THE FLOW ASSOCIATED TO WEAKLY DIFFERENTIABLE VECTOR FIELDS: RECENT RESULTS AND OPEN PROBLEMS

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Abstract. We illustrate some recent developments of the theory of flows associated to weakly differentiable vector fields, listing the regularity/structural conditions considered so far, extensions to state spaces more general than Euclidean and open problems.

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1. Introduction. In the last few years quite some progress has been made on the well-posedness of the continuity and transport equation

$$\partial_t w_t + \nabla \cdot (\boldsymbol{b}_t w_t) = 0, \tag{1.1}$$

$$\partial_t f_t + \boldsymbol{b}_t \cdot \nabla f_t = 0 \tag{1.2}$$

and the relation of these well-posedness results with the existence and stability of the flow $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}$, namely the family of solutions to

$$\dot{x}(t) = \boldsymbol{b}_t(x(t)) \qquad \text{for a.e. } t \in (0,T). \tag{1.3}$$

Here $\boldsymbol{b}(t,x) = \boldsymbol{b}_t(x) : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ is a possibly nonautonomous Borel velocity field in $\mathbb{R}^d$. In order to fix the ideas and to avoid global issues, I will make the standing assumption that $\boldsymbol{b}$ is globally bounded and I will focus, in the same spirit of [4], on the continuity equation (1.1) only, departing a bit from the seminal paper [42]: the conservative form is more amenable for nonsmooth vector fields, possibly having also an unbounded divergence (and in the case of bounded divergence there is no essential difference between (1.1) and (1.2), provided we allow right hand sides of the form cw_t, cf_t).

Let us recall the definition of Regular Lagrangian Flow (RLF in short) associated to $\boldsymbol{b}$:

DEFINITION 1.1 ($\mathscr{L}^d$-RLF in $\mathbb{R}^d$). *Let $\boldsymbol{X}(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. We say that $\boldsymbol{X}(t,x)$ is a $\mathscr{L}^d$-RLF in $\mathbb{R}^d$ (relative to $\boldsymbol{b}$) if the following two conditions are fulfilled:*

(i) for $\mathscr{L}^d$-a.e. x, the function $t \mapsto \boldsymbol{X}(t,x)$ is an absolutely continuous integral solution to the ODE (1.3) in $[0,T]$ with $\boldsymbol{X}(0,x) = x$;

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(ii) $\boldsymbol{X}(t,\cdot)_\sharp\mathscr{L}^d \le C\mathscr{L}^d$ *for all* $t \in [0,T]$*, for some constant* C *independent of* t*.*

Here and in the sequel I use the notation $\mathscr{L}^d$ for the Lebesgue measure in $\mathbb{R}^d$ and I use $f_\sharp$ to denote the push forward operator between measures induced by f, namely $f_\sharp\mu(E) = \mu(f^{-1}(E))$.

Notice that, while (i) imposes a natural condition on the *individual* paths $\boldsymbol{X}(\cdot,x)$, (ii) should be understood as a *global* condition on this family of paths: heuristically it means that we are selecting paths which do not concentrate too much, and we don't rule out the possibility of the existence of concentrating paths (see in particular the illustration of the square root example in [4]). The following result is proved in [4, Theorem 19] for the part concerning existence and in [4, Theorem 16, Remark 17] for the part concerning uniqueness.

THEOREM 1.1 (Existence and uniqueness of the $\mathscr{L}^d$-RLF). *Assume that* (1.1) *has (forward) existence and uniqueness in* $L^\infty_+\bigl([0,T];L^1\cap L^\infty(\mathbb{R}^d)\bigr)$*. Then the* $\mathscr{L}^d$*-RLF* $\boldsymbol{X}$ *exists and is unique.*

Here uniqueness is understood in the following sense: if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathscr{L}^d$-RLF's, then $\boldsymbol{X}(\cdot,x) = \boldsymbol{Y}(\cdot,x)$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.

In the next sections I will address several questions relative to Theorem 1.1 and some recent progress made in this area. The main question is: which assumptions on $\boldsymbol{b}$ ensure that (1.1) has existence and uniqueness in $L^\infty_+\bigl([0,T];L^1\cap L^\infty(\mathbb{R}^d)\bigr)$?

2. Regularity of the vector field. DiPerna and Lions proved in [42], among many other things, that (1.1) has existence and uniqueness in the class

$$L^\infty\bigl([0,T];L^1\cap L^\infty(\mathbb{R}^d)\bigr)$$

provided $\boldsymbol{b}_t \in W^{1,1}_{\rm loc}$ for a.e. $t \in (0,T)$, with:

(i) $\int_{B_R} |\nabla\boldsymbol{b}_t|\,dx$ integrable in $(0,T)$ for all $R > 0$;

(ii) $\|\nabla\cdot\boldsymbol{b}_t\|_\infty \in L^1(0,T)$.

