per}}(\mathbb{R})$. The transport equation associated with the velocity field (3.5) is

$$u_t + a\left(\frac{y}{\varepsilon}\right) u_x = 0\,, \tag{3.6}$$

and describes a phenomenon of propagation in which the velocity oscillates in a transverse direction with respect to the direction of propagation.

Let us consider now the more general case

$$u_t + a_\varepsilon(y) u_x = 0\,, \tag{3.7}$$

with a_ε equibounded functions. If the coefficients a_ε do not converge pointwise (and this is indeed the case when $a_\varepsilon(y) = a(y/\varepsilon)$ for some periodic function a), then the sequence of solutions u^ε to (3.7) is not compact in the strong $L^2(\Omega)$-topology. This motivates the study of the Γ-limit of the functionals F_ε in the sequential weak $L^2(\Omega)$ topology. This study has been carried on in [3], where the functionals

$$F_\varepsilon(u) = \int_\Omega [u_t - a_\varepsilon(y) u_x]^2 \, dxdydt \tag{3.8}$$

are considered. The sequential $\Gamma(w\text{-}L^2(\Omega))$-limit F^w is found, under the hypotheses that Ω has a Lipschitz boundary and that a_ε converges to a Young measure $dy\,\nu_y$, see Theorem 3.20. The interesting feature is that such a Γ-limit has not an integral representation, but can be expressed in terms of a suitable inf-convolution operator and the Young limit ν_y. In the same hypotheses, the solutions u^ε to (3.7) are only (sequentially) weakly-$L^2(\Omega)$ convergent and the limit equation is not of the same kind of the original equation, in that memory effects induced by homogenization appear [36, 4].

Remark 3.11. *By the properties of Γ-convergence, any sequential weak $L^2(\Omega)$-limit of solutions of*

$$u_t - a_\varepsilon(y)u_x = 0$$

belongs to $\{F^w = 0\}$. On the other hand, if $F^w(u) = 0$, we only know that there exist a sequence (u^ε) weakly converging to u in $L^2(\Omega)$ and a sequence (v^ε) strongly converging to 0 in $L^2(\Omega)$ such that

$$u_t^\varepsilon - a_\varepsilon(y)u_x^\varepsilon = v^\varepsilon \qquad \text{in } \Omega\,.$$

3.1. **The case without space derivatives.** Let us start with a simpler case. Consider the family of functionals

$$\mathcal{F}_\varepsilon(u) := \int_\Omega (u_t - a_\varepsilon(y)u)^2\,dydt, \tag{3.9}$$

where $\Omega \subset \mathbb{R}_y \times \mathbb{R}_t \subset \mathbb{R}^2$ is a product of open intervals,

$$a(y) := \begin{cases} +1 & \text{if } [y] \text{ is even;} \\ -1 & \text{if } [y] \text{ is odd} \end{cases} \tag{3.10}$$

($[y]$ denotes the integer part of y), and $a_\varepsilon(y) := a(y/\varepsilon)$. The natural domain of $\mathcal{F}_\varepsilon$ is the space

$$\mathcal{V} = \{u \in L^2(\Omega)\,:\, u_t \in L^2(\Omega)\}.$$

We have

Theorem 3.12. *The family $(\mathcal{F}_\varepsilon)$ in (3.9) $\Gamma(w\text{-}L^2(\Omega))$-converges in $\mathcal{V}$ to the functional*

$$\mathcal{F}^w(u) = \inf_{h\in\mathcal{V}} \int_\Omega (u_t - h)^2 + (u - h_t)^2 dy\,dt\,. \tag{3.11}$$

Remark 3.13. *The functional $\mathcal{F}^w$ vanishes exactly on distributional solutions in $\mathcal{V}$ of the equation $u_{tt} = u$ (i.e. $u \in \mathcal{V}$ such that $\int_\Omega u_t\phi_t dydt = -\int_\Omega u\phi dydt$ for all $\phi \in \mathcal{V}$). Indeed, if $u \in \mathcal{V}$ is such that $\mathcal{F}^w(u) = 0$, then there exists $\tilde h \in \mathcal{V}$ such that $\tilde h = u_t$ and $\tilde h_t = u$, and hence $u_{tt} = u$. Conversely, if $u \in \mathcal{V}$ is such that $u_{tt} = u$ in distributional sense, taking $h = \int^t u$ we have $h_t = u = u_{tt}$ and $h = u_t$, so that $\mathcal{F}^w(u) = 0$.*

Remark 3.14. *The functional $\mathcal{F}^w$ in (3.11) can also be represented as*

$$\mathcal{F}^w(u) = \min\left\{\int_\Omega (u_t - h_t)^2 + (u - h)^2 dt\,dy\,:\, h \in \mathcal{V},\, h_{tt} = h\right\}, \qquad u \in \mathcal{V}. \tag{3.12}$$

Indeed, we know that minimizing functions in (3.11) satisfy the equation $h_{tt} = h$ and thus, taking $k = h_t$, we find that (3.12) is equal to (3.11).

Equation (3.11) does not seem suitable for generalizations, and in [3] an alternative representation is provided, based on the inf-convolution. We recall the definition and the main properties of the inf-convolution (see, for instance, [34]).

