A POINTWISE GRADIENT BOUND FOR ELLIPTIC EQUATIONS ON COMPACT MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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The goal of this note is to prove the following result:

**Theorem 1.** Let $M$ be a smooth, compact, Riemannian manifold with non-negative Ricci curvature and let $f \in C^1(\mathbb{R})$.

Let $u \in C^3(M)$ be a solution of

$$\Delta_g u + f(u) = 0$$

on $M$, with $m := \inf_M u$, $M := \sup_M u$, and let $F$ be a primitive of $f$.

Then,

$$\frac{1}{2} |\nabla_g u(x)|^2 \leq \sup_{r \in [m, M]} F(r) - F(u(x)),$$

for any $x \in M$.

Also, if equality in (1) holds at some point $x_0 \in \{\nabla_g u \neq 0\}$, then:

- equality in (1) holds at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains $x_0$,
- $\text{Ric}_{g}(\nabla_g v, \nabla_g v)$ vanishes at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains $x_0$.


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The notation used here above is the standard one, namely $\nabla_g$ is the Riemannian gradient and $\Delta_g$ is the Laplace-Beltrami operator, that is, in local coordinates,

$$(\nabla_g \phi)^i = g^{ij} \partial_j \phi$$

and

$$\Delta_g \phi = \text{div}_g (\nabla_g \phi) = \frac{1}{|g|} \partial_t \left( \sqrt{|g|} g^{ij} \partial_j \phi \right),$$

for any smooth function $\phi : M \to \mathbb{R}$.

We remark that when equality in (1) holds on a connected open set $U$, then $u$ is an isoparametric function in $U$, see pages 541–548 of [8]. In particular, any level set $\sigma$ of $u$ has constant mean curvature along $\sigma \cap U$. For a comprehensive description of isoparametric functions, see also [10].

The pointwise estimate of Theorem 1 may be seen as an extension of the one obtained in [6], where a similar result was proven in the case of $\mathbb{R}^n$.

We observe that if $F$ is bounded, (1) implies the following universal estimate:

$$\frac{1}{2} |\nabla_g u(x)|^2 \leq \sup_{r \in \mathbb{R}} F(r) - \inf_{r \in \mathbb{R}} F(r).$$

The proof we give here of Theorem 1 uses the technique of [2], where important strengthenings of the work of [6] were performed in the degenerate and singular Euclidean case.

The proof is based on the “$P$-function technique”, i.e. in a convenient use of the maximum principle, applied to a function which solves a degenerate PDE (see [7, 9]).

For related results in the Euclidean setting, see also [4].

**Proof of Theorem 1.** We recall that, if $\phi \in C^3(M)$,

$$\frac{1}{2} \Delta_g |\nabla_g \phi|^2 = |H_\phi|^2 + \langle \nabla_g \Delta_g \phi, \nabla_g \phi \rangle + \text{Ric}_g(\nabla_g \phi, \nabla_g \phi).$$

(2)

Here above, $H_\phi$ is the Hessian of $\phi$: note that (2) is the so-called Bochner-Weitzenböck formula (see, for instance, [1, 11] and references therein).

Moreover, we have that

$$|H_\phi|^2 \geq |\nabla_g \nabla_g \phi|^2 \quad \text{almost everywhere.}$$

(3)

See, for instance, [3] for the simple proof of this fact.

Also, we observe that, since $M$ is compact, if $v \in C^2(M)$ then there exists $x(v) \in M$ which minimizes $v$, and so

$$\nabla_g v(x(v)) = 0.$$  

(4)

We now define

$$G(t) := \sup_{r \in [m, M]} F(r) - F(t).$$

(5)

We remark that

$$G(t) \geq 0$$

(6)

for any $t \in [m, M]$. 

We also fix $\alpha \in (0, 1)$, say $\alpha = 1/2$, and set $[u]_{C^\alpha(M)}$ to be the $\alpha$-seminorm of $u$, which is finite by assumption.

