The Nonlinear Sieve Problem and Applications to Thin Films

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Abstract We consider variational problems defined on domains 'weakly' connected through a separation hyperplane ('sieve plane') by an ε-periodically distributed 'contact zone'. We study the asymptotic behaviour as ε tends to 0 of integral functionals in such domains in the nonlinear and vector-valued case, showing that the asymptotic memory of the sieve is described by a nonlinear 'capacitary-type' formula. In particular we treat the case when the integral energies on both sides of the sieve plane satisfy different growth conditions. We also study the case of thin films, with flat profile and thickness ε, connected by the same sieve plane.

1 Introduction

In this paper we study the asymptotic behaviour of energies defined on domains connected by a 'finely-perforated' separation interface. The model problem we have in mind is that of the so-called 'Neumann sieve', which consists in studying the irrotational flow of an incompressible fluid through a sieve. In mathematical terms, we consider Ω a bounded open subset of $\mathbb{R}^n$ and a trasversal hyperplane Σ such that Ω is divided in two open subsets Ω^+ and Ω^-. We assume that Ω^+ and Ω^- are connected through an ε-periodic perforation of Σ; *i.e.*, we consider a union of ε-periodically distributed sets (the 'holes' of the sieve) which we denote by T_ε.

We then study the following boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \widehat{\Omega}_\varepsilon \\ u_\varepsilon \in H^1(\widehat{\Omega}_\varepsilon) & \\ \frac{\partial}{\partial n} u_\varepsilon = 0 & \text{on both sides of } (\Omega \cap \Sigma) \setminus T_\varepsilon \\ \frac{\partial}{\partial n} u_\varepsilon = 0 & \text{on } \partial\Omega\,; \end{cases} \tag{1.1}$$

where $\widehat{\Omega}_\varepsilon =: \Omega^+ \cup \Omega^- \cup T_\varepsilon$, n is the outer normal and u_ε is the potential function of the velocity (see e.g. [16] and [17]). Note that in this problem the main role is played by the Neumann boundary condition on $(\Omega \cap \Sigma) \setminus T_\varepsilon$, while the boundary condition on $\partial\Omega$ can be replaced by any variational condition.

The Neumann sieve problem was proposed by Sanchez Palencia in [23], who gave a formal asymptotic expansion of the solution u_ε. The problem was then studied by Attouch-Damlamian-Murat-Picard (see [13], [20] and [21]) essentially in the case where the perforations are unions of open balls $B^{n-1}_{i,\varepsilon}$ in $\mathbb{R}^{n-1}$ with center $x_i^\varepsilon = i\varepsilon$, for $i \in \mathbb{Z}^{n-1}$, and radius ρ_ε of order $\varepsilon^{(n-1)/(n-2)}$ (if $n \geq 3$). They proved (among others results) that the sequence u_ε of solutions of (1.1) converges weakly in $H^1(\Omega^+) \times H^1(\Omega^-)$ to a solution (u^+, u^-) of the following limit problem

$$\begin{cases} -\Delta u^\pm = f & \text{in } \Omega^\pm \quad (f \in L^2(\Omega)) \\ u^\pm \in H^1(\Omega^\pm) & \\ \frac{\partial}{\partial n_+} u^+ = -\frac{\partial}{\partial n_-} u^- = \frac{c}{4}(u^+ - u^-) & \text{in } \Omega \cap \Sigma \\ \frac{\partial}{\partial n} u^\pm = 0 & \text{on } \partial\Omega\ , \end{cases}$$

where c is the 2-capacity of a rescaled perforation, $B_1^{n-1}(0) = \{(x_\alpha, 0) \in \mathbb{R}^n : |x_\alpha| < 1\}$, with respect to $\mathbb{R}^n$; *i.e.,*

$$c = \inf\Big\{ \int_{\mathbb{R}^n} |D\psi|^2 dx : \psi \in H^1(\mathbb{R}^n),\ \psi = 1 \text{ on } B_1^{n-1}(0) \Big\}.$$

Attouch-Damlamian-Murat-Picard observed also that their result corresponds, in terms of Γ-convergence, to proving that

$$\begin{aligned} &\Gamma\text{-}\lim_{\varepsilon\to 0} \Big(\frac{1}{2} \int_{\Omega^+\cup\Omega^-} |Du|^2\, dx + I_{\{u \in H^1(\widehat{\Omega}_\varepsilon)\}} \Big) \\ = \ &\frac{1}{2} \Big(\int_{\Omega^+} |Du^+|^2\, dx + \int_{\Omega^-} |Du^-|^2\, dx + \frac{c}{4} \int_{\Omega\cap\Sigma} |u^+ - u^-|^2\, dx_\alpha \Big)\,, \end{aligned}$$

where I denotes the indicator function. This result implies that u_ε is also solution of the minimum problem

$$\min\Big\{ \int_{\Omega^+\cup\Omega^-} |Du|^2\, dx - 2 \int_{\Omega^+\cup\Omega^-} f u\, dx : u \in H^1(\widehat{\Omega}_\varepsilon) \Big\}$$

and the limit (u^+, u^-) solves

$$\min\Big\{\int_{\Omega^+}|Du^+|^2\,dx+\int_{\Omega^-}|Du^-|^2\,dx-2\Big(\int_{\Omega^+}fu^+\,dx+\int_{\Omega^-}fu^-\,dx\Big)$$
$$+\frac{c}{4}\int_{\Omega\cap\Sigma}|u^+-u^-|^2\,dx_\alpha:\ (u^+,u^-)\in H^1(\Omega^+)\times H^1(\Omega^-)\Big\},$$

by the Γ-convergence's properties of convergence of minima and stability with respect to continuous perturbations. For related problems to this subject see also Attouch-Picard [4], Conca [9] [10] [11], Del Vecchio [14] and Sanchez Palencia [22] [24].

In this paper we generalize the Γ-convergence result above to the non-convex vector-valued case, considering in addition the interaction through the separating surface of two different energies, possibly satisfying also different growth conditions, defined in Ω^+ and Ω^-, respectively. More precisely, let ω be an open bounded subset of $\mathbb{R}^{n-1}$, for the sake of simplicity we take $\Sigma=\{x_n=0\}$, so that $\omega\times\{0\}=\Omega\cap\{x_n=0\}$, $\Omega^+=\Omega\cap\{x_n>0\}$ and $\Omega^-=\Omega\cap\{x_n<0\}$. Let $m,n\in\mathbb{N}$ with $m\geq 1$ and let $p,q>1$ with $\min\{p,q\}<n$ be fixed (the case $\min\{p,q\}=n$ differing in technical details only and the case $\min\{p,q\}>n$ being trivial). We suppose $p<q$ (the case $p=q$ being treated similarly). For all $\varepsilon>0$ we define

$$\omega_\varepsilon=\bigcup_{i\in\mathbb{Z}^{n-1}}B^{n-1}_{i,\varepsilon}\cap\omega,$$

where $B^{n-1}_{i,\varepsilon}$ is defined as above; hence,

$$\widehat{\Omega}_\varepsilon=\Omega^+\cup\Omega^-\cup(\omega_\varepsilon\times\{0\})$$

(note that ω_ε is the union of the perforations, including those intersecting $\partial\omega$).

Let $V^{p,q}(\widehat{\Omega}_\varepsilon;\mathbb{R}^m)=W^{1,p}(\widehat{\Omega}_\varepsilon;\mathbb{R}^m)\cap W^{1,q}(\Omega^-;\mathbb{R}^m)$ and let $W_p,W_q:\mathbb{M}^{m\times n}\mapsto[0,+\infty)$ be Borel functions satisfying a growth condition of order p and q, respectively. With fixed a sequence (ε_j) of positive numbers converging to 0 we define

$$F_j(u)=\begin{cases}\displaystyle\int_{\Omega^+}W_p(Du)\,dx+\int_{\Omega^-}W_q(Du)\,dx & \text{if } u\in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)\\ +\infty & \text{otherwise}.\end{cases}$$

Note that the choice of the functions W_p,W_q with $p<q$ corresponds to considering two different (possibly nonlinear) media in Ω^+ and Ω^- connected through the perforations of the surface Σ. The main part of this work is devoted to proving that if the radii ρ_{ε_j} of the perforation satisfy

$$\lim_{j\to+\infty}\frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon_j^{n-1}}=R<+\infty \tag{1.2}$$

then, upon extracting a subsequence, F_j Γ-converge to

$$F(u^+,u^-)=\int_{\Omega^+} QW_p(Du^+)\,dx+\int_{\Omega^-} QW_q(Du^-)\,dx+R\int_\omega \varphi(u^+-u^-)\,dx_\alpha$$

on $W^{1,p}(\Omega^+;\mathbb{R}^m)\times W^{1,q}(\Omega^-;\mathbb{R}^m)$ with respect to a suitable convergence that we introduce in Definition 3.1, where QW_p and QW_q denote the quasiconvexification of W_p and W_q, respectively, and φ is given by the formula

$$\varphi(z)=\inf\Big\{\int_{\mathbb{R}^n_+}\widehat{W}_p(D\zeta)\,dx:\zeta-z\in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m),\quad \zeta=0 \text{ on } B_1^{n-1}(0)\Big\} \tag{1.3}$$

which generalizes the classical p-capacity of $B_1^{n-1}(0)$ with respect to $\mathbb{R}^n$. Note that the larger exponent q does not appear in the definition of φ (the formula is slightly different in the case $p=q$); for example, if $W_p(\xi)=|\xi|^p$ and $W_q(\xi)=|\xi|^q$ then $\varphi(z)=c|z|^{p\wedge q}$. The function $\widehat{W}_p$ is the pointwise limit of a sequence that we obtain by scaling QW_p, to the end of studying the behaviour of the infima of integral functionals on domains independent of the parameter ε_j; *i.e.,*

$$\widehat{W}_p(A)=\lim_j \rho^p_{\varepsilon_j} QW_p\Big(\rho^{-1}_{\varepsilon_j}A\Big)\,.$$

We show that this limit always exists upon passing to a subsequence. This passage to a subsequence cannot be avoided; hence, φ may depend on (the subsequence of) (ε_j). For notational simplicity we do not treat the case $n=p$, which can be dealt with similarly; for the necessary changes in the statements see e.g. [20].

The proof of our result (Theorem 3.2) consists in a direct computation of the Γ-limit; it is based on a technical result (Lemma 3.4) that allows us to modify a sequence (u_j), on suitable n-dimensional annuli surrounding the $(n-1)$-dimensional perforations $B^{n-1}_{i,\varepsilon}$, and to study the behaviour of F_j along the new modified sequence. It gives rise to three terms in the Γ-limit F. The first two terms represent the contribution of the new sequence 'far' from the $B^{n-1}_{i,\varepsilon}$; more precisely, they are the Γ-limit of the two uncoupled problems defined on $W^{1,p}(\Omega^+;\mathbb{R}^m)$ and $W^{1,q}(\Omega^-;\mathbb{R}^m)$, respectively. The third one describes, by the nonlinear capacitary formula φ, the contribution 'near' to $B^{n-1}_{i,\varepsilon}$. This approch follows the method introduced by Ansini-Braides [2] to study the asymptotic behaviour of periodically-perforated nonlinear domains (see also Ansini-Braides [3] for an applications to periodic microstructures); Lemma 3.4 is a suitable variant, for the sieve problem, of Lemma 3.1 in [2].

The second goal of the paper is to study the case of nonlinearly elastic thin films connected by a periodically perforated sieve; that is, we consider the domain

$$\Omega_{\varepsilon_j}=\big(\omega\times(-\varepsilon_j,0)\big)\cup\big(\omega\times(0,\varepsilon_j)\big)\cup\big(\omega_{\varepsilon_j}\times\{0\}\big)\,,$$

with ω_{ε_j} defined as above; hence, the thickness is not fixed but it is equal to the parameter ε_j. In this case the sequence of functionals that we consider is

$$F_j(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\Big(\displaystyle\int_{\Omega^+_{\varepsilon_j}} W_p(Du)\,dx + \int_{\Omega^-_{\varepsilon_j}} W_q(Du)\,dx\Big) & \text{if } u \in V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m) \\ +\infty & \text{otherwise} \end{cases}$$

where $\Omega^{\pm}_{\varepsilon_j} = \Omega_{\varepsilon_j} \cap \{\pm x_n > 0\}$ and $V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m)$ is defined as above with Ω_{ε_j} in place of $\widehat{\Omega}_{\varepsilon_j}$ and $\Omega^- = \omega \times (-1, 0)$.

The two integral terms in F_j, separately considered, represent the analytic description of the energies of nonlinearly elastic thin films in the domains $\Omega^+_{\varepsilon_j}$ and $\Omega^-_{\varepsilon_j}$, respectively; their Γ-convergence has been proved by Le Dret-Raoult in [18] to $(n-1)$-dimensional integral functionals whose energy densities $\widetilde{W}_p$ and $\widetilde{W}_q$ are completely described by the following formulas

$$\widetilde{W}_p(\overline{F}) = Q_{n-1}\overline{W}_p(\overline{F}) \quad , \quad \widetilde{W}_q(\overline{F}) = Q_{n-1}\overline{W}_q(\overline{F})$$

where $\overline{W}_p(\overline{F}) = \inf_{F_n} W_p(\overline{F}, F_n)$, $\overline{W}_q(\overline{F}) = \inf_{F_n} W_q(\overline{F}, F_n)$; here Q_{n-1} denotes the operation of $(n-1)$-quasiconvexification and $\overline{F} = (F_1, \ldots, F_{n-1}) \in \mathbb{M}^{m\times n-1}$.

