A characterization of graphs which can be approximated in area by smooth graphs

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We show that graphs of vector valued maps can be approximated weakly as currents and in area by graphs of smooth maps if, in case of dimension $n \geq 3$, the 2-dimensional graph of the restriction to a.e. 2-plane intersecting the domain is approximable by smooth graphs.

This paper deals with the relaxed extension of the nonparametric area functional for vector valued maps.

Let Ω be a bounded domain in $\mathbf{R}^n$, $n, N \geq 2$, and $u : \Omega \to \mathbf{R}^N$ be a smooth map. The area of the graph of u over Ω is given by

$$\mathrm{A}(u, \Omega) := \int_\Omega |M(Du)| \, dx \tag{1}$$

where

$$|M(Du)| := (1 + |Du|^2 + |M_{(2)}(Du)|^2 + \cdots + |M_{(\underline{n})}(Du)|^2)^{1/2}$$

is the square root of the sum of the squares of the determinants of all minors of the Jacobian matrix Du up to the order $\underline{n} := \min(n, N)$.

In the same spirit as Lebesgue's area for continuous functions, the *relaxed area* of the "graph" of an L^1- function $u : \Omega \to \mathbf{R}^N$ is defined by

$$\overline{\mathrm{A}}(u, \Omega) := \inf\{ \liminf_{k\to+\infty} \mathrm{A}(u_k, \Omega) \mid u_k \in C^1(\Omega, \mathbf{R}^N), \, u_k \to u \text{ in } L^1 \} .$$

Denote by $\mathcal{A}^1(\Omega, \mathbf{R}^N)$ the class of maps $u \in W^{1,1}(\Omega, \mathbf{R}^N)$ such that all minors of Du are summable in Ω. If $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$, we can still define $\mathrm{A}(u, \Omega)$ by (1), so that due to the lower semicontinuity of $\mathrm{A}(u, \Omega)$ w.r.t. the L^1-convergence, see e.g. [1], one has

$$\mathrm{A}(u, \Omega) \leq \overline{\mathrm{A}}(u, \Omega) .$$

For $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$, it can be well defined the integer multiplicity (i.m.) rectifiable current $G_u \in \mathcal{R}_n(\Omega \times \mathbf{R}^N)$ "carried" by the graph of u, see [3], [7] and [8, Vol. I, Ch. 3]. More precisely, if $Id \bowtie u : \Omega \to \Omega \times \mathbf{R}^N$ is the join map $(Id \bowtie u)(x) := (x, u(x))$, taking approximate derivatives we have

$$G_u := (Id \bowtie u)_\# [\![\Omega]\!]$$

with mass equal to the area of the graph of u, i.e.,

$$\mathbf{M}(G_u) = \mathrm{A}(u, \Omega) .$$

The homological condition that the boundary of G_u in $\Omega \times \mathbf{R}^N$ is non zero

$$\partial G_u \llcorner \Omega \times \mathbf{R}^N \neq 0$$

is clearly an obstruction to the weak approximation in the sense of currents of G_u by smooth graphs, compare [4] and [5] , i.e., we cannot find C^1 maps $u_k : \Omega \to \mathbf{R}^N$ such that

$$G_{u_k}(\omega) \to G_u(\omega) \qquad \forall\, \omega \in \mathcal{D}^n(\Omega \times \mathbf{R}^N)$$

as $k \to +\infty$, where $\mathcal{D}^n(\Omega \times \mathbf{R}^N)$ is the space of all smooth n-forms with compact support in $\Omega \times \mathbf{R}^N$. Consequently, it is an obstruction also to the approximation in mass.

However, even in the special case of *Cartesian maps* $u \in \operatorname{cart}^1(\Omega, \mathbf{R}^N)$, where

$$\operatorname{cart}^1(\Omega, \mathbf{R}^N) := \{u \in W^{1,1}(\Omega, \mathbf{R}^N) \mid |M(Du)| \in L^1,\, \partial G_u \llcorner \Omega \times \mathbf{R}^N = 0\}\,,$$

it is shown in [6], see also [10], that in general

$$\mathrm{A}(u, \Omega) < \overline{\mathrm{A}}(u, \Omega)\,.$$

More precisely, there exist maps $u \in \operatorname{cart}^1(B^2, \mathbf{R}^2)$, B^2 being the unit ball in $\mathbf{R}^2$, which are smooth, C^∞, outside the origin, such that for any sequence of smooth maps $u_k : \Omega \to \mathbf{R}^2$, with graphs G_{u_k} weakly converging to G_u , one has

$$\liminf_{k \to +\infty} \mathbf{M}(G_{u_k}) > \mathbf{M}(G_u)\,.$$

These facts lead us to introduce the notion of relaxed area of the graph of a map $u \in \operatorname{cart}^1(\Omega, \mathbf{R}^N)$ w.r.t. the weak convergence as currents

$$\begin{aligned}\widetilde{\mathrm{A}}(u, \Omega) := \inf\{ \liminf_{k \to +\infty} \mathrm{A}(u_k, \Omega) \quad | \quad & u_k \in C^1(\Omega, \mathbf{R}^N) \quad \text{and} \\ & G_{u_k} \rightharpoonup G_u \quad \text{weakly in } \mathcal{D}_n(\Omega \times \mathbf{R}^N)\,\}\,,\end{aligned}$$

so that clearly

$$\mathrm{A}(u, \Omega) \le \overline{\mathrm{A}}(u, \Omega) \le \widetilde{\mathrm{A}}(u, \Omega)\,.$$

In this paper we give a characterization of the class of maps $u : \Omega \to \mathbf{R}^N$, Ω being a bounded domain in $\mathbf{R}^n$, $n \ge 3$, $N \ge 2$, for which $\mathrm{A}(u, \Omega) = \widetilde{\mathrm{A}}(u, \Omega)$, i.e., which can be approximated weakly as currents and in area by graphs of smooth maps. It comes out that the obstructions to approximation can be read on the 2-dimensional slices of the domain Ω . To this aim, let E_2 be any coordinate 2-dimensional plane of $\mathbf{R}^n$ and, for $x \in (E_2)^\perp$, let Π_x denote the affine plane parallel to E_2 and containing x , $\Pi_x := x + E_2$. Also, in case $\widetilde{\Omega}_x := \Pi_x \cap \Omega \neq \emptyset$, denote by $u_{|\widetilde{\Omega}_x} : \widetilde{\Omega}_x \to \mathbf{R}^N$ the restriction of u to $\widetilde{\Omega}_x$. We are going to prove the following

Main Result *Let* $u : \Omega \to \mathbf{R}^N$ *be a vector valued map defined on a bounded domain* $\Omega \subset \mathbf{R}^n$*, with* $n \ge 3$ *and* $N \ge 2$ *. Then,* u *can be approximated weakly as currents and in area by graphs of smooth maps if and only if* u *is in* $\operatorname{cart}^1(\Omega, \mathbf{R}^N)$ *and for any coordinate plane* E_2 *of* $\mathbf{R}^n$*, for*

$\mathcal{H}^{n-2}$*-a.e.* $x \in (E_2)^\perp$, *the 2-dimensional graph of* $u_{|\widetilde{\Omega}_x}$ *can be approximated by smooth graphs, i.e.,*

$$\mathrm{A}(u_{|\widetilde{\Omega}_x}, \widetilde{\Omega}_x) = \widetilde{\mathrm{A}}(u_{|\widetilde{\Omega}_x}, \widetilde{\Omega}_x)\,. \tag{2}$$

Results in this direction are given by the author in [11] and [12], whereas in [13] it is shown that there exist Cartesian maps in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$ such that for any sequence of smooth maps with graphs weakly converging to G_u, then the masses cannot be equibounded. It is an open question to give necessary and sufficient conditions to approximation in case of dimension $n = 2$, compare Remark 1 in Sec. 1.

The rest of the paper divides in four sections. In Sec. 1 we prove the "only if" part and study some local properties of Cartesian maps for which (2) holds. In Sec. 2 we give the geometric description of the local approximation argument. In Sec. 3 we recall the notion of *Whitney family* of cubes associated to a bounded open set $U \subset \mathbf{R}^n$ and describe a suitable partition of U related to such a family of cubes and to a given mapping $u : \Omega \to \mathbf{R}^N$, $\Omega \supset U$, in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$. In Sec. 4 we finally prove the "if" part.

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1. Necessary condition and local properties of approximable maps

Let $u : \Omega \to \mathbf{R}^N$, be a vector valued map defined on a bounded domain $\Omega \subset \mathbf{R}^n$, with $n \geq 3$ and $N \geq 2$. Suppose that $\mathrm{A}(u, \Omega) = \widetilde{\mathrm{A}}(u, \Omega)$, i.e., there exists a sequence $u_k : \Omega \to \mathbf{R}^N$ of smooth maps for which

$$\begin{cases} G_{u_k} \rightharpoonup G_u & \text{weakly in } \mathcal{D}_n(\Omega \times \mathbf{R}^N) \\ \lim\limits_{k \to +\infty} \mathbf{M}(G_{u_k}) = \mathbf{M}(G_u)\,. \end{cases} \tag{3}$$

Then, clearly $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$. Moreover, as noticed in [1], a theorem due to Reshetnyak, see [14], yields that (3) is equivalent to

$$\begin{cases} u_k \to u & \text{strongly in } L^1 \\ M(Du_k) \to M(Du) & \text{strongly in } L^1\,. \end{cases} \tag{4}$$

Then, by Fubini's theorem, if E_2 is any coordinate plane of $\mathbf{R}^n$, $x \in (E_2)^\perp$ and $\Pi_x := x + E_2$, see notations in the introduction, by (4) one has

$$\lim_{k \to +\infty} \int_{\widetilde{\Omega}_x} (|u_k - u| + |M(Du_k) - M(Du)|)\, d\mathcal{H}^2 = 0 \tag{5}$$

for $\mathcal{H}^{n-2}$-a.e. $x \in (E_2)^\perp$, where $\widetilde{\Omega}_x := \Pi_x \cap \Omega \neq \emptyset$. Then by equivalence of (3) and (4) one has (2) and therefore the "only if" part holds. Also, for any such x, denote by $\widetilde{v} := v_{|\widetilde{\Omega}_x} : \widetilde{\Omega}_x \to \mathbf{R}^N$ the restriction of v to $\widetilde{\Omega}_x$, where $v = u_k, u$, thus in particular $\widetilde{u} \in \mathrm{cart}^1(\widetilde{\Omega}_x, \mathbf{R}^N)$. For fixed

$x_0 \in \widetilde{\Omega}_x$, if $\widetilde{B}_\rho(x_0)$ denotes the disk in $\Pi_{\widetilde{x}}$ of radius ρ centered at x_0 , $\widetilde{B}_\rho(x_0) := B^n(x_0,\rho)\cap\Pi_{\widetilde{x}}$, and $\partial\widetilde{B}_\rho(x_0)$ its boundary, by the coarea formula [3, 3.2.11], for $\mathcal{H}^1$-a.e. $0<\rho<\widetilde{\rho}$, $\widetilde{\rho} := \mathrm{dist}(x_0,\partial\widetilde{\Omega}_x)$, (5) yields

$$\lim_{k\to+\infty} \int\limits_{\partial\widetilde{B}_\rho(x_0)} (|\widetilde{u}_k-\widetilde{u}|+|D\widetilde{u}_k-D\widetilde{u}|)\, d\mathcal{H}^1 = 0\,, \tag{6}$$

with $\widetilde{u}_{|\partial\widetilde{B}_\rho(x_0)}$ absolutely continuous. In particular, by a.e. convergence $\widetilde{u}_k\to\widetilde{u}$ in $\partial\widetilde{B}_\rho(x_0)$, and by equi-integrability of $\{D\widetilde{u}_{k\,|\partial\widetilde{B}_\rho(x_0)}\}$, one has

$$\lim_{k\to+\infty} \|\widetilde{u}_k-\widetilde{u}\|_{\infty,\partial\widetilde{B}_\rho(x_0)} = 0 \tag{7}$$

uniformly in C^0.

Denote now by $\widetilde{h}_k : [0,1]\times\partial\widetilde{B}_\rho(x_0)\to\mathbf{R}^N$ the affine homotopy map between the traces of $\widetilde{u}$ and $\widetilde{u}_k$ over the boundary $\partial\widetilde{B}_\rho(x_0)$, i.e.,

$$\widetilde{h}_k(t,x) := t\,\widetilde{u}_k(x)+(1-t)\,\widetilde{u}(x)\,, \qquad 0\le t\le 1\,, \quad x\in\partial\widetilde{B}_\rho(x_0)\,.$$

Also, for $k>1/\rho$, denote by $\widetilde{w}_k : \widetilde{B}_\rho(x_0)\to\mathbf{R}^N$ the map

$$\widetilde{w}_k(x) := \begin{cases} \widetilde{u}_k(x_0+\frac{k\rho}{k\rho-1}(x-x_0)) & \text{if} \quad |x-x_0|\le\rho-1/k \\ \widetilde{h}_k(k(\rho-|x-x_0|),\, x_0+\rho\frac{x-x_0}{|x-x_0|}) & \text{if} \quad \rho-1/k\le|x-x_0|\le\rho\,. \end{cases}$$

With the previous notations we have

Lemma 1 *The following two facts hold:*

i) *for any $k\in\mathbf{N}$, $k>1/\rho$, the map $\widetilde{w}_k$ is in $\mathrm{cart}^1(\widetilde{B}_\rho(x_0),\mathbf{R}^N)$, continuous up to the closure of $\widetilde{B}_\rho(x_0)$, with boundary equal to the boundary of $\widetilde{u}$ in $\widetilde{B}_\rho(x_0)$*

$$\partial(G_{\widetilde{w}_k}\llcorner\widetilde{B}_\rho(x_0)\times\mathbf{R}^N) = \partial(G_{\widetilde{u}}\llcorner\widetilde{B}_\rho(x_0)\times\mathbf{R}^N) \tag{8}$$

in particular $\widetilde{w}_{k\,|\partial\widetilde{B}_\rho(x_0)} = \widetilde{u}_{|\partial\widetilde{B}_\rho(x_0)}$. Moreover, for each small $\eta>0$

$$\int\limits_{\widetilde{B}_\rho(x_0)} |M(D\widetilde{w}_k)|\, d\mathcal{H}^2 \le \int\limits_{\widetilde{B}_\rho(x_0)} |M(D\widetilde{u})|\, d\mathcal{H}^2+\eta \tag{9}$$

for k sufficiently large, where $|M(D\widetilde{u})|$ denotes the 2-dimensional Jacobian of the mapping $Id\bowtie\widetilde{u} : \widetilde{B}_\rho(x_0)\to\widetilde{B}_\rho(x_0)\times\mathbf{R}^N$ (and similarly for $\widetilde{w}_k$) ;

ii) *in case of codimension $N\ge 3$, if*

$$T_k := \widetilde{w}_{k\#}[\![\,\widetilde{B}_\rho(x_0)\,]\!] - \widetilde{u}_{\#}[\![\,\widetilde{B}_\rho(x_0)\,]\!]\,, \tag{10}$$

there exists an i.m. rectifiable current $R_k\in\mathcal{R}_3(\mathbf{R}^N)$ with boundary T_k , $\partial R_k = T_k$, and arbitrary small mass, i.e.,

$$\mathbf{M}(R_k)\le\eta \tag{11}$$

for k sufficiently large.