Let

$$\mathcal{F} := \left\{ v \in C^2(M) \text{ solutions of } \Delta_g v = G'(v) \text{ in } M \text{ with } \mathfrak{m} \leq v \leq \mathfrak{M} \right\},$$

and $[v]_{C^\alpha(M)} \leq [u]_{C^\alpha(M)}$. (7)

Also, given $v \in \mathcal{F}$, following [2], we define

$$P(v, x) := |\nabla_g v(x)|^2 - 2G(v(x)).$$

We now claim that, for any $v \in \mathcal{F}$ and any $x \in M$,

$$|\nabla_g v(x)|^2 \Delta_g P(v, x) - 2G'(v(x))(\nabla_g v(x), \nabla_g P(v, x)) \geq \frac{|\nabla_g P(v, x)|^2}{2}. \quad (9)$$

We remark that (9) may be considered the Riemannian analogue of formula (2.7) in [2], where a similar equality was found in the Euclidean setting.

To prove (9), we use (2) and (7) to obtain that

$$|\nabla_g v|^2 \Delta_g P - 2G'(v)\langle \nabla_g v, \nabla_g P \rangle - \frac{|\nabla_g P|^2}{2}$$

$$= |\nabla_g v|^2 \left( \Delta_g |\nabla_g v|^2 - 2\Delta_g (G(v)) \right) + 2f(v)\langle \nabla_g v, \nabla_g P \rangle - \frac{|\nabla_g P|^2}{2}$$

$$= 2|\nabla_g v|^2 \left( |H_v|^2 + \langle \nabla_g \Delta_g v, \nabla_g v \rangle + \text{Ric}_g(\nabla_g v, \nabla_g v) - \text{div}_g(G'(v)\nabla_g v) \right)$$

$$+ 2f(v)\langle \nabla_g v, \nabla_g P \rangle - \frac{|\nabla_g P|^2}{2}$$

$$= 2|\nabla_g v|^2 \left( |H_v|^2 - \langle \nabla_g (f(v)), \nabla_g v \rangle \right.$$

$$+ \text{Ric}_g(\nabla_g v, \nabla_g v) - G''(v)|\nabla_g v|^2 - G'(v)\Delta_g v)$$

$$+ 2f(v)\left( \langle \nabla_g v, \nabla_g |\nabla_g v|^2 \rangle - 2\langle \nabla_g v, \nabla_g (G(v)) \rangle \right)$$

$$- \frac{|\nabla_g |\nabla_g v|^2 - 2\nabla_g (G(v))|^2}{2}$$

$$= 2|\nabla_g v|^2 \left( |H_v|^2 - f'(v)|\nabla_g v|^2 \right.$$

$$+ \text{Ric}_g(\nabla_g v, \nabla_g v) + f'(v)|\nabla_g v|^2 - (f(v))^2)$$

$$+ 2f(v)\left( \langle \nabla_g v, \nabla_g |\nabla_g v|^2 \rangle + 2f(v)|\nabla_g v|^2 \right)$$

$$- \frac{|\nabla_g |\nabla_g v|^2 + 2f(v)|\nabla_g v|^2}{2}$$

$$= 2|\nabla_g v|^2 \left( |H_v|^2 + \text{Ric}_g(\nabla_g v, \nabla_g v) \right) - \frac{|\nabla_g |\nabla_g v|^2|^2}{2}. \quad (9)$$
Hence, recalling (3) and the fact that the Ricci curvature is nonnegative, we obtain that

\[ |\nabla_g v|^2 \Delta_g P - 2G'(v)\langle \nabla_g v, \nabla_g P \rangle - \frac{|\nabla_g P|^2}{2} = 2|\nabla_g v|^2 \left( |H_v|^2 - |\nabla_g \nabla_g v|^2 + \text{Ric}_g(\nabla_g v, \nabla_g v) \right) \]

\[ \geq 2|\nabla_g v|^2 \text{Ric}_g(\nabla_g v, \nabla_g v). \]

(10)

We observe that the above quantity is nonnegative, and this proves (9).

Now, we define

\[ P_o := \sup_{v \in \mathcal{F}} \sup_{x \in M} P(v, x). \]

(11)

We observe that, if \( v \in \mathcal{F} \),

\[ |f(v(x)) - f(v(y))| \leq \|f\|_{C^1([m, M])} |v(x) - v(y)| \]

\[ \leq \|f\|_{C^1([m, M])} \|v\|_{C^\alpha(M)} |x - y|^\alpha \]

\[ \leq \|f\|_{C^1([m, M])} \|u\|_{C^\alpha(M)} |x - y|^\alpha \]

for any \( x, y \in M \).

