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AN IMPROVED HARDY-SOBOLEV INEQUALITY AND ITS APPLICATION

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September 29, 2000

Abstract. For $\Omega \subset \mathbb{R}^n, n \geq 2$, a bounded domain, and for $1 < p < n$, we improve the Hardy-Sobolev inequality, by adding a term with a singular weight of the type $(\frac{1}{\log(1/|x|)})^2$. We show that this weight function is optimal in the sense that the inequality fails for any other weight function more singular than this one. Moreover, we show that a series of finite terms can be added to improve the Hardy-Sobolev inequality, which answers a question of Brezis-Vazquez. Finally, we use this result to analyze the behaviour of the first eigenvalue of the operator $L_\mu u := -(\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u)$ as μ increases to $\left(\frac{n-p}{p}\right)^p$ for $1 < p < n$.

1. Introduction and Results

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$, with $0 \in \Omega$. For any $1 < p < n$, the well known Hardy-Sobolev inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx \tag{1.1}$$

holds for $u \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{1,p,\Omega} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u|^p dx\right)^{1/p}$. It is known that there is no function $u \in W_0^{1,p}(\Omega)$, for which the best constant $\beta_{n,p} := \left(\frac{n-p}{p}\right)^p$ is achieved. So, one anticipates to have an estimate of the error term on the right hand side of the inequality (1.1). For the case $p = 2$, such improved Hardy-Sobolev inequalities are known. For example, Brezis-Vazquez [BV] have proved that there exists a constant $C > 0$ depending only on $n,\ q$ and Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx + C\left(\int_\Omega |u(x)|^q\, dx\right)^{2/q} \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 35J30.

Key words and phrases. Hardy-Sobolev inequality, eigenvalue, p-laplacian.

The second author was supported in part by CSIR, India.

The third author acknowledges the funding from Indo-French Center for Promotion of Advanced Research, under the project 1901-02

Typeset by $\mathcal{AMS}$-TEX

holds for any $u \in H_0^1(\Omega)$ and $1 < q < \dfrac{2n}{n-2}$. Chaudhuri and Ramaswamy [CR] have improved the above inequality by introducing a weight function $f(x) = |x|^{-\beta}$, for $0 \le \beta < 2$:

$$\int_\Omega |\nabla u|^2 dx \ge \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx + C \left(\int_\Omega \frac{|u(x)|^q}{|x|^\beta}\, dx\right)^{2/q} \tag{1.3}$$

for any $u \in H_0^1(\Omega)$ and $1 < q < 2^*_\beta := \dfrac{2(n-\beta)}{n-2}$. Recently, Cabre and Martel have considered critical potentials in their study of existence versus instantaneous explosion for the heat equation with singular potentials [CM2], where they have an improved inequality similar to (1.4) below for the case $n = 2 = p$.

Our aim in this article, is to achieve an "optimal" improvement of the inequality (1.1), by adding second term involving the singular weight $(\frac{1}{\log(1/|x|)})^2$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one. The improvement of the inequality (1.1) for the case $1 < p < 2$, is more delicate than for the case $p \ge 2$. As far our understanding goes, it is because of the fact that for $1 < p < 2$, $(1+x)^p$, $x \ge -1$, does not have a global estimate in terms of either x^2 or $|x|^p$.

Finally, we use our improved inequality, to determine when exactly the first eigenvalue of the weighted eigenvalue problem for the operator $L_\mu u := -(\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u)$ will tend to 0 as μ increases to $\beta_{n,p}$. Note that the operator L_μ is a positive operator for $0 < \mu \le \beta_{n,p}$, thanks to Hardy-Sobolev inequality and is unbounded from below for $\mu > \beta_{n,p}$ (see [GP], Lemma 3.1). Before stating our main results let us introduce the following notations: $\log^{(1)}(.) := \log(.)$, and $\log^{(k)}(.) := \log\left(\log^{k-1}(.)\right)$ for $k \ge 2$. Our main results are

Theorem 1.1. *Let $R \ge \sup_\Omega (|x|\ e^{2/p})$ and $1 < p \le n$, then there exists $C > 0$, depending on n, p and R such that*

$$\int_\Omega |\nabla u|^p dx \ge \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \tag{1.4}$$

for any $u \in W_0^{1,p}(\Omega)$ if and only if

(i) $\gamma \ge 2$ *when* $1 < p < n$,

(ii) $\gamma \ge n$ *when* $p = n$.

More generally, for $2 \le p < n$ and for any $1 < q < p^*_\beta := \dfrac{p(n-\beta)}{n-p}$, $0 \le \beta < p$, there exists $C_1 > 0$ depending on n, p, q, R and Ω such that

$$\int_\Omega |\nabla u|^p dx \ge \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx$$

$$+ C_1 \left(\int_\Omega \frac{|u(x)|^q}{|x|^\beta}\, dx\right)^{p/q} \tag{1.5}$$

for any $u \in W_0^{1,p}(\Omega)$.

One can show using this theorem

Corollary 1.1. *Let $1 < p < n$ and $R = (\sup_\Omega |x|) \left(e^{e^{e^{\cdot^{\cdot^{\cdot^{e\,(k-times)}}}}}} \right)^{2/p}$. Then there exists $C > 0$ depending on n, p and R such that*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + C \sum_{j=1}^{k} \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|} \right)^{-2} dx$$

for any $u \in W_0^{1,p}(\Omega)$.

Remark 1.1. One of the open problem mentioned in [BV] (Problem 2), is whether the two terms on the RHS of the the inequality (1.2) for $q = 2$, are just the first two terms of a series or not. The Corollary 1.1 for the particular case $p = 2$, shows that, indeed the series continues till a certain k-th term, which depends only on the choice of R.

Remark 1.2. Adimurthi and Sandeep in [AS], has been proved that the best constant in the inequality (1.4) for the case $2 \leq p < n$, is $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

Corollary 1.2. *Let $1 < p < n$ and let*

$$F_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{loc}(\Omega \setminus \{0\}) \text{ with } \limsup_{|x|\to 0} |x|^p f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}.$$

If $f \in F_p$, then there exists $\lambda(f) > 0$ such that for all $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + \lambda(f) \int_\Omega |u(x)|^p\, f(x)\, dx\,. \tag{1.6}$$

If $f \notin F_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then no inequality of the type (1.6) can hold.