Actually an inspection of the proof in [42] (or the stability under time reversal of the assumptions) shows that (ii) ensures both forward and backward uniqueness. On the other hand, bounds on the negative part of the divergence suffice to obtain forward well-posedness, and it was pointed out in [2] that only forward uniqueness is needed for the uniqueness of the (forward) $\mathscr{L}^d$-RLF $\boldsymbol{X}$. Also, existence and uniqueness in the smaller class $L^\infty_+\bigl([0,T];L^1\cap L^\infty(\mathbb{R}^d)\bigr)$ of nonnegative solutions is sufficient to that purpose. We shall comment more on the bounds on divergence in the next section, and focus here on the regularity of $\boldsymbol{b}$:

• (vector fields in LD) It was noticed in [26] that the *isotropic* smoothing scheme of [42], on which the uniqueness proof relies, works under the only assumption that the symmetric part $Du + {}^tDu$ of the distributional derivative is absolutely continuous. This vector space, usually denoted by LD in the theory of linear elasticity [60], can be strictly larger than $W^{1,1}$. Notice

however that $Du + {}^t Du \in L^p_{\rm loc}$ for some $p > 1$ implies $u \in W^{1,p}_{\rm loc}$ by a local version of Korn's inequality.
• (vector fields in BV) Bouchut has been the first one to achieve in [18] this extension, for Hamiltonian vector fields of the form $\boldsymbol{b}_t(x,p) = (p, V_t(x))$, with $V_t \in BV_{\rm loc}$. The proof uses a clever *anisotropic* smoothing scheme, where mollification in the "bad" variables x occurs at a faster rate. In [2] (see also [30], [29] for intermediate results) the scheme has been improved and extended to all $BV_{\rm loc}$ vector fields (with global integrability in time of the total variations $|D\boldsymbol{b}_t|(B_R)$ for all $R > 0$), under the assumptions that $D \cdot \boldsymbol{b}_t \ll \mathscr{L}^d$ and that the density ${\rm div}\boldsymbol{b}_t$ satisfies

$$\|[{\rm div}\boldsymbol{b}_t]^-\|_\infty \in L^1(0,T). \tag{2.1}$$

• (vector fields in BD) Recall that BD consists in the space of functions u such that $(Du + {}^t Du)$ are representable by measures. The extension to BD vector fields is still an open problem: indeed, one is tempted to use symmetric mollifiers as in [26]. But, we know that even for BV vector fields anisotropic mollifiers are needed to get the result. In [6] we follow a different path and we achieve the result for SBD vector fields $\boldsymbol{b}_t$ (namely we need that $(D\boldsymbol{b}_t + {}^t D\boldsymbol{b}_t)$ has no "Cantor" part).
• (vector fields representable as singular integrals) Recently Bouchut and Crippa achieved in [24] a very nice extension of the theory to vector fields $\boldsymbol{b}_t$ that can be represented as a singular integral

$$\boldsymbol{b}_t(x) = \int_{\mathbb{R}^d} K(x-y) F_t(y)\, dy \tag{2.2}$$

with $F_t \in [L^1(\mathbb{R}^d)]^d$. Here K is a matrix-valued map whose components satisfy the standard assumption of the theory of singular integral operators (in particular $|K(x)| \sim |x|^{1-d}$ as $|x| \to 0$), so that weak L^1 estimates are available. The proof uses a very clever improvement of the maximal estimates used in [10] and [33] (see Section 4 below) to obtain regularity properties of the flow and effective stability estimates: the main new ingredient is the use of a suitable family of maximal operators, instead of the standard maximal operator on balls.
Notice that the class (2.2) includes $W^{1,1}$ functions g (and vector fields), because the solution to $\Delta w = \nabla \cdot F$ is representable by (with appropriate boundary conditions)

$$w(x) = c(d) \int_{\mathbb{R}^d} |y-x|^{2-d} \nabla \cdot F(y)\, dy = c(d)(d-2) \int_{\mathbb{R}^d} \frac{y-x}{|y-x|^d} F(y)\, dy.$$

Choosing $F = \nabla g$ gives $w = g$ and hence the desired representation of g.
It is also easily seen that the class of vector fields (2.2) is not contained in $W^{1,1}$ or in BV, so that definitely [24] provides new results (and also new applications to stability of solutions to incompressible Euler equations with vorticity in L^1). In this direction, it would be very nice to have an extension

of this result to the case when F_t is just a measure, and not necessarily an L^1 function. This would include BV and even BD vectorfields, by the same elliptic PDE argument illustrated above.