Although the thickness of Ω_{ε_j} depends on ε_j, we observe that the construction of the annuli, that we make in Lemma 3.4 to separate the contribution near the balls $B^{n-1}_{i,\varepsilon}$ and far from them, may be applied unchanged since the annuli are still contained in the strips $\omega \times (0, \varepsilon_j)$ and $\omega \times (-\varepsilon_j, 0)$; hence, we have just to define the convergence of a sequence $u_j \in V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m)$ to $(u^+, u^-) \in W^{1,p}(\omega^+;\mathbb{R}^m) \times W^{1,q}(\omega^-;\mathbb{R}^m)$ (see Definition 8.1) and repeat the proof of Lemma 3.4.

The meaningful scaling for the radii of the perforations in this case turns out to be $\varepsilon_j^{n/(n-p)}$, and we get that the Γ-limit is given by

$$F(u^+, u^-) = \int_\omega \widetilde{W}_p(D_\alpha u^+)\,dx_\alpha + \int_\omega \widetilde{W}_q(D_\alpha u^-)\,dx_\alpha + T\int_\omega \varphi(u^+ - u^-)\,dx_\alpha$$

where $T = \lim_{j\to+\infty} \frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon_j^n} < +\infty$ and φ is described by the same formula (1.3).

2 Notation and Preliminaries

In all that follows $m, n \in \mathbb{N}$ with $m \geq 1$ and $p, q > 1$ with $p = \min\{p, q\} < n$ are fixed . If $x \in \mathbb{R}^n$ then $x_\alpha = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ is the vector of the first $n-1$ components of x, and $D_\alpha = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$. We denote $\mathbb{R}^n_{\pm} = \{(x_\alpha, x_n) \in \mathbb{R}^n : \pm x_n > 0\}$, respectively.

The notation $\mathbb{M}^{m\times n}$ stands for the space of $m \times n$ matrices. Given a matrix $F \in \mathbb{M}^{m\times n}$, and following the notation introduced in [18], we write $F = (\overline{F}|F_n)$, where F_i denotes the i-th column of F, $1 \leq i \leq n$, and $\overline{F} = (F_1, \ldots, F_{n-1}) \in \mathbb{M}^{m\times n-1}$ is the matrix of the first $n-1$ columns of F.

The Hausdorff k-dimensional measure is denoted as $\mathcal{H}^k$. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $|E|$ is its Lebesgue measure. If E is a subset of $\mathbb{R}^n$ then χ_E is its *characteristic function.*

We use standard notation for Lebesgue and Sobolev spaces $L^s(U;\mathbb{R}^m)$ and $W^{1,s}(U;\mathbb{R}^m)$. The letter c will stand for an arbitrary fixed strictly-positive constant.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let ω be a bounded open subset of $\mathbb{R}^{n-1}$ such that

$$\omega \times \{0\} = \Omega \cap \{x \in \mathbb{R}^n : x_n = 0\}\,, \tag{2.1}$$

we denote

$$\Omega^+ = \Omega \cap \{x \in \mathbb{R}^n : x_n > 0\} \qquad \Omega^- = \Omega \cap \{x \in \mathbb{R}^n : x_n < 0\}\,. \tag{2.2}$$

If u is a function defined on Ω^+ or Ω^-, we use the same symbol u to indicate its trace on $\omega \times \{0\}$. We denote

$$B_1^{n-1}(0) = \{(x_\alpha, 0) \in \mathbb{R}^n : \, |x_\alpha| < 1\}\,,\, C_{1,N} = \Big\{(x_\alpha, 0) \in \mathbb{R}^n : \, 1 \leq |x_\alpha| < N\Big\}\,, \tag{2.3}$$

$B_r(x)$ is the open ball in $\mathbb{R}^n$ of center x and radius r, $B_r^{\pm}(x_\alpha, 0) = B_r(x_\alpha, 0) \cap \{\pm x_n > 0\}$. We denote by $(u_j^{\pm})$ the restriction of a given sequence (u_j) to $\Omega^{\pm}$, respectively; or, when no confusion may arise, a sequence defined on $\Omega^{\pm}$, respectively.

For $p \geq 1$, we denote the p-capacity of $B_1^{n-1}(0)$ with respect to $B_N(0)$ by

$$\begin{aligned} C_p(B_1^{n-1}(0); B_N(0)) \quad = \quad \inf\Big\{ \int_{B_N(0)} |D\psi|^p \, dx : \psi \in W_0^{1,p}(B_N(0)), \\ \psi = 1 \text{ on } B_1^{n-1}(0) \Big\}\,, \end{aligned}$$

and the p-capacity of $B_1^{n-1}(0)$ with respect to $\mathbb{R}^n$ by

$$\begin{aligned} C_p(B_1^{n-1}(0); \mathbb{R}^n) \quad = \quad \inf\Big\{ \int_{\mathbb{R}^n} |D\psi|^p \, dx : \psi \in W^{1,p}(\mathbb{R}^n), \\ \psi = 1 \text{ on } B_1^{n-1}(0) \Big\}\,. \end{aligned}$$

2.1 Quasiconvexity

If $h : \mathbb{M}^{m\times n} \to [0, +\infty)$ is a Borel function, the $(W^{1,s}\text{-})$*quasiconvexification of* h is given by the formula

$$Qh(A) = \inf\Big\{ \int_{(0,1)^n} h(A + Du)\, dx : u \in W_0^{1,s}((0,1)^n; \mathbb{R}^m) \Big\} \tag{2.4}$$

for $A \in \mathbb{M}^{m\times n}$. We say that h is ($W^{1,s}$-)*quasiconvex* if $Qh = h$ (see [19], [5], [7]). If $h : \mathbb{M}^{m\times n-1} \to [0,+\infty)$ we denote $Q_{n-1}h$ the ($W^{1,s}$-) quasiconvexification of h.

We recall that if h is a Borel function as above, and there exist constants $c_1, c_2 > 0$ such that $c_1(|A|^s - 1) \le h(A) \le c_2(|A|^s + 1)$, then the function Qh is quasiconvex (see [7] Proposition 6.7) and the functional

$$\mathcal{H}(u) = \int_\Omega Qh(Du)\,dx$$

is the lower-semicontinuous envelope of the functional

$$H(u) = \int_\Omega h(Du)\,dx$$

on $W^{1,s}(\Omega;\mathbb{R}^m)$ with respect to the $L^s(\Omega;\mathbb{R}^m)$ convergence (see e.g. [7]).

2.2 Γ-convergence

Let U be an open subset of $\mathbb{R}^n$. We recall the definition of Γ-convergence of a sequence (Φ_j) of functionals defined on $W^{1,s}(U;\mathbb{R}^m)$ (with respect to the $L^s(U;\mathbb{R}^m)$-convergence). We say that (Φ_j) Γ-converges to Φ_0 on $W^{1,s}(U;\mathbb{R}^m)$ if for all $u \in W^{1,s}(\Omega;\mathbb{R}^m)$ we have:

(i) (*liminf inequality*) for all (u_j) sequences of functions in $W^{1,s}(U;\mathbb{R}^m)$ converging to $u \in W^{1,s}(U;\mathbb{R}^m)$ in $L^s(U;\mathbb{R}^m)$ we have

$$\Phi_0(u) \le \liminf_j \Phi_j(u_j);$$

(ii) (*limsup inequality: existence of a recovery sequence*) for all $\eta > 0$ there exists a sequence (u_j) of functions in $W^{1,s}(U;\mathbb{R}^m)$ converging to $u \in W^{1,s}(U;\mathbb{R}^m)$ in $L^s(U;\mathbb{R}^m)$ such that

$$\Phi_0(u) \ge \limsup_j \Phi_j(u_j) - \eta.$$

If (i) and (ii) hold we write $\Phi_0(u) = \Gamma\text{-}\lim_j \Phi_j(u)$

We will say that a family (Φ_ε) Γ-converges to Φ_0 if for all sequences (ε_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with Φ_{ε_j} in place of Φ_j.

We recall the following property and fundamental theorem (see e.g. [7] Remark 7.4, Theorem 7.2).

Remark 2.1 If $\Phi_j = \Phi$ for all $j \in \mathbb{N}$ then the Γ-limit Φ_0 is the lower- semicontinuous envelope of the functional Φ on $W^{1,s}(U;\mathbb{R}^m)$ with respect to $L^s(U;\mathbb{R}^m)$ convergence.

Theorem 2.2 *Let Φ_j Γ-converge to Φ_0 on $W^{1,s}(U;\mathbb{R}^m)$. Let there exist a compact set $K \subset W^{1,s}(U;\mathbb{R}^m)$ with respect to the $L^s(U;\mathbb{R}^m)$ convergence, such that* $\inf \Phi_j = \inf_K \Phi_j$ *for all $j \in \mathbb{N}$. Then there exists* $\min \Phi_0 = \lim_j \inf \Phi_j$. *Moreover, if (j_k) is an increasing sequence of integers and (u_k) is a converging sequence such that* $\lim_k \Phi_{j_k}(u_k) = \lim_j \inf \Phi_j$ *then its limit is a minimum point for Φ_0.*

For an introduction to Γ-convergence we refer to [12], [6] and Part II of [7].

3 Domains connected by a periodically perforated interface

Given a sequence (ε_j) of positive number converging to 0, we consider the lattice $\varepsilon_j\mathbb{Z}^{n-1}$ whose points will be denoted by $x_i^\varepsilon = i\varepsilon_j$ ($i \in \mathbb{Z}^{n-1}$). Moreover, for all $i \in \mathbb{Z}^{n-1}$

$$B_{i,\varepsilon}^{n-1} = B(x_i^\varepsilon, \rho_{\varepsilon_j})$$

denotes the open ball in $\mathbb{R}^{n-1}$ of center x_i^ε and radius ρ_{ε_j}. Hence, we define

$$\omega_{\varepsilon_j} = \bigcup_{i\in\mathbb{Z}^{n-1}} B_{i,\varepsilon}^{n-1} \cap \omega$$

and

$$\widehat{\Omega}_{\varepsilon_j} =: \Omega^+ \cup \Omega^- \cup \Big(\omega_{\varepsilon_j} \times \{0\}\Big), \tag{3.1}$$

where ω and $\Omega^\pm$ are given by (2.1) and (2.2).

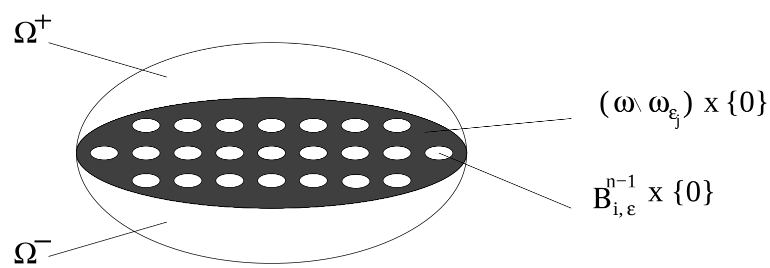

Figure 1: The domain $\widehat{\Omega}_{\varepsilon_j}$

Let $\widehat{\Omega}_{\varepsilon_j}$ be defined by (3.1) (see Figure 1) and let (u_j) be a sequence of functions defined on such domain; since $\widehat{\Omega}_{\varepsilon_j}$ varies with ε_j we have to precise the meaning of 'converging sequence'.

Definition 3.1 *Let*

$$V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m) = W^{1,p}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m) \cap W^{1,q}(\Omega^-;\mathbb{R}^m)\,.$$

We say that $u_j \in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)$ *converge to* $(u^+,u^-) \in W^{1,p}(\Omega^+;\mathbb{R}^m)\times W^{1,q}(\Omega^-;\mathbb{R}^m)$, $u_j \to (u^+,u^-)$, *if*

$$\begin{aligned} u_{j\,|\Omega^+} &= u_j^+ \to u^+ \ \textit{in}\ L^p(\Omega^+;\mathbb{R}^m) \\ u_{j\,|\Omega^-} &= u_j^- \to u^- \ \textit{in}\ L^q(\Omega^-;\mathbb{R}^m)\,. \end{aligned}$$

We say that (u_j) *converges weakly to* (u^+,u^-), $u_j \rightharpoonup (u^+,u^-)$, *if*

$$\begin{aligned} u_{j\,|\Omega^+} &= u_j^+ \rightharpoonup u^+ \ \textit{weakly in}\ W^{1,p}(\Omega^+;\mathbb{R}^m) \\ u_{j\,|\Omega^-} &= u_j^- \rightharpoonup u^- \ \textit{weakly in}\ W^{1,q}(\Omega^-;\mathbb{R}^m)\,. \end{aligned}$$

In this paper we prove the following result for domains with a periodically perforated interface (we state it in the case $p<q$; for the changes in the case $p=q$ see Remark 3.3 (a)).