Consider the weighted eigenvalue problem with a singular weight,

$$\begin{aligned} -\left(\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u\right) &= \lambda |u|^{p-2}u\, f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

where $f \in \Im_p$, $\Im_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x|\to 0} |x|^p f(x) = 0 \text{ with } f \in L^\infty_{loc}(\Omega \setminus \{0\}) \right\}$, $1 < p < n$, $0 \leq \mu < \beta_{n,p}$, and $\lambda \in \mathbb{R}$. We look for a weak solution $u \in W_0^{1,p}(\Omega)$ of this problem and study the asymptotic behaviour of the first eigenvalues for different singular weights as μ increases to $\beta_{n,p}$, after which the operator L_μ is no more bounded from below.

Theorem 1.2. *The above problem, admits a positive weak solution $u \in W_0^{1,p}(\Omega)$ for all $1 < p < n$, corresponding to the first eigenvalue $\lambda = \lambda_\mu^1(f) > 0$. Moreover, as μ increases to $\beta_{n,p}$, $\lambda_\mu^1(f) \to \lambda(f) \geq 0$, for all $f \in \Im_p$ and the limit $\lambda(f) > 0$ if $f \in F_p$. If $f \notin F_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then the limit $\lambda(f) = 0$.*

For the case $p = 2$, problem (1.7), has been studied with singular weights f like $(1/|x|^\beta), 0 < \beta < 2$, in [CR] and for the radial problem with $f = 1$ in [CM1].

2. Proof of Theorem 1.1

For the proof of the theorem we will need the following lemmas. The lemma 2.2 is an improved 2-dimensional Hardy's inequality and a similar one for the case $k = 1$ appears in [BrM] (inequality (A.4)), Lemma 2.3 is a n-dimensional Hardy inequality. Proof of Lemma 2.1 is deferred to the appendix.

Lemma 2.1. *If $p \geq 2$, then there exist positive constants B and C such that*

$$(1+x)^p \geq 1 + px + Cx^2 + B|x|^p \quad \forall\ x \geq -1 \tag{2.1}$$

If $1 < p < 2$ and $M \geq 1$,

$$(1+x)^p \geq \begin{cases} 1 + px + 2^{p-4}p(p-1)M^{p-2}\,x^2, & \text{for } -1 \leq x \leq M \\ 1 + px + 2^{p-3}p(p-1)\,x^p, & \text{for } x \geq M. \end{cases} \tag{2.2}$$

Lemma 2.2 (2-dimensional Hardy inequality). *For any $h \in C^1[a,T]$, $0 \leq a < T$ and for $R > T$, the following inequality holds for $k = k(R)$:*

$$\begin{aligned} \int_a^T |h'(r)|^2 r\ dr \ \geq\ & \frac{1}{4}\sum_{j=1}^{k}\int_a^T \frac{|h(r)|^2}{\left(r\ \prod_{i=1}^{j}\log^{(i)} R/r\right)^2}\, r\, dr \\ & + \frac{1}{2}\sum_{j=1}^{k}\left\{ h^2(a)\left(\prod_{i=1}^{j}\log^{(i)} R/a\right)^{-1} - h^2(T)\left(\prod_{i=1}^{j}\log^{(i)} R/T\right)^{-1}\right\} \end{aligned} \tag{2.3}$$

where $k(R)$ is the first positive integer for which $0 < \log^{(k)} R/T \leq 1$.

Proof. Let h be in $C^1[a,T]$. For $r \in [a,T]$, define $\psi_1(r) := (\log R/r)^{-1/2}h(r)$. Then

$$\begin{aligned} \int_a^T |h'(r)|^2 r\ dr \ &= \frac{1}{4}\int_a^T \frac{|\psi_1(r)|^2}{(r\ \log R/r)}dr + \int_a^T |\psi_1'(r)|^2(\log R/r)\, r\, dr \\ &\quad - \int_a^T \psi_1(r)\,\psi_1'(r)dr \\ &= \frac{1}{4}\int_a^T \frac{|h(r)|^2}{(r\ \log R/r)^2}\, r\, dr + \int_a^T |\psi_1'(r)|^2(\log R/r)\, r\, dr \\ &\quad + \frac{1}{2}\left\{\psi_1^2(a) - \psi_1^2(T)\right\}. \end{aligned}$$

To estimate the middle term on the RHS, we introduce the iterative function $\psi_2(r) := \psi_1(r)\left(\log^{(2)} R/r\right)^{-1/2}$. A simple calculations shows that

$$\begin{aligned} \int_a^T |h'(r)|^2 r\ dr \ &= \frac{1}{4}\sum_{j=1}^{2}\int_a^T \frac{|h(r)|^2}{\left(r\ \prod_{i=1}^{j}\log^{(i)} R/r\right)^2}\, r\, dr \\ &\quad + \int_a^T |\psi_2'(r)|^2(\log R/r)\,(\log\log R/r)\, r\, dr \\ &\quad + \frac{1}{2}\sum_{j=1}^{2}\left\{ h^2(a)\left(\prod_{i=1}^{j}\log^{(i)} R/a\right)^{-1} - h^2(T)\left(\prod_{i=1}^{j}\log^{(i)} R/T\right)^{-1}\right\}. \end{aligned}$$

We can again estimate the middle term on the RHS, by introducing $\psi_3(r) := \psi_2(r)\left(\log^{(3)} R/r\right)^{-1/2}$. This process can be continued till we reach $k = k(R)$ for which $0 < \log^{(k)} R/T \leq 1$ and by defining, $\psi_k(r) := \psi_{k-1}(r)\left(\log^{(k)} R/r\right)^{-1/2}$, ignoring the integral involving the term $|\psi_k'(r)|^2$ we obtain the inequality (2.3). □

By the standard approximation argument we have the following corollary.

Corollary 2.1. *Given any $T > 0$ and for $R > T$, the inequality*

$$\int_0^T |h'(r)|^2 r\, dr \;\geq\; \frac{1}{4}\sum_{j=1}^{k}\int_0^T \frac{|h(r)|^2}{\left(r\ \prod_{i=i}^{j}\log^{(i)} R/r\right)^2}\, r\, dr \tag{2.4}$$

holds for any radial function h in $H_0^1(B)$, where B is the ball of radius T centered at origin, where $k = k(R)$ is the first integer for which $0 < \log^{(k)} R/T \leq 1$.

Remark 2.1. In fact (2.4) holds for any h in $H_0^1(B)$ with $R = e^{e^{e^{\cdot^{\cdot^{\cdot^{e\,(k-times)}}}}}} T$, by using a symmetrization argument as in the proof of Theorem 1.1 below. This choice of R enables one to conclude that the singular weights $\left(r\prod_{i=1}^{j}\log^{(i)} R/r\right)^{-2}$ are decreasing functions of r for $1 \leq j \leq k$.