• (vector fields having a special structure) In this research field it is difficult to imagine a result better than the others. Indeed, sometimes the special structure of the vector field of $\boldsymbol{b}$ helps a lot in getting well-posedness, under very mild regularity conditions. I will illustrate this by two examples.

The first example concerns bounded, divergence-free autonomous vector fields $\boldsymbol{b}$ in the plane; they can obviously can be represented as rotations of ∇H, for some Lipschitz potential function H, and solutions to the ODE should preserve H. This suggests a factorization of the dynamics on the level sets of H. This argument has been used by Bouchut and Desvillettes in [22] (see also [28], [45] for related results) to obtain well-posedness under an additional local "regularity" assumption on $\boldsymbol{b}$, and the assumption that H maps the critical set $\Sigma := \{\nabla H = 0\}$ into a $\mathscr{L}^1$-negligible set (notice that by Sard's theorem this condition always holds if $\boldsymbol{b}$ is smooth). A much more detailed analysis, performed by Alberti, Bianchini and Crippa in [1], reveals that no extra regularity of $\boldsymbol{b}$ is needed, and that the "weak Sard" condition

$$H_\sharp(\chi_\Sigma \mathscr{L}^2) \ll \mathscr{L}^1$$

suffices for well-posedness of (1.1). Also, it turns out that a further refinement of this condition, involving also the topology of the level sets of H, is *necessary and sufficient* for well-posedness.

The second example concerns vector fields $\boldsymbol{B}_t$ of the form

$$\boldsymbol{B}_t(x,y) := \big(\boldsymbol{b}_t(x), \nabla \boldsymbol{b}_t(x) y\big) \tag{2.3}$$

with $\boldsymbol{b}_t \in W^{1,1}_{\rm loc}$. Here we see that the last d components of $\boldsymbol{B}_t$ have no regularity in x, but they are very regular in y (on the other hand the divergence of $\boldsymbol{B}_t$ is nice as long as the divergence of $\boldsymbol{b}_t$ is nice, since $\nabla\cdot\boldsymbol{B}_t = 2\nabla\cdot\boldsymbol{b}_t$). Lions and Le Bris used in [48] this particular structure and adapted Bouchut's scheme [18] to obtain well-posedness of (1.1) (see also [49] for related results in a BV framework). Here the function space where well-posedness occurs is adapted to $\boldsymbol{B}_t$:

$$L^\infty_+\big([0,T]; L^1\cap L^\infty(\mathbb{R}^{2d})\big) \cap L^\infty\left([0,T]; L^\infty_{\rm loc}(\mathbb{R}^d_x; L^1(\mathbb{R}^d_y))\right).$$

The reason for the restriction to this smaller space is the fact that $|\boldsymbol{B}_t|/(1+|x|+|y|)$ in general does not belong to

$$L^1\left([0,T]; L^1(\mathbb{R}^d_x\times\mathbb{R}^d_y)\right) + L^1\left([0,T]; L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_y)\right),$$

because the last d components do not tend to 0 as $|y|\to\infty$ while x is kept fixed (and their limit is possibly unbounded as a function of x). For

this reason, a weaker growth condition on $\boldsymbol{B}_t$ turns into a stronger growth condition on w, still compatible with existence of solutions.
As we will see later on, the vector fields (2.3) occur in the study of the differentiability properties of the $\mathscr{L}^d$-RLF flow $\boldsymbol{X}$ associated to $\boldsymbol{b}_t$.

3. Bounds on divergence. The bound (2.1) on the divergence might be too restrictive for some applications, where even a singular divergence might appear. As an application, let us consider the multidimensional version of the Keyfitz-Kranzer system considered in [25], [7], [5]:

$$\partial_t u + \sum_{i=1}^{d} \partial_i \bigl(\boldsymbol{f}(|u|)u\bigr) = 0$$

with $u : (0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^k$ and $\boldsymbol{f} : \mathbb{R}_+ \to \mathbb{R}^k$ smooth. The system can formally be decoupled into a scalar conservation law and a transport equation, in the polar variables $u = \rho\theta$, $\rho = |u|$:

$$\partial_t \rho + \nabla \cdot \bigl(\boldsymbol{f}(\rho)\rho\bigr) = 0, \qquad \partial_t \theta + \boldsymbol{f}(\rho) \cdot \nabla\theta = 0.$$