Remark 1 Clearly Lemma 1 holds even in case of dimension $n = 2$ (with $\widetilde{\Omega}_x = \Omega$). Moreover, with an obvious adaptation of the "if" part, one clearly infers that, in case $n = 2$, u is approximable if it is in cart^1 and for each suitable small rectangle F, one can substitute u on F by a continuous map which has the same boundary and quite the same area of u on F, compare i) and (16) in Sec. 2. Note, anyway, that condition i) in Sec. 2 obviously does not hold for the non approximable maps of [6], compare [11].

Before proving Lemma 1, we give the following

Definition 1 We say that a map $w : U \subset \mathbf{R}^n \to \mathbf{R}^N$, where U is either a polyhedral domain F of dimension $k \leq n$ in $\mathbf{R}^n$ or its boundary ∂F or $U = [0,1] \times \partial F$, is a Cartesian map in $\mathrm{cart}^1(U, \mathbf{R}^N)$, if it is in $W^{1,1}(U, \mathbf{R}^N)$, with all minors of the Jacobian matrix of Dw in L^1, where Dw is the differential of w w.r.t. an O.N. basis for U, and also, if

$$G_w = G_w \llcorner U \times \mathbf{R}^N = (Id \bowtie w)_\# [\![U]\!]$$

is the i.m. rectifiable current in $\mathcal{R}_k(\mathbf{R}^n \times \mathbf{R}^N)$ carried by the graph $Id \bowtie w(U)$, where Id is the identity map on $\mathbf{R}^n$, we have $\partial G_w \llcorner \mathrm{int}(U) \times \mathbf{R}^N = 0$ in case $\partial U \neq \emptyset$, where $\mathrm{int}(U) = \mathrm{clos}(U) \setminus \partial U$, and $\partial G_w = 0$ in case U is a boundary, $U = \partial F$. Then, if $|M(Dw)|$ denotes the k-dimensional Jacobian of the join map $Id \bowtie w : U \to U \times \mathbf{R}^N$, with $k = \dim(U)$, i.e.,

$$|M(Dw)| := (1 + |Dw|^2 + |M_{(2)}(Dw)|^2 + \cdots + |M_{(\underline{n})}(Dw)|^2)^{1/2}$$

where $|M_{(j)}(Dw)|^2$ is the sum of the squares of the determinants of the minors of order j of the Jacobian matrix of Dw, for $j \leq \underline{n} := \min(k, N)$, we have

$$\mathbf{M}(G_w \llcorner U \times \mathbf{R}^N) = \int_U |M(Dw)| \, d\mathcal{H}^k.$$

In particular, if the restriction of w to the boundary of U is a map in $\mathrm{cart}^1(\partial U, \mathbf{R}^N)$, one has

$$\partial(G_w \llcorner U \times \mathbf{R}^N) = G_w \llcorner \partial U \times \mathbf{R}^N = (Id \bowtie w)_\# [\![\partial U]\!] \in \mathcal{R}_{k-1}(\mathbf{R}^n \times \mathbf{R}^N)$$

and

$$\partial w_\# [\![U]\!] = w_\# [\![\partial U]\!], \tag{12}$$

compare 3.2.5 and 3.2.3 of [8, Vol. I].

Proof of Lemma 1: To show (9), by changing variables and using (5)

$$\begin{aligned} \int_{\widetilde{B}_{\rho-1/k}(x_0)} |M(D\widetilde{w}_k)| \, d\mathcal{H}^2 &\leq \int_{\widetilde{B}_\rho(x_0)} |M(D\widetilde{u}_k)| \, d\mathcal{H}^2 \\ &\leq \int_{\widetilde{B}_\rho(x_0)} |M(D\widetilde{u})| \, d\mathcal{H}^2 + \eta \end{aligned} \tag{13}$$

for k large. Moreover, if $|M_{(2)}(D\widetilde{h}_k)|$ denotes the 2-dimensional Jacobian of the differential of $\widetilde{h}_k$ w.r.t. t and to the tangential direction τ to $\partial\widetilde{B}_\rho(x_0)$, we have

$$\int\limits_{\widetilde{B}_\rho(x_0)\setminus\widetilde{B}_{\rho-1/k}(x_0)} |M(D\widetilde{w}_k)|\,d\mathcal{H}^2 \le \int\limits_0^1 dt \int\limits_{\partial\widetilde{B}_\rho(x_0)} \sqrt{|\widetilde{u}_k-\widetilde{u}|^2+|M_{(2)}(D\widetilde{h}_k)|^2}\,d\mathcal{H}^1 + O(1/k) \tag{14}$$

which is small by (6) and (7), for k large, since for a.e. $(t,x)\in[0,1]\times\partial\widetilde{B}_\rho(x_0)$

$$|M_{(2)}(D\widetilde{h}_k(t,x))| \le |\widetilde{u}_k(x)-\widetilde{u}(x)|\,|t\,D_\tau\widetilde{u}_k(x)+(1-t)\,D_\tau\widetilde{u}(x)|\,.$$

Then (13) and (14) yield (9), whereas (8) holds since by the homotopy formula [15, 26.22] and (12)

$$\begin{aligned}\partial\widetilde{h}_{k\#}([\![0,1]\!]\times[\![\partial\widetilde{B}_\rho(x_0)]\!]) &= \widetilde{u}_{k\#}[\![\partial\widetilde{B}_\rho(x_0)]\!]-\widetilde{u}_{\#}[\![\partial\widetilde{B}_\rho(x_0)]\!]\\ &= \partial\,\widetilde{u}_{k\#}[\![\widetilde{B}_\rho(x_0)]\!]-\partial\,\widetilde{u}_{\#}[\![\widetilde{B}_\rho(x_0)]\!]\end{aligned}$$

and then, see Definition 1,

$$\begin{aligned}\partial(G_{\widetilde{w}_k}\llcorner\widetilde{B}_\rho(x_0)\times\mathbf{R}^N) &= (Id\bowtie\widetilde{w}_k)_\#[\![\partial\widetilde{B}_\rho(x_0)]\!]\\ &= (Id\bowtie\widetilde{u})_\#[\![\partial\widetilde{B}_\rho(x_0)]\!]\,.\end{aligned}$$

Finally, to show (11) we make use of an extension of [15, 31.2]. Fix $M>0$ and an increasing sequence $\{r_j\}$ of positive radii, with $r_j\to+\infty$ as $j\to+\infty$. For $r>0$ denote by C_r the complement of the N-cube of side $2r$ of $\mathbf{R}^N$

$$C_r := \{y=(y_1,\dots,y_N)\in\mathbf{R}^N \mid \max_{1\le i\le N}|y_i|>r\}\,.$$

Also set

$$\mathcal{I} := \{T\in\mathcal{R}_2(\mathbf{R}^N) \mid \mathbf{M}(T)\le M,\ \partial T=0,\ \mathbf{M}(T\llcorner C_{r_j})\le 1/j\quad\forall j\in\mathbf{N}\}$$

and for T_1, $T_2\in\mathcal{R}_2(\mathbf{R}^N)$, with $\partial T_1=\partial T_2=0$,

$$d(T_1,T_2) := \inf\{\mathbf{M}(R)\mid R\in\mathcal{R}_3(\mathbf{R}^N)\,,\quad \partial R=T_1-T_2\}\,.$$

Lemma 2 *If $\{T_k\}$, $T\in\mathcal{I}$, then $T_k\rightharpoonup T$ weakly in $\mathcal{D}_2(\mathbf{R}^N)$ if and only if $d(T_k,T)\to 0$ as $k\to+\infty$.*

PROOF: The "if" part is trivial. To show the "only if" part, we claim that $\mathcal{I}$ is totally bounded w.r.t. the metric d. More precisely, for each small $\varepsilon>0$, we can find $m=m(\varepsilon,M)\in\mathbf{N}$ and $P_i\in\mathcal{R}_2(\mathbf{R}^N)$, $i\in\{1,\dots,m\}$, with $\partial P_i=0$ and such that

$$\mathcal{I}\subset\sum_{i=1}^m\{S\in\mathcal{I}\mid d(S,P_i)\le\varepsilon\}\,.$$

In fact, by the deformation theorem [15, 29.3], for fixed $\delta > 0$, to each $T \in \mathcal{I}$ it corresponds an integral polyhedral 2-chain P, with zero boundary $\partial P = 0$, supported in the 2-skeleton $\mathcal{L}_2$ of the grid of mesh δ in $\mathbf{R}^N$

$$\mathcal{L} = \mathcal{L}(\delta) := \bigcup_{z \in \mathbf{Z}^N} \delta\,(z + C)\,, \qquad C = [0,1]^N$$

and an i.m. rectifiable current $R \in \mathcal{R}_3(\mathbf{R}^N)$ such that

$$T = P + \partial R$$

with mass estimates

$$\mathbf{M}(P) \leq c\,\mathbf{M}(T)\,, \qquad \mathbf{M}(R) \leq c\,\delta\,\mathbf{M}(T)$$

for some absolute constant $c > 0$. If $\delta > 0$ is sufficiently small we have

$$\mathbf{M}(R) \leq c\,\delta\,M \leq \varepsilon\,.$$

Moreover, P is given by

$$P = \sum_{F \in \mathcal{L}_2} \beta_F\,[\![\, F \,]\!]\,, \qquad \beta_F \in \mathbf{Z}\,.$$

Making use of the deformation technique due to Almgren [2, 1.15], one also obtains the following local mass estimates for P

$$|\beta_F|\,\delta^2 = \mathbf{M}(P \llcorner F) \leq c\,\mathbf{M}(T \llcorner \cup \operatorname{nbs}(F)) \tag{15}$$

for each 2-face $F \in \mathcal{L}_2$. Here $\operatorname{nbs}(F)$ is the family of N-cubes of the grid $\mathcal{L}$ which contain F in their boundary. Clearly, if $F \subset C_r$, then $\cup \operatorname{nbs}(F) \subset C_{r-\delta}$, so that if $j > c\,\delta^{-2}$, with $c > 0$ given by (15), and $r > r_j + \delta$ is fixed, one has

$$\mathbf{M}(T \llcorner \cup \operatorname{nbs}(F)) \leq \mathbf{M}(T \llcorner C_{r_j}) \leq 1/j\,.$$

Then $\beta_F = 0$ for each $F \subset C_r$ and therefore $\operatorname{spt} P \cap C_r = \emptyset$. Since there is only a finite number of polyhedral 2-chains P of $\mathbf{R}^N$ with boundary $\partial P = 0$, with mass $\mathbf{M}(P) \leq c\,M$ and support

$$\operatorname{spt} P \subset \bigcup \{F \in \mathcal{L}_2 \mid F \subset \mathbf{R}^N \setminus C_r\}\,,$$

the total boundedness property holds. The rest of the proof is similar to [15, 31.2]. □

To prove (11), by Lemma 2 it suffices to show that $\{T_k\} \in \mathcal{I}$. In fact, by (10) and (9), with $\eta = 1$,

$$\mathbf{M}(T_k) \leq 2\,\mathbf{M}(G_{\widetilde{u}} \llcorner \widetilde{B}_\rho(x_0) \times \mathbf{R}^N) + 1 = M$$

for k sufficiently large, whereas by (8) and (12) $\partial T_k = 0$. Moreover, (13) and (14) yield that $T_k \rightharpoonup 0$ as $k \to +\infty$. Also, by L^1-convergence of $\widetilde{w}_k$ to $\widetilde{u}$ we have

$$\sup_{k \in \mathbf{N}} \operatorname{meas}(\{x \in \widetilde{B}_\rho(x_0) \mid |\widetilde{w}_k(x)| > r\}) \to 0$$

as $r \to +\infty$, thus by equi-integrability we infer

$$\sup_{k\in\mathbf{N}} \int_{\{|\widetilde{w}_k(x)|>r\}} |M_{(2)}(D\widetilde{w}_k)|\,dx \le \varepsilon$$

for $r \ge r(\varepsilon)$. Then, for a suitable increasing sequence $\{r_j\}$, we have $\sup_{k\in\mathbf{N}} \mathbf{M}(T_k \llcorner C_{r_j}) \le 1/j$ for each $j \in \mathbf{N}$, as required. $\square$

2. Local approximation argument

Let now u be a map in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$ which can be approximated by smooth maps. In this section we describe how to substitute the graph of u over small pieces of the domain Ω by the graph of a nice map v without creating discontinuities and with a suitable control of the area, compare (19) .

To this aim, let $Q := [-\delta, \delta]^n$ denote the standard n-dimensional cube of side 2δ , for fixed small $0 < \delta < 1$. Also, let $Q_{(k)}$, $k \in \{0, \dots, n\}$, denote the k-dimensional skeleton of Q . The n-cube is supposed to be in *generic position* w.r.t. the map $u : \Omega \to \mathbf{R}^N$ in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$, where we suppose $Q \subset \Omega$. This means that for each $(n-j)$-face K of the $(n-j)$-skeleton $Q_{(n-j)}$, $j \in \{1, \dots, n-2\}$, the restriction of u to K , $u_K := u_{|K} : K \to \mathbf{R}^N$, is a map in $\mathrm{cart}^1(K, \mathbf{R}^N)$. In particular, $\partial G_{u_K} \llcorner \mathrm{int}(K) \times \mathbf{R}^N = 0$ and

$$\int_K |M(Du_K)|\, d\mathcal{H}^{n-j} < +\infty$$

(compare Definition 1 with $U = K$, $w = u_K$ and $k = n - j$) . Also, for each 1-face I of $Q_{(1)}$, we require that the restriction $u_I := u_{|I}$ is a map in $W^{1,1}(I, \mathbf{R}^N)$.