Lemma 2.3 (n-dimensional Hardy inequality). *Let $n \geq 2$, $\Omega = B(0,T) \subset \mathbb{R}^n$, be the open ball of radius T, centered at origin and $R > T$. Then*

$$\int_\Omega |\nabla h|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|h|^n}{|x|^n}\left(\log R/|x|\right)^{-n} dx \tag{2.4'}$$

for all radial $h \in W_0^{1,n}(\Omega)$.

Proof. For any $h \in C_0^1(\Omega)$, $h > 0$, radially nonincreasing, we define $\psi(r) := (\log R/r)^{-(n-1)/n} h(r)$, for $r \in [0,T]$. Then

$$\begin{aligned} h'(r) &= \psi'(r)(\log R/r)^{(n-1)/n} - \frac{n-1}{n\,r}\psi(r)(\log R/r)^{-1/n} \\ &= -\frac{n-1}{n\,r}\psi(r)(\log R/r)^{-1/n}\left(1 - \frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right). \end{aligned}$$

Since $h'(r) \leq 0$, we have $\left(1 - \frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\right) \geq 0$ and hence

$$\begin{aligned} |h'(r)|^n &= \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1}\left(1 - \frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right)^n \\ &\geq \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1}\left(1 - n\,\frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right) \\ &= \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1} - n\left(\frac{n-1}{n\,r}\right)^{n-1}\psi'(r)\psi^{n-1}(r). \end{aligned}$$

Since $\psi(r) \to 0$ as $r \to 0$ and $\psi(T) = 0$, we obtain

$$\begin{aligned}\int_\Omega |\nabla h|^n dx &= \omega_n \int_0^T |h'(r)|^n r^{n-1} dr \\ &\geq \omega_n \left(\frac{n-1}{n}\right)^n \int_0^T |h(r)|^n \left(\log R/r\right)^{-n} \frac{dr}{r} \\ &- \omega_n \left(\frac{n-1}{n}\right)^{n-1} \int_0^T \left(\frac{d}{dr}\psi^n(r)\right) dr \\ &= \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|h(x)|^n}{|x|^n} \left(\log R/r\right)^{-n} dx.\end{aligned}$$

For any radial function h in $W_0^{1,n}(\Omega)$, approximate h by smooth radial h_m. Then using strong convergence of h_m to h in the gradient norm and also Fatou's lemma, the above inequality holds for all radial h in $W_0^{1,n}(\Omega)$. $\square$

Now we are in a position to give a proof of Theorem 1.1 and we organize the proof in following manner: First we prove the validity of the inequality (1.4) for the cases $2 \leq p < n$ and $1 < p < 2$ separately and finally we show the optimality the inequality (1.4).

Proof of the Theorem 1.1. : Let us assume, $1 < p < n$ and $\gamma \geq 2$. Since the function $\gamma \mapsto (\log R/r)^{-\gamma}$, is monotonically decreasing on $[2, \infty)$, it is enough to prove the inequality (1.4), only for $\gamma = 2$. For both the cases $p \geq 2$ and $p < 2$, we shall first prove the inequalities for smooth positive radially nonincreasing functions defined on a ball B, centered at 0 and of radius T. For $u \in C_0^2(B)$, $u > 0$, radially nonincreasing, we define

$$v(r) := u(r)\, r^{(n-p)/p}\ ,\ r = |x|. \tag{2.5}$$

Here without loss of generality we as well assume $u'(r) < 0$, (replacing u by $u + \epsilon(T - r)$ for $\epsilon > 0$, sufficiently small). Now we observe that,

$$\begin{aligned}\int_B |\nabla u|^p dx - \beta_{n,p} \int_B \frac{|u(x)|^p}{|x|^p} dx &= \omega_n \int_0^T \left|\frac{n-p}{p} r^{-n/p} v(r) - r^{1-n/p} v'(r)\right|^p r^{n-1} dr \\ &- \beta_{n,p}\, \omega_n \int_0^T \frac{v^p(r)}{r}\, dr \\ &= \omega_n \beta_{n,p} \int_0^T v^p(r) \left\{\left|1 - \frac{p v'(r)\, r}{(n-p) v(r)}\right|^p - 1\right\} \frac{dr}{r}. \qquad (2.6)\end{aligned}$$

Since u is a decreasing function, we have from (2.5), $v'(r) - \dfrac{(n-p)v(r)}{p\, r} < 0$ and we call $x(r) := -\dfrac{p v'(r)\, r}{(n-p)v(r)}$ so that, $x(r) > -1$. Now we consider the following cases:

Case $p \geq 2$: By the inequality (2.1) in Lemma 2.1, and from (2.6), we obtain

$$\begin{aligned}\int_B |\nabla u|^p dx - \beta_{n,p} \int_B \frac{|u(x)|^p}{|x|^p} dx &\geq C\,\omega_n \left(\frac{n-p}{p}\right)^{p-2} \int_0^T v^{p-2}(r)|v'(r)|^2 r\,dr \\ &\quad + B\,\omega_n \int_0^T |v'(r)|^p r^{p-1}\,dr - p\,\omega_n \int_0^T v^{p-1}(r)v'(r)\,dr \\ &= \frac{4C\,\omega_n}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \int_0^T \left|\left(v^{p/2}(r)\right)'\right|^2 r\,dr \\ &\quad + B\,\omega_n \int_0^T |v'(r)|^p r^{p-1}\,dr. \qquad (2.7)\end{aligned}$$

since $v \in C_0^1(0,T)$. Now by corollary 2.1,

$$\begin{aligned}\int_0^T \left|\left(v^{p/2}(r)\right)'\right|^2 r\,dr &\geq \frac{1}{4}\int_0^T \left(\frac{v^{p/2}(r)}{r\log R/r}\right)^2 r\,dr \\ &= \frac{1}{4}\int_0^T \frac{u^p(r)}{r^p}\left(\log R/r\right)^{-2} r^{n-1} dr \\ &= \frac{1}{4}\int_B \frac{|u(x)|^p}{|x|^p}\left(\log R/|x|\right)^{-2} dx\,. \qquad (2.8)\end{aligned}$$

Now for $0 \leq \beta < p$, we have

$$\int_B \frac{|u(x)|^q}{|x|^\beta} dx = \omega_n \int_0^T |v(r)|^q r^{n-\beta-1-q(n-p)/p} dr.$$