If the initial condition $\bar{\rho} = |\bar{u}|$ is sufficiently nice, say $BV_{\rm loc}$ and L^∞, then Kruzkhov theory provides us with the unique entropy solution $t \mapsto \rho_t$ of the scalar conservation law, and this solution is locally BV on bounded sets of $(0, +\infty) \times \mathbb{R}^d$. The vector field $\boldsymbol{b}_t := \boldsymbol{f}(\rho_t)$ appearing in the transport equation for θ is bounded and BV, but its distributional divergence need not be bounded or even absolutely continuous (for instance this can be seen by computing the divergence on shocks, where ρ is discontinuous, if $\boldsymbol{f}$ is injective). In [7], [5] this difficulty has been bypassed by considering the autonomous, divergence-free and $(d + 1)$-dimensional vector field

$$\boldsymbol{B}(t, x) := \bigl(\rho_t(x), \boldsymbol{f}(\rho_t(x))\rho_t(x)\bigr)$$

and building from the flow of $\boldsymbol{B}$ a "natural" flow of $\boldsymbol{b}_t$ (by a reparameterization).
More generally, we may think that whenever we have a nonnegative function ρ satisfying

$$\partial_t \rho_t + \nabla \cdot (\boldsymbol{b}_t \rho_t) = 0 \tag{3.1}$$

then we should think to $\rho\mathscr{L}^d$ as our new reference measure and try to obtain well-posedness, at least in the regions where ρ does not vanish. This is particularly clear if ρ is independent of time: in this case $\nabla \cdot (\boldsymbol{b}_t \rho) = 0$ precisely corresponds to the fact that the ρ-divergence of $\boldsymbol{b}_t$, namely the $L^2(\rho)$ adjoint of the gradient, vanishes.
This point of view has been used in [8] (and more recently in [13]). The uniqueness of the flow and the well-posedness of the PDE when $\boldsymbol{b}_t \in BV$ and the function ρ in (3.1) belongs to L^∞ are still open problems: the main result of [8] is to answer these questions affirmatively when $\boldsymbol{b}_t \in SBV$, and

to relate this problem to a compactness conjecture of Bressan [25]: we prove that if the limit field in Bressan's conjecture is BV, then compactness hold (see also [32] for a positive answer to the conjecture under $W^{1,p}$ bounds, $p > 1$).

4. Differentiability of the flow and effective stability results. In this section we start from the discussion of the differentiability properties of the $\mathscr{L}^d$-RLF associated to $W^{1,1}$ vector fields, briefly describing the results obtained in [48]. On the other hand, in the $W^{1,p}$ case, with $p > 1$, much stronger results are available [33], as we will see. The main theorem in [48] is the following:

THEOREM 4.1. *Let* $\boldsymbol{b} \in L^1\left((0,T); W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\right)$ *be satisfying*

(i) $\dfrac{|\boldsymbol{b}|}{1+|x|} \in L^1\left((0,T); L^1(\mathbb{R}^d)\right) + L^1\left((0,T); L^\infty(\mathbb{R}^d)\right)$;

(ii) $[\mathrm{div}\boldsymbol{b}_t] \in L^1\left((0,T); L^\infty(\mathbb{R}^d)\right)$;

and let $\boldsymbol{X}(t,x)$ *be the corresponding* $\mathscr{L}^d$*-Lagrangian flow. Then for a.e.* $t \in (0,T)$ *there exists a measurable function* $\boldsymbol{W}_t : \mathbb{R}^d \to M^{d\times d}$ *such that*

$$\frac{\boldsymbol{X}(t,x+\varepsilon y) - \boldsymbol{X}(t,x)}{\varepsilon} \to \boldsymbol{W}_t(x)y \quad \textit{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y \tag{4.1}$$

as $\varepsilon \downarrow 0$*, uniformly in time.*

Actually in [48] it is not proved, or stated, that the limit of difference quotients is linear in y, but this can be proved by an approximation argument, using the stability properties of flows: indeed, it turns out that the difference quotients of $\boldsymbol{X}$, together with $\boldsymbol{X}$, are solutions to the ODE relative to

$$B^\varepsilon_t(x,y) := \big(\boldsymbol{b}_t(x), \frac{\boldsymbol{b}_t(x+\varepsilon y) - \boldsymbol{b}_t(x)}{\varepsilon}\big)$$

whose limit is precisely the vector field $\boldsymbol{B}_t$ in (2.3). Hence, $\boldsymbol{W}_t(x)y$ can be recovered from the flow of $\boldsymbol{B}_t$ and it is clear that the linear structure is preserved by a smooth approximation of $\boldsymbol{b}_t$ (and of $\boldsymbol{B}_t$ as well). In [12] we called $\boldsymbol{W}_t(x)$, as defined by (4.1), "derivative in measure": though strictly weaker than many other differentiability concept (even approximate differentiability, as proved in [12]), (4.1) is the only differentiability property known in the $W^{1,1}$ case, and unknown in the BV case.