Theorem 3.2 *Let* (ε_j) *and* (ρ_{ε_j}) *be sequences of strictly positive numbers converging to* 0 *such that*

$$0 < \lim_{j\to+\infty} \frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^{n-1}} = R < +\infty\,.$$

Let ω *be a bounded open subset of* $\mathbb{R}^{n-1}$ *defined by* (2.1) *such that* $\mathcal{H}^{n-1}(\partial\omega)=0$ *and let* Ω^+, Ω^- *and* $\widehat{\Omega}_{\varepsilon_j}$ *be defined by* (2.2) *and* (3.1). *Let* $1<p<q$ *and let* $W_p, W_q : \mathbb{M}^{m\times n} \mapsto [0,+\infty)$ *be Borel functions satisfying a growth condition of order* p *and* q*, respectively: there exists a constant* $c_1>0$ *such that*

$$|A|^p - 1 \le W_p(A) \le c_1(1+|A|^p) \tag{3.2}$$

and there exists a constant $c_2>0$ *such that*

$$|A|^q - 1 \le W_q(A) \le c_2(1+|A|^q) \tag{3.3}$$

for all $A\in\mathbb{M}^{m\times n}$. *Then, upon possibly extracting a subsequence, for all* $A\in\mathbb{M}^{m\times n}$ *there exists the limit*

$$\widehat{W}_p(A) = \lim_j \rho_{\varepsilon_j}^p QW_p\Big(\rho_{\varepsilon_j}^{-1}A\Big), \tag{3.4}$$

where QW_p *denotes the quasiconvexification of* W_p*, so that the value*

$$\varphi(z) = \inf\Big\{\int_{\mathbb{R}^n_+} \widehat{W}_p(D\zeta)\,dx : \zeta - z \in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m), \quad \zeta = 0 \ \textit{on}\ B_1^{n-1}(0)\Big\} \tag{3.5}$$

is well defined for all $z \in \mathbb{R}^m$*. Moreover, the functionals defined by*

$$F_j(u) = \begin{cases} \displaystyle\int_{\Omega^+} W_p(Du)\,dx + \int_{\Omega^-} W_q(Du)\,dx & \text{if } u \in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m) \\ +\infty & \text{otherwise} \end{cases}$$

Γ*-converge to the functional defined by*

$$F(u^+,u^-) = \int_{\Omega^+} QW_p(Du^+)\,dx + \int_{\Omega^-} QW_q(Du^-)\,dx + R\int_\omega \varphi(u^+ - u^-)\,dx_\alpha$$

on $W^{1,p}(\Omega^+;\mathbb{R}^m) \times W^{1,q}(\Omega^-;\mathbb{R}^m)$ *with respect to the convergence introduced in Definition* 3.1*.*

Remark 3.3 (a) Let us denote U_p the function that in F_j plays the role of W_q but with $q = p$. In this case the proof of Theorem 3.2 is the same but we get a different formula for the function φ:

$$\begin{aligned} \varphi(z) \quad = \quad & \inf\Big\{\int_{\mathbb{R}^n_+} \widehat{W}_p(D\zeta)\,dx + \int_{\mathbb{R}^n_-} \widehat{U}_p(D\zeta)\,dx : \; \zeta \in W^{1,p}(\mathbb{R}^n_{+,-} \cup B_1^{n-1}(0);\mathbb{R}^m) \\ & \qquad \zeta - \frac{z}{2} \in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m), \; \zeta + \frac{z}{2} \in W^{1,p}(\mathbb{R}^n_-;\mathbb{R}^m)\Big\} \end{aligned}$$

where $\mathbb{R}^n_{+,-} = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ (see Section 7).

(b) We prove Theorem 3.2 when W_p, W_q are quasiconvex functions; the generalization to the arbitrary W_p, W_q Borel functions can be treated by preliminary relaxation as in [2] with slight modifications of the proof (see Remark 2.1 and Section 2.1).

To compute the Γ-limit of functionals F_j, following the definition of Γ-convergence (see Section 2.2), we have to study the behaviour of $F_j(u_j)$ with (u_j) converging to (u^+,u^-). In analogy with the method introduced by Ansini-Braides in [2], we wish to separate the contribution due to Du_j near the balls $B^{n-1}_{i,\varepsilon}$ and far from them. This is possible since we can repeat the proof of Lemma 3.1 in [2], with suitable variants for the sieve problem. Since the sequence (u_j) is not defined in $(\omega \setminus \omega_{\varepsilon_j}) \times \{0\}$ in order to isolate the two contributions (near and far from $B^{n-1}_{i,\varepsilon}$) we have to construct a suitable annuli surrounding the perforations in $\Omega^+ \cup \Omega^-$ (instead of $\omega \times \{0\}$). Even the modifications are technical and not substantial, we include the proof of the Lemma for sieve problem for the reader convenience.

Lemma 3.4 *Let* (u_j) *be bounded in* $W^{1,p}(\Omega^+ \cup \Omega^-;\mathbb{R}^m) \cap W^{1,q}(\Omega^-;\mathbb{R}^m)$ *and let* $N, k \in \mathbb{N}$*. Let* (ε_j) *be a sequence of positive numbers converging to* 0 *and let*

$$Z_j = \{i \in \mathbb{Z}^{n-1} : \; \operatorname{dist}((x_i^\varepsilon, 0), \partial\Omega) > \varepsilon_j\}.$$

Let (ρ_{ε_j}) be a sequence of positive numbers with $N\rho_{\varepsilon_j} < \varepsilon_j/2$. For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \mathbb{R}^n : \ 2^{-k_i-1} N\rho_{\varepsilon_j} < |x - (x_i^\varepsilon, 0)| < 2^{-k_i} N\rho_{\varepsilon_j}\Big\}, \tag{3.6}$$

$$u_j^{i\pm} = \fint_{C_i^j \cap \{\pm x_n > 0\}} u_j \, dx \quad (\textit{the } \text{mean value } \textit{of } u_j \ \textit{on } C_i^j \cap \{\pm x_n > 0\} \ ), \tag{3.7}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i} N\rho_{\varepsilon_j}, \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{3.8}$$

there exists a sequence (w_j) such that

$$w_j = u_j \qquad \textit{on } \Omega \setminus \bigcup_{i \in Z_j} C_i^j \, , \tag{3.9}$$

$$w_j(x) = u_j^{i\pm} \qquad \textit{if } |x - (x_i^\varepsilon, 0)| = \rho_j^i \ \textit{ and } \pm x_n > 0, \ \textit{respectively, for } i \in Z_j \tag{3.10}$$

and

$$\sum_{i \in Z_j} \Big(\int_{C_i^j \cap \{x_n > 0\}} \Big(W_p(Dw_j) + W_p(Du_j) \Big) \, dx + \int_{C_i^j \cap \{-x_n > 0\}} \Big(W_q(Dw_j) + W_q(Du_j) \Big) \, dx \Big) \le \frac{c}{k}. \tag{3.11}$$

Moreover, if $\rho_{\varepsilon_j}^n = o\Big(\varepsilon_j^{n-1}\Big)$ and the sequences $(|Du_j|^p)$, $(|Du_j|^q)$ are equi-integrable in $\Omega^\pm$, respectively, then we can choose $k_i = 0$ for all $i \in Z_j$ and

$$\lim_{j \to +\infty} \Big(\sum_{i \in Z_j} \int_{C_i^j \cap \{x_n > 0\}} \Big(W_p(Dw_j) + W_p(Du_j) \Big) \, dx + \int_{C_i^j \cap \{-x_n > 0\}} \Big(W_q(Dw_j) + W_q(Du_j) \Big) \, dx \Big) = 0. \tag{3.12}$$

Proof. For all $j \in \mathbb{N}$, $i \in Z_j$ and $h \in \{0, ..., k-1\}$ let

$$C_{i,h}^j = \Big\{x \in \mathbb{R}^n : \ 2^{-h-1} N\rho_{\varepsilon_j} < |x - (x_i^\varepsilon, 0)| < 2^{-h} N\rho_{\varepsilon_j}\Big\},$$

and let

$$(u_j^{i,h})^\pm = \fint_{C_{i,h}^j \cap \{\pm x_n > 0\}} u_j \, dx,$$

and

$$\rho_j^{i,h} = \frac{3}{4} 2^{-h} N\rho_{\varepsilon_j}.$$

Consider a function $\phi = \phi^j_{i,h} \in \mathrm{C}^\infty_0(C^j_{i,h})$ such that $\phi = 1$ on $\partial B_{\rho^{i,h}_j}(x^\varepsilon_i, 0)$ and $|D\phi| \leq c/2^{-h}N\rho_{\varepsilon_j} = c/\rho^{i,h}_j$. Let $w^{i,h}_j$ be defined on $C^j_{i,h}$ by

$$w^{i,h}_j = (u^{i,h}_j)^\pm\phi + (1-\phi)u_j \quad \text{ on } C^j_{i,h} \cap \{\pm x_n > 0\}, \text{ respectively}$$

with $\phi = \phi^j_{i,h}$ as above. We then have,

$$\begin{aligned} & \int_{C^j_{i,h}\cap\{x_n>0\}} W_p(Dw^{i,h}_j)\,dx + \int_{C^j_{i,h}\cap\{-x_n>0\}} W_q(Dw^{i,h}_j)\,dx \\ \leq \; & c\Big(\int_{C^j_{i,h}\cap\{x_n>0\}} (1 + |D\phi|^p|u_j - (u^{i,h}_j)^+|^p + |Du_j|^p)\,dx \\ & + \int_{C^j_{i,h}\cap\{-x_n>0\}} (1 + |D\phi|^q|u_j - (u^{i,h}_j)^-|^q + |Du_j|^q)\,dx\Big) \end{aligned}$$

By the Poincaré inequality and its scaling properties we have

$$\int_{C^j_{i,h}\cap\{\pm x_n>0\}} |u_j - (u^{i,h}_j)^\pm|^s\,dx \leq c(\rho^{i,h}_j)^s \int_{C^j_{i,h}\cap\{\pm x_n>0\}} |Du_j|^s\,dx, \tag{3.13}$$

so that, recalling that $|D\phi| \leq c/\rho^{i,h}_j$, by (3.13) with $s = p$ and $s = q$, respectively, we have

$$\int_{C^j_{i,h}\cap\{x_n>0\}} W_p(Dw^{i,h}_j)\,dx \leq c\int_{C^j_{i,h}\cap\{x_n>0\}} (1 + |Du_j|^p)\,dx \tag{3.14}$$

and

$$\int_{C^j_{i,h}\cap\{-x_n>0\}} W_q(Dw^{i,h}_j)\,dx \leq c\int_{C^j_{i,h}\cap\{-x_n>0\}} (1 + |Du_j|^q)\,dx\,. \tag{3.15}$$

Since by summing up in h we trivially have

$$\begin{aligned} & \sum_{h=0}^{k-1} \int_{C^j_{i,h}\cap\{x_n>0\}} (1 + |Du_j|^p)\,dx + \int_{C^j_{i,h}\cap\{-x_n>0\}} (1 + |Du_j|^q)\,dx \\ \leq \; & |B_{N\rho_{\varepsilon_j}}| + \int_{B^+_{N\rho_{\varepsilon_j}}} |Du_j|^p\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}} |Du_j|^q\,dx \end{aligned}$$

where $B^\pm_{N\rho_{\varepsilon_j}} = B_{N\rho_{\varepsilon_j}}(x^\varepsilon_i, 0) \cap \{\pm x_n > 0\}$; there exists $k_i \in \{0, \dots, k-1\}$ such that

$$\begin{aligned} & \int_{C^j_{i,k_i}\cap\{x_n>0\}} (1 + |Du_j|^p)\,dx + \int_{C^j_{i,k_i}\cap\{-x_n>0\}} (1 + |Du_j|^q)\,dx \\ \leq \; & \frac{1}{k}\Big(|B_{N\rho_{\varepsilon_j}}| + \int_{B^+_{N\rho_{\varepsilon_j}}} |Du_j|^p\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}} |Du_j|^q\,dx\Big)\,. \end{aligned} \tag{3.16}$$

There follows that

$$\begin{aligned}&\int_{C^j_{i,k_i}\cap\{x_n>0\}} W_p(Dw_j^{i,k_i})\,dx + \int_{C^j_{i,k_i}\cap\{-x_n>0\}} W_q(Dw_j^{i,k_i})\,dx\\ \le\ & \frac{c}{k}\Big(|B_{N\rho_{\varepsilon_j}}| + \int_{B^+_{N\rho_{\varepsilon_j}}} |Du_j|^p\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}} |Du_j|^q\,dx\Big)\,. \end{aligned} \tag{3.17}$$

By (3.16), (3.17) we get

$$\begin{aligned}&\sum_{i\in Z_j}\int_{C^j_{i,k_i}\cap\{x_n>0\}}\Big(W_p(Dw_j^{i,k_i})+W_p(Du_j)\Big)\,dx\\ &+\int_{C^j_{i,k_i}\cap\{-x_n>0\}}\Big(W_q(Dw_j^{i,k_i})+W_q(Du_j)\Big)\,dx\\ \le\ & \frac{c}{k}\Big(|\Omega| + \int_{\Omega^+}|Du_j|^p\,dx + \int_{\Omega^-}|Du_j|^q\,dx\Big)\,.\end{aligned}$$

Note that if $(|Du_j|^p)$ and $(|Du_j|^q)$ are equi-integrable in $\Omega^\pm$, respectively, we may simply choose $k_i = 0$ for all $i \in Z_j$; hence, by (3.14) and (3.15), we get (3.12).

With this choice of k_i for all $i \in Z_j$, conditions (3.9)–(3.11) are satisfied by choosing $h = k_i$ in the definitions above, i.e. with $C^j_i = C^j_{i,k_i}$ $u_j^{i\pm} = (u_j^{i,k_i})^\pm$, $\rho^i_j = \rho^{i,k_i}_j$, and w_j defined by (3.9) and

$$w_j = u_j^{i\pm}\phi + (1-\phi)u_j \text{ on } C^j_i \cap \{\pm x_n > 0\},\ \text{respectively}$$

with $\phi = \phi^j_{i,k_i}$.

□

Remark 3.5 Note that if $u_j \to (u^+, u^-)$ and $\sup_j F_j(u_j) < +\infty$ then (u_j) converges weakly to (u^+, u^-) in the sense of Definition 3.1. Moreover if (w_j) is defined as in Lemma 3.4 then $w_j \to (u^+, u^-)$ (see e.g. [2] Lemma 3.1) and, since (w_j) is bounded in $W^{1,p}(\Omega^+ \cup \Omega^-;\mathbb{R}^m) \cap W^{1,q}(\Omega^-;\mathbb{R}^m)$, we get that also (w_j) converges weakly to (u^+, u^-) in the sense of Definition 3.1.