Notice that, for C^1 radial function $v(r)$ vanishing at T,

$$\begin{aligned}|v(r)| &= \left|\int_r^T v'(t)dt\right| = \left|\int_r^T \left(v'(t)t^{(p-1)/p}\right) t^{-(p-1)/p} dt\right| \\ &\leq \left(\int_r^T |v'(t)|^p t^{p-1} dt\right)^{1/p} \left(\log\frac{T}{r}\right)^{(p-1)/p}\end{aligned}$$

and hence,

$$\begin{aligned}\int_0^T |v(r)|^q r^{\sigma-1} dr &\leq \left(\int_0^T |v'(r)|^p r^{p-1} dr\right)^{q/p} \int_0^T r^{\sigma-1}\left(\log\frac{T}{r}\right)^{q(p-1)/p} dr \\ &= \left(\int_0^T |v'(r)|^p r^{p-1} dr\right)^{q/p} \int_0^\infty e^{-\sigma\, r} r^{q(p-1)/p} dr,\end{aligned}$$

where $\sigma := n - \beta - q(n-p)/p$. Now the second integral on the right hand side is convergent iff $\sigma > 0$ i.e., $1 < q < \dfrac{p(n-\beta)}{n-p}$. Call

$$C_1 := B(\omega_n)^{1-p/q}\left(\int_0^\infty e^{-\sigma\, r} r^{q(p-1)/p} dr\right)^{-p/q}.$$

Then from these, we get $B\,\omega_n \int_0^T |v'(r)|^p r^{p-1} dr \geq C_1 \left(\int_B \frac{|u(x)|^q}{|x|^\beta} dx \right)^{p/q}$. Combining this with (2.8), and (2.7) we get the inequality (1.5) and hence (1.4) for radially decreasing, smooth positive functions. Now by density arguments, inequality (1.5) is valid for any $u \in W_0^{1,p}(B)$, $u \geq 0$, and radially nonincreasing. For a general domain Ω, we use symmetrization arguments. Let B_T be a ball having same volume as Ω with $T = (n|\Omega|/\omega_n)^{1/n}$, where $\omega_n = |S^{n-1}|$ and let $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. Now observe that, for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(B_T)$ and $|u|^* > 0$ and radially nonincreasing and hence inequality (1.5) holds for $|u|^*$. It is well known that the symmetrization does not change the L^p- norm, decreases gradient norm and increases the integrals $\int_\Omega \frac{|u(x)|^p}{|x|^p} dx$ and $\int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|} \right)^{-2} dx$, since both the singular weights are decreasing functions of $|x|$ under our assumption on R, thanks to Hardy's inequality for rearrangements. Hence (1.5) holds also for any $u \in W_0^{1,p}(\Omega)$. $\square$

Case $1 < p < 2$: Since for the case $1 < p < 2$, $(1+x)^p$, does not have global estimate in terms of either x^2 or $|x|^p$, we do the following decomposition of the interval $[0, T]$. Take $M \geq \max\left\{1, \frac{1}{(n-p)\,(\log\, R/T)}\right\}$ and denote, $\alpha := \frac{n-p}{p}$. Now we define the following sets, $A := \left\{ r \in [0,T] : x(r) = -\frac{p v'(r)\, r}{(n-p) v(r)} < M \right\}$, $B := \{r \in [0,T] : x(r) >\ M\}$ and $\Gamma := \left\{ r \in [0,T] : \frac{v'(r)}{v(r)} = -\frac{\alpha\, M}{r} \right\}$ so that $[0,T] = A \cup B \cup \Gamma$. Now observe that on the set A, $\frac{v'(r)}{v(r)} > -\frac{\alpha\, M}{r}$, i.e., *iff* $\left(\log v(r)\, r^{\alpha\, M}\right)' > 0$, i.e, *iff* the function $v\, r^{\alpha\, M}$ is monotonically increasing. So, if we define the function $g(r)$ to be $v\, r^{\alpha\, M}$, for $0 \leq r \leq T$, then $A = \{r \in [0,T] : g(r) \text{ is monotonically increasing}\}$ and $B = \{r \in [0,T] : g(r) \quad \text{is monotonically decreasing}\}$. We now prove the theorem through following steps:

Step I: Let us assume that, $\Gamma = \left\{ r \in [0,T] : \frac{u'(r)}{u(r)} = -\frac{\alpha(M+1)}{r} \right\}$ is a finite set. Now observe that critical points of the function $v\, r^{\alpha\, M}$ is precisely Γ and hence its cardinality will be odd. So let, $\Gamma = \{a_1, a_2, , , , a_{2m+1}\}$ with $a_0 := 0 < a_1 < .....a_{2m+1} < a_{2m+2} := T$. Hence $[0,T]$ can be decomposed as $[0,T] = A \cup B \cup \Gamma$, with $A = \cup_{i=0}^m (a_{2i}, a_{2i+1})$ and $B = \cup_{i=0}^m (a_{2i+1}, a_{2i+2})$. We denote, $C_1(p) := 2^{p-4} p(p-1)$. Hence from (2.6) and by the inequality (2.2) in Lemma 2.1, we get

$$
\begin{aligned}
X(u) &= \omega_n \beta_{n,p} \left[\int_A v^p(r) \left\{ \left| 1 - \frac{v'(r)\, r}{\alpha\, v(r)} \right|^p - 1 \right\} \frac{dr}{r} + \int_B v^p(r) \left\{ \left| 1 - \frac{v'(r)\, r}{\alpha\, v(r)} \right|^p - 1 \right\} \frac{dr}{r} \right] \\
&\geq \omega_n \beta_{n,p} \left[\frac{C_1(p) M^{p-2}}{\alpha^2} \int_A v^{p-2}(r) \left(v'(r) \right)^2 r\, dr + \frac{2 C_1(p)}{\alpha^p} \int_B |v'(r)|^p\, r^{p-1} dr \right]
\end{aligned}
$$

$$= \omega_n \left[\frac{C_1(p)}{(M\alpha)^{2-p}} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} v^{p-2}(r) \left(v'(r)\right)^2 r\, dr + 2C_1(p) \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr \right]$$

(2.9)

$$= \omega_n \left[\frac{4\, C_1(p)}{p^2\, (M\alpha)^{2-p}} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \left| \left(v^{p/2}(r)\right)' \right|^2 r\, dr + 2C_1(p) \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr \right].$$

For $r \in [0, T]$, define $w(r) := v^p(r) \left(\log R/r\right)^{-1}$. Now by lemma 2.2, we have for $0 \le i \le m$

(2.10)

$$\int_{a_{2i}}^{a_{2i+1}} \left| \left(v^{p/2}(r)\right)' \right|^2 r\, dr \ge \frac{1}{4} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r \log R/r\right)^2} r\, dr - \frac{1}{2} \left\{ w(a_{2i+1}) - w(a_{2i}) \right\}.$$