By an obvious generalization of Lemma 1, up to a slight displacement of the faces of the cube Q , one infers that for each 2-face F of $Q_{(2)}$ and for each small $\eta > 0$, denoting by $u_F := u_{|F} : F \to \mathbf{R}^N$ the restriction of u to F, the following conditions hold:

i) there exists a map $v_F : F \to \mathbf{R}^N$ in $\mathrm{cart}^1(F, \mathbf{R}^N)$, continuous up to the closure of F, with boundary equal to the boundary of u_F

$$\partial(G_{v_F} \llcorner F \times \mathbf{R}^N) = \partial(G_{u_F} \llcorner F \times \mathbf{R}^N)$$

in particular $v_{F|\partial F} = u_{F|\partial F}$, such that

$$\int_F |M(Dv_F)|\, d\mathcal{H}^2 \le \int_F |M(Du_F)|\, d\mathcal{H}^2 + \eta\,; \tag{16}$$

ii) in case of codimension $N \ge 3$, there exists an i.m. rectifiable current $R_F \in \mathcal{R}_3(\mathbf{R}^N)$ with boundary

$$\partial R_F = v_{F\#}[\![\, F \,]\!] - u_{F\#}[\![\, F \,]\!] \tag{17}$$

and small mass

$$\mathbf{M}(R_F) \le \eta\,. \tag{18}$$

Remark 2 In (16), $|M(Dv_F)|$ denotes the 2-dimensional Jacobian of the differential of the join map $Id \bowtie v_F$ (the same for u_F), see Definition 1. Moreover, the i.m. rectifiable current $[\![F]\!] \in \mathcal{R}_2(\mathbf{R}^n)$ is equipped with the natural orientation induced by the canonical basis of $\mathbf{R}^n$. Also, (17) makes sense since u_F and v_F are Cartesian maps with the same boundary, thus by (12)

$$\partial v_{F\#}[\![F]\!] = v_{F\#}[\![\partial F]\!] = u_{F\#}[\![\partial F]\!] = \partial u_{F\#}[\![F]\!] .$$

We show that to define a suitable approximating map for u on Q, it suffices, by i), to substitute u on the 2-faces of Q and, by ii), to fill the "holes" this way produced. In fact, thanks to the previous hypotheses i) and ii), we infer the following

Proposition 1 *For each small $\varepsilon > 0$ there exists a map $v : Q \to \mathbf{R}^N$ in $\mathrm{cart}^1(Q, \mathbf{R}^N)$, continuous up to the closure of Q, such that the restriction of v to each $(n-j)$-face K of $Q_{(n-j)}$, $j \in \{1, \ldots, n-2\}$, is a map in $\mathrm{cart}^1(K, \mathbf{R}^N)$ which only depends on the restriction $u_K := u_{|K}$ of u to K, v coincides with u on the 1-skeleton $Q_{(1)}$ of Q and the following integral estimate holds*

$$\begin{aligned} \mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) \;\; &\leq \;\; \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + \varepsilon \\ &+ \;\; c \sum_{j=1}^{n-2} \delta^j \sum_{K \in Q_{(n-j)}} \int_K |M(Du_K)| \, d\mathcal{H}^{n-j} \end{aligned} \tag{19}$$

where $c > 0$ is an absolute constant.

Remark 3 It comes out from the proof (Lemma 3, case $N < k$) that, in case $N < n$, (19) can be improved by taking off the term $\mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N)$ and by limiting the summation to $j \in \{n - N, \ldots, n-2\}$.

To prove Proposition 1, we make use of the following generalization of [16, Thm. 1]

Lemma 3 *Let $v : \partial Q \to \mathbf{R}^N$ be a continuous map in $\mathrm{cart}^1(\partial Q, \mathbf{R}^N)$ defined on the boundary of the k-dimensional cube $Q := [-1,1]^k$, with $k \geq 3$ and $N \geq 2$. Also, suppose that for any j-dimensional face F of ∂Q, the restriction $v_{|F}$ is in $\mathrm{cart}^1(F, \mathbf{R}^N)$, for each $j \in \{1, \ldots, k-2\}$ (see Definition 1). Let T be an i.m. rectifiable k-current in $\mathbf{R}^k \times \mathbf{R}^N$, $T \in \mathcal{R}_k(\mathbf{R}^k \times \mathbf{R}^N)$, such that*

$$\partial T = G_v \llcorner \partial Q \times \mathbf{R}^N = (Id \bowtie v)_{\#}[\![\partial Q]\!] . \tag{20}$$

Then, for each small $\varepsilon > 0$, there exists a map $w : Q \to \mathbf{R}^N$ in $\mathrm{cart}^1(Q, \mathbf{R}^N)$, continuous up to the closure of Q, with boundary

$$\partial(G_w \llcorner Q \times \mathbf{R}^N) = G_v \llcorner \partial Q \times \mathbf{R}^N,$$

in particular $w_{|\partial Q} = v$, such that, if $N \geq k$, its mapping area is comparable to the mass of T, i.e.,

$$\int_Q |M_{(k)}(Dw)| \, dx \leq \mathbf{M}(T) + \varepsilon ,$$

where $|M_{(k)}(Dw)|$ is the k-dimensional Jacobian of w, see Definition 1 ($|detDw|$ if $N = k$), whereas, if $N < k$, $|M_{(k)}(Dw)| \equiv 0$, thus the k-dimensional mapping area of w is zero.

PROOF OF LEMMA 3: By [16, Thm. 1 and Prop. 1], if $f : \partial Q \to \mathbf{R}^{k+N}$ is a Lipschitz map and $\widetilde{T} \in \mathcal{R}_k(\mathbf{R}^{k+N})$ is an i.m. rectifiable current with $\partial \widetilde{T} = f_{\#}[\![\partial Q]\!]$, it can be performed a Lipschitz map $g : Q \to \mathbf{R}^{k+N}$ with boundary values f, $g_{|\partial Q} = f$, and with k-dimensional mapping area comparable to the mass of $\widetilde{T}$, i.e.,

$$\int_Q J_k g \, dx \le \mathbf{M}(\widetilde{T}) + \frac{\varepsilon}{2} ,$$

where $J_k g$ is the k-dimensional Jacobian of g. Then it suffices to perform a continuous homotopy map $H : [0,1] \times \partial Q \to \mathbf{R}^{k+N}$ from $Id \bowtie v$ to a Lipschitz map f

$$H(0,x) = (Id \bowtie v)(x) , \qquad H(1,x) = f(x) , \qquad x \in \partial Q$$

such that H is in $\mathrm{cart}^1([0,1] \times \partial Q, \mathbf{R}^{k+N})$, with boundary given by the homotopy formula

$$\begin{aligned} \partial H_{\#}([\![0,1]\!] \times [\![\partial Q]\!]) &= H_{\#}(\delta_1 \times [\![\partial Q]\!]) - H_{\#}(\delta_0 \times [\![\partial Q]\!]) \\ &= f_{\#}[\![\partial Q]\!] - (Id \bowtie v)_{\#}[\![\partial Q]\!] \end{aligned} \tag{21}$$

and k-dimensional mapping area less than $\varepsilon/2$. In fact, taking

$$\widetilde{T} := T + H_{\#}([\![0,1]\!] \times [\![\partial Q]\!]) ,$$

by (21) and (20) $\partial \widetilde{T} = f_{\#}[\![\partial Q]\!]$, so that if g is defined as before, we set

$$w(x) := \begin{cases} \widetilde{\pi} \circ g(2x) & \text{if} \quad \|x\| \le 1/2 \\ \widetilde{\pi} \circ H(2 - 2\|x\|, \frac{x}{\|x\|}) & \text{if} \quad 1/2 \le \|x\| \le 1 \end{cases}$$

where $\|x\| := \max_{1 \le i \le k} |x_i|$ and $\widetilde{\pi} : \mathbf{R}^k \times \mathbf{R}^N \to \mathbf{R}^N$ is the orthogonal projection onto the last N coordinates.

To define H, denote by $\mathcal{L}_j$ the j-dimensional skeleton of the grid $\mathcal{L}$ of mesh 2δ in $\mathbf{R}^{k+N}$

$$\mathcal{L} = \mathcal{L}(\delta) := \bigcup_{z \in \mathbf{Z}^{k+N}} \delta \, (z + [-1,1]^{k+N}) .$$

Similarly to [15, Sec. 29], we can define a deformation map Ψ of $\mathbf{R}^{k+N}$ onto the $(k-1)$-skeleton $\mathcal{L}_{k-1}$ such that Ψ is Lipschitz continuous on $(Id \bowtie v)(\partial Q)$ (since v is continuous on ∂Q) and such that if $h : [0,1] \times \partial Q \to \mathbf{R}^{k+N}$ is the affine homotopy between $Id \bowtie v$ and $\Psi \circ (Id \bowtie v)$

$$h(t,x) := t \cdot \Psi \circ (Id \bowtie v)(x) + (1-t) \cdot (Id \bowtie v)(x) , \qquad 0 \le t \le 1 , \quad x \in \partial Q$$

then the k-dimensional mapping area of h is small with δ. Also, by means of compositions of central projections from each j-face of the skeleton $\mathcal{L}_j$ onto its boundary, for decreasing j from $k-1$ to 1, the deformation map Ψ can be performed so that if

$$\widetilde{f} := \Psi \circ (Id \bowtie v) : \partial Q \longrightarrow \mathcal{L}_{k-1} ,$$

and $Q_{(j)}$ denotes the j-dimensional skeleton of the k-cube Q, then, for each $j \in \{0, \dots, k-1\}$, $\widetilde{f}$ takes each j-face F of $Q_{(j)}$ onto the j-skeleton $\mathcal{L}_j$ whereas (for $j \geq 1$)

$$\int_{(Id \bowtie v)(F)} J_j \Psi \, d\mathcal{H}^j \leq c \, (\mathbf{M}(G_{v_{|F}} \llcorner F \times \mathbf{R}^N) + \delta \, \mathbf{M}(\partial(G_{v_{|F}} \llcorner F \times \mathbf{R}^N))) \tag{22}$$

for some absolute constant $c > 0$, where $J_j \Psi$ is the j-dimensional Jacobian of the restriction of Ψ to $(Id \bowtie v)(F)$ and $v_{|F}$ is the restriction of v to F.

Moreover, by (22), it can be defined a continuous homotopy map $\widetilde{h} : [0,1] \times \partial Q \to \mathcal{L}_{k-1}$ from $\widetilde{f}$ to a Lipschitz map $f : \partial Q \to \mathcal{L}_{k-1}$

$$\widetilde{h}(0,x) = \widetilde{f}(x) \,, \qquad \widetilde{h}(1,x) = f(x) \,, \qquad x \in \partial Q$$

such that $\widetilde{h}$ is in $\mathrm{cart}^1([0,1] \times \partial Q, \mathbf{R}^{k+N})$ (and $\widetilde{h}$ has zero k-dimensional mapping area).
To this aim, at j^{th} step, $j \in \{1, \dots, k-1\}$, for each j-face F of $Q_{(j)}$, we can define a continuous homotopy map $\widetilde{h}_F : [0,1] \times F \to \mathcal{L}_j$ from $\widetilde{f}_{|F} : F \to \mathcal{L}_j$ to a Lipschitz map (by (22)) $f_F : F \to \mathcal{L}_j$ such that $\widetilde{h}_F([0,1] \times F) \subset \widetilde{f}(F)$, $\widetilde{h}_F$ is in $\mathrm{cart}^1([0,1] \times F, \mathbf{R}^{k+N})$ and

(a) if $j = 1$, $\widetilde{h}_F$ is constant on $[0,1] \times \partial F$;

(b) if $j > 1$, for each $(j-1)$-face I of the boundary of F, the restriction of $\widetilde{h}_F$ to $[0,1] \times I$ is equal to the homotopy $\widetilde{h}_I : [0,1] \times I \to \mathcal{L}_{j-1}$ defined at the previous step, with

$$\partial(\widetilde{h}_{F\#}([\![0,1]\!] \times [\![F]\!])) = \widetilde{h}_{F\#}(\delta_1 \times [\![F]\!]) - \widetilde{h}_{F\#}(\delta_0 \times [\![F]\!]) + \sum_{\substack{I \in Q_{(j-1)} \\ I \subset \partial F}} \sigma_j \, \widetilde{h}_{I\#}([\![0,1]\!] \times [\![I]\!])$$

if $[\![\partial F]\!] = \sum \{\sigma_j \, [\![I]\!] \mid I \in Q_{(j-1)}, I \subset \partial F\}$ for suitable $\sigma_j \in \{+1, -1\}$, $[\![F]\!]$ and $[\![I]\!]$ being equipped with the natural orientation.

Finally take

$$H(t,x) := \begin{cases} h(2t,x) & \text{if} \quad 0 \leq t \leq 1/2 \,, \quad x \in \partial Q \\ \widetilde{h}(2t-1,x) & \text{if} \quad 1/2 \leq t \leq 1 \,, \quad x \in \partial Q \,. \end{cases}$$

□

PROOF OF PROPOSITION 1: For fixed $0 < \lambda < 1$, denote by $\lambda \, Q := [-\lambda\delta, +\lambda\delta]^n$ the λ-homothetic of the n-cube $Q = [-\delta, \delta]^n$ and by $Q_{(k)}$ and $\lambda \, Q_{(k)}$ the k-skeleton of Q and $\lambda \, Q$, $k \in \{0, \dots, n\}$. Also, denote by $\Sigma_{(k)}$, for $k \in \{1, \dots n\}$, the k-dimensional subset of $Q \setminus \lambda \, Q$ given by the union of the convex hull of the corresponding $(k-1)$-faces of $Q_{(k-1)}$ and $\lambda \, Q_{(k-1)}$, i.e.,

$$\Sigma_{(k)} := \bigcup_{F \in Q_{(k-1)}} \mathrm{co}(F \cup \lambda \, F) = \{t \, x \mid \lambda \leq t \leq 1, \, x \in Q_{(k-1)}\} \,,$$

so that $Q_{(n)} = Q$, $Q_{(n-1)} = \partial Q$ and $\Sigma_{(n)} = Q \setminus \mathrm{int}(\lambda \, Q)$. For the sake of simplicity, $Q_{(k)}$ will denote either the set of points of the k-skeleton or the family of k-faces of Q.

We are going to define the approximating map v in n steps on $\Sigma_{(k)} \cup Q_{(k)}$, for growing $k \in \{1, \dots, n\}$.