Definition 3.15. *Let X be a Banach space and $F : X \to (-\infty, +\infty]$ be a function. The polar of F is the function $F^* : X^* \to (-\infty, +\infty]$ defined by*

$$F^*(u) := \sup_{v \in X} \{< v, u > -F(v)\} \ .$$

F^* is convex and lower semicontinuous, and if F is convex and lower semicontinuous, then $F = (F^*)^*$.

Definition 3.16. *Let X be a Banach space and $F_i : X \to (-\infty, +\infty]$ be convex functions, $i = 1, \dots, m$. The inf-convolution $F_1 \Box \dots \Box F_m$ of $F_1, \dots, F_m$ is defined as*

$$(F_1 \Box \dots \Box F_m)(u) := \inf\{F_1(u_1) + \dots + F_m(u_m) \, : \, u_1 + ... + u_m = u\} \, .$$

The terminology arises from the fact that, when only two functions (say F and G) are involved, $\Box$ can be expressed by

$$(F \Box G)(u) = \inf_v \{F(v) + G(u - v)\}$$

and this is analogous to the classical formula for convolution. In case $G(\cdot) = \lambda|\cdot|^2$, the Yosida transform of F is obtained.

The inf-convolution is a functional operation which corresponds to the addition of epigraphs as sets: if $E_1, \dots, E_m$ denote the epigraphs of $F_1, \dots, F_m$, and $E := E_1 + \dots + E_m$, then

$$(F_1 \Box \dots \Box F_m)(u) = \inf\{\mu \, : \, (u, \mu) \in E\} \, .$$

The inf-convolution is convex but not necessarily lower semicontinuous, and is dual to the operation of addition of convex functions, that is

$$(F_1 \Box \dots \Box F_m)^* = F_1^* + \dots + F_m^* \, .$$

Proposition 3.17. *The functional $\mathcal{F}^w$ in (3.11) can also be represented as*

$$\mathcal{F}^w(u) = F_1 \Box F_2(u) \, , \qquad u \in \mathcal{V} \, , \tag{3.13}$$

where

$$F_1(v) := 2 \int_\Omega (v_t - v)^2 dy dt \, , \qquad F_2(v) := 2 \int_\Omega (v_t + v)^2 dy \, dt \, , \qquad v \in \mathcal{V} \, .$$

Remark 3.18. *Let $u \in L^2(\mathbb{R}^2)$. The function $\mathcal{F}^w(u, \cdot)$, if considered as a function of the domain of integration, is not subadditive, therefore $\mathcal{F}^w$ cannot be represented by an integral. Take for instance the function*

$$u(y,t) := \begin{cases} e^t & \text{if } t < 0 \, , \\ e^{-t} & \text{if } t \geq 0 \, . \end{cases}$$

For every $\varepsilon \in (0, 1)$, $\mathcal{F}^w\,(u, (0, 1) \times (-1, -\varepsilon)) = 0$, since u is a (strong) solution of $u_{tt} = u$ in $(0, 1) \times (-1, -\varepsilon)$. Similarly, $\mathcal{F}^w(u, (0, 1) \times (\varepsilon, 1)) = 0$. Moreover, since $(u_t)^2 + u^2 \leq 2$ we have $\mathcal{F}^w(u, \Omega) \leq 2\mathcal{L}(\Omega)$, and therefore $\mathcal{F}^w\,(u, (0, 1) \times (-2\varepsilon, 2\varepsilon)) \leq$

8ε. *Observe now that* $\mathcal{F}^w(u,(0,1)\times(-1,1))$ *is strictly positive, since* u *is not a solution of* $u_{tt} = u$ *in* $(0,1)\times(-1,1)$ *(consider for instance any* $\phi \in C^\infty$ *with support in* $(a/2,a)\times(-a,a)$, $a<1$*). Then, for* ε *sufficiently small, we have*

$$\mathcal{F}^w(u,(0,1)\times(-1,1)) > \mathcal{F}^w(u,(0,1)\times(-1,-\varepsilon)) + \mathcal{F}^w(u,(0,1)\times(-2\varepsilon,2\varepsilon)) + \mathcal{F}^w(u,(0,1)\times(\varepsilon,1))\,.$$

3.2. **The case with space derivatives.** Let us study now the asymptotic behaviour of the functionals (3.8) where Ω is a bounded convex open set in $\mathbb{R}_x\times\mathbb{R}_y\times\mathbb{R}_t$, $a_\varepsilon(y)$ is again given by $a_\varepsilon(y) = a(y/\varepsilon)$, with $a(y)$ defined in (3.10), and the domain of F_ε is

$$V = \{u \in L^2(\Omega) \,:\, u_x, u_t \in L^2(\Omega)\}\,.$$

This example is related to some conjectures by De Giorgi [17], and has been studied in [3].