Therefore from (2.9) and (2.10), we obtain

$$\begin{aligned} X(u) &\ge \frac{\omega_n\, C_1(p)}{p^2\, (M\alpha)^{2-p}} \left[\sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r \log R/r\right)^2} r\, dr - 2 \sum_{i=0}^{m} \left\{ w(a_{2i+1}) - w(a_{2i}) \right\} \right] \\ &\quad + 2\omega_n C_1(p) \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr \\ &= \frac{\omega_n\, C_1(p)}{p^2\, (M\alpha)^{2-p}} \left[\sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r \log R/r\right)^2} r\, dr \right. \\ &\quad \left. + 2 \left\{ \sum_{i=0}^{m-1} [w(a_{2i+2}) - w(a_{2i+1})] - w(a_{2m+1}) \right\} \right] \\ &\quad + 2\omega_n C_1(p) \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr. \end{aligned} \tag{2.11}$$

Let us observe that $w'(r) = pv^{p-1}(r)v'(r) \left(\log R/r\right)^{-1} + \frac{1}{r} v^p(r) \left(\log R/r\right)^{-2}$, and hence for $0 \le i \le m$, $(w(a_{2m+2}) = w(T) = 0)$ we have

$$\begin{aligned} w(a_{2i+2}) - w(a_{2i+1}) &= \int_{a_{2i+1}}^{a_{2i+2}} w'(r)\, dr \\ &= \int_{a_{2i+1}}^{a_{2i+2}} v^p(r) (r \log R/r)^{-2} r\, dr \\ &\quad + p \int_{a_{2i+1}}^{a_{2i+2}} v^{p-1}(r) v'(r) \left(\log R/r\right)^{-1} dr \end{aligned} \tag{2.12}$$

Hence from (2.11) and (2.12) we have

$$
\begin{aligned}
X(u) &\geq \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}} \left[\sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{(r\,\log\,R/r)^2} r\,dr + 2 \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} \frac{v^p(r)}{(r\,\log\,R/r)^2} r\,dr \right] \\
&\quad + 2\omega_n C_1(p) \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} \left\{ |v'(r)|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r)\,(\log\,R/r)^{-1} \right\} dr \\
&\geq \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}} \int_0^T \frac{v^p(r)}{(r\,\log\,R/r)^2} r\,dr \\
&\quad + 2\omega_n C_1(p) \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} \left\{ |v'(r)|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r)\,(\log\,R/r)^{-1} \right\} dr
\end{aligned} \tag{2.13}
$$

Now, since $v'(r) < -\dfrac{\alpha\, M\, v(r)}{r}$, on (a_{2i+1}, a_{2i+2}) for each $0 \leq i \leq m$, we have

$$
\begin{aligned}
Y(v) &:= |v'(r)|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r)\,(\log\,R/r)^{-1} \\
&= (-v'(r))^p\, r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r)\,(\log\,R/r)^{-1} \\
&= (-v'(r))\, v^{p-1}(r) \left[\left(\frac{-v'(r)\, r}{v(r)} \right)^{p-1} - \frac{(M\alpha)^{p-2}}{p}\,(\log\,R/r)^{-1} \right] \\
&\geq (-v'(r))\, v^{p-1}(r) \left[(M\alpha)^{p-1} - \frac{(M\alpha)^{p-2}}{p}\,(\log\,R/T)^{-1} \right] \\
&\geq 0, \quad \text{iff, } (M\alpha)^{p-1} - \frac{(M|\alpha)^{p-2}}{p}\,(\log\,R/T)^{-1} \geq 0.
\end{aligned}
$$

Since $M \geq \dfrac{1}{n-p}\,(\log\,R/T)^{-1} = \frac{1}{p\alpha}\,(\log\,R/T)^{-1}$, $Y(v) \geq 0$ on each of the intervals (a_{2i+1}, a_{2i+2}), $0 \leq i \leq m$, from (2.13) we get

$$
\begin{aligned}
X(u) &\geq \omega_n \frac{(M\alpha)^{p-2}\, C_1(p)}{p^2} \int_0^T \frac{v^p(r)}{(r\,\log\,R/r)^2} r\,dr \\
&= \frac{(M\alpha)^{p-2}\, C_1(p)}{p^2} \int\limits_{B_T} \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|} \right)^{-2} dx.
\end{aligned} \tag{2.14}
$$

Step II: If the set Γ is not finite, we proceed as follows. Define the function $\phi(r) := r\,u'(r) + \delta\,u(r)$, for $r \in [0,T]$, $\delta := \alpha\,(M+1)$. Observe that $\phi \in C^1[0,T]$, $\phi(0) = \delta\,u(0) > 0$ and $\phi(T) = Tu'(T) < 0$ and $\Gamma = \phi^{-1}\{0\}$. By Sards Theorem we know that regular values of ϕ are dense in $\mathbb{R}$, so there exists $\epsilon_k\,\delta > 0$, regular values of ϕ such that $\epsilon_k \to 0$, as $k \to \infty$. Thus for $\epsilon_k < u(0)$, the set $\phi^{-1}\{\epsilon_k\,\delta\} = \left\{ r \in [0,T] : \dfrac{(u(r)-\epsilon_k)'}{u(r)-\epsilon_k} = -\dfrac{\delta}{r} \right\}$ is finite. Denoting $u_k(r) := u(r) - \epsilon_k$, then there exists $T_k > 0$ such that $u_k' < 0$ on $[0, T_k]$, $u_k(T_k) = 0$

and $T_k \to T$ as $k \to \infty$. Now by step I, we have the inequality (2.14) for u_k on $[0, T_k]$. Since the constant in (2.14) does not depend on k, passing through the limit as $k \to \infty$ we get the inequality (2..14) for any smooth, positive and strictly radially decreasing functions in $W_0^{1,p}(\Omega)$.