1^{st} *step:* definition of v on $\Sigma_{(1)} \cup Q_{(1)}$.
Define v on $\Sigma_{(1)}$ by taking

$$v(t\,x) = u(x)\,, \qquad \lambda \le t \le 1\,, \quad x \in Q_{(0)}$$

and on $Q_{(1)}$ by

$$v(x) = u(x)\,, \qquad x \in Q_{(1)}\,.$$

2^{nd} *step:* definition of v on $\Sigma_{(2)} \cup Q_{(2)}$.
Define v on $\Sigma_{(2)}$ by taking

$$v(t\,x) = u(x)\,, \qquad \lambda \le t \le 1\,, \quad x \in Q_{(1)}\,.$$

To define v on $Q_{(2)}$, for each 2-face F of $Q_{(2)}$ take

$$v(x) = v_F(x)\,, \qquad x \in F$$

where $v_F : F \to \mathbf{R}^N$ is the continuous map in $\mathrm{cart}^1(F, \mathbf{R}^N)$ given by i), so that by (16), if $v_{|F}$ is the restriction of v to F,

$$\int_F |M(Dv_{|F})|\, d\mathcal{H}^2 \le \int_F |M(Du_F)|\, d\mathcal{H}^2 + \eta\,. \tag{23}$$

Remark 4 Note that if $F_1, F_2 \in Q_{(2)}$ are such that $\partial F_1 \cap \partial F_2 = I \in Q_{(1)}$, then the common boundary of the graphs of the $v_{|F_i}$'s coincides, i.e.,

$$\partial(G_{v_{|F_1}} \llcorner \mathrm{int}(F_1) \times \mathbf{R}^N) \llcorner I \times \mathbf{R}^N = -\partial(G_{v_{|F_2}} \llcorner \mathrm{int}(F_2) \times \mathbf{R}^N) \llcorner I \times \mathbf{R}^N.$$

3^{rd} *step:* definition of v on $\Sigma_{(3)} \cup Q_{(3)}$.
To define v on $\Sigma_{(3)}$, consider a 2-face F of $Q_{(2)}$, for simplicity

$$F = \{x \in \mathbf{R}^n \mid x = (\widetilde{x}, \widehat{\delta}\,),\ \|\widetilde{x}\|_2 \le \delta\}\,.$$

Here we denote $\widetilde{x} := (x_1, x_2) \in \mathbf{R}^2$, $\widehat{\delta} := (\delta, \dots, \delta) \in \mathbf{R}^{n-2}$, $\|\widetilde{x}\|_2 := \max_{1 \le i \le 2} |x_i|$. Making use of the maps $u_F := u_{|F}$, restriction of u to F, and v_F, see 2^{nd} step, we are going to define v on the convex hull

$$\mathrm{co}(F \cup \lambda\, F) = \{(\widetilde{x}, \widehat{t}\,) \mid \lambda\delta \le t \le \delta\,, \quad \|\widetilde{x}\|_2 \le t\}\,.$$

To this aim divide in three parts

$$\mathrm{co}(F \cup \lambda\, F) = \bigcup_{i=1}^{3} \Delta_F^i$$

and define v on the Δ_F^i's by means of implicit maps $G^{(i)}$. Set

$$\Delta_F^1 := \{(\widetilde{x}, \widehat{t}\,) \mid \delta\,(\frac{\lambda+1}{2}) < t \le \delta\,, \quad \|\widetilde{x}\|_2 \le \delta\, f(t)\} \tag{24}$$

where $f : [\delta\,(\frac{\lambda+1}{2}), \delta] \to [0,1]$ is the affine map

$$f(t) := \frac{1}{\lambda - 1}(\lambda + 1 - \frac{2}{\delta}\,t) \tag{25}$$

such that $f(\delta\,(\frac{\lambda+1}{2})) = 0\,,\; f(\delta) = 1\,.$ Moreover, using the map v_F given by i), define v on Δ^1_F by means of the implicit map

$$\begin{gathered} G^{(1)} :]\delta\,(\tfrac{\lambda+1}{2}), \delta] \times [-\delta, \delta]^2 \longrightarrow \Delta^1_F \times \mathbf{R}^N \\ G^{(1)}(t, \widetilde{x}) := ((f(t)\,\widetilde{x}, \widehat{t}\,), v_F(\,\widetilde{x}, \widehat{\delta}\,))\,. \end{gathered}$$

We then infer

$$\int_{\Delta^1_F} |M(Dv)|\, d\mathcal{H}^3 \le \sqrt{2}\,\delta \int_F |M(Dv_F)|\, d\mathcal{H}^2 + O(1-\lambda)\,. \tag{26}$$

In fact, by the area formula [3, 3.2.3]

$$\int_{\Delta^1_F} |M(Dv)|\, d\mathcal{H}^3 = \int_{\delta\,(\frac{\lambda+1}{2})}^{\delta} dt \int_{[-\delta,\delta]^2} J_3 G^{(1)}\, d\mathcal{H}^2$$

where $J_3 G^{(1)}$ is the 3-dimensional Jacobian of $G^{(1)}$. By direct computation

$$\begin{aligned} (J_3 G^{(1)})^2 &= (n-2)\, f(t)^4 + (f(t)\,\tfrac{2}{\delta\,(1-\lambda)}\, < \widetilde{x}, Dv_F >)^2 \\ &+ (n-2)\, f(t)^2\, |Dv_F|^2 + (\tfrac{2}{\delta\,(1-\lambda)})^2\, ((x_1)^2 + (x_2)^2)\, |M_{(2)}(Dv_F)|^2 \\ &+ (n-2)\, |M_{(2)}(Dv_F)|^2. \end{aligned}$$

Since $|< \widetilde{x}, Dv_F >|^2 \le ((x_1)^2 + (x_2)^2)\, |Dv_F|^2$, for λ close to 1 we obtain (26). Set

$$\Delta^2_F := \{(\widetilde{x}, \widehat{t}\,) \mid \delta\lambda \le t < \delta\,(\frac{\lambda+1}{2})\,, \quad \|\widetilde{x}\|_2 \le \delta\, f(t)\} \tag{27}$$

where $f : [\delta\lambda, \delta\,(\frac{\lambda+1}{2})] \to [0, \lambda]$ is the affine map

$$f(t) := \frac{\lambda}{\delta\,(1-\lambda)}(\delta\,(\lambda+1) - 2t) \tag{28}$$

such that $f(\delta\lambda) = \lambda$ and $f(\delta\,(\frac{\lambda+1}{2})) = 0\,.$ Moreover, using the restriction u_F of the given map u to F, define v on Δ^2_F by means of the implicit map

$$\begin{gathered} G^{(2)} : [\delta\lambda, \delta\,(\tfrac{\lambda+1}{2})[\times[-\delta, \delta]^2 \longrightarrow \Delta^2_F \times \mathbf{R}^N \\ G^{(2)}(t, \widetilde{x}) := ((f(t)\,\widetilde{x}, \widehat{t}\,), u_F(\,\widetilde{x}, \widehat{\delta}\,))\,. \end{gathered}$$

By similar computations one infers

$$\int_{\Delta^2_F} |M(Dv)|\, d\mathcal{H}^3 \le \sqrt{2}\,\delta \int_F |M(Du_F)|\, d\mathcal{H}^2 + O(1-\lambda)\,. \tag{29}$$

Set

$$\Delta_F^3 := \{(\widetilde{x},\widehat{t}\,) \mid \delta\lambda \le t \le \delta\,, \quad \delta\, f(t) < \|\widetilde{x}\|_2 \le t\} \tag{30}$$

where $f : [\delta\lambda, \delta] \to [0, \lambda]$ is the gluing of the previous maps

$$f(t) := \begin{cases} \frac{\lambda}{\delta\,(1-\lambda)}(\delta\,(\lambda+1) - 2t) & \text{if} \quad \delta\lambda \le t \le \delta\,(\frac{\lambda+1}{2}) \\ \frac{1}{\lambda-1}(\lambda + 1 - \frac{2}{\delta}\,t) & \text{if} \quad \delta\,(\frac{\lambda+1}{2}) \le t \le \delta\,. \end{cases}$$

Moreover, define v on Δ_F^3 by means of the trace of u on the boundary of F, as follows. For fixed $(\widetilde{x}, \delta) \in \partial F$, i.e., $\|\widetilde{x}\|_2 = \delta$, and for each $\sigma \in]0,1]$, v takes the constant value $u(\widetilde{x}, \widehat{\delta})$ on the parallel line segments connecting the points $(\lambda\sigma\widetilde{x}, \frac{1}{2}[(1+\lambda) + \sigma(\lambda-1)]\,\widehat{\delta}\,)$ and $(\sigma\widetilde{x}, \frac{1}{2}[(1+\lambda) - \sigma(\lambda-1)]\,\widehat{\delta}\,)$. More precisely, v is defined by the implicit map

$$G^{(3)} : \{\widetilde{x} \in \mathbf{R}^2 \mid \|\widetilde{x}\|_2 = \delta\} \times [\lambda, 1] \times]0,1] \longrightarrow \Delta_F^3 \times \mathbf{R}^N$$
$$G^{(3)}(\widetilde{x}, t, \sigma) := ((t\sigma\widetilde{x}, \tfrac{1}{2}\,[(1+\lambda) + \sigma(2t - (1+\lambda))]\,\widehat{\delta}\,), u(\,\widetilde{x}, \widehat{\delta}\,))\,.$$

Denoting by u_I the restriction of u to the 1-face I of the boundary of F, one easily infers, for some absolute constant $c > 0$,

$$\int_{\Delta_F^3} |M(Dv)|\, d\mathcal{H}^3 \le c\,\delta^2\,(1-\lambda) \sum_{\substack{I \in Q_{(1)} \\ I \subset \partial F}} \int_I |M(Du_I)|\, d\mathcal{H}^1 \tag{31}$$

which is small with λ close to 1, since u_I is in $\mathrm{cart}^1(I, \mathbf{R}^N)$.

By (16), (26), (29) and (31), we then obtain

$$\int_{\mathrm{co}(F \cup \lambda F)} |M(Dv)|\, d\mathcal{H}^3 \le 2\sqrt{2}\,\delta \int_F |M(Du_F)|\, d\mathcal{H}^2 + \eta + O(1-\lambda)\,. \tag{32}$$

Repeating the argument for each 2-face F of $Q_{(2)}$ (up to a rotation of Q), by (32) we finally infer

$$\int_{\Sigma_{(3)}} |M(Dv)|\, d\mathcal{H}^3 \le 2\sqrt{2}\,\delta \sum_{F \in Q_{(2)}} \int_F |M(Du_F)|\, d\mathcal{H}^2 + \varepsilon \tag{33}$$

for any fixed small $\varepsilon > 0$, taking λ close to 1 and $\eta > 0$ small.

We are now able to define v on the 3-skeleton $Q_{(3)}$.
Consider first the case $n = 3$. Then, since $Q = Q_{(3)}$ and $\Sigma_{(3)} = Q \setminus \mathrm{int}(\lambda\, Q)$, we set

$$v(x) = u(x/\lambda) \qquad \text{if } x \in \lambda\, Q\,. \tag{34}$$

Since $Q_{(2)} = \partial Q$, (23) yields (for $\eta > 0$ small)

$$\mathbf{M}(G_v \llcorner \partial Q \times \mathbf{R}^N) \le \sum_{F \in Q_{(2)}} \int_F |M(Du_F)|\, d\mathcal{H}^2 + \varepsilon \tag{35}$$

whereas by (33), (34) and the area formula

$$\mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) \leq \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + 2\sqrt{2}\,\delta \sum_{F \in Q_{(2)}} \int_F |M(Du_F)|\, d\mathcal{H}^2 + \varepsilon\,. \tag{36}$$

Unfortunately, v is not continuous in Q and, in case of codimension $N \geq 3$, in general

$$\partial G_v \llcorner \operatorname{int}(Q) \times \mathbf{R}^N \neq 0\,.$$

More precisely, if $[\![\, \partial Q \,]\!] = \sum_{i,j=1}^{3,2} (-1)^{i+j+1} [\![\, F_i^j \,]\!]$, where F_i^j, $j = 1, 2$, are the 2-faces parallel to the plane $\{x \in \mathbf{R}^3 \mid x_i = 0\}$, taking $p_i^j := (-1)^j \,\delta\, \frac{\lambda+1}{2}\, e_i$, $\{e_i\}_{i=1}^3$ being the canonical basis of $\mathbf{R}^3$, one infers that v is not continuous on the p_i^j's and (taking account of Remark 4 in 2^{nd} step), if δ_p denotes the Dirac mass centered at p,

$$\partial G_v \llcorner \operatorname{int}(Q) \times \mathbf{R}^N = \sum_{i,j=1}^{3,2} (-1)^{i+j}\, \delta_{p_i^j} \times (v_{F_i^j \#} [\![\, F_i^j \,]\!] - u_{F_i^j \#} [\![\, F_i^j \,]\!])\,.$$

Let now $R_{F_i^j} \in \mathcal{R}_3(\mathbf{R}^N)$ be the i.m. rectifiable current for which (17) and (18) hold w.r.t. $F = F_i^j$ and with $\eta \leq \varepsilon/6$. Taking

$$T := G_v \llcorner Q \times \mathbf{R}^N - \sum_{i,j=1}^{3,2} (-1)^{i+j}\, \delta_{p_i^j} \times R_{F_i^j}$$

($T = G_v \llcorner Q \times \mathbf{R}^N$ in case $N = 2$), we then obtain $\partial T \llcorner \operatorname{int}(Q) \times \mathbf{R}^N = 0$ and

$$\partial T = G_v \llcorner \partial Q \times \mathbf{R}^N = (Id \bowtie v)_\# [\![\, \partial Q \,]\!]\,, \tag{37}$$

with $v_{|\partial Q} : \partial Q \to \mathbf{R}^N$ continuous map in $\operatorname{cart}^1(\partial Q, \mathbf{R}^N)$ for which (35) holds, with restrictions $v_{|I}$ in $\operatorname{cart}^1(I, \mathbf{R}^N)$ for each 1-face I of $Q_{(1)}$, whereas

$$\begin{aligned} \mathbf{M}(T) &= \mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) + \sum_{i,j=1}^{3,2} \mathbf{M}(R_{F_i^j}) \\ &\leq \mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) + \varepsilon\,. \end{aligned} \tag{38}$$

Then, by Lemma 3 (with $k = 3$) there exists a map $w : Q \to \mathbf{R}^N$ in $\operatorname{cart}^1(Q, \mathbf{R}^N)$, continuous up to the closure and with boundary

$$\partial(G_w \llcorner Q \times \mathbf{R}^N) = G_v \llcorner \partial Q \times \mathbf{R}^N,$$

such that

$$\int_Q |M_{(3)}(Dw)|\, dx \leq \mathbf{M}(T) + \varepsilon \tag{39}$$

if $N \geq 3$, $|M_{(3)}(Dw)| \equiv 0$ if $N = 2$.