If $(|Du_j|^p)$ and $(|Du_j|^q)$ are equi-integrable in $\Omega^\pm$, respectively, then also $(|Dw_j|^p)$ and $(|Dw_j|^q)$ are equi-integrable.

4 Some preliminary results

In this section we prove some preliminary results which allow us to define the function φ and to prove Theorem 3.2 (see Propositions 5.2 and 6.1).

We consider $1 < p < q$ and the functions $g^p_j, g^q_j : \mathbb{M}^{m\times n} \mapsto [0, +\infty)$ defined by

$$g^p_j(A) = \rho^p_{\varepsilon_j} W_p(\rho^{-1}_{\varepsilon_j} A) \quad , \quad g^q_j(A) = \rho^q_{\varepsilon_j} W_q(\rho^{-1}_{\varepsilon_j} A)\,.$$

Since W_p is quasiconvex and satisfies a growth condition of order p it is locally Lipschitz continuous on $\mathbb{M}^{m\times n}$: there exists C depending only on c_1, p such that

$$|W_p(A) - W_p(B)| \leq C(1 + |A|^{p-1} + |B|^{p-1})|A - B|$$

for all $A, B \in \mathbb{M}^{m\times n}$ (see [7] Remark 4.13); by definition of g_j^p we get that

$$|g_j^p(A) - g_j^p(B)| \leq C(\rho_{\varepsilon_j}^{p-1} + |A|^{p-1} + |B|^{p-1})|A - B| \,. \tag{4.1}$$

Hence, there exists a subsequence (not relabeled) converging pointwise to some limit function $\widehat{W}_p$; *i.e.*,

$$\lim_{j\to+\infty} g_j^p(A) = \widehat{W}_p(A) \tag{4.2}$$

for all $A \in \mathbb{M}^{m\times n}$. Note that, if there exists the limit

$$\lim_{t\to+\infty} \frac{W_p(tA)}{t^p} = \widehat{W}_p(A) \tag{4.3}$$

(it has that the limit is independent of subsequences) then $A \mapsto \widehat{W}_p(A)$ is positively homogeneous of degree p.

We consider the functionals defined on $L^p(B_N(0) \setminus C_{1,N}; \mathbb{R}^m)$ by

$$G_j(u) = \begin{cases} \displaystyle\int_{B_N^+(0)} g_j^p(Du)\,dy + \rho_{\varepsilon_j}^{p-q} \int_{B_N^-(0)} g_j^q(Du)\,dy & u \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m) \\ & u = z \text{ on } \partial B_N^+(0), \\ & u = 0 \text{ on } \partial B_N^-(0) \\ +\infty & \text{otherwise}\,, \end{cases}$$

where $C_{1,N}$ is defined as in (2.3) (see Figure 2).

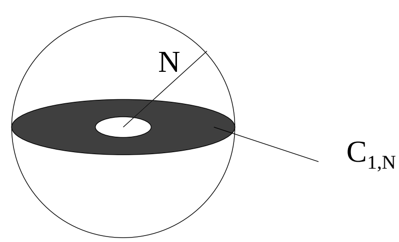

Figure 2: The domain $B_N(0) \setminus C_{1,N}$

The reason why we are interested in studying the Γ-convergence of (G_j) and, as a consequence, the convergence of minimum problem (see Proposition 4.1 and Corollary 4.2) is that (G_j) is the sequence of functionals that we obtain studying the contribution near the balls $B_{i,\varepsilon}^{n-1}$ and it gives rise to the term in φ (see in the following (5.5), (6.5)).

Proposition 4.1 *Let (G_j) be given as above and let (ρ_{ε_j}) be a sequence of positive numbers converging to 0; then there exists the Γ-limit with respect to the L^p-convergence*

$$\Gamma\text{-}\lim_{j\to+\infty} G_j(u) = \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx$$

for all $u \in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m)$ such that $u = z$ on $\partial B_N^+(0)$ and $u = 0$ on $B_N^-(0)$.

Proof. We first deal with the liminf inequality. Let $u_j \to u$ in $L^p(B_N(0)\setminus C_{1,N};\mathbb{R}^m)$, such that $u_j = z$ on $\partial B_N^+(0)$, $u_j = 0$ on $\partial B_N^-(0)$ and

$$\liminf_{j\to+\infty} G_j(u_j) < c\,; \tag{4.4}$$

in particular, $u_j \rightharpoonup u$ in $W^{1,q}(B_N^-(0);\mathbb{R}^m)$. Since $p < q$, for every $K > 0$ there exists j_K such that $\rho_{\varepsilon_j}^{p-q} > K$ for every $j > j_K$; hence, by the standard growth condition (3.3), we have

$$\liminf_{j\to+\infty} G_j(u_j) \geq \liminf_{j\to+\infty} \int_{B_N^+(0)} g_j^p(Du_j)\,dx + K \liminf_{j\to+\infty}\Big(\int_{B_N^-(0)} |Du_j|^q\,dx - |B_N^-|\rho_{\varepsilon_j}^q\Big)\,.$$

By (4.2) and [7] Proposition 12.8, we have

$$\Gamma\text{-}\lim_{j\to+\infty} \int_{B_N^+(0)} g_j^p(Du)\,dx = \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx \tag{4.5}$$

for every $u \in W^{1,p}(B_N^+(0);\mathbb{R}^m)$. Hence,

$$\liminf_{j\to+\infty} \int_{B_N^+(0)} g_j^p(Du_j)\,dx \geq \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx$$

and, by the lower semicontinuity of $\int_{B_N^-(0)} |Du|^q\,dx$, we get that

$$\liminf_{j\to+\infty} G_j(u_j) \geq \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx + K \int_{B_N^-(0)} |Du|^q\,dx$$

for every $K > 0$. By (4.4), there follows that

$$\int_{B_N^-(0)} |Du|^q\,dx \leq \frac{1}{K}\Big(c - \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx\Big)$$

for every $K > 0$; hence, passing to the limit as K tends to $+\infty$, we get that $u = 0$ on $B_N^-(0)$, and

$$\liminf_{j\to+\infty} G_j(u_j) \geq \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx\,,$$

which proves the liminf inequality.

Now we pass to compute the limsup inequality. Let $u \in W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m)$ such that $u = z$ on $\partial B_N^+(0)$ and $u = 0$ on $B_N^-(0)$. By the standard growth condition (3.2), the sequence of functionals $\int_{B_N^+(0)} g_j^p(Du)\,dx$ satisfies the L^p-fundamental estimate (see [7] Proposition 12.2); hence, by [7] Proposition 11.7, there exists a sequence $(v_j) \in W^{1,p}(B_N^+(0);\mathbb{R}^m)$ converging to u in $L^p(B_N^+(0);\mathbb{R}^m)$ such that $v_j = z$ on $\partial B_N^+(0)$ and $v_j = 0$ on $B_1^{n-1}(0)$ and

$$\lim_{j\to+\infty} \int_{B_N^+(0)} g_j^p(Dv_j)\,dx = \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx\,.$$

We can define $(\tilde{v}_j)$ on $B_N(0) \setminus C_{1,N}$ extending v_j on $B_N^-(0)$ such that

$$\tilde{v}_j(x) = \begin{cases} v_j(x) & \text{if } x \in B_N^+(0) \\ 0 & \text{if } x \in B_N^-(0) \end{cases} \;;$$

the new sequence $(\tilde{v}_j)$ belongs to $W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m)$, it converges to u in $L^p(B_N(0) \setminus C_{1,N};\mathbb{R}^m)$ and satisfies the limsup inequality

$$\begin{aligned} \limsup_{j\to+\infty} G_j(\tilde{v}_j) &= \limsup_{j\to+\infty}\Big(\int_{B_N^+(0)} g_j^p(Dv_j)\,dx + \rho_{\varepsilon_j}^p \int_{B_N^-(0)} W^q(0)\,dx\Big) \\ &= \int_{B_N^+(0)} \widehat{W}_p(Du)\,dx, \end{aligned}$$

which concludes the proof. □

Corollary 4.2 *(Convergence of minimum problems) The minimum values*

$$\begin{aligned} \varphi_{N,j}(z) = \inf\Big\{ &\int_{B_N^+(0)} g_j^p(D\zeta)\,dy + \rho_{\varepsilon_j}^{p-q} \int_{B_N^-(0)} g_j^q(D\zeta)\,dy : \\ &\zeta \in W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m) \quad \zeta = z \text{ on } \partial B_N^+(0),\ \zeta = 0 \text{ on } \partial B_N^-(0)\Big\} \end{aligned} \tag{4.6}$$

converge to

$$\begin{aligned} \varphi_N(z) = \inf\Big\{ &\int_{B_N^+(0)} \widehat{W}_p(D\zeta)\,dy : \zeta \in W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m) \\ &\zeta = z \text{ on } \partial B_N^+(0), \quad \zeta = 0 \text{ on } B_N^-(0)\Big\} \end{aligned} \tag{4.7}$$

as j tends to $+\infty$.

Proof. By Proposition 4.1, the sequence of functionals (G_j) Γ-converges to $\int_{B_N^+(0)} \widehat{W}_p(Du)\,dx$ for all $u \in W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m)$ such that $u = z$ on $\partial B_N^+(0)$ and $u = 0$ on $B_N^-(0)$. Since $\inf_X G_j < +\infty$ for every $j \in \mathbb{N}$, where

$$X = \{u \in W^{1,p}(B_N(0) \setminus C_{1,N};\mathbb{R}^m) :\ u = z \text{ on } \partial B_N^+(0),\ u = 0 \text{ on } \partial B_N^-(0)\}\,,$$

there exists $c > 0$ such that $c > \inf_X G_j$ for every $j \in \mathbb{N}$; hence, we can apply the Theorem 2.2 with

$$\begin{aligned} K &= \Big\{ u \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m) : \; u = z \text{ on } \partial B_N^+(0), \\ &\qquad\qquad u = 0 \text{ on } \partial B_N^-(0), \quad \int_{B_N^+} |Du|^p \, dx \le c \Big\}. \end{aligned}$$

□

4.1 Some particular choices of W_p and W_q

In the following we show some examples of φ_N for a particular choice of W_p and W_q. In the cases (b) and (c) the formula describing φ_N involves also the classical p-capacity (see (4.9) and (4.11)).

(a) If $\widehat{W_p}$ is homogeneous of degree p, (e.g. $A \mapsto W_p(A)$ satisfies (4.3)), then

$$\begin{aligned} \varphi_N(z) &= |z|^p \inf \Big\{ \int_{B_N^+(0)} \widehat{W_p}(D\zeta) \, dy : \zeta \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m) \\ &\qquad \zeta = \frac{z}{|z|} \text{ on } \partial B_N^+(0), \quad \zeta = 0 \text{ on } B_N^-(0) \Big\}. \end{aligned} \tag{4.8}$$

(b) If $W_p(A) = |A|^p$ and $W_q(A) = |A|^q$ with $p < q$, then

$$\begin{aligned} \varphi_N(z) &= |z|^p \inf \Big\{ \int_{B_N^+(0)} |D\zeta|^p \, dy : \zeta \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m) \\ &\qquad \zeta = \frac{z}{|z|} \text{ on } \partial B_N^+(0), \quad \zeta = 0 \text{ on } B_N^-(0) \Big\}. \end{aligned}$$

Since $\varphi_N(z)$ is invariant by rotations, we can fix, for example, $\frac{z}{|z|} = e_1$ and restrict our attention to the following class of functions without increasing the infimum:

$$\begin{aligned} \frac{\varphi_N(z)}{|z|^p} &= \inf \Big\{ \int_{B_N^+(0)} |D\zeta|^p \, dx : \; \zeta = \psi e_1, \quad \psi \in W^{1,p}(B_N(0) \setminus C_{1,N}) \\ &\qquad \psi = 1 \text{ on } \partial B_N^+(0), \quad \psi = 0 \text{ on } B_N^-(0) \Big\} \\ &= \frac{1}{2} \inf \Big\{ \int_{B_N(0)} |D\psi|^p \, dx : \psi - 1 \in W_0^{1,p}(B_N(0)), \, \psi = 0 \text{ on } B_1^{n-1}(0) \Big\} \\ &= \frac{1}{2} C_p(B_1^{n-1}(0); B_N(0)), \end{aligned} \tag{4.9}$$

where $B_1^{n-1}(0)$ is defined as in (2.3).