Step III : Now the remaining proof of the inequality (1.4) will follow from (2.14) together with the symmetrization arguments as in the case of $p \geq 2$. □

Case $p = n$**:** In this case the inequality (1.4), follows immediately from the Lemma 2.3 together with symmetization arguments. □

Optimality. Now suppose $1 < p < n$ and $0 \leq \gamma < 2$. Since $\beta_{n,p}$ is the best constant for the inequality (1.1), the theorem follows for the case $\gamma = 0$. So we assume $0 < \gamma < 2$. The theorem will follow if we can prove for the unit ball B that,

$$\inf_{u \in W_0^{1,p}(B)\setminus\{0\}} I_\gamma(u) := \frac{\int\limits_B \left(|\nabla u|^p - \beta_{n,p} \frac{|u(x)|^p}{|x|^p} \right) dx}{\int\limits_B \frac{|u(x)|^p}{|x|^p} \left(\log R/|x| \right)^{-\gamma} dx} = 0. \tag{2.15}$$

Now, our aim is to construct a family $u_\epsilon \in W_0^{1,p}(B)$, $\epsilon > 0$, such that

$$\lim_{\epsilon \to 0} I_\gamma(u_\epsilon) = 0 \tag{2.16}$$

and then (2.15) will follow. For $\epsilon > 0$, sufficiently small, let us define

$$u_\epsilon(r) := \begin{cases} 0, & \text{for } r \leq \epsilon^2 \\ \dfrac{\log r/\epsilon^2}{r^{(n-p)/p} \log 1/\epsilon}, & \text{for } \epsilon^2 \leq r \leq \epsilon \\ \dfrac{\log 1/r}{r^{(n-p)/p} \log 1/\epsilon}, & \text{for } \epsilon \leq r \leq 1. \end{cases}$$

Clearly, u_ϵ is continuous and differentiable a.e. and its derivative is given by

$$u_\epsilon'(r) = \begin{cases} 0, & \text{for } 0 \leq r \leq \epsilon^2 \\ \dfrac{1}{r^{n/p} \log 1/\epsilon} \left[1 - \dfrac{n-p}{p} \log r/\epsilon^2 \right], & \text{for } \epsilon^2 \leq r \leq \epsilon \\ -\dfrac{1}{r^{n/p} \log 1/\epsilon} \left[1 + \dfrac{n-p}{p} \log 1/r \right], & \text{for } \epsilon \leq r \leq 1. \end{cases}$$

Then we have

$$\begin{aligned} \int\limits_B \frac{|u_\epsilon(x)|^p}{|x|^p} dx &= \frac{\omega_n}{(\log 1/\epsilon)^p} \left[\int_{\epsilon^2}^{\epsilon} \left(\log r/\epsilon^2 \right)^p \frac{dr}{r} + \int_\epsilon^1 (\log 1/r)^p \frac{dr}{r} \right] \\ &= \frac{\omega_n}{(p+1)(\log 1/\epsilon)^p} \left[\int_{\epsilon^2}^{\epsilon} \frac{d}{dr} \left(\log r/\epsilon^2 \right)^{p+1} dr - \int_\epsilon^1 \frac{d}{dr} (\log 1/r)^{p+1} dr \right] \\ &= \frac{2\omega_n}{(p+1)} (\log 1/\epsilon). \end{aligned} \tag{2.17}$$

Since $\epsilon > 0$ is sufficiently small, we have the following estimates, after a change of variables and the use of Neumann series:

$$\begin{aligned}
\int_B |\nabla u_\epsilon|^p dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\left[\int_{\epsilon^2}^{\epsilon} \left|\alpha \log r/\epsilon^2 - 1\right|^p \frac{dr}{r} + \int_{\epsilon}^{1} |1+\alpha \log 1/r|^p \frac{dr}{r}\right] \\
&= \frac{\omega_n}{\alpha\,(\log 1/\epsilon)^p}\left[\int_0^1 r^p\,dr + \int_0^{\alpha \log 1/\epsilon - 1} r^p\,dr + \int_1^{\alpha \log 1/\epsilon + 1} r^p\,dr\right] \\
&= \frac{\beta_{n,p}\omega_n}{(p+1)} \log 1/\epsilon \left[\left(1 - \frac{1}{\alpha \log 1/\epsilon}\right)^{p+1} + \left(1 + \frac{1}{\alpha \log 1/\epsilon}\right)^{p+1}\right] \\
&= \frac{2\beta_{n,p}\omega_n}{(p+1)} \log 1/\epsilon + O(\frac{1}{\log 1/\epsilon}). \qquad (2.18)
\end{aligned}$$

Therefore from (2.17) and (2.18), we conclude that

$$\int_B |\nabla u_\epsilon|^p dx - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u_\epsilon(x)|^p}{|x|^p}\,dx = O\left(\frac{1}{\log 1/\epsilon}\right). \qquad (2.19)$$

Finally, for sufficiently small $\epsilon > 0$, let us estimate the following integral,

$$\begin{aligned}
\int_B \frac{|u_\epsilon(x)|^p}{|x|^p} (\log R/|x|)^{-\gamma}\,dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\left[\int_{\epsilon^2}^{\epsilon} \frac{\left(\log r/\epsilon^2\right)^p}{r\,(\log R/r)^\gamma} dr + \int_{\epsilon}^{1} \frac{(\log 1/r)^p}{r\,(\log R/r)^\gamma} dr\right] \\
&\geq \frac{\omega_n}{(\log 1/\epsilon)^p}\left[\left(\log R/\epsilon^2\right)^{-\gamma} \int_{\epsilon^2}^{\epsilon} \left(\log r/\epsilon^2\right)^p \frac{dr}{r}\right. \\
&\quad \left. + (\log R/\epsilon)^{-\gamma} \int_{\epsilon}^{1} (\log 1/r)^p \frac{dr}{r}\right] \\
&\geq \frac{\omega_n}{(p+1)} \log 1/\epsilon \left[\left(\log R/\epsilon^2\right)^{-\gamma} + (\log R/\epsilon)^{-\gamma}\right] \\
&\geq \frac{\omega_n}{(p+1)} \log 1/\epsilon \left[\left(2 \log 1/\epsilon^2\right)^{-\gamma} + (\log 1/\epsilon)^{-\gamma}\right] \\
&= C\,(\log 1/\epsilon)^{1-\gamma}. \qquad (2.20)
\end{aligned}$$

Since $0 < \gamma < 2$, $I_\gamma(u_\epsilon) \to 0$, as $\epsilon \to 0$ and hence the theorem for the case $1 < p < n$.

For the case $p = n$, by taking the same test functions u_ϵ with $p = n$, it is easy to verify that $\int_B |\nabla u_\epsilon|^n dx = \frac{2\omega_n}{(n+1)} (\log 1/\epsilon)^{1-n}$ and $\int_B \frac{|u_\epsilon(x)|^n}{|x|^n} (\log R/|x|)^{-\gamma}\,dx \geq C\,(\log 1/\epsilon)^{1-\gamma}$ and hence the optimality. □

Proof of Corollary 1.1. In the proof of Theorem 1.1, we have used only the first term of the series in 2-dimensional Hardy inequality. By making use of Lemma 2.2 and Corollary 2.1 in its full generality, the Corollary will follow. In particular, in (2.8), we use the inequality (2.4) for the case $2 \leq p < n$ and in (2.10), we use the inequality (2.3) for the case $1 < p < 2$. But in this case, the proof is a bit more cumbersome because of the presence of more boundary terms in (2.10), which can be again handled by considering iterative functions similar to the ones defined in the proof of Lemma 2.2 and following the same line of proof of Theorem 1.1. □

Proof of Corollary 1.2. If $f \in F_p$, then

$$\lim_{\epsilon \to 0} \sup_{x \in B_\epsilon} f(x)|x|^p(\log(1/|x|)^2 \ < \ \infty$$

and hence for sufficiently small ϵ, in B_ϵ

$$f(x) \ < \ \frac{C}{|x|^p(\log 1/|x|)^2} \ .$$

Outside B_ϵ, both are bounded functions and hence C can be so chosen that this inequality holds in Ω. Then (1.6) will follow from (1.4).