For fixed $0 < \sigma < 1$, define now $v_\varepsilon : Q \to \mathbf{R}^N$ by

$$v_\varepsilon(x) := \begin{cases} w(x/\sigma) & \text{if} \quad \|x\|_3 \le \sigma\delta \\ v(\delta \frac{x}{\|x\|_3}) & \text{if} \quad \sigma\delta \le \|x\|_3 \le \delta \end{cases}$$

where $\|x\|_3 := \max_{1 \le i \le 3} |x_i|$, so that v_ε is continuous in Q and in $\mathrm{cart}^1(Q, \mathbf{R}^N)$. Moreover, by changing variable

$$\int_{\{\|x\|_3 \le \sigma\delta\}} |M_{(j)}(Dv_\varepsilon)|\, dx = \sigma^{3-j} \int_Q |M_{(j)}(Dw)|\, dx$$

for each $j \in \{1, \ldots, 3\}$, so that for σ sufficiently small

$$\int_{\{\|x\|_3 \le \sigma\delta\}} |M(Dv_\varepsilon)|\, dx \le \int_Q |M_{(3)}(Dw)|\, dx + \varepsilon \tag{40}$$

($\le \varepsilon$ if $N = 2$). Also, since $v_\varepsilon(\{\sigma\delta \le \|x\|_3 \le \delta\})$ is (at most) 2-dimensional

$$\int_{\{\sigma\delta \le \|x\|_3 \le \delta\}} |M_{(3)}(Dv_\varepsilon)|\, dx = 0$$

whereas, since for $\mathcal{H}^3$-a.e. $\sigma\delta \le \|x\|_3 \le \delta$ and for $j = 1, 2$

$$|M_{(j)}(D_x v_\varepsilon(x))| \le c\, (\frac{\delta}{\|x\|_3})^j\, |M_{(j)}(D_\tau v(y))| \tag{41}$$

where $y = \delta \frac{x}{\|x\|_3}$, τ is an O.N. basis for the boundary ∂Q and $c > 0$ is an absolute constant, by the coarea formula and changing variable

$$\begin{aligned} \int_{\{\sigma\delta \le \|x\|_3 \le \delta\}} |M(Dv_\varepsilon)|\, dx &\le c \int_{\sigma\delta}^{\delta} d\rho \int_{\{\|x\|_3 = \rho\}} \sqrt{1 + (\frac{\delta}{\rho})^2\, |D_\tau v(y)|^2 + (\frac{\delta}{\rho})^4\, |M_{(2)}(D_\tau v(y))|^2}\, d\mathcal{H}^2 \\ &\le c \int_{\sigma\delta}^{\delta} d\rho \int_{\{\|x\|_3 = \delta\}} \sqrt{1 + |D_\tau v|^2 + |M_{(2)}(D_\tau v)|^2}\, d\mathcal{H}^2 \\ &= c\, \delta\, (1 - \sigma)\, \mathbf{M}(G_v \llcorner \partial Q \times \mathbf{R}^N)\,. \end{aligned} \tag{42}$$

By (35), (36), (38), (39), (40) and (42) we thus obtain

$$\mathbf{M}(G_{v_\varepsilon} \llcorner Q \times \mathbf{R}^N) \le \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + (2\sqrt{2} + c)\, \delta \sum_{F \in Q_{(2)}} \int_F |M(Du_F)|\, d\mathcal{H}^2 + (4 + c\,\delta)\, \varepsilon$$

as required in (19).

Suppose now $n > 3$. Let K be a 3-face of $Q_{(3)}$, and choose for simplicity

$$K = \{x \in \mathbf{R}^n \mid x = (\widetilde{x}, \widehat{\delta}),\ \|\widetilde{x}\|_3 \le \delta\}\,,$$

where now $\widetilde{x} := (x_1, x_2, x_3) \in \mathbf{R}^3$, $\widehat{\delta} := (\delta, \ldots, \delta) \in \mathbf{R}^{n-3}$, $\|\widetilde{x}\|_3 := \max_{1\le i\le 3} |x_i|$. We have $[\![\, \partial K \,]\!] = \sum_{i,j=1}^{3,2} (-1)^{i+j+1} [\![\, F_i^j \,]\!]$, where F_i^j, $j = 1, 2$, are the 2-faces parallel to the plane $\{x \in \mathbf{R}^n \mid x_h = 0 \text{ if } h = i, 4, \ldots, n\}$, for $1 \le i \le 3$. Since the map v is already given on the subset of $\Sigma_{(3)}$

$$\bigcup_{i,j=1}^{3,2} \mathrm{co}(F_i^j \cup \lambda\, F_i^j) = \{(t\,\widetilde{x}, t\,\widehat{\delta}) \mid \lambda \le t \le 1, \quad \|\widetilde{x}\|_3 = \delta\}$$

we define v on K by

$$v(\widetilde{x}, \widehat{\delta}) = u(\widetilde{x}/\lambda, \widehat{\delta}) \qquad \text{if} \quad \|\widetilde{x}\|_3 \le \delta\,\lambda$$

whereas on $K \bigcap \{(t\,\widetilde{x}, \widehat{\delta}) \mid \lambda \le t \le 1\,,\ \|\widetilde{x}\|_3 = \delta\}$ we take v as in $\mathrm{co}(F_i^j \cup \lambda\, F_i^j)$, i.e.,

$$v(t\,\widetilde{x}, \widehat{\delta}) = v(t\,\widetilde{x}, t\,\widehat{\delta})\,, \qquad \lambda \le t \le 1\,, \quad \|\widetilde{x}\|_3 = \delta\,.$$

Then the mass estimates (35) and (36) clearly hold again (with K instead of Q and $F \in Q_{(2)}$, $F \subset \partial K$) whereas, in case $N \ge 3$, setting

$$T := G_v \llcorner K \times \mathbf{R}^N - \sum_{i,j=1}^{3,2} (-1)^{i+j}\, \delta_{p_i^j} \times R_{F_i^j}$$

($T = G_v \llcorner K \times \mathbf{R}^N$ if $N = 2$), with $p_i^j := (-1)^j\,\delta\,\frac{\lambda+1}{2}\,e_i$, $\{e_i\}_{i=1}^n$ canonical basis of $\mathbf{R}^n$, where $R_{F_i^j} \in \mathcal{R}_3(\mathbf{R}^N)$ is given by (17) and (18), again (37) holds (with K instead of Q).
Applying Lemma 3 with $Q = K$, working similarly to the case $n = 3$ and repeating the argument for each $K \in Q_{(3)}$, we then define a continuous map $v : Q_{(3)} \to \mathbf{R}^N$ such that v_K is in $\mathrm{cart}^1(K, \mathbf{R}^N)$ with

$$\int_K |M(Dv_K)|\, d\mathcal{H}^3 \le \int_K |M(Du_K)|\, d\mathcal{H}^3 + c\,\delta \sum_{\substack{F \in Q_{(2)} \\ F \subset \partial K}} \int_F |M(Du_F)|\, d\mathcal{H}^2 + \eta \tag{43}$$

for each 3-face K of $Q_{(3)}$, where v_K, $u_K : K \to \mathbf{R}^N$ denote the restrictions of v and u to K.

Remark 5 Note that if $K_1, K_2 \in Q_{(3)}$ are such that $\partial K_1 \cap \partial K_2 = F \in Q_{(2)}$, then the common boundary of the graphs of the $v_{|K_i}$'s coincides with the graph of the restriction $v_{|F}$ of v to F, i.e.,

$$\partial(G_{v_{K_i}} \llcorner \mathrm{int}(K_i) \times \mathbf{R}^N) \llcorner F \times \mathbf{R}^N = \sigma_i\, G_{v_{|F}} \llcorner F \times \mathbf{R}^N$$

for $i = 1, 2$, where $\sigma_1, \sigma_2 = \pm 1$, according to the orientations.

In case $n > 3$, and for each $k \in \{4, \ldots, n\}$, we have the following

k^{th} *step:* definition of v on $\Sigma_{(k)} \cup Q_{(k)}$.

At the $(k-1)^{th}$ step, v has been defined on $\Sigma_{(k-1)}$ so that for each $(k-2)$-face F of $Q_{(k-2)}$

$$\begin{aligned}\int\limits_{\mathrm{co}(F\cup\lambda F)} |M(Dv)|\, d\mathcal{H}^{k-1} \;&\leq\; 2\sqrt{k-2}\,\delta \int\limits_{F} |M(Du_F)|\, d\mathcal{H}^{k-2} + \eta + O(1-\lambda) \\ &+\; c\,(n,k-1) \sum_{j=1}^{k-4} \delta^{j+1} \sum_{\substack{I\,\in\, Q_{(k-2-j)} \\ I\,\subset\,\partial F}} \int\limits_{I} |M(Du_I)|\, d\mathcal{H}^{k-2-j}\end{aligned} \tag{44}$$

(see (32) for $k=4$), where $c\,(n,k-1)>0$ is an absolute constant, $\eta>0$ is small and $u_F := u_{|F}$, $u_I := u_{|I}$ denote the restrictions of u to the corresponding faces. Moreover, v has been defined on $Q_{(k-1)}$ so that, for each $(k-1)$-face K of $Q_{(k-1)}$

$$\begin{aligned}\int\limits_{K} |M(Dv_K)|\, d\mathcal{H}^{k-1} \;&\leq\; \int\limits_{K} |M(Du_K)|\, d\mathcal{H}^{k-1} + \eta \\ &+\; c \sum_{j=1}^{k-3} \delta^{j} \sum_{\substack{I\,\in\, Q_{(k-1-j)} \\ I\,\subset\,\partial K}} \int\limits_{I} |M(Du_I)|\, d\mathcal{H}^{k-1-j}\end{aligned} \tag{45}$$

(see (43) for $k=4$), where $c>0$ is an absolute constant and $v_K := v_{|K}$, $u_K := u_{|K}$.

Remark 6 Finally, Remark 5 of 3^{rd} step holds again with $K_i \in Q_{(k-1)}$ and $F \in Q_{(k-2)}$.

To define v on $\Sigma_{(k)}$, let us fix a $(k-1)$-face K of $Q_{(k-1)}$, for simplicity

$$K = \{x \in \mathbf{R}^n \mid x = (\widetilde{x}, \widehat{\delta}\,),\ \|\widetilde{x}\|_{k-1} \leq \delta\}$$

where now $\widetilde{x} := (x_1, \ldots, x_{k-1}) \in \mathbf{R}^{k-1}$, $\widehat{\delta} := (\delta, \ldots, \delta) \in \mathbf{R}^{n-k+1}$, $\|\widetilde{x}\|_{k-1} := \max_{1\leq i\leq k-1} |x_i|$, and define v on the convex hull

$$\mathrm{co}(K \cup \lambda\, K) = \{(\widetilde{x}, \widehat{t}\,) \mid \lambda\delta \leq t \leq \delta\,, \quad \|\widetilde{x}\|_{k-1} \leq t\}\,.$$

To this aim, following 3^{rd} step, we divide in three parts

$$\mathrm{co}(K \cup \lambda\, K) = \bigcup_{i=1}^{3} \Delta_K^i$$

and define v on the Δ_K^i's by means of implicit maps $G^{(i)}$. The Δ_K^i's are given similarly to (24), (27) and (30), respectively, with $\widetilde{x} \in \mathbf{R}^{k-1}$, $\widehat{t} \in \mathbf{R}^{n-k+1}$ and $\|\cdot\|_{k-1}$ instead of $\|\cdot\|_2$.

Using the trace v_K of v to K given at the $(k-1)^{th}$ step, see (45), define v on Δ_K^1 by means of the implicit map

$$\begin{gathered} G^{(1)} :]\delta\,(\tfrac{\lambda+1}{2}), \delta] \times [-\delta, \delta]^{k-1} \longrightarrow \Delta_K^1 \times \mathbf{R}^N \\ G^{(1)}(t, \widetilde{x}) := ((f(t)\,\widetilde{x}, \widehat{t}\,), v_K(\,\widetilde{x}, \widehat{\delta}\,))\,, \end{gathered}$$

with f given by (25). We then infer

$$\int_{\Delta_K^1} |M(Dv)|\, d\mathcal{H}^k \leq \sqrt{k-1}\,\delta \int_K |M(Dv_K)|\, d\mathcal{H}^{k-1} + O(1-\lambda)\,. \tag{46}$$

In fact, by the area formula

$$\int_{\Delta_K^1} |M(Dv)|\, d\mathcal{H}^k = \int_{\delta\,(\frac{\lambda+1}{2})}^{\delta} dt \int_{[-\delta,\delta]^{k-1}} J_k G^{(1)}\, d\mathcal{H}^{k-1}$$

where $J_k G^{(1)}$ is the k-dimensional Jacobian of $G^{(1)}$. By direct computation

$$\begin{aligned}(J_k G^{(1)})^2 \;&\leq\; (n-k+1)\, f(t)^{2\,(k-1)} \\ &+\; \sum_{j=1}^{k-1} f(t)^{2\,(k-1-j)}\, [(n-k+1) + (\frac{2}{\delta\,(1-\lambda)})^2\, |\widetilde{x}|^2\,] \cdot |M_{(j)}(Dv_K)|^2\end{aligned}$$

where $|M_{(j)}(Dv_K)| = 0$ if $j > N$ and $|\widetilde{x}|^2 = (x_1)^2 + \cdots + (x_{k-1})^2$. Then, for λ sufficiently close to 1 we obtain (46) .

Using the trace u_K of the given map u to K , define v on Δ_K^2 by means of the implicit map

$$\begin{gathered}G^{(2)} : [\delta\lambda, \delta\,(\tfrac{\lambda+1}{2})[\times[-\delta,\delta]^{k-1} \longrightarrow \Delta_K^2 \times \mathbf{R}^N \\ G^{(2)}(t,\widetilde{x}) := ((f(t)\,\widetilde{x}, \widehat{t}\,), u_K(\,\widetilde{x}, \widehat{\delta}\,))\,,\end{gathered}$$

with f given by (28). By similar computations one then infers

$$\int_{\Delta_K^2} |M(Dv)|\, d\mathcal{H}^k \leq \sqrt{k-1}\,\delta \int_K |M(Du_K)|\, d\mathcal{H}^{k-1} + O(1-\lambda)\,. \tag{47}$$

Finally, define v on Δ_K^3 by means of the trace of v given at the $(k-1)^{th}$ step on the subset of $\Sigma_{(k-1)}$

$$\bigcup_{i=1}^{2\,(k-1)} \mathrm{co}(F_i \cup \lambda\, F_i) = \{(t\,\widetilde{x}, t\,\widehat{\delta}\,) \mid \lambda < t < 1\,, \quad \|\widetilde{x}\|_{k-1} = \delta\}$$

where $F_i \in Q_{(k-2)}$ are such that $\partial K = \bigcup_{i=1}^{2\,(k-1)} F_i$. For fixed $(\widetilde{x}, \widehat{\delta}) \in \partial K$, i.e., $\|\widetilde{x}\|_{k-1} = \delta$, and for each $\sigma \in]0,1]$, v describes with $t \in (\lambda, 1)$, on the parallel line segments connecting the points $(\lambda\sigma\widetilde{x}, \frac{1}{2}[(1+\lambda) + \sigma(\lambda-1)]\,\widehat{\delta}\,)$ and $(\sigma\widetilde{x}, \frac{1}{2}[(1+\lambda) - \sigma(\lambda-1)]\,\widehat{\delta}\,)$, the trace of v given at the $(k-1)^{th}$ step on the line segments connecting $(\lambda\widetilde{x}, \lambda\widehat{\delta})$ with $(\widetilde{x}, \widehat{\delta})$. More precisely, v is defined on Δ_K^3 by the implicit map

$$\begin{gathered}G^{(3)} : \{\widetilde{x} \in \mathbf{R}^{k-1} \mid \|\widetilde{x}\|_{k-1} = \delta\} \times [\lambda, 1]\times]0,1] \longrightarrow \Delta_K^3 \times \mathbf{R}^N \\ G^{(3)}(\widetilde{x}, t, \sigma) := ((t\sigma\widetilde{x}, \tfrac{1}{2}\,[(1+\lambda) + \sigma(2t - (1+\lambda))]\,\widehat{\delta}\,), v(\,t\,\widetilde{x}, t\,\widehat{\delta}\,))\,.\end{gathered}$$