(c) If $q = p$ and $W_p(A) = W_q(A) = |A|^p$ then $\varphi_{N,j}(z) = \varphi_N(z)$ for every $j \in \mathbb{N}$. Reasoning as in the case (b), we can fix $\frac{z}{|z|} = e_1$; hence,

$$\begin{aligned}
\frac{\varphi_N(z)}{|z|^p} &= \inf\Big\{\int_{B_N^+(0)} |D\zeta|^p\,dx + \int_{B_N^-(0)} |D\zeta|^p\,dx : \zeta \in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m) \\
&\qquad \zeta = e_1 \text{ on } \partial B_N^+(0),\ \zeta = 0 \text{ on } \partial B_N^-(0)\Big\} \\
&= \inf\Big\{\int_{B_N^+(0)} |D\psi|^p\,dx + \int_{B_N^-(0)} |D\psi|^p\,dx : \psi \in W^{1,p}(B_N(0)\setminus C_{1,N}) \\
&\qquad \psi = 0 \text{ on } \partial B_N^+(0),\ \psi = 1 \text{ on } \partial B_N^-(0)\Big\}. \qquad (4.10)
\end{aligned}$$

Let ψ_1 be the unique solution of the minimum problem defined in (4.10); since also $\psi_2(x) = 1 - \psi_1(x)$ is a solution of the same minimum problem, by the uniqueness, we have that $1 - \psi_1(x) = \psi_1(x)$. In particular,

$$1 - \psi_1(x_\alpha, 0) = \psi_1(x_\alpha, 0)$$

which implies that $\psi_1(x_\alpha, 0) = 1/2$ for every $(x_\alpha, 0) \in B_1^{n-1}(0)$. Hence, we get

$$\begin{aligned}
\frac{\varphi_N(z)}{|z|^p} &= \inf\Big\{\int_{B_N^+(0)} |D\psi|^p\,dx + \int_{B_N^-(0)} |D\psi|^p\,dx : \psi \in W^{1,p}(B_N(0)\setminus C_{1,N}) \\
&\qquad \psi = 0 \text{ on } \partial B_N^+(0),\ \psi = \frac{1}{2} \text{ on } B_1^{n-1}(0),\ \psi = 1 \text{ on } \partial B_N^-(0)\Big\} \\
&= 2\,\inf\Big\{\int_{B_N^+(0)} |D\psi|^p\,dx : \psi \in W^{1,p}(B_N(0)\setminus C_{1,N}) \\
&\qquad \psi = 0 \text{ on } \partial B_N^+(0),\ \psi = \frac{1}{2} \text{ on } B_1^{n-1}(0)\Big\} \\
&= 2\,\frac{1}{2^p}\,\inf\Big\{\int_{B_N^+(0)} |D\psi|^p\,dx : \psi \in W^{1,p}(B_N(0)\setminus C_{1,N}) \\
&\qquad \psi = 0 \text{ on } \partial B_N^+(0),\ \psi = 1 \text{ on } B_1^{n-1}(0)\Big\} \\
&= \frac{1}{2^p}\,C_p(B_1^{n-1}(0); B_N(0)). \qquad (4.11)
\end{aligned}$$

Remark 4.3 By comparing (b) and (c) we note the discontinuous dependence of φ_N on q as $q \to p$. (see (4.9) and (4.11)).

4.2 Properties of the functions $\varphi_{N,j}$ and φ_N

By the pointwise convergence of $(\varphi_{N,j})_j$ to φ_N (see Corollary 4.2), we can define φ as the pointwise limit of φ_N; *i.e.*,

$$\varphi(z) = \inf_N \varphi_N(z).$$

Since pointwise convergence is not sufficient to prove that our sequence of functionals (F_j) Γ-converges to F (Theorem 3.2), to get this result we prove the uniform convergence on compact sets of $\mathbb{R}^m$ by Ascoli Arzela's Theorem; hence, we start from the property of equi-continuity.

(1) Equi-continuity of $(\varphi_{N,j})_j$
For all $N \in \mathbb{N}$ and $N > 2$ there exists c_N such that

$$|\varphi_{N,j}(z) - \varphi_{N,j}(w)| \quad \leq \quad c\,|w-z|\Big(\rho_{\varepsilon_j}^{p-1} c_N + |z|^{p-1} + |w|^{p-1}\Big) \tag{4.12}$$

for all $z, w \in \mathbb{R}^m$ and j, where $c_N = (1 + |B_N(0)|^{(p-1)/p})$.

By definition of $\varphi_{N,j}(z)$, fixed $\eta > 0$ there exists $\zeta \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m)$ such that $\zeta = 0$ on $\partial B_N^+(0)$, $\zeta = -z$ on $\partial B_N^-(0)$ and

$$\int_{B_N^+(0)} g_j^p(D\zeta)\,dy + \rho_{\varepsilon_j}^{p-q} \int_{B_N^-(0)} g_j^q(D\zeta)\,dy \leq \varphi_{N,j}(z) + \eta\,. \tag{4.13}$$

Let $\varphi \in C_0^\infty(B_2^+(0))$ be a cut-off function such that $\varphi = 1$ on $B_1^+(0)$ and $|D\varphi| \leq c$.

Let $w \in \mathbb{R}^m$, if we define

$$\tilde{\zeta} = \begin{cases} \zeta + (1-\varphi)(w-z) & \text{on } B_N^+(0) \cup B_1^{n-1}(0) \\ \\ \zeta & \text{on } B_N^-(0) \end{cases}$$

then $\tilde{\zeta} \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m)$ such that $\tilde{\zeta} = w - z$ on $\partial B_N^+(0)$ and $\tilde{\zeta} = -z$ on $\partial B_N^-(0)$; hence, it is a test function for $\varphi_{N,j}(w)$ and we can estimate the difference with $\varphi_{N,j}(z)$ in the following way by (4.1)

$$\begin{aligned} & \varphi_{N,j}(w) - \varphi_{N,j}(z) \\ \leq \quad & \int_{B_N^+(0)} \Big(g_j^p(D\tilde{\zeta}) - g_j^p(D\zeta)\Big)\,dx + \eta \\ \leq \quad & \int_{B_N^+(0)} c\Big(\rho_{\varepsilon_j}^{p-1} + |D\tilde{\zeta}|^{p-1} + |D\zeta|^{p-1}\Big)\Big|D\tilde{\zeta} - D\zeta\Big|\,dx + \eta\,. \end{aligned} \tag{4.14}$$

Note that since $\tilde{\zeta} = \zeta$ on $B_N^-(0)$ we lose the contribution of $B_N^-(0)$; hence, by (4.14) and Hölder's inequality, we have

$$\begin{aligned} & \varphi_{N,j}(w) - \varphi_{N,j}(z) \\ \leq \quad & \int_{B_N^+(0)} c\Big(\rho_{\varepsilon_j}^{p-1} + 2|D\zeta|^{p-1} + |w-z|^{p-1}|D\varphi|^{p-1}\Big)|w-z||D\varphi|\,dx + \eta \\ \leq \quad & c|w-z|\Big(\rho_{\varepsilon_j}^{p-1}\int_{B_N^+(0)} |D\varphi|\,dx + \Big(\int_{B_N^+(0)} |D\zeta|^p\,dx\Big)^{(p-1)/p}\Big(\int_{B_N^+(0)} |D\varphi|^p\,dx\Big)^{1/p}\Big) \\ & + |w-z|^p \int_{B_N^+(0)} |D\varphi|^p\,dx + \eta\,. \end{aligned} \tag{4.15}$$

Since $N > 2$ we have that $\int_{B_N^+(0)} |D\varphi|\,dx$ and $\int_{B_N^+(0)} |D\varphi|^p\,dx$ are constant independent from N; moreover, by the standard growth condition (3.2) and (4.13) we get

$$\begin{aligned}\int_{B_N^+(0)} |D\zeta|^p\,dx &\leq \int_{B_N^+(0)} g_j^p(D\zeta)\,dx + \rho_{\varepsilon_j}^p |B_N| \\ &\leq \varphi_{N,j}(z) + \eta + \rho_{\varepsilon_j}^p |B_N|\,. \end{aligned} \tag{4.16}$$

Since $\varphi_{N,j}(z) \leq \varphi_N(z)$, for all $j \in \mathbb{N}$, and $|A|^p \leq \widehat{W}_p(A) \leq c_1 |A|^p$, we have that

$$\begin{aligned}\varphi_{N,j}(z) \quad \leq \quad c_1 \inf\Big\{ \int_{B_N^+(0)} |D\phi|^p\,dx : \phi \in W^{1,p}(B_N(0) \setminus C_{1,N}; \mathbb{R}^m) \\ \phi = \frac{z}{|z|} \text{ on } \partial B_N^+(0), \quad \phi = 0 \text{ on } B_N^-(0) \Big\}\,;\end{aligned}$$

hence, reasoning as in Section 4.1 case (b), we get that

$$\varphi_{N,j}(z) \leq c_1\,\frac{|z|^p}{2} C_p(B_1^{n-1}(0); B_N(0)) \leq c_1\,\frac{|z|^p}{2} C_p(B_1^{n-1}(0); \mathbb{R}^n)$$

for every $N > 2$.

By (4.16) we then have

$$\int_{B_N^+(0)} |D\zeta|^p\,dx \leq c\,(\rho_{\varepsilon_j}^p |B_N| + |z|^p) + \eta$$

which implies, togheter with (4.15), that

$$\begin{aligned} & \varphi_{N,j}(w) - \varphi_{N,j}(z) \\ \leq\ & c|w-z| \Big(\rho_{\varepsilon_j}^p + \rho_{\varepsilon_j}^{p-1} |B_N(0)|^{(p-1)/p} + |z|^{p-1} + \eta^{(p-1)/p} \Big) \\ & + \tilde{c}|w-z|^p + \eta \\ \leq\ & c\,|w-z| \Big(\rho_{\varepsilon_j}^{p-1}(1 + |B_N(0)|^{(p-1)/p}) + |z|^{p-1} + |w|^{p-1} + \eta^{(p-1)/p} \Big) + \eta\,, \end{aligned}$$

and by the arbitrariness of η we get then (4.12).

(2) Uniform convergence of $(\varphi_{N,j})_j$
From (4.12) we deduce that

$$\varphi_{N,j} \to \varphi_N \qquad \text{uniformly} \tag{4.17}$$

on compact sets of $\mathbb{R}^m$ by Ascoli Arzela's Theorem.

(3) Equi-continuity of φ_N
Passing to the limit in (4.12), as j tends to $+\infty$, we get

$$|\varphi_N(z) - \varphi_N(w)| \leq c\,|w-z|\,(|z|^{p-1} + |w|^{p-1}) \tag{4.18}$$

for all $z, w \in \mathbb{R}^m$.

(4) Uniform convergence of φ_N
From (4.18) we deduce that

$$\varphi_N \to \varphi \qquad \text{uniformly} \tag{4.19}$$

on compact sets of $\mathbb{R}^m$ by Ascoli Arzela's Theorem.

Proposition 4.4 *Let (u_j) be a sequence converging to (u^+, u^-) weakly in the sense of Definition* 3.1 *and bounded in $L^\infty(\Omega^+ \cup \Omega^-; \mathbb{R}^m)$. Let $(u_j^{i\pm})$ be defined by* (3.7) *and let ψ_j be defined by*

$$Q^\varepsilon_{i,n-1} = (x_i^\varepsilon, 0) + \Big(-\frac{\varepsilon_j}{2}, \frac{\varepsilon_j}{2}\Big)^{n-1}, \qquad \psi_j = \sum_{i \in Z_j} \varphi_{N,j}(u_j^{i+} - u_j^{i-})\chi_{Q^\varepsilon_{i,n-1}}. \tag{4.20}$$

Then we have

$$\lim_{j \to +\infty} \int_\omega \Big|\psi_j - \varphi_N(u^+ - u^-)\Big|\, dx_\alpha = 0. \tag{4.21}$$

Proof. Reasoning as in [2] Proposition 4.3; if $|z| \le \sup_j(\|u_j^+\|_\infty + \|u_j^-\|_\infty)$ then we have, by (4.17),

$$|\varphi_{N,j}(z) - \varphi_N(z)| \le o(1)$$

as $j \to +\infty$, uniformly in z. Set

$$\hat\psi_j = \sum_{i \in Z_j} \varphi_N(u_j^{i+} - u_j^{i-})\chi_{Q^\varepsilon_{i,n-1}}\,, \tag{4.22}$$

we deduce that the limit in (4.21) is equal to the limits

$$\begin{aligned}
& \lim_j \int_\omega \Big|\hat\psi_j - \varphi_N(u^+ - u^-)\Big|\, dx_\alpha \\
= \ & \lim_j \Big(\sum_{i \in Z_j} \int_{Q^\varepsilon_{i,n-1}} \Big|\varphi_N(u_j^{i+} - u_j^{i-}) - \varphi_N(u^+ - u^-)\Big|\, dx_\alpha\Big) \\
\le \ & c \lim_{j \to +\infty} \Big(\sum_{i \in Z_j} \int_{Q^\varepsilon_{i,n-1}} \Big|u_j^{i-} - u^-\Big| + \Big|u_j^{i+} - u^+\Big|\, dx_\alpha\Big)
\end{aligned} \tag{4.23}$$

by (4.18).