If $f \ \notin \ F_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then we can write $f(x) = h(x)/|x|^p(\log 1/|x|)^2$, where $h(x)$ tends to infinity as x tends to 0. Then from the calculations of theorem 1.1, for $\epsilon > 0$ sufficiently small

$$\begin{aligned}
\int_B \frac{|u_\epsilon(x)|^p \ h(x)}{|x|^p(\log(1/|x|)^2)} \ dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\Bigg[\int_{B_\epsilon \setminus B_{\epsilon^2}} \frac{\left(\log |x|/\epsilon^2\right)^p \ h(x)}{|x|^n \ (\log R/|x|)^2} \ dx \\
&\quad + \int_{B_1 \setminus B_\epsilon} \frac{(\log 1/|x|)^p \ h(x)}{|x|^n \ (\log R/|x|)^2} dx\Bigg] \\
&\geq \frac{\omega_n \ \inf_{B_\epsilon \setminus B_{\epsilon^2}} h(x)}{(\log 1/\epsilon)^p} \left(\log R/\epsilon^2\right)^{-2} \int_{\epsilon^2}^{\epsilon} \left(\log r/\epsilon^2\right)^p \frac{dr}{r} \\
&\geq \frac{\omega_n \ m(\epsilon) \log 1/\epsilon}{(p+1)\left(\log R/\epsilon^2\right)^2} \\
&\geq \frac{\omega_n \ m(\epsilon) \log 1/\epsilon}{(p+1)\left(2\log 1/\epsilon^2\right)^2} \\
&= C\frac{m(\epsilon)}{\log 1/\epsilon}.
\end{aligned}$$

Here m_ϵ stands for $\inf_{B_\epsilon \setminus B_{\epsilon^2}} \ h(x)$. Since m_ϵ tends to ∞ as $\epsilon \to 0$, we conclude that $I_f(u_\epsilon) \to 0$ as $\epsilon \to 0$ and the inequality (1.6) cannot hold for such $f \notin F_p$. $\square$

3. Proof of Theorem 1.2

In defining the eigenvalues for the operator $(\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u)$, in a given domain $\Omega \subseteq \mathbb{R}^n$, we interpret the first equation in (1.7) in the following weak sense:

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle dx = \lambda \int_\Omega |u|^{p-2}u\phi dx \tag{3.1}$$

whenever $\phi \in W_0^{1,p}(\Omega)$.

In order to prove the theorem we need the following two lemmas, the first one is due to Boccardo and Murat [BM] and the second one is a standard result from measure theory (see for example, [S], Chapter 1, section 4). The use of lemma 3.1 here is inspired by the arguments in [GP].

Lemma 3.1(see Theorem 2.1 in [BM]) If $(u_m)_{m\in\mathbf{N}} \subset W_0^{1,p}(\Omega)$ be such that as $k \to \infty$, $u_m \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and satisfy

$$-\big(\mathrm{div}(|\nabla u_m|^{p-2}\nabla u_m)\big) \;=\; f_m \;+\; g_m$$

in $D'(\Omega)$ where $f_m \to 0$ strongly in $W^{-1,p'}(\Omega)$ and g_m is bounded in $M(\Omega)$, the space of Radon measures, i.e

$$|\; <\; g_m, \phi\; >\; | \le C_K\; \|\phi\|_\infty$$

for all $\phi \in D(\Omega)$ with $\mathrm{supp}(\phi) \subset K$, then there exists a subsequence say, u_{m_k} such that

$$u_{m_k} \to u \;\text{ strongly }\; \text{in } W_0^{1,q}(\Omega)\;\forall\; q < p.$$

Lemma 3.2. Let $(g_m)_{m\in\mathbf{N}} \subset L^p(\Omega)$, $1 \le p < \infty$, be such that, as $m \to \infty$, (i) $g_m \rightharpoonup g$ weakly in $L^p(\Omega)$ and (ii) $g_m(x) \to g(x)$ a.e. in Ω, then

$$\lim_{m\to\infty}\big[\|g_m\|_p^p - \|g_m - g\|_p^p\big] = \|g\|_p^p$$

Proof of theorem 1.2. We look for the critical points of the functional

$$J_\mu(u) := \frac{1}{p}\int\limits_\Omega |\nabla u|^p - \frac{\mu}{p}\int\limits_\Omega \frac{|u(x)|^p}{|x|^p}dx$$

which is continuous, Gateaux differentiable and coercive on $W_0^{1,p}(\Omega)$ thanks to Hardy-Sobolev inequality. We minimize this functional J_μ over the manifold, $M = \left\{u \in W_0^{1,p}(\Omega)\Big|\; \int_\Omega |u(x)|^p f(x)dx = 1\right\}$ and let λ_μ^1 be the infimum. It is clear that $\lambda_\mu^1 > 0$. By standard arguments, we can choose a special minimizing sequence $(u_m)_{m\in\mathbf{N}} \subset M$ with $J_\mu(u_m) \to \lambda_\mu^1$ and the component of $DJ_\mu(u_m)$ restricted to M, tends to 0 strongly in $W^{-1,p'}(\Omega)$. The coercivity of J_μ implies that $(u_m)_{m\in\mathbf{N}}$, is a bounded sequence and hence we have for a subsequence, as $k \to \infty$

$$\begin{aligned} &u_{m_k} \rightharpoonup u \quad \text{weakly in } \; W_0^{1,p}(\Omega)\\ &u_{m_k} \rightharpoonup u \quad \text{weakly in } \; L^p(\Omega, |x|^{-p})\\ &u_{m_k} \to u \quad \text{strongly in } \; L^p(\Omega)\,. \end{aligned} \tag{3.2}$$