Theorem 3.19. *The family* (3.8) $\Gamma(w\text{-}L^2(\Omega))$*-converges in* V *to the functional*

$$F^w(u) = \inf_{h\in V}\int_\Omega (u_t-h_x)^2 + (u_x-h_t)^2 dx\,dy\,dt\,, \tag{3.14}$$

Moreover, we have

$$F^w = F_+\square\, F_- \qquad \text{in } V,$$

where

$$F_\pm(u) := \begin{cases} 2\displaystyle\int_\Omega (u_x\pm u_t)^2\,dx\,dt\,dy & \text{if } u_x\pm u_t \in L^2(\Omega),\\ +\infty, & \text{otherwise.}\end{cases}$$

The functional F^w in (3.14) is convex, quadratic and its zero set consists of all distributional solutions of the linear wave equation $u_{tt} = u_{xx}$ (no derivatives with respect to y). Indeed, if $u \in V$ is such that $F^w(u) = 0$, then there exists $\tilde h \in V$ such that $\tilde h_x = u_t$ and $\tilde h_t = u_x$, and hence $u_{tt} = u_{xx}$. Conversely, if u is such that $u_{tt} = u_{xx}$, taking $h = \int^t u_x dt$ we have $h_t = u_x$ and then $h_{xt} = h_{tx} = u_{xx} = u_{tt}$, so that $h_x = u_t$ and $F^w(u) = 0$. Arguing as in Remark 3.18, it can be seen that $F^w(u,\cdot)$ is not subadditive with respect to Ω

Let us describe now the sequential weak Γ-limit of the functionals (3.8) in the general case, assuming a to be bounded. We assume that $\Omega_y = \{(x,t) \,:\, (x,y,t)\in\Omega\}$ is a Lipschitz domain with compact boundary for almost every $y \in (0,1)$, and $\Omega_y = \emptyset$ if $y \notin (0,1)$.

Let $A \subset \mathbb{R}^2$ be an open set, and let $u \in L^2(\mathbb{R}^2)$. Denote by $J \subset \mathbb{R}$ a compact interval containing all the values of a_ε. For $j \in J$, define

$$\mathcal{H}(j,u,A) := \begin{cases} \int_A |u_t - j u_x|^2 dx dt, & \text{if } u_t - j u_x \in L^2(A);\\ +\infty, & \text{otherwise.}\end{cases}$$

Observe that

$$F_h(u) = \int_\Omega (u_t - a_h(y)u_x)^2\; dx\,dy\,dt = \int_I H(a_h(y), u(y,\cdot), \Omega_y)dy$$

Given any probability measure ν in J, define also

$$\tilde{F}(u,\nu,A) := \inf\left\{\int_J \mathcal{H}(j,w(j),A)d\nu(j) \ : \ w \in L^2(J,\nu;L^2(\mathbb{R}^2)), \int_J w(j)d\nu(j) = u\right\}.$$

The functional $\tilde{F}$ can be viewed as the "inf-convolution", weighted by ν, of the family of functionals $\mathcal{H}(j,\cdot,A)$ indexed by $j \in J$.

The main result in [3] is the following. Let I, $J \subset \mathbb{R}$ be compact intervals. Let $\mathcal{L}$ denote the Lebesgue measure. We say that a sequence of Borel functions $a_h : I \to J$ Y-converges to the measure $\mathcal{L} \otimes \nu_y$ if

$$\lim_{h\to\infty} \int_I \phi(y, a_h(y))dy = \int_I \left(\int_J \phi(y,j)d\nu_y(j)\right) dy$$

for any map $\phi \in L^1(I, C(J))$, namely, continuous in j for almost every y, measurable in y, such that $||\phi(y,j)||_{C(J)} \in L^1(I)$.

Theorem 3.20. *If the functions a_ε Y-converge to $\mathcal{L}\otimes\nu_y$, then the $\Gamma(w\text{-}L^2(\Omega))$-limit of F_ε is given by*

$$F^w(u,\Omega) = \int_0^1 \tilde{F}(u(y,\cdot),\nu_y,\Omega_y)dy$$

for every $u \in L^2(\Omega)$.

An essential tool in the proof of Theorem 3.20 is a lower semicontinuity result for convex integrals with respect to weak convergence of measure. It may be applied thanks to the convexity of $\mathcal{H}$ in u. The authors in [3] remark that the proof cannot be extended to nonlinear cases, since if the equation is nonlinear the analogue of $\mathcal{H}$ need not be convex in u.

4. Least squares method and scalar conservation laws

In this section, we give some details on the application of the least squares approach for the variational characterization of weak solutions to the one-dimensional scalar conservation laws, reporting some recent results from [6]. For simplicity, we will assume periodicity in space, in order to match with the results in Section 5.

Let $\mathbb{T}$ be the one-dimensional torus of unitary length. For fixed $T > 0$, consider the scalar conservation law (1.8) with $(t,x) \in [0,T] \times \mathbb{T}$, where u takes values in $[-1,1]$, f is assumed to be smooth in $[-1,1]$ and not affine.

We denote by U the space of measurable functions $u : \mathbb{T} \to [-1,1]$, equipped with the $H^{-1}(\mathbb{T})$ metric, and we let $\mathcal{U}$ be the set $C\big([0,T];U\big)$ endowed with the sup-norm.