We now estimate

$$\int_{Q^\varepsilon_{i,n-1}} \Big|u_j^{i+} - u^+\Big|\, dx_\alpha \le \int_{Q^\varepsilon_{i,n-1}} \Big|u_j^{i+} - u_j^+\Big|\, dx_\alpha + \int_{Q^\varepsilon_{i,n-1}} \Big|u_j^+ - u^+\Big|\, dx_\alpha\,;$$

by [1] Theorem 6.2

$$\lim_j \sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^+ - u^+ \right| dx_\alpha = 0$$

while, by Hölder's inequalities,

$$\sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i+} - u_j^+ \right| dx_\alpha \le \Big(\sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i+} - u_j^+ \right|^p dx_\alpha \Big)^{1/p} . \tag{4.24}$$

By [1] Lemma 5.19, we have that

$$\int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i+} - u_j^+ \right|^p dx_\alpha \le c \Big(\frac{1}{\varepsilon_j} \int_{Q^\varepsilon_{i,n}} |u_j^{i+} - u_j^+|^p \, dx + \varepsilon_j^{p-1} \int_{Q^\varepsilon_{i,n}} |Du_j^+|^p \, dx \Big) \tag{4.25}$$

where

$$Q^\varepsilon_{i,n} = (x^\varepsilon_i, 0) + \Big(\Big(-\frac{\varepsilon_j}{2}, \frac{\varepsilon_j}{2} \Big)^{n-1} \times (0, \varepsilon_j) \Big)$$

and, by Poincaré's inequality, we get

$$\int_{Q^\varepsilon_{i,n}} |u_j^{i+} - u_j^+|^p \, dx \le c\, \varepsilon_j^p \int_{Q^\varepsilon_{i,n}} |Du_j^+|^p \, dx \, . \tag{4.26}$$

Taking (4.25) and (4.26) into account, we get

$$\Big(\sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i+} - u_j^+ \right|^p dx_\alpha \Big)^{1/p} \le c\, \varepsilon_j^{(p-1)/p} \sup_j \Big(\int_{\Omega^+} |Du_j^+|^p \, dx \Big)^{1/p}$$

which implies, by (4.24), that

$$\lim_j \sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i+} - u_j^+ \right| dx_\alpha = 0 \, .$$

Reasoning as above, we get

$$\lim_j \sum_{i\in Z_j} \int_{Q^\varepsilon_{i,n-1}} \left| u_j^{i-} - u_j^- \right| dx_\alpha = 0 \, ;$$

hence, by (4.23), (4.21) is proved. □

5 Liminf inequality

Let $u_j \to (u^+, u^-)$ be such that $\sup_j F_{\varepsilon_j}(u_j) < +\infty$. We fix $k, N \in \mathbb{N}$ with $N > 2^k$, and define (w_j) as in Lemma 3.4 with

$$\rho^n_{\varepsilon_j} = o(\varepsilon_j^{n-1}) \, . \tag{5.1}$$

Let

$$E_j^{\pm} = \bigcup_{i \in Z_j} B_i^{j\pm}, \qquad \text{where} \qquad B_i^{j\pm} = B_{\rho_j^i}(x_i^\varepsilon, 0) \cap \{\pm x_n > 0\} \tag{5.2}$$

for all $i \in Z_j$.

The following proposition shows that the 'contribution far from' the balls $B_{i,\varepsilon}^{n-1}$ can be estimated by the Γ-limit of the two uncoupled problems

$$F_j^+(u) = \int_{\Omega^+} W_p(Du)\, dx\,, \qquad F_j^-(u) = \int_{\Omega^-} W_q(Du)\, dx \tag{5.3}$$

for all $j \in \mathbb{N}$. Since W_p and W_q are quasiconvex we have that

$$\Gamma\text{-}\lim_j F_j^+(u) = \int_{\Omega^+} W_p(Du)\, dx\,, \quad \Gamma\text{-}\lim_j F_j^-(u) = \int_{\Omega^-} W_q(Du)\, dx$$

(see Remark 2.1 and Section 2.1).

Proposition 5.1 *We have*

$$\begin{aligned} & \liminf_{j \to +\infty} \Big(\int_{\Omega^+} W_p(Du_j)\, dx + \int_{\Omega^-} W_q(Du_j)\, dx \Big) \\ \geq \; & \int_{\Omega^+} W_p(Du^+)\, dx + \int_{\Omega^-} W_q(Du^-)\, dx \\ & + \liminf_{j \to +\infty} \Big(\int_{E_j^+} W_p(Dw_j)\, dx + \int_{E_j^-} W_q(Dw_j)\, dx \Big) - \frac{c}{k}\,. \end{aligned} \tag{5.4}$$

Proof. Let us define

$$v_j^+ = \begin{cases} u_j^{i+} & \text{on } B_i^{j+},\ i \in Z_j \\ w_j & \text{on } \Omega^+ \setminus E_j^+\,, \end{cases} \qquad\qquad v_j^- = \begin{cases} u_j^{i-} & \text{on } B_i^{j-},\ i \in Z_j \\ w_j & \text{on } \Omega^- \setminus E_j^-\,. \end{cases}$$

By Remark 3.5 (v_j^+) is bounded in $W^{1,p}(\Omega^+;\mathbb{R}^m)$ and (v_j^-) in $W^{1,q}(\Omega^-;\mathbb{R}^m)$. Moreover, by (5.1) we have that $\lim_{j\to+\infty} |E_j^{\pm}| = 0$; hence, $\lim_{j\to+\infty} |\{x \in \Omega^{\pm} : w_j \neq v_j^{\pm}\}| = 0$ which implies that $v_j^+ \rightharpoonup u^+$ in $W^{1,p}(\Omega^+;\mathbb{R}^m)$ and $v_j^- \rightharpoonup u^-$ in $W^{1,q}(\Omega^-;\mathbb{R}^m)$, so that, by Lemma 3.4

$$\begin{aligned} & \liminf_{j \to +\infty} \Big(\int_{\Omega^+ \setminus E_j^+} W_p(Du_j)\, dx + \int_{\Omega^- \setminus E_j^-} W_q(Du_j)\, dx \Big) + \frac{c}{k} \\ \geq \; & \liminf_{j \to +\infty} \Big(\int_{\Omega^+ \setminus E_j^+} W_p(Dw_j)\, dx + \int_{\Omega^- \setminus E_j^-} W_q(Dw_j)\, dx \Big) \\ = \; & \liminf_{j \to +\infty} \Big(\int_{\Omega^+} W_p(Dv_j^+)\, dx + \int_{\Omega^-} W_q(Dv_j^-)\, dx \Big) \\ \geq \; & \int_{\Omega^+} W_p(Du^+)\, dx + \int_{\Omega^-} W_q(Du^-)\, dx\,. \end{aligned}$$

$\square$

Let us estimate the contribution on $E_j^+ \cup E_j^-$. With fixed $j \in \mathbb{N}$ and $i \in Z_j$ we define

$$C^i_{1,\frac{3}{4}2^{-k_i}N} = \Big\{(x_\alpha, 0) \in \mathbb{R}^n : 1 \leq |x_\alpha| < \frac{3}{4}2^{-k_i}N\Big\}$$

and

$$\zeta(x) = \begin{cases} w_j((x_i^\varepsilon, 0) + \rho_{\varepsilon_j} x) - u_j^{i-} & \text{if } x \in B_{\frac{3}{4}2^{-k_i}N}(0) \setminus C^i_{1,\frac{3}{4}2^{-k_i}N} \\ u_j^{i+} - u_j^{i-} & \text{if } x \in B_N^+(0) \setminus B^+_{\frac{3}{4}2^{-k_i}N}(0) \\ 0 & \text{if } x \in B_N^-(0) \setminus B^-_{\frac{3}{4}2^{-k_i}N}(0) \ . \end{cases}$$

By a change of variables and (4.6) we obtain

$$\begin{aligned} & \int_{B_i^{j+}} W_p(Dw_j)\, dx + \int_{B_i^{j-}} W_q(Dw_j)\, dx + (W_p(0) + W_q(0))\, |B^+_{N\rho_{\varepsilon_j}} \setminus B_i^{j+}| \\ = & \int_{B_N^+(0)} \rho^n_{\varepsilon_j} W_p(\rho^{-1}_{\varepsilon_j} D\zeta)\, dx + \int_{B_N^-(0)} \rho^n_{\varepsilon_j} W_q(\rho^{-1}_{\varepsilon_j} D\zeta)\, dx \\ = & \ \rho^{n-p}_{\varepsilon_j} \int_{B_N^+(0)} g_j^p(D\zeta)\, dx + \rho^{n-q}_{\varepsilon_j} \int_{B_N^-(0)} g_j^q(D\zeta)\, dx \\ = & \ \rho^{n-p}_{\varepsilon_j} \Big(\int_{B_N^+(0)} g_j^p(D\zeta)\, dx + \rho^{p-q}_{\varepsilon_j} \int_{B_N^-(0)} g_j^q(D\zeta)\, dx \Big) \\ \geq & \ \rho^{n-p}_{\varepsilon_j}\, \varphi_{N,j}(u_j^{i+} - u_j^{i-})\,; \end{aligned} \tag{5.5}$$

hence, we get

$$\begin{aligned} & \liminf_{j\to+\infty} \int_{E_j^+} W_p(Dw_j)\, dx + \int_{E_j^-} W_q(Dw_j)\, dx \\ \geq & \ \liminf_{j\to+\infty} \Big(\frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon_j^{n-1}} \Big) \sum_{i \in Z_j} \varepsilon_j^{n-1}\, \varphi_{N,j}(u_j^{i+} - u_j^{i-}) \\ = & \ \lim_{j\to+\infty} \Big(\frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon_j^{n-1}} \Big) \liminf_{j\to+\infty} \sum_{i \in Z_j} \varepsilon_j^{n-1}\, \varphi_{N,j}(u_j^{i+} - u_j^{i-})\,. \end{aligned} \tag{5.6}$$

We use this inequality to prove the following liminf inequality.

Proposition 5.2 *Let (ρ_{ε_j}) be a sequence of positive numbers converging to* 0 *such that*

$$0 < \lim_{j\to+\infty} \frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon_j^{n-1}} = R < +\infty\,;$$

then for every sequence $(u_j) \in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)$ *converging to* (u^+,u^-)*, in the sense of Definition* 3.1*, we have*

$$\begin{aligned}\liminf_{j\to+\infty} F_j(u_j) \quad &\geq \quad \int_{\Omega^+} W_p(Du^+)\,dx + \int_{\Omega^-} W_q(Du^-)\,dx \\ &+R\int_\omega \varphi(u^+-u^-)\,dx_\alpha\,.\end{aligned}$$

Proof. Let $u_j \to (u^+,u^-)$. We can always assume, up to a subsequence, that there exists the limit

$$\lim_j F_j(u_j) < +\infty,$$

so that $u_j \rightharpoonup (u^+,u^-)$ in the sense of Definition 3.1. By [8] Lemma 3.5, upon passing to a further subsequence, for all $M\in\mathbb{N}$ and $\eta>0$ there exists $R_M > M$ and a Lipschitz function Φ_M of Lipschitz constant 1 such that $\Phi_M(z) = z$ if $|z| < R_M$ and $\Phi_M(z) = 0$ if $|z| > 2R_M$, and

$$\lim_j F_j(u_j) \geq \liminf_j F_j(\Phi_M(u_j)) - \eta. \tag{5.7}$$

Note that $\Phi_M(u_j) \in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)\cap L^\infty(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)$ and

$$\Phi_M(u_j) \to (\Phi_M(u^+),\Phi_M(u^-))\,.$$

Moreover $\Phi_M(u^+) \rightharpoonup u^+$ in $W^{1,p}(\Omega^+;\mathbb{R}^m)$ and $\Phi_M(u^-) \rightharpoonup u^-$ in $W^{1,q}(\Omega^-;\mathbb{R}^m)$ as M tends to $+\infty$, which implies that

$$\Phi_M(u^+) \to u^+ \qquad , \qquad \Phi_M(u^-) \to u^- \qquad \text{in} \qquad L^1(\omega;\mathbb{R}^m)\,.$$

Note that the L^1-convergence of the traces is sufficient to our aims since we use it just when we apply inequality (4.18) to prove that

$$\lim_j \int_\omega \varphi(\Phi_M(u^+)-\Phi_M(u^-))\,dx_\alpha = \int_\omega \varphi(u^+-u^-)\,dx_\alpha\,. \tag{5.8}$$

Reasoning as in [2] Proposition 5.2, if we apply Lemma 3.4, (5.6), (5.4) and Proposition 4.4 to $(\Phi_M(u_j))$ in place of (u_j), we get that

$$\begin{aligned}\liminf_j F_j(\Phi_M(u_j)) \quad &\geq \quad \int_{\Omega^+} W_p(D\Phi_M(u^+))\,dx + \int_{\Omega^-} W_q(D\Phi_M(u^-))\,dx \\ &+R\int_\omega \varphi(\Phi_M(u^+)-\Phi_M(u^-))\,dx_\alpha\,.\end{aligned}$$

By the lower semicontinuity of $\int_{\Omega^+} W_p(D\zeta)\,dx$ and $\int_{\Omega^-} W_q(D\zeta)\,dx$ with respect to the weak convergence and (5.8), we get the liminf inequality. □

Remark 5.3 Note that $0 < \lim_{j\to+\infty} \frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^{n-1}} = R < +\infty$ is the only meaningful scaling for the radii of the perforation. In fact, if $R = 0$, *i.e.* if ρ_{ε_j} tends to zero faster than $\varepsilon_j^{(n-1)/(n-p)}$, then we obtain two uncoupled problems in Ω^+ and Ω^-; while, if $R = +\infty$, *i.e.* if ρ_{ε_j} tends to zero more slowly than $\varepsilon_j^{(n-1)/(n-p)}$, then $u^+ = u^-$ on $\omega \times \{0\}$ and the limit function $(u^+, u^-) \in W^{1,p}(\Omega^+;\mathbb{R}^m) \times W^{1,q}(\Omega^-;\mathbb{R}^m)$ defines a unique function in $W^{1,p}(\Omega;\mathbb{R}^m)$.

6 Limsup inequality

For every $(u^+, u^-) \in W^{1,p}(\Omega^+;\mathbb{R}^m) \times W^{1,q}(\Omega^-;\mathbb{R}^m)$ the limsup inequality is obtained by suitably modifying the function $v = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-}$ to get a recovery sequence defined on $\widehat{\Omega}_{\varepsilon_j}$. Note that if we remove the quasiconvexity assumptions on W_p and W_q, we have to consider the recovery sequences for the Γ-limits of $F_j^+(u) = \int_{\Omega^+} W_p(Du)\,dx$ and $F_j^-(u) = \int_{\Omega^-} W_q(Du)\,dx$, in place of u^+ and u^-, respectively (see Remark 3.3).