Since $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega, f(x))$ (see Proposition 2.1 in [CR]), it follows that M is weakly closed and hence $u \in M$. Further u_m satisfies in $D'(\Omega)$

$$-\left(\mathrm{div}(|\nabla u_m|^{p-2}\nabla u_m) + \frac{\mu}{|x|^p}|u_m|^{p-2}u_m\right) = \lambda_m(|u_m|^{p-2}u_m\, f\,) + f_m$$

where $f_m \to 0$ strongly in $W^{-1,p'}(\Omega)$ and $\lambda_m \to \lambda$ as $m \to \infty$. Calling $g_m = \dfrac{\mu}{|x|^p}|u_m|^{p-2}u_m + \lambda_m(|u_m|^{p-2}u_m\, f\,)$, one can check that g_m is bounded in

$M(\Omega)$. Then we can use lemma 3.1 to conclude a.e convergence of ∇u_{m_k} to ∇u in Ω and then apply lemma 3.2 to u_{m_k} and also to ∇u_{m_k} to obtain,

$$\begin{aligned}\lambda_\mu^1 &= \|\nabla(u_{m_k}-u)\|_p^p - \mu\|u_{m_k}-u\|^p_{L^p(\Omega,|x|^{-p})} + \|\nabla u\|_p^p - \mu\|u\|^p_{L^p(\Omega,|x|^{-p})} + o(1)\\ &\geq (\beta_{n,p}-\mu)\,\|u_{m_k}-u\|^p_{L^p(\Omega,|x|^{-p})} + \lambda_\mu^1 + o(1)\end{aligned}$$

where, $o(1) \to 0$ as $k \to \infty$. As, $\mu < \beta_{n,p}$, we conclude that $\|(u_{m_k} - u)\|^p_{L^p(\Omega,|x|^{-p})} \to 0$ as $k \to \infty$ and also $\|\nabla(u_{m_k} - u)\|^p_{L^p(\Omega)} \to 0$ as $k \to \infty$ and hence we have $J_\mu(u) = \lambda_\mu^1$ and $\lambda = \lambda_\mu^1$. Since $J_\mu(|u|) = J_\mu(u)$, we can take $u > 0$ in Ω. Then using lemma 3.1, u is a distribution solution of (1.7) and since $u \in W_0^{1,p}(\Omega)$, it is a weak solution to the eigenvalue problem (1.7), corresponding to $\lambda = \lambda_\mu^1$. The simplicity of λ_μ^1, can be proved exactly as in [PL], where the eigenvalue problem with $f = 1$ is studied. Moreover, if $f \in F_p$, by corollary 1.2, we have

$$\lambda_\mu^1(f) \to \lambda(f) = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega \left(|\nabla u|^p - \beta_{n,p}\frac{|u(x)|^p}{|x|^p}\right)dx}{\int\limits_\Omega |u(x)|^p f(x)dx} \;>\; 0 \;\text{ as }\; \mu \to \beta_{n,p}.$$

If $f \notin F_p$, then again by corollary 1.2, inequality of the type (1.5) cannot hold and hence $\lambda(f) = 0$. □

4. Appendix

Lemma 2.1. *If $p \geq 2$, then there exist positive constants B and C such that*

$$(1+x)^p \geq 1 + px + Cx^2 + B|x|^p \quad \forall\ x \geq -1. \tag{4.1}$$

If $1 < p < 2$ and $M \geq 1$,

$$(1+x)^p \geq \begin{cases} 1 + px + 2^{p-4}p(p-1)M^{p-2}\,x^2, & \text{for } -1 \leq x \leq M\\ 1 + px + 2^{p-3}p(p-1)\,x^p, & \text{for } x \geq M.\end{cases} \tag{4.2}$$

Proof. First let us assume $p \geq 2$, for $x \geq -1$, define $h_x(t) := (1+tx)^p$, for all $0 \leq t \leq 1$. Then $h \in C^2[0,1]$ and hence by Taylors expansion we have

$$h_x(t) = h_x(0) + t\,h_x'(0) + t^2\int_0^1 h_x''(\theta t)(1-\theta)d\theta$$

therefore

$$\begin{aligned} h_x(1) &= 1 + px + p(p-1)\,x^2\int_0^1 (1+tx)^{p-2}(1-t)dt\\ &\geq 1 + px + p(p-1)\,x^2\int_0^1 (1-t)^{p-1}dt\\ &= 1 + px + (p-1)\,x^2 \end{aligned} \tag{4.3}$$

Let $0 < \alpha < 1$ and choose $M > 1$, such that for $x \geq M$ such that, $\alpha x^p \geq 1 + px + (\alpha/2)x^2$. Then for $-1 \leq x \leq M$

$$\begin{aligned}(1+x)^p &\geq 1 + px + (p-1)\,x^2 \\ &= 1 + px + (\alpha/2)x^2 + (p-1-\alpha/2)\,x^2 \\ &\geq 1 + px + (\alpha/2)x^2 + M^{2-p}\,(p-1-\alpha/2)\,|x|^p\end{aligned}$$

and for $x \geq M$, $(1+x)^p \geq x^p = \alpha x^p + (1-\alpha)x^p \geq 1 + px + (\alpha/2)x^2 + (1-\alpha)x^p$. Combining these inequalities by replacing $C = \alpha/2$ and $B = \min\left\{1-\alpha, M^{2-p}(p-1-\alpha)\right\}$ to we obtain (4.1).

Let us consider the case $1 < p < 2$ and take the function h_x for $x > -1$, as above. Notice that $1 + tx \neq 0$ and hence h_x'' exists and we have

$$(1+x)^p = 1 + px + p(p-1)\,x^2 \int_0^1 (1+tx)^{p-2}(1-t)dt \tag{4.4}$$

Now for the case $-1 < x \leq M$, from (4.4) we get

$$\begin{aligned}(1+x)^p &\geq 1 + px + p(p-1)\,x^2 \int_0^{1/2} (1+tx)^{p-2}(1-t)dt \\ &\geq 1 + px + p(p-1)\,x^2 \int_0^{1/2} (2M)^{p-2}(1-t)dt \\ &= 1 + px + \frac{p(p-1)}{4}\,(2M)^{p-2}x^2 \\ &= 1 + px + 2^{p-4}p(p-1)M^{p-2}\,x^2. \qquad (4.5)\end{aligned}$$

Since $M \geq 1$, for the case $x \geq M$, we have from (4.4)

$$\begin{aligned}(1+x)^p &\geq 1 + px + p(p-1)\,x^2 \int_0^1 (2x)^{p-2}(1-t)dt \\ &= 1 + px + 2^{p-3}p(p-1)\,x^p. \qquad (4.6)\end{aligned}$$

Observe that for $x = -1$, the inequalities (4.5) and (4.6) are trivially true and hence the lemma. □

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