Definition 4.1. *An element $u \in \mathcal{U}$ is a weak solution to* (1.8) *if for each $\varphi \in C_c^\infty\big((0,T)\times\mathbb{T}\big)$ it satisfies*

$$\langle\langle u, \varphi_t\rangle\rangle + \langle\langle f(u), \varphi_x\rangle\rangle = 0\,,$$

$\langle\langle\cdot,\cdot\rangle\rangle$ denoting the inner product in $L^2([0,T]\times\mathbb{T})$. A weak solution to (1.8) *is called* entropic *if and only if for each entropy-entropy flux pair (η, q) with η convex of class*

C^2, and q defined as $q(u) := \int^u \eta'(v)\, f'(v) dv$, the η-entropy production $\wp_{\eta,u}$, which is a distribution defined as

$$\wp_{\eta,u}(\varphi) := -\langle\langle \eta(u), \varphi_t \rangle\rangle - \langle\langle q(u), \varphi_x \rangle\rangle\,, \qquad \varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times\mathbb{T}\big), \tag{4.1}$$

is nonpositive.

The classical theory [15] shows existence and uniqueness in $C\big([0,T];L^1(\mathbb{T})\big)$ of the entropic solution to the Cauchy problem associated with (1.8). A possible approach to construct entropic solutions is to consider the viscous approximation (1.9) where D, assumed smooth and uniformly positive, represents the diffusion coefficient and $\varepsilon > 0$ the viscosity. Indeed, as $\varepsilon \to 0$ equibounded solutions to (1.9) converge in $L^1([0,T]\times\mathbb{T})$ to entropic solutions to (1.8).

As proven in [6], it is possible to characterize the asymptotic cost of non-entropic weak solutions to (1.8) in terms of the parabolic problem (1.9). The parabolic cost functional $I_\varepsilon : \mathcal{U} \to [0,+\infty]$ is defined as follows. For $\varepsilon > 0$, $u \in \mathcal{U}$ such that $u_x \in L^2_{\mathrm{loc}}([0,T]\times\mathbb{T})$ we set

$$I_\varepsilon(u) := \sup_{\varphi\in C_{\mathrm{c}}^\infty((0,T)\times\mathbb{T})} \Big[-\langle\langle u, \varphi_t\rangle\rangle - \langle\langle f(u), \varphi_x\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u)u_x, \varphi_x\rangle\rangle - \frac{1}{2}\langle\langle \sigma(u)\varphi_x, \varphi_x\rangle\rangle \Big]$$

and $I_\varepsilon := +\infty$ elsewhere in $\mathcal{U}$.

The family of functionals (I_ε) is equicoercive on $\mathcal{U}$, and $I_\varepsilon(u)$ vanishes if and only if $u \in \mathcal{U}$ is a weak solution to (1.9). Moreover, when $I_\varepsilon(u) < +\infty$, then $u \in C\big([0,T];L^1(\mathbb{T})\big)$. The first Γ-convergence result in [6] is the identification of the Γ-limit of (I_ε) on $\mathcal{U}$, obtained in a Young measures' setting.

Denote now by $\mathcal{X}_T$ the set $C([0,T];U)$ endowed with the norm $\|u-v\|_{L_1([0,T]\times\mathbb{T})} + \sup_{t\in[0,T]} \|u(t)-v(t)\|_{H^{-1}(\mathbb{T})}$. (Equi)-coercivity on $\mathcal{X}$ implies (equi)-coercivity on $\mathcal{U}$ and, conversely, lower semicontinuity on $\mathcal{U}$ implies lower semicontinuity on $\mathcal{X}$.

The second result in [6] concerns the Γ-convergence of (H_ε), which turn out to be equicoercive on $\mathcal{X}_T$. In order to illustrate this second result, we need to introduce the class of *entropy-measure solutions*.

Let $C_{\mathrm{c}}^{2,\infty}\big([-1,1]\times(0,T)\times\mathbb{T}\big)$ be the set of compactly supported maps $\vartheta : [-1,1]\times(0,T)\times\mathbb{T} \ni (v,t,x) \mapsto \vartheta(v,t,x) \in \mathbb{R}$, that are twice differentiable in the v variable, with derivatives continuous up to the boundary of $[-1,1]\times(0,T)\times\mathbb{T}$, and that are infinitely differentiable in the (t,x) variables. For $\vartheta \in C_{\mathrm{c}}^{2,\infty}\big([-1,1]\times(0,T)\times\mathbb{T}\big)$ let ϑ' and ϑ'' denote its partial derivatives with respect to the v variable. We say that $\vartheta \in C_{\mathrm{c}}^{2,\infty}\big(-1,1]\times(0,T)\times\mathbb{T}\big)$ is an *entropy sampler*, and its *conjugated entropy flux sampler* $Q : [-1,1]\times(0,T)\times\mathbb{T}$ is defined up to an additive function of (t,x) by $Q(u,t,x) := \int^u \vartheta'(v,t,x) f'(v) dv$.

Given a weak solution u to (1.8), the *ϑ-sampled entropy production* $P_{\vartheta,u}$ is the real number

$$P_{\vartheta,u} := -\int \Big[\big(\partial_t\vartheta\big)\big(u(t,x),t,x\big) + \big(\partial_x Q\big)\big(u(t,x),t,x\big) \Big] dt dx\,. \tag{4.2}$$

If $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some entropy η and some $\varphi \in C_{\mathrm{c}}^\infty\big((0,T)\times\mathbb{T}\big)$, then $P_{\vartheta,u} = \wp_{\eta,u}(\varphi)$.