One then infers

$$\int_{\Delta_K^3} |M(Dv)|\, d\mathcal{H}^k \le c \sum_{j=1}^{k-3} \delta^{j+1} \sum_{\substack{I \in Q_{(k-1-j)} \\ I \subset \partial K}} \int_I |M(Du_I)|\, d\mathcal{H}^{k-1-j} + \eta + O(1-\lambda) \tag{48}$$

for some absolute constant $c > 0$. In fact, for example, on the subset

$$\Delta_K^3 \cap \{(\widetilde{x}, \widehat{t}\,) \mid \delta\lambda \le t \le \delta,\, |x_i| \le x_1 \quad \forall i = 2, \dots, k-1\}$$

corresponding to the $(k-2)$-face F of ∂K

$$F = \{(\delta, x_2, \dots, x_{k-1}, \widehat{\delta}\,) \mid |x_i| \le \delta \quad \forall i = 2, \dots, k-1\},$$

the area of the graph of v is lower than the area of v on $\mathrm{co}(F \cup \lambda\, F)$ times the length of its projection onto the orthogonal section to $\mathrm{co}(F \cup \lambda\, F)$, i.e., lower than

$$\delta \int_{\mathrm{co}(F \cup \lambda\, F)} |M(Dv)|\, d\mathcal{H}^{k-1}.$$

Then

$$\int_{\Delta_K^3} |M(Dv)|\, d\mathcal{H}^k \le \delta \sum_{\substack{F \in Q_{(k-2)} \\ F \subset \partial K}} \int_{\mathrm{co}(F \cup \lambda\, F)} |M(Dv)|\, d\mathcal{H}^{k-1}$$

and by (44) one obtains (48). By (46), (47), (48) and (45) one then infers

$$\begin{aligned} \int_{\mathrm{co}(K \cup \lambda\, K)} |M(Dv)|\, d\mathcal{H}^k \quad &\le \quad 2\sqrt{k-1}\,\delta \int_K |M(Du_K)|\, d\mathcal{H}^{k-1} + \eta + O(1-\lambda) \\ &+ \quad c\,(n,k) \sum_{j=1}^{k-3} \delta^{j+1} \sum_{\substack{I \in Q_{(k-1-j)} \\ I \subset \partial K}} \int_I |M(Du_I)|\, d\mathcal{H}^{k-1-j} \end{aligned}$$

(compare (44)). Repeating the argument for each $(k-1)$-face K of $Q_{(k-1)}$, we finally obtain

$$\begin{aligned} \int_{\Sigma_{(k)}} |M(Dv)|\, d\mathcal{H}^k \quad &\le \quad 2\sqrt{k-1}\,\delta \sum_{K \in Q_{(k-1)}} \int_K |M(Du_K)|\, d\mathcal{H}^{k-1} + \varepsilon \\ &+ \quad \widetilde{c}\,(n,k) \sum_{j=1}^{k-3} \delta^{j+1} \sum_{I \in Q_{(k-1-j)}} \int_I |M(Du_I)|\, d\mathcal{H}^{k-1-j} \end{aligned} \tag{49}$$

for λ close to 1 and $\eta > 0$ small, where $c\,(n,k)$, $\widetilde{c}\,(n,k)$ are absolute constant and $\varepsilon > 0$ is small.

We are now able to define v on the k-skeleton $Q_{(k)}$.

Consider first the case $k = n$. Then, since $Q = Q_{(n)}$ and $\Sigma_{(n)} = Q \setminus \mathrm{int}(\lambda\, Q)$, we set

$$v(x) = u(x/\lambda) \qquad \text{if } x \in \lambda\, Q\,. \tag{50}$$

Since $Q_{(n-1)} = \partial Q$, (45) yields (for $\eta > 0$ small)

$$\begin{aligned} \mathbf{M}(G_v \llcorner \partial Q \times \mathbf{R}^N) \;\; &\leq \;\; \sum_{K \in Q_{(n-1)}} \int_K |M(Du_K)|\, d\mathcal{H}^{n-1} + \varepsilon \\ &+ \;\; c \sum_{j=1}^{n-3} \delta^j \sum_{I \in Q_{(n-1-j)}} \int_I |M(Du_I)|\, d\mathcal{H}^{n-1-j} \end{aligned} \tag{51}$$

whereas by (49), (50) and the area formula

$$\mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) \leq \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + c \sum_{j=1}^{n-2} \delta^j \sum_{I \in Q_{(n-j)}} \int_I |M(Du_I)|\, d\mathcal{H}^{n-j} + \varepsilon\,. \tag{52}$$

Moreover v is not continuous on Q and, in case $N \geq 3$, in general

$$\partial G_v \llcorner \mathrm{int}(Q) \times \mathbf{R}^N \neq 0\,.$$

In fact, if F is a 2-face of $Q_{(2)}$, for example

$$F = \{(x_1, x_2, \delta, \ldots, \delta) \in \mathbf{R}^n \mid \max\{|x_1|, |x_2|\} \leq \delta\}\,,$$

and Q_F is the corresponding subset of Q given by

$$Q_F := \{x \in Q \mid \min\{x_3, \ldots, x_n\} \geq \max\{|x_1|, |x_2|\}\}$$

then one can check that v is not continuous on S_F and

$$\partial G_v \llcorner \mathrm{int}(Q_F) \times \mathbf{R}^N = -[\![\, S_F \,]\!] \times (v_{F\#}[\![\, F \,]\!] - u_{F\#}[\![\, F \,]\!])\,,$$

where S_F is the $(n-3)$-dimensional subset of the boundary of the $(n-2)$-cube

$$\{x \in \mathbf{R}^n \mid x_1 = x_2 = 0\,, \quad 0 \leq x_i \leq \delta\, \frac{\lambda + 1}{2} \quad \forall\, i = 3, \ldots, n\}$$

given by

$$S_F := \{x \in \mathbf{R}^n \mid x_1 = x_2 = 0\,, \quad x_i \geq 0 \quad \forall\, i = 3, \ldots, n\,, \quad \max_{3 \leq i \leq n} x_i = \delta\, \frac{\lambda + 1}{2}\}$$

and the i.m. rectifiable current $[\![\, S_F \,]\!] \in \mathcal{R}_{n-3}(\mathbf{R}^n)$ is equipped with the natural orientation induced by such $(n-2)$-cube onto its boundary. If $R_F \in \mathcal{R}_3(\mathbf{R}^N)$ is the i.m. rectifiable current for which (17) and (18) hold, taking account of [15, 26.17] we also infer

$$\partial(G_v \llcorner \mathrm{int}(Q_F) \times \mathbf{R}^N) = G_v \llcorner \partial Q_F \times \mathbf{R}^N - (-1)^{n-3} \partial([\![\, S_F \,]\!] \times R_F)\,.$$

Similarly, if Q_{F_i}, $[\![S_{F_i}]\!] \in \mathcal{R}_{n-3}(\mathbf{R}^n)$ and $R_{F_i} \in \mathcal{R}_3(\mathbf{R}^N)$ are defined in analogous way w.r.t. each 2-face F_i of $Q_{(2)}$ (up to a rotation of Q so that F_i coincides with F), for suitable $\sigma_i \in \{+1,-1\}$ we obtain

$$\partial(G_v \llcorner \operatorname{int}(Q_{F_i}) \times \mathbf{R}^N) = G_v \llcorner \partial Q_{F_i} \times \mathbf{R}^N - (-1)^{n-3} \sigma_i \, \partial([\![S_{F_i}]\!] \times R_{F_i}) .$$

Then we take

$$T = G_v \llcorner Q \times \mathbf{R}^N + (-1)^{n-3} \sum_{F_i \in Q_{(2)}} \sigma_i \, [\![S_{F_i}]\!] \times R_{F_i}$$

($T = G_v \llcorner Q \times \mathbf{R}^N$ if $N = 2$), so that (taking account of Remark 6) $\partial T \llcorner \operatorname{int}(Q) \times \mathbf{R}^N = 0$. Also

$$\partial T = G_v \llcorner \partial Q \times \mathbf{R}^N = (Id \bowtie v)_{\#} [\![\partial Q]\!] ,$$

where $v_{|\partial Q} : \partial Q \to \mathbf{R}^N$ is a continuous map in $\operatorname{cart}^1(\partial Q, \mathbf{R}^N)$ for which (51) holds, with restrictions $v_{|I}$ in $\operatorname{cart}^1(I, \mathbf{R}^N)$ for each j-face I of $Q_{(j)}$, $j \in \{1, \ldots, n-2\}$, whereas since $\mathcal{H}^{n-3}(S_{F_i}) = c(n) \, [\delta \, (\frac{\lambda+1}{2})]^{n-3}$ and the masses $\mathbf{M}(R_{F_i})$ are taken arbitrary small

$$\mathbf{M}(T) \leq \mathbf{M}(G_v \llcorner Q \times \mathbf{R}^N) + c(n) \, \delta^{n-3} \, \varepsilon . \tag{53}$$

Then, by Lemma 3 (with $k = n$) there exists a map $w : Q \to \mathbf{R}^N$ in $\operatorname{cart}^1(Q, \mathbf{R}^N)$, continuous up to the closure of Q and with boundary

$$\partial(G_w \llcorner Q \times \mathbf{R}^N) = G_v \llcorner \partial Q \times \mathbf{R}^N,$$

such that

$$\int_Q |M_{(n)}(Dw)| \, dx \leq \mathbf{M}(T) + \varepsilon \tag{54}$$

if $N \geq n$, $|M_{(n)}(Dw)| \equiv 0$ if $N < n$.

Setting again, for fixed $0 < \sigma < 1$, $v_\varepsilon : Q \to \mathbf{R}^N$ by

$$v_\varepsilon(x) := \begin{cases} w(x/\sigma) & \text{if} \quad \|x\|_n \leq \sigma\delta \\ v(\delta \, \frac{x}{\|x\|_n}) & \text{if} \quad \sigma\delta \leq \|x\|_n \leq \delta \end{cases}$$

where $\|x\|_n := \max_{1 \leq i \leq n} |x_i|$, then v_ε is continuous on Q and in $\operatorname{cart}^1(Q, \mathbf{R}^N)$. Moreover, similarly to the case $n = 3$ of 3^{rd} step, for each $j \in \{1, \ldots, n\}$ one has

$$\int_{\{\|x\|_n \leq \sigma\delta\}} |M_{(j)}(Dv_\varepsilon)| \, dx = \sigma^{n-j} \int_Q |M_{(j)}(Dw)| \, dx$$

so that for σ sufficiently small

$$\int_{\{\|x\|_n \leq \sigma\delta\}} |M(Dv_\varepsilon)| \, dx \leq \int_Q |M_{(n)}(Dw)| \, dx + \varepsilon \tag{55}$$

($\leq \varepsilon$ if $N < n$) whereas, since

$$\int_{\{\sigma\delta \leq \|x\|_n \leq \delta\}} |M_{(n)}(Dv_\varepsilon)|\, dx = 0$$

and (41) holds again for each $j \in \{1, \ldots, n-1\}$ and for $\mathcal{H}^n$-a.e. x, with $\|\cdot\|_n$ instead of $\|\cdot\|_3$, similarly to (42) one infers

$$\int_{\{\sigma\delta \leq \|x\|_n \leq \delta\}} |M(Dv_\varepsilon)|\, dx \leq c\,\delta\, \mathbf{M}(G_v \llcorner \partial Q \times \mathbf{R}^N)\,. \tag{56}$$

Therefore, by (51), (52), (53), (54), (55) and (56) we obtain (as $\delta < 1$)

$$\begin{aligned} \mathbf{M}(G_{v_\varepsilon} \llcorner Q \times \mathbf{R}^N) \quad &\leq \quad \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + c_1\,\varepsilon \\ &+ \quad c_2 \sum_{j=1}^{n-2} \delta^j \sum_{I \in Q_{(n-j)}} \int_I |M(Du_I)|\, d\mathcal{H}^{n-j} \end{aligned}$$

for some absolute constants $c_1,\, c_2 > 0$, as required in (19).

Finally let us define v on $Q_{(k)}$ in the case $k < n$. Let L be a k-face of $Q_{(k)}$ and choose for simplicity

$$L = \{x \in \mathbf{R}^n \mid x = (\widetilde{x}, \widehat{\delta}),\ \|\widetilde{x}\|_k \leq \delta\}\,,$$

where now $\widetilde{x} := (x_1, \ldots, x_k) \in \mathbf{R}^k$, $\widehat{\delta} := (\delta, \ldots, \delta) \in \mathbf{R}^{n-k}$, $\|\widetilde{x}\|_k := \max_{1 \leq i \leq k} |x_i|$. If $\partial L = \bigcup_{j=1}^{2k} K_j$, with $K_j \in Q_{(k-1)}$, the map v is already defined on the subset of $\Sigma_{(k)}$

$$\bigcup_{j=1}^{2k} \mathrm{co}(K_j \cup \lambda\, K_j) = \{(t\,\widetilde{x}, t\,\widehat{\delta}) \mid \lambda \leq t \leq 1\,, \quad \|\widetilde{x}\|_k = \delta\}\,.$$

Define now v on L by

$$v(\widetilde{x}, \widehat{\delta}) = u(\widetilde{x}/\lambda, \widehat{\delta}) \qquad \text{if} \quad \|\widetilde{x}\|_k \leq \delta\,\lambda$$

whereas on $L \bigcap \{(t\,\widetilde{x}, \widehat{\delta}) \mid \lambda \leq t \leq 1\,,\ \|\widetilde{x}\|_k = \delta\}$ we take v as in $\mathrm{co}(K_j \cup \lambda\, K_j)$, i.e., set

$$v(t\,\widetilde{x}, \widehat{\delta}) = v(t\,\widetilde{x}, t\,\widehat{\delta})\,, \qquad \lambda \leq t \leq 1\,, \quad \|\widetilde{x}\|_k = \delta\,.$$

The mass estimates (51) and (52) hold again (with L and k instead of Q and n, respectively, $K \subset \partial L$ and $I \subset \partial L$) whereas v is not continuous on K and (for $N \geq 3$) in general

$$\partial G_v \llcorner \mathrm{int}(K) \times \mathbf{R}^N \neq 0\,.$$

In fact, if $F \in Q_{(2)}$ is a 2-face of the boundary of L, $F \subset \partial L$, for example

$$F = \{(x_1, x_2, \delta, \ldots, \delta) \in \mathbf{R}^n \mid \max\{|x_1|, |x_2|\} \leq \delta\}\,,$$

let $\widetilde{Q}_F$ be the corresponding subset of L given by

$$\widetilde{Q}_F := \{(\widetilde{x}, \widehat{\delta}) \in \mathbf{R}^n \mid \min\{x_3, \ldots, x_k\} \geq \max\{|x_1|, |x_2|\}\}\,,$$