On the other hand, in the $W^{1,p}$ case, $p > 1$, stronger results are available. The first results relative to the approximative differentiability of the flow have been obtained in [10], and then improved substantially in [33]. An important fact is that the approach is completely different from the one of [48]: the main idea is to estimate the difference quotient of $\boldsymbol{b}_t$ in terms of maximal functions of the modulus of $\nabla\boldsymbol{b}_t$. More precisely, for all $g \in W^{1,1}_{\rm loc}$ we have the *pointwise* inequality

$$|g(x) - g(y)| \le c_d|x-y|\left(M|\nabla g|(x) + M|\nabla g|(y)\right)$$

at all Lebesgue points x, y of g. The heuristic idea is to use this, in conjunction with maximal inequalities and the no-concentration property ((ii) in Definition 1.1) to get regularity properties of the flow.
The starting point of the estimates given in [33] is already present in [10] (and, at least in a formal way, in the introduction of [42]): in a smooth context, since we can differentiate in space the ODE to get

$$\frac{d}{dt}\nabla \boldsymbol{X}(t,x) = \nabla \boldsymbol{b}_t\big(\boldsymbol{X}(t,x)\big)\nabla \boldsymbol{X}(t,x)$$

(i.e. the spatial gradient satisfies a *linear* ODE), so that we can control from above the time derivative $\frac{d}{dt}\log\left(|\nabla \boldsymbol{X}|\right)$ with $|\nabla \boldsymbol{b}_t|(\boldsymbol{X})$. The latter quantity belongs to L^p thanks to the regularity of the flow. The strategy of [10] allows to make this remark rigorous: it is possible to consider some integral quantities which contain a discretization of the space gradient of the flow, more stable by approximation (by the concavity of the logarithm, which results in a lack of lower semicontinuity, there is no way to pass to the limit in the differential inequality, as stated above, from smooth to nonsmooth flows).
Now we state simplified versions of the results of [33], referring to that paper for the most general statements.
THEOREM 4.2 (Lipschitz estimates). *Let $p > 1$ and let $\boldsymbol{b}_t \in W^{1,p}_{\rm loc}$ be divergence-free, uniformly bounded, with $\int_0^T \int_{B_R} |\nabla \boldsymbol{b}_t|^p \, dxdt < \infty$ for all $R > 0$. Then, for every ε, $R > 0$, we can find a compact set $K \subset B_R(0)$ such that*

(i) $\mathscr{L}^d(B_R(0) \setminus K) \leq \varepsilon$;

(ii) the restriction of $\boldsymbol{X}$ to $[0,T] \times K$ is Lipschitz continuous.

This result is remarkable: it shows that we can recover somehow the standard Cauchy-Lipschitz theory provided we remove sets of small measure (the optimal statement, not given here, is quantitative). We conclude this section showing another result obtained in [33] with techniques which are very similar to the ones described so far: it provides a logarithmic error estimate, implying in particular an effective stability result for RLF's. Of course this result yields uniqueness of the flow as a consequence, in this respect see also [46].
THEOREM 4.3 (Quantitative stability). *Let $\boldsymbol{b}$ and $\tilde{\boldsymbol{b}}$ be vector fields as in the previous theorem, and let $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ be the respective flows. Then, for every $\tau \in [0,T]$, we have*

$$\|\boldsymbol{X}(\tau,\cdot) - \tilde{\boldsymbol{X}}(\tau,\cdot)\|_{L^1(B_r(0))} \leq C \left|\log\left(\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\|_{L^1([0,\tau]\times B_R(0))}\right)\right|^{-1},$$

where $R - r > 0$ and C depend only on the supremum and Sobolev bounds on $\boldsymbol{b}$ and $\tilde{\boldsymbol{b}}$.