We denote by $M\big((0,T)\times\mathbb{T}\big)$ the set of Radon measures on $(0,T)\times\mathbb{T}$ that we consider equipped with the vague topology. Moreover, for $\rho\in M\big((0,T)\times\mathbb{T}\big)$ we denote by $\rho^{\pm}$ the positive and negative part of ρ.

Definition 4.2. *Let $u\in\mathcal{X}_T$ be a weak solution to* (1.8)*. If there exists a bounded measurable map $\rho_u:[-1,1]\ni v\to\rho_u(v;dt,dx)\in M\big((0,T)\times\mathbb{T}\big)$ such that for any entropy sampler ϑ*

$$P_{\vartheta,u}=\int\vartheta''(v,t,x)\rho_u(v;dt,dx)dv\,,\tag{4.3}$$

then u is called an entropy-measure solution *to* (1.8)*. The set of entropy-measure solutions to* (1.8) *is denoted by $\mathcal{E}_T$.*

Remark 4.3. *Entropy solutions are entropy-measure solutions such that $\rho_u(v;dt,dx)$ is a negative Radon measure for each $v\in[-1,1]$. Moreover entropy-measure solutions to* (1.8) *are in $C\big([0,T];L^1(\mathbb{T})\big)$.*

We are now in a position to introduce the candidate Γ-limit $H_T:\mathcal{X}_T\to[0,+\infty]$ of the functionals H_ε. We have

$$H_T(u):=\begin{cases}\displaystyle\int\frac{1}{\chi(v)}\rho_u^+(v;dt,dx)dv & \text{if } u\in\mathcal{E}_T\,,\\ +\infty & \text{otherwise}\,,\end{cases}\tag{4.4}$$

where the *susceptibility* χ is defined as $\chi(v):=\sigma(v)/D(v)$, $v\in[-1,1]$.

Remark 4.4. *H_T turns out to be coercive, lower semicontinuous, and $H_T(u)=0$ if and only if u is an entropic solution to* (1.8)*. In addition, $H_T(u)=0$ if and only if u is a limit point of a sequence $(u^\varepsilon)\subset\mathcal{X}_T$ such that $I_\varepsilon(u^\varepsilon)=0$. In* [6] *the authors also prove that if u is a weak solution to* (1.8) *and $H_T(u)<+\infty$, then $H_T(u)=\sup_\vartheta P_{\vartheta,u}$, where the supremum is taken over the entropy samplers ϑ such that $0\le\sigma(v)\vartheta''(v,t,x)\le D(v)$, for each $(v,t,x)\in[-1,1]\times[0,T]\times\mathbb{R}$.*

The results concerning the Γ-convergence of H_ε are collected in the next theorem. We need a further definition.

Definition 4.5. *We say that an entropy-measure solution $u\in\mathcal{E}$ is* entropy-splittable *if there exist two closed sets $E^+,E^-\subset[0,T]\times\mathbb{R}$ such that*

- (i) *For a.e. $v\in[0,1]$, the support of $\rho_u^+(v;dt,dx)$ is contained in E^+, and the support of $\rho_u^-(v;dt,dx)$ is contained in E^-.*
- (ii) *For each $L>0$, the set $\Big\{t\in[0,T]\,:\,\big(\{t\}\times[-L,L]\big)\cap E^+\cap E^-\neq\emptyset\Big\}$ is such that the closure of its interior is empty.*

The set of entropy-splittable solutions is denoted by $\mathcal{S}$. An entropy-splittable solution $u\in\mathcal{S}$ such that $H(u)<+\infty$ and

- (iii) *For each $L>0$ there exists $\delta_L>0$ such that $\sigma(u(t,x))\ge\delta_L$ for a.e. $(t,x)\in[0,T]\times[-L,L]$.*

is called nice with respect to σ, *and we write $u\in\mathcal{S}_\sigma$.*

Theorem 4.6. *We have* Γ-$\liminf_\varepsilon H_\varepsilon \geq H_T$ *on* $\mathcal{X}_T$*, and* Γ-$\limsup_\varepsilon H_\varepsilon \leq \overline{H}_T$ *on* $\mathcal{X}_T$*, where*

$$\overline{H}_T(u) := \inf \left\{ \liminf H_T(u_n),\, \{u_n\} \subset \mathcal{S}_\sigma \,:\, u_n \to u \text{ in } \mathcal{X}_T \right\}.$$

If $u \in \mathcal{X}_T$ is a weak solution with locally bounded variation, Vol'pert chain rule gives a formula for $H(u)$ in terms of the normal traces $u^\pm$ of u on its jump set J_u. In particular, for Burgers' equation and $D \equiv \sigma \equiv 1$ we get

$$H_T(u) = \int_{J_u^a} \frac{(u^+ - u^-)^3}{12} |n^x| d\mathcal{H}^1 \,,$$

n_x being the x-component of a unit normal vector to J_u, and $J_u^a = \{u^- < u^+\}$ the set of non-entropic shocks.

We conclude this section by pointing out that studying the properties of the functions belonging to the class $\mathcal{E}_T$ is essentially equivalent to study the solutions to a family of conservation laws of the form $\eta(u)_t + q(u)_x = \mu_\eta$, where, for each smooth η, μ_η is a Radon measure with finite total variation.