Consequently, by elliptic regularity (see, e.g., [5]), any \( v \in \mathcal{F} \) satisfies

\[ \|v\|_{C^{2,\alpha}(M)} \leq C_o \]

(12)

for a suitable \( C_o > 0 \) independent on \( v \) (more precisely, \( C_o \) only depends on \( f, m, M \) and \( \|u\|_{C^\alpha(M)} \)).

In particular, the sup in (11) is finite.

We claim that

\[ P_o \leq 0. \]

(13)

To check (13), we argue by contradiction. We suppose that

\[ P_o > 0 \]

(14)

\( v_k \in \mathcal{F} \) and \( x_k \in M \) in such a way that

\[ P_o - \frac{1}{k} \leq P(v_k, x_k) \leq P_o. \]

(15)

Since \( M \) is compact, we may suppose that \( x_k \) converges to some \( x_\infty \in M \), up to subsequence.

Also, by (12), \( v_k \) converges in \( C^2(M) \), up to subsequence, to some \( v_\infty \).

Notice that \( v_\infty \in \mathcal{F} \) by construction.

Therefore, (15) gives that

\[ P(v_\infty, x_\infty) = P_o. \]

(16)

From (6), (14) and (16), we obtain that

\[ |\nabla_g v_\infty(x_\infty)|^2 \geq |\nabla_g v_\infty(x_\infty)|^2 - 2G(v_\infty(x_\infty)) = P_o > 0 \]

and therefore

\[ \nabla_g v_\infty(x_\infty) \neq 0. \]
In light of (9), (16) and (17), the Strong Maximum Principle gives that
\[ P(v, x) = P_o \quad \text{for any } x \in M. \]  
(18)

In particular, recalling (4) and using (6), (14) and (18), we conclude that
\[
0 = |\nabla_g v(x(v_\infty))|^2 \geq |\nabla_g v_\infty(x(v_\infty))|^2 - 2G(v_\infty(x(v_\infty)))
\]
\[ = P(v_\infty, x(v_\infty)) = P_o > 0. \]

Since this is a contradiction, the proof of (13) is complete.

Then, by (5) and (13),
\[
0 \geq P_o = \sup_{v \in \mathcal{F}, x \in M} P(v, x) \geq P(u, x) = |\nabla_g u(x)|^2 - 2G(u(x))
\]
\[ = |\nabla_g u(x)|^2 - 2 \left( \sup_{r \in [m, \infty]} F(r) - F(u(x)) \right), \]
that is (1).

We now suppose that equality in (1) holds at some point
\[ x_o \in \{\nabla_g u \neq 0\} \]  
(19)
and we prove that it holds at all the points of the connected component of \( M \cap \{\nabla_g u \neq 0\} \) that contains \( x_o \).

For this, let \( M' \) be such connected component. We notice that, by (5) and (13),
\[
0 \geq P_o \geq P(u, x_o) = |\nabla_g u(x_o)|^2 - 2 \left( \sup_{r \in [m, \infty]} F(r) - F(u(x_o)) \right) = 0,
\]
and so
\[ P(u, x_o) = \max_{x \in M} P(u, x) = 0. \]

Thus, (19) and the Strong Maximum Principle gives that
\[ P(u, x) = 0 \quad \text{for any } x \in M'. \]  
(20)

This shows that equality in (1) holds at all the points of \( M' \).

Furthermore, from (10) and (20),
\[
0 = |\nabla_g v|^2 \Delta_g P - 2G'(v)\langle \nabla_g v, \nabla g P \rangle - \frac{|\nabla g P|^2}{2} \geq 2|\nabla_g v|^2 \text{Ric}_g(\nabla_g v, \nabla_g v)
\]
and so
\[ |\nabla_g v|^2 \text{Ric}_g(\nabla_g v, \nabla_g v) = 0 \]

at all the points of \( M' \).

Since \( \nabla_g v \neq 0 \) in \( M' \), this gives that \( \text{Ric}_g(\nabla_g v, \nabla_g v) \) vanishes identically in \( M' \).

This completes the proof of Theorem 1. \( \square \)
Remark 2. We observe that, from (3), (10) and (20), we have also proved that if equality in (1) holds at some point $x_0 \in \{\nabla_g u \neq 0\}$, then $|H_v| = |\nabla_g \nabla_g v|$ at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains $x_0$.

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