5. Infinite-dimensional spaces. In this final section I will illustrate two examples of extension of the theory to infinite-dimensional spaces. Here the main difficulty is that requiring that a reference measure ($\mathscr{L}^d$ in the Euclidean case) is invariant or quasi-invariant under the flow leads to a severe restriction on the direction of the vector field. This seems to be a limitation in the attempt to apply this theory to PDE's (viewed as ODE's in an infinite-dimensional space): the only result I am aware of is [57]. In [16], see also the announcement [15], in the case when the state space is $\mathscr{P}(\mathbb{R}^d)$ (Borel probability measures in $\mathbb{R}^d$) we have been able to use a weaker regularity condition on the flow; however for the moment our results apply only to the *linear* continuity equation

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{b}_t \mu_t) = 0 \tag{5.1}$$

which, in this abstract perspective, should be seen as a *constant coefficient* ODE in $\mathscr{P}(\mathbb{R}^d)$ (with rough coefficients if $\boldsymbol{b}_t$ is rough).

Flows in Wiener spaces. Let us first introduce, briefly, the structure of Gaussian Wiener space. We consider a separable Banach space X and a nondegenerate, centered, Gaussian measure $\gamma \in \mathscr{P}(X)$. The Cameron-Martin space $H \subset X$ is the vector subspace of all $h \in H$ such that $(\tau_h)_\sharp \gamma \ll \gamma$, where $\tau_h(x) = x + h$. It turns out that $\gamma(H) = 0$ whenever X is infinite-dimensional. The maps

$$R^*(x^*) := \langle x^*, \cdot \rangle_{X^*,X}, \qquad R(f) := \int_X f(x) x \, d\gamma(x)$$

(here the integral is understood in Bochner's sense) provide canonical embeddings of X^* in $L^2(X,\gamma)$ and of $L^2(X,\gamma)$ in X respectively, and it can be proved that, denoting by $\mathscr{H}$ the closure of R^*X^* in $L^2(X,\gamma)$, the restriction of R to $\mathscr{H}$ is injective and $R\mathscr{H} = H$. With this notation we can endow H with the L^2 distance inherited from $\mathscr{H}$ and we have the integration by parts formula (for sufficiently nice f and g)

$$\int_X f \partial_h g \, d\gamma = -\int_X g \partial_h f \, d\gamma + \int_X \hat{h} f g \, d\gamma \qquad \forall h = R\hat{h} \in H. \tag{5.2}$$

This formula corresponds precisely, in the standard (product) Gaussian space $(\mathbb{R}^d, \gamma_d)$ with variance 1 in all coordinates, to

$$\int_{\mathbb{R}^d} f \partial_i g \, d\gamma_d = -\int_{\mathbb{R}^d} g \partial_i f \, d\gamma_d + \int_{\mathbb{R}^d} x_i f g \, d\gamma_d,$$

easy to obtain because γ_d is a constant multiple of $e^{-|x|^2/2}\mathscr{L}^d$.

Using (5.2) it is not hard to define Sobolev spaces $W^{1,p}$ and we may expect that a theory analogous to the finite-dimensional one could be developed for vector fields $\boldsymbol{b}_t : X \to H$ (the restriction on the target of $\boldsymbol{b}_t$ being due to quasi-invariance). This is the goal that we achieve in [14], assuming bounds on the (intrinsic) divergence and on the Hilbert-Schmidt norm of the gradient, now a linear operator from H to H. It is not easy to compare the well-posedness results with those obtained, for instance, in [17] (see also [34], [35], [36], [56]): therein the much weaker operator norm is considered, but the norm is assumed to be *exponentially* integrable. So, the result in [14] seem closer to the finite-dimensional theory and are perfectly consistent with it (changing the reference measure from $\mathscr{L}^d$ to γ_d in the same spirit of Section 3).

Flows in $\mathscr{P}(\mathbb{R}^d)$. As I anticipated, we may view (5.1) as an infinite-dimensional ODE in $\mathscr{P}(\mathbb{R}^d)$ and try to obtain existence and uniqueness results for (5.1) in the same spirit of the finite-dimensional theory, starting from the simple observation that $t \mapsto \delta_{\boldsymbol{X}(t,x)}$ solves (5.1) whenever $t \mapsto \boldsymbol{X}(t,x)$ solves (1.3). We may expect that, if we fix a "good" measure $\boldsymbol{\nu}$ in the space $\mathscr{P}(\mathbb{R}^d)$ of initial data, then existence, uniqueness $\boldsymbol{\nu}$-a.e. and stability hold. Moreover, for $\boldsymbol{\nu}$-a.e. μ, the unique and stable solution of (5.1) starting from μ should be given by

$$\boldsymbol{\mu}(t,\mu) := \int \delta_{\boldsymbol{X}(t,x)}\, d\mu(x) \qquad \forall\, t \in [0,T],\ \mu \in \mathscr{P}(\mathbb{R}^d). \tag{5.3}$$