5. The quasi-potential problem for Burgers' equation

We define the functionals V_T, $V : U \times U \to [0, +\infty]$ as

$$\begin{aligned} V_T(u_i, u_f) &:= \inf \left\{ H_T(u) \,:\, u \in \mathcal{X}_T,\, u(0) = u_i,\, u(T) = u_f \right\}, \\ V(u_i, u_f) &:= \inf_{T \geq 0} V_T(u_i, u_f)\,. \end{aligned} \tag{5.1}$$

Remark 5.1. *The infimum in the definition of* V_T *is a minimum. This follows from the lower semicontinuity and the coercivity of* H_T *on* $\mathcal{X}_T$ *(see Remark 4.4), and from the closure of the subset* $\{u \in \mathcal{X}_T \,:\, u(0) = u_i,\, u(T) = u_f\}$*. Equivalently, we can substitute* $\mathcal{E}_T$ *in place of* $\mathcal{X}_T$ *in* (5.1)*. It will follow from our analysis that the set* $\{u \in \mathcal{E}_T \,:\, u(0) = u_i, u(T) = u_f\}$ *is nonempty, provided* $u_i(0) \equiv m$ *and* T *is large enough.*

From now on we suppose $f(u) = u^2/2$ (Burgers' equation), $D \equiv \sigma \equiv 1$ and $u_i \equiv m$. Our aim is to announce an expression for V and to indicate some of the main points in the proof. In a forthcoming paper we will provide all mathematical details, under less restrictive hypotheses.

Remark 5.2. *If* $\int_{\mathbb{T}} u_i dx \neq \int_{\mathbb{T}} u_f dx$ *then, for each* $T > 0$,

$$V_T(u_i, u_f) = V(u_i, u_f) = +\infty$$

since weak solutions to (1.8) *conserve the total mass. Hereafter we will always consider the case* $\int_{\mathbb{T}} u_i dx = \int_{\mathbb{T}} u_f dx =: m \in (-1, 1)$*, since the cases* $m = \pm 1$ *are trivial.*

Observe that H_T has not a quadratic structure, usually exploited to provide explicit representations of quasi-potential functionals. However, we will take advantage of the following space-time symmetry: if $u(t, x) \in \mathcal{X}_T$ is a weak (resp.

entropy-measure) solution to (1.8), then $u(T-t,-x)$ is a weak (resp. entropy-measure)solution too. More precisely, if $u_f \in U$, and if for $v \in U$ we define $\tilde{v}(x) := v(-x)$, then

$$V_T(m, u_f) = V_T(\tilde{u}_f, m) + \int_{\mathbb{T}} (u_f - m)^2 dx\,. \tag{5.2}$$

As explained in the Introduction, the assumption of constant initial profile is natural, in view of the following result (see, e.g. [15] Ch. 11, [25]).

Theorem 5.3. *Let $u_0 \in U$, let $\bar{u}$ be the entropic solution to* (1.8) *with initial datum u_0 and let $m := \int_{\mathbb{T}} u_0(x)dx$. For each $\delta > 0$, there exists a time $\tau'(\delta) > 0$ independent of u_0 such that*

$$\sup_{x\in\mathbb{T}} |\bar{u}(t,x) - m| \le \delta$$

for each $t \ge \tau'(\delta)$.

Our main result is the following.

Theorem 5.4. *Let $m \in [-1,1]$. Then*

$$V(m, u_f) = \int_{\mathbb{T}} (u_f - m)^2 dx\,, \qquad u_f \in U,\ \int_{\mathbb{T}} u_f\, dx = m$$

Note that from (5.2) it follows $V(m,u_f) \ge \int_{\mathbb{T}} (u_f - m)^2 dx$. The converse inequality is obtained by proving that for each $\delta > 0$ and $u_f \in U$ with $\int_{\mathbb{T}} u_f\ dx = m$, there exists $T(\delta) > 0$ such that $V_T(\tilde{u}_f, m) \le \delta$ for every $T \ge T(\delta)$, where $\tilde{u}_f(x) = u_f(-x)$, and then by using (5.2). Indeed, we exhibit a time $T(\delta)$ and, for every $T \ge T(\delta)$, a function $u \in \mathcal{E}_T$ such that $u(0,x) = \tilde{u}_f(x)$, $u(T,x) = m$ and $H_T(u) \le \delta$. This result is first established for a piecewise constant function u_f. The proof is then completed by exploiting the lower semicontinuity and coerciveness of H_T.