$\widetilde{S}_F$ be the $(k-3)$-dimensional subset (depending on L) of the boundary of the $(k-2)$-cube

$$\{(\widetilde{x},\widehat{\delta}) \mid x_1 = x_2 = 0\,, \quad 0 \leq x_i \leq \delta\,\frac{\lambda+1}{2} \quad \forall\, i = 3,\ldots,k\}$$

$$\widetilde{S}_F := \{(\widetilde{x},\widehat{\delta}) \mid x_1 = x_2 = 0\,, \quad x_i \geq 0 \quad \forall\, i = 3,\ldots,k\,, \quad \max_{3\leq i\leq k} x_i = \delta\,\frac{\lambda+1}{2}\}$$

and $[\![\,\widetilde{S}_F\,]\!] \in \mathcal{R}_{k-3}(\mathbf{R}^n)$ be equipped with the natural orientation induced by such $(k-2)$-cube onto its boundary. Then, similarly to the case $k=n$, v is not continuous on $\widetilde{S}_F$ and

$$\partial(G_v \llcorner \operatorname{int}(\widetilde{Q}_F) \times \mathbf{R}^N) = G_v \llcorner \partial\widetilde{Q}_F \times \mathbf{R}^N - (-1)^{k-3}\partial([\![\,\widetilde{S}_F\,]\!] \times R_F)\,,$$

where $R_F \in \mathcal{R}_3(\mathbf{R}^N)$ is the i.m. rectifiable current for which (17) and (18) hold. If $\widetilde{Q}_{F_i}$, $[\![\,\widetilde{S}_{F_i}\,]\!] \in \mathcal{R}_{k-3}(\mathbf{R}^n)$ and $R_{F_i} \in \mathcal{R}_3(\mathbf{R}^N)$ are defined in analogous way for each 2-face F_i of $Q_{(2)}$ contained in ∂L, for suitable $\sigma_i \in \{+1,-1\}$ we take

$$T = G_v \llcorner L \times \mathbf{R}^N + (-1)^{k-3} \sum_{\substack{F_i \in Q_{(2)} \\ F_i \subset \partial L}} \sigma_i\,[\![\,\widetilde{S}_{F_i}\,]\!] \times R_{F_i}$$

$(T = G_v \llcorner L \times \mathbf{R}^N$ if $N=2)$, so that $\partial T \llcorner \operatorname{int}(L) \times \mathbf{R}^N = 0$ and

$$\partial T = G_v \llcorner \partial L \times \mathbf{R}^N = (Id \bowtie v)_{\#}[\![\,\partial L\,]\!]\,.$$

Applying Lemma 3 with $Q = L$, working similarly to the case $k = n$ and repeating the argument for each $L \in Q_{(k)}$, we then define a continuous map $v : Q_{(k)} \to \mathbf{R}^N$ such that v_L is in $\operatorname{cart}^1(L,\mathbf{R}^N)$ with

$$\begin{aligned}\int_L |M(Dv_L)|\,d\mathcal{H}^k \quad &\leq \quad \int_L |M(Du_L)|\,d\mathcal{H}^k + \eta \\ &+ \quad c\sum_{j=1}^{k-2}\delta^j \sum_{\substack{I \in Q_{(k-j)} \\ I \subset \partial L}} \int_I |M(Du_I)|\,d\mathcal{H}^{k-j}\end{aligned}$$

for each k-face $L \in Q_{(k)}$, where $v_L := v_{|L}$, $u_L := u_{|L}$ and $c > 0$ is an absolute constant, compare (45). Finally, Remark 6 holds with $K_i \in Q_{(k)}$ and $F \in Q_{(k-1)}$, as required. $\square$

3. Partition of a bounded open set

In this section we define a suitable partition of a bounded open set U of $\mathbf{R}^n$, which will be used to define the approximating sequence. To this aim we recall the notion of *Whitney family* of cubes $\mathrm{WF}(U)$ associated to the bounded open set $U \subset \mathbf{R}^n$, compare [2] or [12].

Denote by $\mathbf{K}(0)$ the family of closed unit n-dimensional cubes associated to the integer grid $\mathbf{Z}^n$ of $\mathbf{R}^n$. For each integer $l \in \mathbf{Z}$, denote by $\mathbf{K}(l)$ the family of standard cubes

$$\mathbf{K}(l) := \{2^{-l}\,K \mid K \in \mathbf{K}(0)\}\,.$$

Cubes of $\mathbf{K}(l)$ are said *of level* l and we set $\mathrm{level}(K) = l$ if $K \in \mathbf{K}(l)$. Finally set

$$\mathbf{K} := \bigcup_{l \in \mathbf{Z}} \mathbf{K}(l)\,.$$

Consider the *edge metric*

$$\delta\,(x, y) := \sup\{|x_i - y_i| \mid i \in \{1, \dots, n\}\}$$

for $x, y \in \mathbf{R}^n$, so that for $A, B \subset \mathbf{R}^n$ one has

$$\delta\,(A, B) := \inf\{\delta(a, b) \mid (a, b) \in A \times B\}\,.$$

The Whitney family $\mathrm{WF}(U)$ of a bounded open set $U \subset \mathbf{R}^n$ is given by the subclass of n-cubes K of $\mathbf{K}$ such that:

(a) $\delta\,(K, \mathbf{R}^n \setminus U) > 2^{1-\mathrm{level}(K)}$ for each $K \in \mathrm{WF}(U)$;

(b) $\delta\,(L, \mathbf{R}^n \setminus U) \le 2^{1-\mathrm{level}(L)}$ for the unique element L of $\mathbf{K}$ such that L contains K and $\mathrm{level}(L) = \mathrm{level}(K) - 1$.

Note that the size of the cubes of $\mathrm{WF}(U)$ is smaller and smaller according to the distance from the boundary of U. Moreover one has:

(a) $\bigcup \mathrm{WF}(U) = U$;

(b) if $K,\ L \in \mathrm{WF}(U)$, then $\mathrm{int}(K) \cap \mathrm{int}(L) = \emptyset$;

(c) if $K,\ L \in \mathrm{WF}(U)$ are such that $\partial K \cap \partial L \neq \emptyset$, then $|\,\mathrm{level}(K) - \mathrm{level}(L)| \le 1$;

(d) if $K \in \mathrm{WF}(U)$, then $\partial K \subset \bigcup\{\partial L \mid L \in \mathrm{WF}(U)$ and $L \neq K\}$.

We also recall the notion of *cubical complex* $\mathrm{CX}(\mathrm{WF}(U))$ associated to the Whitney family $\mathrm{WF}(U)$. For each integer $j \in \{1, \dots, n\}$ and for fixed $l \in \mathbf{Z}$, denote by $\mathbf{K}_j(l)$ the class of the j-dimensional faces of the cubes of $\mathbf{K}(l)$, say *j-faces of level l*. The cubical complex $\mathrm{CX}(\mathrm{WF}(U))$ is given by the faces F of $\bigcup_{j=1}^{n} \bigcup_{l \in \mathbf{Z}} \mathbf{K}_j(l)$ for which the following conditions hold:

(a) if $F \in \mathbf{K}_j(l)$, then F is a j-face of some cube of $\mathrm{WF}(U)$;

(b) if $F \in \mathbf{K}_j(l)$, $j \le n - 1$, then $l = \mathrm{level}(F) \ge \mathrm{level}(\widetilde{F})$ for each j-face $\widetilde{F}$ of any cube of $\mathrm{WF}(U)$ such that $\mathrm{int}(\widetilde{F}) \cap \mathrm{int}(F) \neq \emptyset$.

Define moreover, for each $j \in \{1, \dots, n\}$,

$$\mathrm{CX}_j(\mathrm{WF}(U)) := \{F \in \mathrm{CX}(\mathrm{WF}(U)) \mid F \text{ is of dimension } j\}$$

so that clearly $\mathrm{CX}_n(\mathrm{WF}(U)) = \mathrm{WF}(U)$. Finally set

$$\mathrm{CX}_0(\mathrm{WF}(U)) := \{p \in \mathbf{R}^n \mid p \in \partial F \text{ and } F \in \mathrm{CX}_1(\mathrm{WF}(U))\}$$

and define the level of a point $p \in \mathrm{CX}_0(\mathrm{WF}(U))$ by the greatest of $\mathrm{level}(F)$, as F ranges over the 1-faces of $\mathrm{CX}_1(\mathrm{WF}(U))$ for which $p \in \partial F$.

The partition of U is related to the cubical complex $\mathrm{CX}(\mathrm{WF}(U))$ and to a given map $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$, where $U \subset \Omega \subset \mathbf{R}^n$, thanks to the following

Proposition 2 *There exists a partition of U made by a countable family of polyhedral regions Q with sides parallel to the coordinate directions and strictly starshaped w.r.t. some interior point $x_0 \in \mathrm{int}(Q)$, i.e., such that for each direction $v \in \mathbf{R}^n$, $|v| = 1$, there is a unique positive $\lambda_v > 0$ such that the point $x_0 + \lambda_v v$ lies in the boundary ∂Q of Q, $\lambda_v \geq \lambda_0 > 0$ and the line segment given by $x_0 + t\lambda_v v$, $0 \leq t < 1$, lies inside $\mathrm{int}(Q)$ for any such v. Moreover, every polyhedral region belongs to one of the disjoint families $\mathcal{F}_k$, for $k \in \{0, \ldots, n\}$, so that for each k there is a $1:1$ correspondence between elements of $\mathrm{CX}_k(\mathrm{WF}(U))$ and polyhedral regions of $\mathcal{F}_k$ (possibly empty for $k \leq n-1$). The union of all the domains of the $\mathcal{F}_k$'s exactly covers the given set U, whereas different domains are pairwise interiorly disjoint. Let Q be one such domain, $Q \in \mathcal{F}_k$, and $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U))$ be the corresponding k-face. Also, let $Q_{(n-j)}$ denote the $(n-j)$-dimensional skeleton of Q, i.e., the union of the $(n-j)$-faces of the boundary of Q. Then, for each $j \in \{1, \ldots, n-2\}$, the following integral estimate holds*

$$\sum_{K \in Q_{(n-j)}} \int_K |M(Du_K)|\, d\mathcal{H}^{n-j} \leq c\,(2^l)^j \int_{\Omega_{\widetilde{Q}}} |M(Du)|\, dx \tag{57}$$

where $u_K := u_{|K} : K \to \mathbf{R}^N$ is the restriction of u to the $(n-j)$-face K, $\Omega_{\widetilde{Q}}$ is a suitable small neighbor of the k-face $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U))$, $c > 0$ is an absolute constant and $l = \mathrm{level}(\widetilde{Q})$. In particular, u_K is in $\mathrm{cart}^1(K, \mathbf{R}^N)$ for any such $(n-j)$-face $K \in Q_{(n-j)}$, $j \in \{1, \ldots, n-2\}$, see Definition 1. Moreover, the domains $\Omega_{\widetilde{Q}}$ are contained in U and intersect in a finite number, i.e., different $\Omega_{\widetilde{Q}}$'s, for $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U))$, are pairwise disjoint if $k = 0$ or $k = n$ whereas, if $1 \leq k \leq n-1$, intersect if and only if the corresponding k-faces $\widetilde{Q}$ do intersect. Finally, for each 1-face I of $Q_{(1)}$, the restriction $u_I := u_{|I}$ is in $W^{1,1}(I, \mathbf{R}^N)$, and then absolutely continuous.

Proof: To well understand the construction in Proposition 2, we first consider the case of a partition of all the space $\mathbf{R}^n$ w.r.t. a standard grid of n-cubes of fixed size $2\delta > 0$

$$\mathcal{L} = \mathcal{L}(\delta) := \bigcup_{z \in \mathbf{Z}^n} \delta\,(z + [-1,1]^n\,).$$

Denote then by $\mathcal{L}_{n-j}$ the $(n-j)$-skeleton of the grid, i.e., the family of $(n-j)$-faces of the n-cubes of $\mathcal{L}$, and suppose $u : \mathbf{R}^n \to \mathbf{R}^N$ is a map locally in cart^1, i.e., such that for each bounded domain $\Omega \subset \mathbf{R}^n$, the restriction $u_{|\Omega}$ is in $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$.
For each vertex $p \in \mathcal{L}_0$, denote by Ω_p the n-cube centered at p, with sides parallel to the coordinate axes and of length δ

$$\Omega_p := p + [-\delta/2, \delta/2]^n.$$

If F is a $(n-j)$-face of $\mathcal{L}_{n-j}$, $j \in \{1, \ldots, n-1\}$, denote by Ω_F the n-parallelepiped given by the convex envelope of the union of the domains Ω_p corresponding to the vertices p of F

$$\Omega_F := \mathrm{co}(\bigcup\{\Omega_p \mid p \in \mathcal{L}_0\,, \quad p \subset \partial F\}).$$

We select, in correspondence to each $(n-1)$-face K of $\mathcal{L}_{n-1}$, an hyperplane Π_K of $\mathbf{R}^n$, parallel to K, intersecting the domain Ω_K corresponding to K, $\Pi_K \cap \Omega_K \neq \emptyset$, so that nice integral estimates hold in $\Pi_K \cap \Omega_K$ and in the subsets $\Pi_K \cap \Omega_F$, with Ω_F corresponding to any $(n-j)$-face $F \in \mathcal{L}_{n-j}$, $j \in \{2, \ldots, n\}$, which lies in the boundary of K, $F \subset \partial K$ (see (60)). Moreover (since $n \geq 3$) the hyperplanes Π_K's, for $K \in \mathcal{L}_{n-1}$, are chosen so that if $F \in \mathcal{L}_{n-j}$ is any $(n-j)$-face, $j \in \{2, \ldots, n-1\}$, and $\{K_i \mid i = 1, \ldots, j\}$ is any family of pairwise orthogonal $(n-1)$-faces of $\mathcal{L}_{n-1}$ such that $F = \bigcap_{i=1}^{j} K_i$, then the $(n-j)$-dimensional affine space Π_F detected by the hyperplanes Π_{K_i}'s, $\Pi_F := \bigcap_{i=1}^{j} \Pi_{K_i}$, is such that nice integral estimates hold in $\Pi_F \cap \Omega_F$ and in the sections $\Pi_F \cap \Omega_I$, with Ω_I corresponding to any $(n-k)$-face $I \in \mathcal{L}_{n-k}$, $k \in \{j+1, \ldots, n\}$, which lies in the boundary of F, $I \subset \partial F$ (see (61)). More precisely, we work in n steps, as follows.