Let us start with a notation and a definition (I use $\mathscr{M}_+(X)$ for the space of positive finite Borel measures in X). Given a nonnegative σ-finite measure $\boldsymbol{\nu} \in \mathscr{M}_+(\mathscr{P}(\mathbb{R}^d))$, I denote by $\mathbb{E}\boldsymbol{\nu} \in \mathscr{M}_+(\mathbb{R}^d)$ its expectation, namely

$$\int_{\mathbb{R}^d} \phi\, d\mathbb{E}\boldsymbol{\nu} = \int_{\mathscr{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \phi\, d\mu\, d\boldsymbol{\nu}(\mu) \qquad \text{for all } \phi \text{ bounded Borel.}$$

Definition 5.1 (Regular measures in $\mathscr{M}_+(\mathscr{P}(\mathbb{R}^d))$). *We say that $\boldsymbol{\nu} \in \mathscr{M}_+(\mathscr{P}(\mathbb{R}^d))$ is* regular *if $\mathbb{E}\boldsymbol{\nu} \le C\mathscr{L}^d$ for some constant C.*

We now observe that Definition 1.1 has a natural (but not perfect) transposition to flows in $\mathscr{P}(\mathbb{R}^d)$:

Definition 5.2 (Regular Lagrangian flow in $\mathscr{P}(\mathbb{R}^d)$). *Let $\boldsymbol{\mu} : [0,T] \times \mathscr{P}(\mathbb{R}^d) \to \mathscr{P}(\mathbb{R}^d)$ and $\boldsymbol{\nu} \in \mathscr{M}_+(\mathscr{P}(\mathbb{R}^d))$. We say that $\boldsymbol{\mu}$ is a $\boldsymbol{\nu}$-RLF in $\mathscr{P}(\mathbb{R}^d)$ (relative to $\boldsymbol{b}$) if*

- *(i) for $\boldsymbol{\nu}$-a.e. μ, $t \mapsto \mu_t := \boldsymbol{\mu}(t,\mu)$ is continuous from $[0,1]$ to $\mathscr{P}(\mathbb{R}^d)$ with $\boldsymbol{\mu}(0,\mu) = \mu$, $|\boldsymbol{b}| \in L^1_{\rm loc}(\mu_t dt)$ and μ_t solves (5.1) in the sense of distributions;*
- *(ii) $\mathbb{E}(\boldsymbol{\mu}(t,\cdot)_\sharp \boldsymbol{\nu}) \le C\mathscr{L}^d$ for all $t \in [0,T]$, for some constant C independent of t.*

Notice that condition (ii) is weaker than $\boldsymbol{\mu}(t,\cdot)_\sharp\boldsymbol{\nu} \le C\boldsymbol{\nu}$, which would be the analogue of (ii) in Definition 1.1, and it is actually sufficient (at least for the special ODE (5.1) in $\mathscr{P}(\mathbb{R}^d)$) and much more flexible for many purposes.

Theorem 5.1 (Existence and uniqueness of the $\boldsymbol{\nu}$-RLF in $\mathscr{P}(\mathbb{R}^d)$). *Assume that (1.1) has uniqueness in $L^\infty_+\big([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$. Then, for all $\boldsymbol{\nu} \in \mathscr{M}_+(\mathscr{P}(\mathbb{R}^d))$ regular, there exists at most one $\boldsymbol{\nu}$-RLF in $\mathscr{P}(\mathbb{R}^d)$. If (1.1) has existence in $L^\infty_+\big([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$, this unique flow is* *given by*

$$\boldsymbol{\mu}(t,\mu) := \int_{\mathbb{R}^d} \delta_{\boldsymbol{X}(t,x)}\, d\mu(x).,\tag{5.4}$$

where $\boldsymbol{X}(t,x)$ *denotes the unique* $\mathscr{L}^d$*-RLF.*

Of course the main point here is about uniqueness rather than existence, since the linear formula (5.3) provides a natural recipe for the solution: it is remarkable that the same condition used for the uniqueness of the $\mathscr{L}^d$-RLF in $\mathbb{R}^d$ provides also uniqueness of the "higher level" flow in $\mathscr{P}(\mathbb{R}^d)$. At the level of existence, on the other hand, one may speculate about situations where a $\boldsymbol{\nu}$-RLF $\boldsymbol{\mu}$ exists in $\mathscr{P}(\mathbb{R}^d)$, with $\boldsymbol{\nu}$ regular, but the flow $\boldsymbol{\mu}$ is not induced by any $\mathscr{L}^d$-RLF $\boldsymbol{X}$ as in (5.3).