Proposition 6.1 *Let (ρ_{ε_j}) be a sequence of positive numbers converging to* 0 *such that*

$$0 < \lim_{j\to+\infty} \frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^{n-1}} = R < +\infty\,;$$

if $\mathcal{H}^{n-1}(\partial\omega) = 0$ then for all $(u^+, u^-) \in W^{1,p}(\Omega^+;\mathbb{R}^m) \times W^{1,q}(\Omega^-;\mathbb{R}^m)$ and for all $\eta > 0$ there exists a sequence $u_j \in V^{p,q}(\widehat{\Omega}_{\varepsilon_j};\mathbb{R}^m)$ converging to (u^+, u^-) such that

$$\begin{aligned}\limsup_{j\to+\infty} F_j(u_j) \quad &\leq \quad \int_{\Omega^+} W_p(Du^+)\,dx + \int_{\Omega^-} W_q(Du^-)\,dx \\ &\quad +R\int_\omega \varphi(u^+ - u^-)\,dx_\alpha + \eta\mathcal{H}^{n-1}(\omega)\,.\end{aligned}$$

Proof. Let

$$v = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-} \in W^{1,p}(\Omega^+ \cup \Omega^-;\mathbb{R}^m) \cap W^{1,q}(\Omega^-;\mathbb{R}^m)\,. \qquad (6.1)$$

With fixed $N \in \mathbb{N}$, by Lemma 3.4 applied with (u_j) and (ρ_{ε_j}) replaced by (v) and $(\frac{4}{3}\rho_{\varepsilon_j})$, respectively, and taking the equi-integrability condition into account we obtain a sequence (w_j) which equals the constants $v_j^{i\pm} = \fint_{C_i^j \cap \{\pm x_n > 0\}} v\,dx$ on $\partial B^\pm_{N\rho_{\varepsilon_j}}$, respectively, for all $i \in Z_j$.

We recall that $B_{N\rho_{\varepsilon_j}}$ denotes $B_{N\rho_{\varepsilon_j}}(x_i^\varepsilon, 0)$ and $B^\pm_{N\rho_{\varepsilon_j}} = B_{N\rho_{\varepsilon_j}} \cap \{\pm x_n > 0\}$.

Reasoning as in [2] Proposition 6.1, we first assume that in addition $(u^+, u^-) \in L^\infty(\Omega^\pm;\mathbb{R}^m)$. We define the sequence (u_j) by

$$u_j = w_j \qquad \text{on} \quad \Omega^\pm \setminus \Big(\bigcup_{i\in\mathbb{Z}^{n-1}} B^\pm_{N\rho_{\varepsilon_j}}\Big);$$

hence,

$$
\begin{aligned}
\limsup_{j\to+\infty} F_j(u_j) \quad \le \quad & \limsup_{j\to+\infty}\Big(\int_{\Omega^+\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^+_{N\rho_{\varepsilon_j}}} W_p(Dw_j)\,dx \\
& \qquad + \int_{\Omega^-\setminus\bigcup_{i\in\mathbb{Z}^{n-1}} B^-_{N\rho_{\varepsilon_j}}} W_q(Dw_j)\,dx\Big) \\
& + \limsup_{j\to+\infty}\Big(\int_{\Omega^+\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^+_{N\rho_{\varepsilon_j}}} W_p(Du_j)\,dx \\
& \qquad + \int_{\Omega^-\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^-_{N\rho_{\varepsilon_j}}} W_q(Du_j)\,dx\Big) \\
= \quad & \int_{\Omega^+} W_p(Du^+)\,dx + \int_{\Omega^-} W_q(Du^-)\,dx \\
& + \limsup_{j\to+\infty}\Big(\int_{\Omega^+\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^+_{N\rho_{\varepsilon_j}}} W_p(Du_j)\,dx \\
& \qquad + \int_{\Omega^-\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^-_{N\rho_{\varepsilon_j}}} W_q(Du_j)\,dx\Big)\,.
\end{aligned}
$$

We now define u_j on $\widehat{\Omega}_{\varepsilon_j}\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B_{N\rho_{\varepsilon_j}}$, and we compute the limit

$$
\limsup_{j\to+\infty}\Big(\int_{\Omega^+\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^+_{N\rho_{\varepsilon_j}}} W_p(Du_j)\,dx + \int_{\Omega^-\cap\bigcup_{i\in\mathbb{Z}^{n-1}} B^-_{N\rho_{\varepsilon_j}}} W_q(Du_j)\,dx\Big)\,.
$$

Let us consider the case $i\in Z_j$. Let

$$
M=\max\{\|u^+\|_{L^\infty},\|u^-\|_{L^\infty}\}\,,
$$

fixed $\eta>0$, by the uniform convergence of $\varphi_{N,j}\to\varphi_N$ and $\varphi_N\to\varphi$ on compact sets of $\mathbb{R}^m$, there exists N such that

$$
\varphi(z)\ge\varphi_N(z)-\frac{\eta}{3} \tag{6.2}
$$

for all $|z|\le M$ and

$$
|\varphi_{N,j}(z)-\varphi_N(z)|\le\frac{\eta}{3} \tag{6.3}
$$

for all $|z|\le M$ and $j\in\mathbb{N}$. Moreover, by (4.6), there exists $\zeta^i_j\in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m)$ such that

$$
\zeta^i_j=\begin{cases} v^{i+}_j-v^{i-}_j & \text{on } \partial B^+_N(0)\\ 0 & \text{on } \partial B^-_N(0)\end{cases}
$$

and

$$\begin{aligned}\int_{B_N^+(0)} g_j^p(D\zeta_j^i)\,dy + \rho_{\varepsilon_j}^{p-q}\int_{B_N^-(0)} g_j^q(D\zeta_j^i)\,dx &\leq \varphi_{N,j}(v_j^{i+}-v_j^{i-}) + \frac{\eta}{3} \\ &\leq \varphi(v_j^{i+}-v_j^{i-}) + \eta \qquad (6.4)\end{aligned}$$

by (6.2) and (6.3). Hence, if we define u_j on $\widehat{\Omega}_{\varepsilon_j} \cap B_{N\rho_{\varepsilon_j}}$ by

$$u_j = \zeta_j^i\Big(\frac{x-(x_i^\varepsilon,0)}{\rho_{\varepsilon_j}}\Big) + v_j^{i-}$$

then, by (6.4), we get

$$\begin{aligned}&\int_{B^+_{N\rho_{\varepsilon_j}}} W_p(Du_j)\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}} W_q(Du_j)\,dx \\ =\ & \rho_{\varepsilon_j}^{n-p}\Big(\int_{B_N^+(0)} g_j^p(D\zeta_j^i)\,dy + \rho_{\varepsilon_j}^{p-q}\int_{B_N^-(0)} g_j^q(D\zeta_j^i)\,dx\Big) \\ \leq\ & \Big(\frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^{n-1}}\Big)\Big(\varepsilon_j^{n-1}\varphi(v_j^{i+}-v_j^{i-}) + \varepsilon_j^{n-1}\eta\Big) \qquad (6.5)\end{aligned}$$

for all $i \in Z_j$.

If $i \notin Z_j$, it is not possible to use the construction above since $B_{N\rho_{\varepsilon_j}}$ might intersect $\partial\Omega$. We then consider a scalar $0 \leq \zeta \leq 1$ on $B_N(0) \setminus C_{1,N}$ such that

$$\zeta(x) = \begin{cases} 1 & \text{on } \partial B_N^+(0) \\ 0 & \text{on } B_N^-(0) \end{cases}$$

and $\tilde{\zeta}(x) = 1 - \zeta(x)$. We can define the extension of $w_j^+ = w_j\chi_{\Omega^+}$ to Ω as the function $w_j^p(x_\alpha, x_n) = w_j^+(x_\alpha, -x_n)$ and the extension of $w_j^- = w_j\chi_{\Omega^-}$ to Ω as the function $w_j^q(x_\alpha, x_n) = w_j^-(x_\alpha, -x_n)$, such that $w_j^p \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $w_j^q \in W^{1,q}(\Omega;\mathbb{R}^m)$.

Hence, u_j is defined by

$$u_j(x) = \zeta\Big(\frac{x-(x_i^\varepsilon,0)}{\rho_{\varepsilon_j}}\Big)w_j^p(x) + \tilde{\zeta}\Big(\frac{x-(x_i^\varepsilon,0)}{\rho_{\varepsilon_j}}\Big)w_j^q(x)$$

on $B_{N\rho_{\varepsilon_j}} \cap \widehat{\Omega}_{\varepsilon_j}$, and

$$Du_j = \begin{cases} \zeta Dw_j^+ + \dfrac{1}{\rho_{\varepsilon_j}}D\zeta(w_j^+ - w_j^q) + \tilde{\zeta}Dw_j^q & \text{on } B^+_{N\rho_{\varepsilon_j}} \cap \Omega^+ \\ Dw_j^- & \text{on } B^-_{N\rho_{\varepsilon_j}} \cap \Omega^- \end{cases}.$$

By the standard growth conditions (3.2), (3.3) and Hölder's inequality, we have

$$
\begin{aligned}
& \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} W_p(Du_j)\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} W_q(Du_j)\,dx \\
\leq\ & c\Big(|B_{N\rho_{\varepsilon_j}}| + \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} |Du_j|^p\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} |Du_j|^q\,dx\Big) \\
\leq\ & c\Big(|B_{N\rho_{\varepsilon_j}}| + \frac{1}{\rho^p_{\varepsilon_j}} \int_{B^+_{N\rho_{\varepsilon_j}}(x^\varepsilon_i,0)\cap\Omega^+} |D\zeta|^p(|w^+_j|^p + |w^q_j|^p)\,dx \\
& + \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} |Dw^+_j|^p\,dx + \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} |Dw^q_j|^p\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} |Dw^-_j|^q\,dx\Big) \\
\leq\ & c\Big(|B_{N\rho_{\varepsilon_j}}| + M^p\rho^{n-p}_{\varepsilon_j} \int_{B^+_N(0)} |D\zeta|^p\,dy + \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} |Dw^+_j|^p\,dx \\
& + \Big(\int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} |Dw^-_j|^q\,dx\Big)^{p/q} |B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-|^{(q-p)/q} + \int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} |Dw^-_j|^q\,dx\Big),
\end{aligned}
$$

where we have also taken into account that $\|w^+_j\|_{L^\infty} + \|w^q_j\|_{L^\infty} \leq 2\,M$.

Let $\omega'_j = \bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} Q^\varepsilon_{i,n-1}$, since

$$
\lim_{j\to+\infty}\Big|\bigcup_{i\in\mathbb{Z}^{n-1}\setminus Z_j} B_{N\rho_{\varepsilon_j}}\cap\Omega\Big| = 0
$$

by the equi-integrability of $|Dw^+_j|^p$ and $|Dw^-_j|^q$ (see Remark 3.5), we get

$$
\begin{aligned}
& \limsup_{j\to+\infty}\Big(\sum_{i\in\mathbb{Z}^{n-1}\setminus Z_j} \int_{B^+_{N\rho_{\varepsilon_j}}\cap\Omega^+} W_p(Du_j)\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}\cap\Omega^-} W_q(Du_j)\,dx\Big) \\
\leq\ & c\lim_{j\to+\infty}\Big(\frac{\rho^{n-p}_{\varepsilon_j}}{\varepsilon^{n-1}_j}\Big)\lim_{j\to+\infty}\mathcal{H}^{n-1}(\omega'_j) \leq c\,R\,\mathcal{H}^{n-1}(\partial\omega) = 0\,. \qquad (6.6)
\end{aligned}
$$

Taking (6.5) and (6.6) into account, by Proposition 4.4, we have

$$
\begin{aligned}
& \limsup_{j\to+\infty}\Big(\sum_{i\in\mathbb{Z}^{n-1}} \int_{B^+_{N\rho_{\varepsilon_j}}(x^\varepsilon_i,0)\cap\Omega^+} W_p(Du_j)\,dx + \int_{B^-_{N\rho_{\varepsilon_j}}(x^\varepsilon_i,0)\cap\Omega^-} W_q(Du_j)\,dx\Big) \\
\leq\ & R\,\Big(\limsup_{j\to+\infty}\sum_{i\in Z_j}\varepsilon^{n-1}_j\varphi(v^{i+}_j - v^{i-}_j) + \eta\,\mathcal{H}^{n-1}(\omega)\Big) \\
=\ & R\int_\omega \varphi(u^+ - u^-)\,dx_\alpha + \eta\,\mathcal{H}^{n-1}(\omega)\,.
\end{aligned}
$$

We conclude the proof of the limsup inequality for arbitrary $(u^+, u^-) \in W^{1,p}(\Omega^+;\mathbb{R}^m)\times W^{1,q}(\Omega^-;\mathbb{R}^m)$, simply noting that u^+ can be approximated by a sequence of functions $v_j^+ \in W^{1,p}(\Omega^+;\mathbb{R}^m) \cap L^\infty(\Omega^+;\mathbb{R}^m)$ and u^- by $v_j^- \in W^{1,q}(\Omega^-;\mathbb{R}^m) \cap L^\infty(\Omega^-;\mathbb{R}^m)$ with respect to the strong convergence of $W^{1,p}$ and $W^{1,q}$, respectively. □

7 The case p=q

If $q = p$ we consider a Borel function U_p satisfying a growth condition of order p in place of W_q. In this case we are in a simpler situation since W_p and U_p are rescaled in the same way and we get

$$\begin{aligned}\varphi_{N,j}(z) &= \inf\Big\{\int_{B_N^+(0)} \rho_{\varepsilon_j}^p W_p(\rho_{\varepsilon_j}^{-1} D\zeta)\, dy + \int_{B_N^-(0)} \rho_{\varepsilon_j}^p U_p(\rho_{\varepsilon_j}^{-1} D\zeta)\, dy : \\ &\qquad \zeta \in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m) \quad \zeta = z \text{ on } \partial B_N^+(0),\ \zeta = 0 \text{ on } \partial B_N^-(0)\Big\}.\end{aligned}$$