Piecewise constant initial profiles have the nice property that the subsequent entropic solution is piecewise affine, and new shocks cannot be generated at any positive time. Let $\bar{u}$ be the (piecewise affine) entropic solution with the initial datum $\bar{u}(0,x) = u_f(-x)$. Fix $\delta > 0$ such that $-1 < m - 3\delta$ and $1 > m + 3\delta$ and let $\tau' = \tau'(\delta)$ be defined as in Theorem 5.3. For $t \in (0, \tau']$ let $u(t,x) := \bar{u}(t,x)$. Let $\bar{x}$ be an arbitrary point in $\mathbb{T}$ and consider the maps $s_\pm : [0,+\infty) \to \mathbb{T}$ as the piecewise C^1 solutions to

$$\dot{s}_\pm(t) = \frac{m \pm 3\delta + \bar{u}(\tau' + t, s_\pm(t))}{2}\,, \qquad s_\pm(0) = \bar{x}\,.$$

It is possible to show that $s_\pm$ are well defined. Since the maps take values on the unitary torus, we have $s_+(t) - s_-(t) \ge 2\delta\, t \mod 1$. Define

$$\tau'' := \inf\{t > 0 \,:\, s_+(t) - s_-(t) = 1\} \le (2\delta)^{-1}\,,$$

and for $t \in (\tau', \tau' + \tau'']$ set

$$u(t,x) := \begin{cases} m + 3\delta & \text{if } \bar{x} + m(t-\tau') < x \le s_+(t-\tau')\,, \\ m - 3\delta & \text{if } \bar{x} + m(t-\tau') > x \ge s_-(t-\tau')\,, \\ \bar{u}(t,x) & \text{otherwise}\,. \end{cases}$$

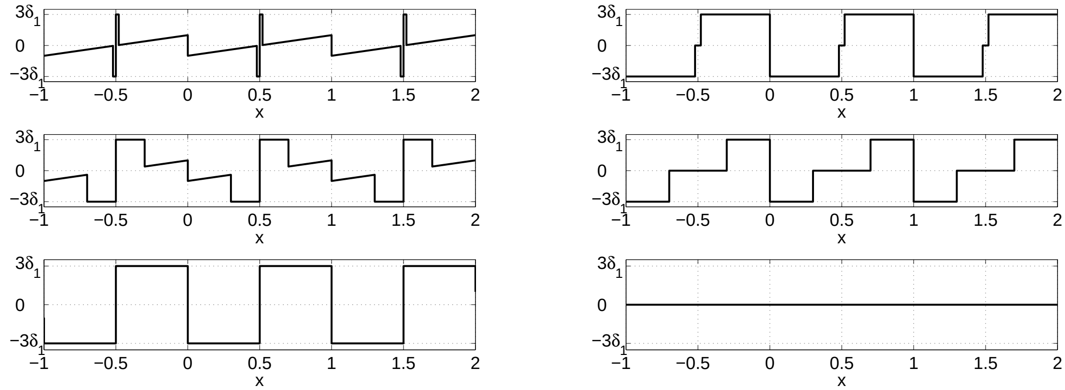

FIGURE 1. On the left, behaviour of $u(t,x)$ for $t \in (\tau', \tau'']$, $\bar{x} = 1/2$. On the right, behaviour of $u(t,x)$ for $t \in (\tau'', \tau]$. The pictures show, from top to bottom, $u(t,x)$ at three subsequent times.

Define now $\tau''' := \frac{1}{3\delta}$, $\tau = \tau(\delta) := \tau' + \tau'' + \tau''' \le \tau' + \frac{5}{6\delta}$, and for $t \in (\tau' + \tau'', \tau]$ set

$$u(t,x) := \begin{cases} m & \text{if } m - \frac{3\delta}{2} \le \frac{x-\bar{x}}{t-\tau'-\tau''} \le m + \frac{3\delta}{2}\,, \\ m + 3\delta & \text{if } \frac{x-\bar{x}}{t-\tau'-\tau''} > m + \frac{3\delta}{2}\,, \\ m - 3\delta & \text{if } m - \frac{3\delta}{2} > \frac{x-\bar{x}}{t-\tau'-\tau''}\,. \end{cases}$$

Observe that $u(\tau, x) \equiv m$. Given $T > \tau$, we finally set $u(t,x) \equiv m$ in $(\tau, T]$.

One checks that $u \in \mathcal{E}_T$, $u(0,x) = \tilde{u}_f(x)$, $u(T,x) = m$, and $H_T(u) \le \frac{3\delta^2}{2}$. For every $\delta > 0$, it is then enough to choose δ such that $\frac{\delta^2}{9} + \frac{3\delta^2}{2} \le \delta$ and take $T(\delta) = \tau(\delta)$.

Example 5.5. *Consider the initial profile $\tilde{u}_f(x) = -1$ if $x < \frac{1}{2}$, $\tilde{u}_f(x) = 1$ if $x > \frac{1}{2}$, so that $m = 0$. The unique entropy solution of Burgers' equation is given by*

$$\bar{u}(t,x) = \begin{cases} -1 & \text{if } \frac{x-\frac{1}{2}}{t} < -1\,, \\ \frac{x-\frac{1}{2}}{t} & \text{if } -1 < \frac{x-\frac{1}{2}}{t} < 1\,, \\ 1 & \text{if } \frac{x-\frac{1}{2}}{t} > 1\,, \end{cases}$$

for $t \in (0, 1/2]$, and $\bar{u}(t,x) = \frac{x-\frac{1}{2}}{t}$ for $t \ge 1$. For fixed $\delta > 0$, the smallest time $\tau' = \tau'(\delta)$ satisfying the requirement of Theorem 5.3 is $(2\delta)^{-1}$. Choosing $\bar{x} = \frac{1}{2}$ we have

$$s_{\pm}(t) = \pm 2 \pm 3\delta t \mp \frac{3}{2}\sqrt{1 + 2\delta t}\,,$$

and $\tau'' = \frac{5+\sqrt{21}}{12\delta}$. Figure 1 shows the behaviour of $u(t,x)$.