The main motivation for the development of the theory in [16] has been an application to semiclassical limits, where not only uniqueness, but also stability of the $\boldsymbol{\nu}$-RLF is relevant. In order to describe this application briefly, let $\alpha \in (0,1)$ and let $\psi^\varepsilon_{x_0,p_0,t} : \mathbb{R}^n \to \mathbb{C}$ be a family of solutions to the Schrödinger equation:

$$\begin{cases} i\varepsilon\partial_t \psi^\varepsilon_{x_0,p_0,t} = -\frac{\varepsilon^2}{2}\Delta\psi^\varepsilon_{x_0,p_0,t} + U\psi^\varepsilon_{x_0,p_0,t} \\ \psi^\varepsilon_{x_0,p_0,0} = \varepsilon^{-n\alpha/2}\phi_0\Big(\frac{x-x_0}{\varepsilon^\alpha}\Big)e^{i(x\cdot p_0)/\varepsilon}. \end{cases}\tag{5.5}$$

Here $\phi_0 \in C^2_c(\mathbb{R}^n)$ and $\int |\phi_0|^2\,dx = 1$. When the potential U is of class C^2, it was proven in [53] that for every (x_0,p_0) the Wigner transforms

$$W_\varepsilon \psi^\varepsilon_{x_0,p_0,t}(x,p) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \psi^\varepsilon_{x_0,p_0,t}(x+\frac{\varepsilon}{2}y)\overline{\psi^\varepsilon_{x_0,p_0,t}(x-\frac{\varepsilon}{2}y)}e^{-ipy}dy$$

converge, in the natural dual space $\mathcal{A}'$ for the Wigner transforms, to $\delta_{\boldsymbol{X}(t,x_0,p_0)}$ as $\varepsilon \downarrow 0$. Here $\boldsymbol{X}(t,x,p)$ if the unique flow in $\mathbb{R}^{2n}$ associated to the Liouville equation

$$\partial_t W + p\cdot\nabla_x W - \nabla U(x)\cdot\nabla_p W = 0.\tag{5.6}$$

In [16] we are able to consider a potential U which can be written as the sum of a repulsive Coulomb potential U_s plus a bounded Lipschitz interation term U_b with $\nabla U_b \in BV_{\rm loc}$. We observe that in this case the equation (5.6) does not even make sense for measure initial data, as ∇U is not continuous (so the product $\nabla U(x)\cdot\nabla_p W$ is not a well-defined distribution, if W is just a measure). Still, we can prove *full* convergence as $\varepsilon \downarrow 0$, namely

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\rho(x_0,p_0)\sup_{t\in[-T,T]} d_{\mathcal{A}'}\big(W_\varepsilon\psi^\varepsilon_{x_0,p_0}(t),\delta_{\boldsymbol{X}(t,x_0,p_0)}\big)dx_0dp_0 = 0 \quad \forall\, T>0\tag{5.7}$$

for all $\rho \in L^1(\mathbb{R}^{2n})\cap L^\infty(\mathbb{R}^{2n})$ nonnegative, where $\boldsymbol{X}(t,x,p)$ if the unique $\mathscr{L}^{2n}$-RLF associated to (5.6) and $d_{\mathcal{A}'}$ is a bounded distance inducing the weak* topology in the unit ball of $\mathcal{A}'$.

The proof of (5.7) relies on an application of the stability properties of the flow in $\mathscr{P}(\mathbb{R}^d)$ to the Husimi transforms of $\psi^\varepsilon_{x_0,p_0}(t)$, namely the convolutions with a Gaussian kernel with variance $\varepsilon/2$ of the Wigner transforms. The scheme is sufficiently flexible to allow more general families of initial conditions: for instance, the limiting case $\alpha = 1$ in (5.5) leads to

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\rho(x_0)\sup_{t\in[-T,T]} d_{\mathcal{A}'}\big(W_\varepsilon\psi^\varepsilon_{x_0,p_0}(t),\boldsymbol{\mu}(t,\mu(x_0,p_0))\big)dx_0 = 0$$

for all $T > 0$, $p_0 \in \mathbb{R}^n$, $\rho \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ nonnegative, with $\boldsymbol{\mu}(t,\mu)$ given by (5.3) and $\mu(x_0,p_0) = \delta_{x_0}\times|\hat{\phi}_0|^2(\cdot - p_0)\mathscr{L}^n$.

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