Reasoning as in Section 4 the limit problem keeps the same boundary conditions on the test function ζ as $j \to +\infty$

$$\begin{aligned}\varphi_N(z) &= \inf\Big\{\int_{B_N^+(0)} \widehat{W}_p(D\zeta)\, dy + \int_{B_N^-(0)} \widehat{U}_p(D\zeta)\, dy : \\ &\qquad \zeta \in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m),\ \zeta = z \text{ on } \partial B_N^+(0), \quad \zeta = 0 \text{ on } \partial B_N^-(0)\Big\} \\ &= \inf\Big\{\int_{B_N^+(0)} \widehat{W}_p(D\zeta)\, dy + \int_{B_N^-(0)} \widehat{U}_p(D\zeta)\, dy : \\ &\qquad \zeta \in W^{1,p}(B_N(0)\setminus C_{1,N};\mathbb{R}^m),\ \zeta = \frac{z}{2} \text{ on } \partial B_N^+(0), \quad \zeta = -\frac{z}{2} \text{ on } \partial B_N^-(0)\Big\};\end{aligned}$$

hence, passing to the limit as $N \to +\infty$, we get that

$$\begin{aligned}\varphi(z) &= \inf\Big\{\int_{\mathbb{R}^n_+} \widehat{W}_p(D\zeta)\, dx + \int_{\mathbb{R}^n_-} \widehat{U}_p(D\zeta)\, dx :\ \zeta \in W^{1,p}(\mathbb{R}^n_{+,-} \cup B_1^{n-1}(0);\mathbb{R}^m) \\ &\qquad \zeta - \frac{z}{2} \in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m),\ \zeta + \frac{z}{2} \in W^{1,p}(\mathbb{R}^n_-;\mathbb{R}^m)\Big\}\end{aligned}$$

where $\mathbb{R}^n_{+,-} = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$. After having precise the definition of function φ, the proof of Theorem 3.2, for $q = p$, follows as in Sections 5 and 6.

8 Thin films connected by a periodically perforated interface

Let us consider the following domain (see Figure 3)

$$\begin{aligned}\Omega_{\varepsilon_j} &= \Big(\omega \times (-\varepsilon_j, 0)\Big) \cup \Big(\omega \times (0, \varepsilon_j)\Big) \cup \Big(\omega_{\varepsilon_j} \times \{0\}\Big) \\ &=: \Omega_{\varepsilon_j}^- \cup \Omega_{\varepsilon_j}^+ \cup \Big(\omega_{\varepsilon_j} \times \{0\}\Big). \end{aligned} \tag{8.1}$$

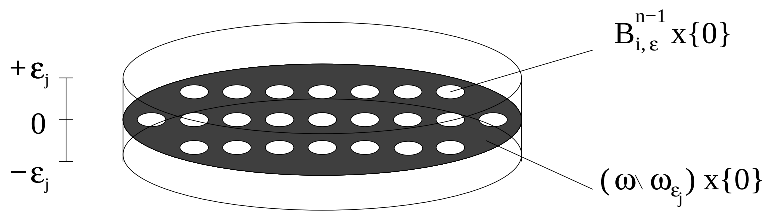

Figure 3: The domain Ω_{ε_j}

In analogy with the notation introduced in Section 2, we denote $\Omega = \omega \times (-1,1)$, $\Omega^+ = \omega \times (0,1)$ and $\Omega^- = \omega \times (-1,0)$.

Definition 8.1 *Let*

$$V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m) = W^{1,p}(\Omega_{\varepsilon_j};\mathbb{R}^m) \cap W^{1,q}(\Omega^-_{\varepsilon_j};\mathbb{R}^m)\,.$$

Given a sequence $(u_j) \in V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m)$, we define $\hat{u}_j(x_\alpha, x_n) = u_j(x_\alpha, \varepsilon_j x_n)$. We say that (u_j) converges to (or converges weakly to) $(u^+, u^-) \in W^{1,p}(\omega;\mathbb{R}^m) \times W^{1,q}(\omega;\mathbb{R}^m)$ if we have

$$\begin{aligned} \hat{u}^+_j &= \hat{u}_{j|\Omega^+} \to u^+ \ \textit{in}\ L^p(\Omega^+;\mathbb{R}^m) \qquad (\textit{or weakly in}\ W^{1,p}(\Omega^+;\mathbb{R}^m)) & (8.2)\\ \hat{u}^-_j &= \hat{u}_{j|\Omega^-} \to u^- \ \textit{in}\ L^q(\Omega^-;\mathbb{R}^m) \qquad (\textit{or weakly in}\ W^{1,q}(\Omega^-;\mathbb{R}^m))\,. & (8.3)\end{aligned}$$

Equivalently: we can define the $2\varepsilon_j$-periodic (in x_n) extensions of $u^\pm_j = u_{j|\Omega^\pm_{\varepsilon_j}}$ as the functions $\tilde{u}^\pm_j$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $W^{1,q}(\Omega;\mathbb{R}^m)$, respectively; such that

$$\tilde{u}^\pm_j(x_\alpha, -x_n) = \tilde{u}^\pm_j(x_\alpha, x_n)$$

and

$$\tilde{u}^\pm_j(x) = u_j(x) \ \textit{on} \quad \Omega^\pm_{\varepsilon_j}\,.$$

Then (8.2) and (8.3) above are equivalent to

$$\tilde{u}^\pm_j \to u^\pm$$

in $L^p(\Omega;\mathbb{R}^m)$ and $L^q(\Omega;\mathbb{R}^m)$, respectively (or weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $W^{1,q}(\Omega;\mathbb{R}^m)$, respectively).

If $v \in L^p(\omega;\mathbb{R}^m)$ we identify it with $v \in L^p(\Omega;\mathbb{R}^m)$ (independent of x_n), similarly for the other spaces L^q, $W^{1,p}$ and $W^{1,q}$.

We prove the following result for thin films with periodically perforated interface in the case $p < q$; $q = p$ can be treated as in Section 7.

Theorem 8.2 *Let* (ε_j) *and* (ρ_{ε_j}) *be sequences of strictly positive numbers converging to* 0 *such that*

$$0 < \lim_{j \to +\infty} \frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^n} = T < +\infty\,.$$

Let ω *be a bounded open subset of* $\mathbb{R}^{n-1}$ *with Lipschitz boundary and let* $\Omega^+_{\varepsilon_j}$, $\Omega^-_{\varepsilon_j}$ *and* Ω_{ε_j} *be defined as in* (8.1). *Let* $1 < p < q$ *and let* $W_p, W_q : \mathbb{M}^{m\times n} \mapsto [0, +\infty)$ *be Borel functions satisfying a growth condition of order* p *and* q, *respectively: there exists a constant* $c_1 > 0$ *such that*

$$|A|^p - 1 \le W_p(A) \le c_1(1 + |A|^p) \tag{8.4}$$

and there exists a constant $c_2 > 0$ *such that*

$$|A|^q - 1 \le W_q(A) \le c_2(1 + |A|^q) \tag{8.5}$$

for all $A \in \mathbb{M}^{m\times n}$. *Then, upon possibly extracting a subsequence, for all* $A \in \mathbb{M}^{m\times n}$ *there exists the limit*

$$\widehat{W}_p(A) = \lim_j \rho^p_{\varepsilon_j} QW_p\Big(\rho^{-1}_{\varepsilon_j} A\Big), \tag{8.6}$$

where QW_p *denotes the quasiconvexification of* W_p, *so that the value*

$$\varphi(z) = \inf\Big\{\int_{\mathbb{R}^n_+} \widehat{W}_p(D\zeta)\,dx : \zeta - z \in W^{1,p}(\mathbb{R}^n_+;\mathbb{R}^m), \quad \zeta = 0 \ \textit{on}\ B_1^{n-1}(0)\Big\} \tag{8.7}$$

is well defined for all $z \in \mathbb{R}^m$. *Moreover, the functionals defined by*

$$F_j(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\Big(\displaystyle\int_{\Omega^+_{\varepsilon_j}} W_p(Du)\,dx + \int_{\Omega^-_{\varepsilon_j}} W_q(Du)\,dx\Big) & \textit{if } u \in V^{p,q}(\Omega_{\varepsilon_j};\mathbb{R}^m) \\ +\infty & \textit{otherwise} \end{cases}$$

Γ*-converge to the functional defined by*

$$F(u^+, u^-) = \int_\omega \widetilde{W}_p(D_\alpha u^+)\,dx_\alpha + \int_\omega \widetilde{W}_q(D_\alpha u^-)\,dx_\alpha + T\int_\omega \varphi(u^+ - u^-)\,dx_\alpha$$

on $W^{1,p}(\omega;\mathbb{R}^m) \times W^{1,q}(\omega;\mathbb{R}^m)$ *with respect to the convergence introduced in Definition* 8.1. *The functions* $\widetilde{W}_p$ *and* $\widetilde{W}_q$ *are given by*

$$\widetilde{W}_p(\overline{F}) = Q_{n-1}\overline{W}_p(\overline{F})\,, \quad \widetilde{W}_q(\overline{F}) = Q_{n-1}\overline{W}_q(\overline{F})\,,$$

for all $\overline{F} \in \mathbb{M}^{m\times n-1}$, *where*

$$\overline{W}_p(\overline{F}) = \inf_{F_n} W_p(\overline{F}, F_n)\,, \quad \overline{W}_q(\overline{F}) = \inf_{F_n} W_q(\overline{F}, F_n)\,,$$

and Q_{n-1} *denotes the operation of* $(n-1)$*-quasiconvexification.*

Proof. To prove the theorem we can follow the lines of the proof of Theorem 3.2. In fact, among the hypothesis of Lemma 3.4, we have that, fixed $N \in \mathbb{N}$, $N\rho_{\varepsilon_j} < \varepsilon_j/2$; hence,

$$\bigcup_{i\in Z_j} C_i^j \cap \{\pm x_n > 0\} \subset \Omega_\varepsilon^\pm$$

where C_i^j are defined in (3.6).

Therefore we can repeat the proof of Lemma 3.4 with $\Omega_{\varepsilon_j}^+ \cup \Omega_{\varepsilon_j}^-$ in place of $\Omega^+ \cup \Omega^-$ and with respect to the convergence introduced in Definition 8.1.

The fact that the thickness ε_j tends to zero, as $j \to +\infty$, does not influence the contribution near the balls $B_{i,\varepsilon}^{n-1}$, except that in the determination of the critical size of the perforations that, in this case, is of order $\varepsilon_j^{n/(n-p)}$. In fact, let us deal with the liminf inequality: reasoning as in Section 5, we get the analog of (5.6) for the contribution on $E_j^+ \cup E_j^-$; *i.e.*,

$$\begin{aligned}
&\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\Big(\int_{E_j^+} W_p(Dw_j)\,dx + \int_{E_j^-} W_q(Dw_j)\,dx\Big)\\
\geq \quad &\lim_{j\to+\infty}\Big(\frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^n}\Big)\liminf_{j\to+\infty}\sum_{i\in Z_j}\varepsilon_j^n\,\varphi_{N,j}(u_j^{i+} - u_j^{i-})\,,
\end{aligned}$$

where (w_j) is defined by the Lemma suitably modified for the case of thin films.

There follows that we have to choose ρ_{ε_j} such that

$$0 < \lim_{j\to+\infty}\frac{\rho_{\varepsilon_j}^{n-p}}{\varepsilon_j^n} = T < +\infty\,,$$

but all the rest is unchanged and it gives rise to the same function φ defined in (8.7). To conclude the proof of the liminf inequality we estimate the contribution far from the balls $B_{i,\varepsilon}^{n-1}$ applying the following Γ-convergence result due to Le Dret-Raoult [18]; *i.e.*, the sequence of functionals

$$F_j^+(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\displaystyle\int_{\Omega_{\varepsilon_j}^+} W_p(Du)\,dx & \text{if } u \in W^{1,p}(\Omega_{\varepsilon_j}^+;\mathbb{R}^m)\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converges, with respect to the $L^p(\Omega^+;\mathbb{R}^m)$ convergence, to

$$F^+(u^+) = \begin{cases} \displaystyle\int_\omega \widetilde{W}_p(D_\alpha u^+)\,dx_\alpha & \text{if } u^+ \in W^{1,p}(\omega;\mathbb{R}^m)\\ +\infty & \text{otherwise}\ ,\end{cases} \tag{8.8}$$

and, similarly,

$$F_j^-(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\displaystyle\int_{\Omega_{\varepsilon_j}^-} W_q(Du)\,dx & \text{if } u \in W^{1,q}(\Omega_{\varepsilon_j}^-;\mathbb{R}^m)\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converges, with respect to the $L^q(\Omega^-;\mathbb{R}^m)$ convergence, to

$$F^-(u^-) = \begin{cases} \displaystyle\int_\omega \widetilde{W}_q(D_\alpha u^-)\,dx_\alpha & \text{if } u^- \in W^{1,q}(\omega;\mathbb{R}^m) \\ +\infty & \text{otherwise}. \end{cases} \tag{8.9}$$

Also for the limsup inequality we can repeat the proof of Proposition 6.1 but in this case we do not apply Lemma 3.4 to the sequence (v), defined in (6.1), but to the sequence

$$v_j = v_j^+ \chi_{\Omega_\varepsilon^+} + v_j^- \chi_{\Omega_\varepsilon^-}$$

where $(v_j^+), (v_j^-)$ are the recovery sequence for the Γ-limits (8.8) and (8.9). $\square$

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