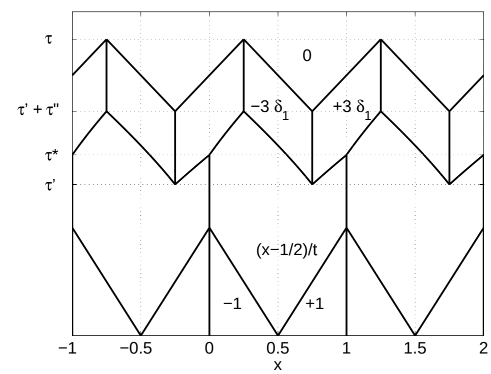

FIGURE 2. $u(x,t)$ in the plane (x,t), $\bar{x}=3/4$.

If $\bar{x}=3/4$ instead of $1/2$, there is a time τ^* at which the curve (t,s_+) crosses a discontinuity of $\bar{u}(t,x)$. In this case we have

$$s_+(t)=\frac{1}{2}(8+12\delta t-5\sqrt{1+2\delta t}) \qquad t\in(\tau',\tau^*]\,,$$

$$s_+(t)=3(1+\delta t)-\frac{73\sqrt{2}+5\sqrt{146}}{8\sqrt{49+5\sqrt{73}}}\sqrt{1+2\delta t} \qquad t\in(\tau^*,\tau'+\tau'']\,,$$

and

$$s_-(t)=\frac{1}{2}(-4-12\delta t+7\sqrt{1+2\delta t}) \qquad t\in(\tau',\tau'+\tau'']\,.$$

Figure 5.5 shows $u(t,x)$ in the plane (x,t) for the case $\bar{x}=3/4$.

6. Least squares method and singular perturbations to mean curvature flow

In this final section we provide some partial information on the Γ-limits for the action functional (1.17) related to the singularly perturbed reaction-diffusion equation (1.16). A first related conjecture [17], concerning the "static" case, is concerned with the asymptotic behaviour of a sequence of functionals which contain a term similar to

$$S_\varepsilon(u):=\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\left(-\varepsilon\Delta u+\frac{1}{\varepsilon}W'(u)\right)^2dx.$$

In [35] (see also [9], [29] for related results, and [7] for a nonlocal situation) it has been proved that, provided $n\in\{2,3\}$, the sequence $(M_\varepsilon+S_\varepsilon)$ $\Gamma(L^1)$-converges, on smooth bounded sets E, to

$$\sigma\int_{\partial E}(1+\kappa^2)d\mathcal{H}^{n-1},$$

where κ is the mean curvature of ∂E and $\sigma:=\int_0^1\sqrt{W(u)}du$.

Going back to the full space-time functionals A_ε, assume that $n=1$. In [23] it has been proved that the $\Gamma(L^2(\mathbb{R}\times[0,T]))$-limit of (A_ε) is given by a functional, that on a suitable subclass of its domain, takes the form

$$\sigma \int_0^T \sum_{i=1}^N (\dot{g}_i(t))^2 dt + 2\sigma \sum_t L(t)^+,$$

where: $g_i \in H^1(0,T)$ denote the boundary points of a finite union of subintervals of the unit interval (the characteristic function of such a union is the limit of the approximating sequences (u_ε)); $L(t)^+$ denotes a contribution which, roughly speaking, is the positive part of the space-localization of the time-jumps of the perimeter measures due to disappearance or nucleations of new phases. Note that in dimension one, there is no contribution of the curvature of the boundaries of the intervals. In [33] it is given the expression of the Γ-limit on the whole domain (it turns out that this functional is nonlocal).

Assume now that $n \in \{2,3\}$. Let $t \in [0,T] \to E(t)$ be a map, where $E(t)$ is a bounded smooth open subset of $\mathbb{R}^n$. Assume that the map $t \in [0,T] \to \partial E(t)$ is smooth up to a finite number of times t_i, where the perimeter jumps from $P(E(t_i^-))$ to $P(E(t_i^+))$. Then the $\Gamma(L^2(\mathbb{R}^n \times [0,T]))$-limit of the sequence (A_ε) is given by

$$\sigma \int_0^T \int_{\partial E(t)} (V-\kappa)^2 \, d\mathcal{H}^{n-1} dt + 2\sigma \sum_i \sup_\psi \big(P(E(t_i^+),\psi) - P(E(t_i^-),\psi) \big)^+$$

where V is the velocity of $\partial E(t)$, $^+$ denotes the positive part, and the supremum is taken over all functions $\psi \in C^1(\mathbb{R}^n;[0,1])$. We recall (see, for instance, [30]) that a uniform bound on the action provides the existence, in the limit, of a square integrable weak mean curvature and a square integrable generalized velocity. Similarly in spirit to what pointed out at the end of Section 4, understanding the properties of the flows in the whole domain of the Γ-limit is related to solve a geometric evolution equation of the form $V - \kappa = g$, for a forcing term g which is in L^2 with respect to a measure concentrated on suitable space-time tracks.

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