1^{st} *step:* Let $K \in \mathcal{L}_{n-1}$ be any $(n-1)$-face parallel to the hyperplane $\{x \in \mathbf{R}^n \mid x_1 = 0\}$ and $\overline{x}_1 \in \mathbf{R}$ be such that

$$K \subset \{x \in \mathbf{R}^n \mid x_1 = \overline{x}_1\}.$$

Since by Fubini's theorem

$$\int_{\Omega_K} |M(Du)|\, dx = \int_{\overline{x}_1-\delta/2}^{\overline{x}_1+\delta/2} d\xi \int_{\Omega_K \cap \{x_1 = \xi\}} |M(Du)|\, d\mathcal{H}^{n-1},$$

for $c > 0$ large, the measure of the set of points $\xi \in (\overline{x}_1 - \delta/2, \overline{x}_1 + \delta/2)$ for which

$$\int_{\Omega_K \cap \{x_1 = \xi\}} |M(Du)|\, d\mathcal{H}^{n-1} \leq \frac{c}{\delta} \int_{\Omega_K} |M(Du)|\, dx \tag{58}$$

is greater than $(1 - 1/c)\,\delta$. Moreover, if $I \in \mathcal{L}_{n-j}$, $j \in \{2, \ldots, n\}$, is any $(n-j)$-face of the boundary of K, $I \subset \partial K$, analogous integral estimates hold with Ω_I instead of Ω_K in (58). Since the number of $(n-j)$-faces of an $(n-1)$-cube is $2^{j-1} \binom{n-1}{n-j}$, choose

$$c > \sum_{j=1}^{n} 2^{j-1} \binom{n-1}{n-j}$$

and select $\xi_1 \in (\overline{x}_1 - \delta/2, \overline{x}_1 + \delta/2)$ so that (58) holds for $\xi = \xi_1$ and

$$\int_{\Omega_I \cap \{x_1 = \xi_1\}} |M(Du)|\, d\mathcal{H}^{n-1} \leq \frac{c}{\delta} \int_{\Omega_I} |M(Du)|\, dx$$

for each $I \in \mathcal{L}_{n-j}$, $I \subset \partial K$, $j \in \{2, \ldots, n\}$.

2^{nd} *step:* Let now $K \in \mathcal{L}_{n-1}$ be any $(n-1)$-face parallel to the hyperplane $\{x \in \mathbf{R}^n \mid x_2 = 0\}$ and $\overline{x}_2 \in \mathbf{R}$ be such that

$$K \subset \{x \in \mathbf{R}^n \mid x_2 = \overline{x}_2\}.$$

Moreover, let $F_i \in \mathcal{L}_{n-2}$, $i = 1,2$, be the $(n-2)$-faces of K parallel to the $(n-2)$-plane $\{x \in \mathbf{R}^n \mid x_1 = x_2 = 0\}$ and Ω_{F_i} be the corresponding domains. Also, let $K_i^h \in \mathcal{L}_{n-1}$, $h = 1,2$, be one of the two $(n-1)$-faces of $\mathcal{L}_{n-1}$ parallel to $\{x \in \mathbf{R}^n \mid x_1 = 0\}$ and such that $K_i^h \bigcap K = F_i$ for each $i,h = 1,2$. Finally, let $\xi_1(i,h) \in \mathbf{R}$ be chosen in 1^{st} step in correspondence to K_i^h. Since

$$\int_{\Omega_{F_i}\cap\{x_1=\xi_1(i,h)\}} |M(Du)|\, d\mathcal{H}^{n-1} = \int_{\overline{x}_2-\delta/2}^{\overline{x}_2+\delta/2} d\xi \int_{\Omega_{F_i}\cap\{x_1=\xi_1(i,h)\}\cap\{x_2=\xi\}} |M(Du)|\, d\mathcal{H}^{n-2},$$

for $c > 0$ large, the measure of the set of points $\xi \in (\overline{x}_2 - \delta/2, \overline{x}_2 + \delta/2)$ for which

$$\int_{\Omega_{F_i}\cap\{x_1=\xi_1(i,h)\}\cap\{x_2=\xi\}} |M(Du)|\, d\mathcal{H}^{n-2} \le \frac{c}{\delta} \int_{\Omega_{F_i}\cap\{x_1=\xi_1(i,h)\}} |M(Du)|\, d\mathcal{H}^{n-1}$$

is greater than $(1-1/c)\,\delta$, and similarly for Ω_I, for each $I \in \mathcal{L}_{n-j}$, $j \in \{3,\ldots,n\}$, $I \subset \partial F_i$, with $i,h = 1,2$. Since the number of $(n-j)$-faces of an $(n-2)$-cube is $2^{j-2}\left(\begin{array}{c} n-2 \\ n-j \end{array}\right)$, choosing

$$c > \sum_{j=1}^{n} 2^{j-1}\left(\begin{array}{c} n-1 \\ n-j \end{array}\right) + 4\sum_{j=2}^{n} 2^{j-2}\left(\begin{array}{c} n-2 \\ n-j \end{array}\right)$$

we select $\xi_2 \in (\overline{x}_2 - \delta/2, \overline{x}_2 + \delta/2)$ so that on one side

$$\int_{\Omega_I\cap\{x_2=\xi_2\}} |M(Du)|\, d\mathcal{H}^{n-1} \le \frac{c}{\delta} \int_{\Omega_I} |M(Du)|\, dx$$

holds for $I = K$ and for each $I \in \mathcal{L}_{n-j}$, $I \subset \partial K$, $j \in \{2,\ldots,n\}$, see 1^{st} step, and on the other side

$$\int_{\Omega_I\cap\{x_1=\xi_1(i,h),\, x_2=\xi_2\}} |M(Du)|\, d\mathcal{H}^{n-2} \le \frac{c}{\delta} \int_{\Omega_I\cap\{x_1=\xi_1(i,h)\}} |M(Du)|\, d\mathcal{H}^{n-1}$$

holds for $I = F_i$ and for any $(n-j)$-face $I \in \mathcal{L}_{n-j}$, $I \subset \partial F_i$, with $j \in \{3,\ldots,n\}$, for each $i,h = 1,2$.

Since $n \ge 3$, for each $k \in \{3,\ldots,n\}$ we have the following

k^{th} step: Let now $K \in \mathcal{L}_{n-1}$ be any $(n-1)$-face parallel to the hyperplane $\{x \in \mathbf{R}^n \mid x_k = 0\}$ and $\overline{x}_k \in \mathbf{R}$ be such that

$$K \subset \{x \in \mathbf{R}^n \mid x_k = \overline{x}_k\}.$$

Moreover, for fixed $l \in \{1,\ldots,k-1\}$ ($l \le n-2$ if $k = n$) and $1 \le i_1 < i_2 < \ldots < i_l \le k-1$, let $F \in \mathcal{L}_{n-l-1}$ be any $(n-l-1)$-face of K parallel to the $(n-l-1)$-plane $\{x \in \mathbf{R}^n \mid x_{i_h} = 0 \ \ \forall\, h = 1,\ldots,l,\ x_k = 0\}$. Also, for $h \in \{1,\ldots,l\}$, let $K_h \in \mathcal{L}_{n-1}$ be any $(n-1)$-face parallel to the hyperplane $\{x \in \mathbf{R}^n \mid x_{i_h} = 0\}$ and such that

$$\bigcap_{h=1}^{l} K_h \cap K = F. \tag{59}$$

If $\xi_{i_h} \in \mathbf{R}$ is chosen at ${i_h}^{th}$ step in correspondence to the $(n-1)$-face K_h, one has

$$\int_{\Omega_F \cap \{x_{i_h}=\xi_{i_h}\ \forall h=1,\dots,l\}} |M(Du)|\, d\mathcal{H}^{n-l} = \int_{\overline{x}_k-\delta/2}^{\overline{x}_k+\delta/2} d\xi \int_{\Omega_F \cap \{x_{i_h}=\xi_{i_h}\ \forall h=1,\dots,l\}\cap\{x_k=\xi\}} |M(Du)|\, d\mathcal{H}^{n-l-1}$$

(the left-hand side integral being finite by the previous steps) and similarly for Ω_I, for each $(n-j)$-face $I \in \mathcal{L}_{n-j}$, $j \in \{l+2,\dots,n\}$, $I \subset \partial F$. Since the number of ordered l-tuples in $(1,\dots,k-1)$ is $\binom{k-1}{l}$, for $l \in \{1,\dots,k-1\}$, the $(n-l-1)$-faces F of K parallel to the same $(n-l-1)$-plane are 2^l, the possible choices of distinct $(n-1)$-faces K_h parallel to $\{x_{i_h}=0\}$ for which (59) holds are 2^l, for each $h \in \{1,\dots,l\}$, and finally the number of $(n-j)$-faces I of the $(n-l-1)$-face F, for $j \in \{l+2,\dots,n\}$, is $2^{j-l-1}\binom{n-l-1}{n-j}$, if we choose

$$c > \sum_{j=1}^{n} 2^{j-1}\binom{n-1}{n-j} + \sum_{l=1}^{k-1}\binom{k-1}{l}\, l\, 2^{2l} \sum_{j=l+1}^{n} 2^{j-l-1}\binom{n-l-1}{n-j}$$

we are then able to select $\xi_k \in (\overline{x}_k-\delta/2, \overline{x}_k+\delta/2)$ so that, on one side,

$$\int_{\Omega_I\cap\{x_k=\xi_k\}} |M(Du)|\, d\mathcal{H}^{n-1} \le \frac{c}{\delta}\int_{\Omega_I} |M(Du)|\, dx \tag{60}$$

holds for $I=K$ and for each $I\in\mathcal{L}_{n-j}$, $I\subset\partial K$, $j\in\{2,\dots,n\}$, see 1^{st} step, and on the other side for each $l \in \{1,\dots,k-1\}$ $(l \le n-2)$, for any choice of $1 \le i_1 < i_2 < \ldots < i_l \le k-1$, for each $(n-l-1)$-face $F \in \mathcal{L}_{n-l-1}$ of K parallel to $\{x_{i_h}=0 \quad \forall h=1,\dots,l,\ x_k=0\}$, for any choice of $K_h \in \mathcal{L}_{n-1}$ parallel to $\{x_{i_h}=0\}$ and for which (59) holds, then

$$\int_{\Omega_I\cap\{x_{i_h}=\xi_{i_h}\ \forall h=1,\dots,l,\ x_k=\xi_k\}} |M(Du)|\, d\mathcal{H}^{n-l-1} \le \frac{c}{\delta}\int_{\Omega_I\cap\{x_{i_h}=\xi_{i_h}\ \forall h=1,\dots,l\}} |M(Du)|\, d\mathcal{H}^{n-l} \tag{61}$$

holds for $I=F$ and for each $I\in\mathcal{L}_{n-j}$, $I\subset\partial F$, $j\in\{l+2,\dots,n\}$, where $\xi_{i_h}\in\mathbf{R}$ is chosen at ${i_h}^{th}$ step in correspondence to the $(n-1)$-face K_h.

Let us now consider the case of a Whitney family of n-cubes $\mathrm{WF}(U)$, associated to a bounded open set $U \subset \mathbf{R}^n$, and obtain analogous integral estimates for a map $u \in \mathrm{cart}^1(\Omega,\mathbf{R}^N)$, $U \subset \Omega \subset \mathbf{R}^n$, according to Proposition 2. Here, the role of the $(n-j)$-skeleton $\mathcal{L}_{n-j}$ is played by the $(n-j)$-dimensional cubical complex $\mathrm{CX}_{n-j}(\mathrm{WF}(U))$.
For each vertex $p \in \mathrm{CX}_0(\mathrm{WF}(U))$, denote by Ω_p the n-cube centered at p, with sides parallel to the coordinate axes and of length $2^{-(l+1)}$, where $\mathrm{level}(p)=l$,

$$\Omega_p := p + [-2^{-(l+2)}, 2^{-(l+2)}]^n.$$

If $F \in \mathrm{CX}_{n-j}(\mathrm{WF}(U))$ is any $(n-j)$-face of the cubical complex, $j \in \{1,\dots,n-1\}$, with $\mathrm{level}(F)=l$, let

$$\{p_i \mid i=1,\dots,2^{n-j}\} \subset \mathrm{CX}_0(\mathrm{WF}(U))$$

denote the set of vertices of F. We distinguish two cases.

If all the vertices p_i of F have the same level, denote by Ω_F the n-parallelepiped given by the convex envelope of the union of the n-cubes Ω_{p_i}

$$\Omega_F := \mathrm{co}(\bigcup_{i=1}^{2^{n-j}} \Omega_{p_i}) \,. \tag{62}$$

If not all the vertices p_i of F have the same level, i.e., level(p_i) is equal to l for some p_i and to $l+1$ for other ones, we denote by $\widetilde{\Omega}_{p_i}$, for each $i = 1, \ldots, 2^{n-j}$, the n-parallelepiped centered at p_i, with sides parallel to the sides of F of length $2^{-(l+1)}$ and sides orthogonal to F of half length $2^{-(l+2)}$ (thus $\widetilde{\Omega}_{p_i}$ depends on F) and define Ω_F as in (62), with $\widetilde{\Omega}_{p_i}$ instead of Ω_{p_i}.

The integral estimates are obtained similarly to the case of the standard grid of mesh 2δ, taking account that for each $K \in \mathrm{CX}_{n-1}(\mathrm{WF}(U))$, the size δ is equal to $2^{-(l+1)}$ or $2^{-(l+2)}$, $l = \mathrm{level}(K)$, according to the length of the sides of Ω_K orthogonal to K. Note that the combinatorial complexity of $\mathrm{CX}(\mathrm{WF}(U))$ is dominated by the case of a standard grid $\mathcal{L}$. We then reach the following conclusions.

For each $k \in \{1, \ldots, n\}$, at k^{th} step, for any $(n-1)$-face K of $\mathrm{CX}_{n-1}(\mathrm{WF}(U))$ parallel to $\{x \mid x_k = 0\}$, say $K \subset \{x \mid x_k = \overline{x}_k\}$, we can select $\xi_k \in (\overline{x}_k - 2^{-(l+2)}, \overline{x}_k + 2^{-(l+2)})$ (or $2^{-(l+3)}$ if the side of Ω_K orthogonal to K is of length $2^{-(l+2)}$), where $l = \mathrm{level}(K)$, such that (60) holds, with $\delta = 2^{-l}$, for $I = K$ and for each $I \in \mathrm{CX}_{n-j}(\mathrm{WF}(U))$, $j \in \{2, \ldots, n\}$, $I \subset \partial K$, where $c > 0$ is an absolute constant. Moreover, in case $k > 1$, for each $l \in \{1, \ldots, k-1\}$ ($l \leq n-2$ if $k = n$), for any choice of $1 \leq i_1 < i_2 < \ldots < i_l \leq k-1$, for each $(n-l-1)$-face F of K in $\mathrm{CX}_{n-l-1}(\mathrm{WF}(U))$, parallel to $\{x \mid x_{i_h} = 0 \quad \forall h = 1, \ldots, l, x_k = 0\}$, for any choice of $K_{i_h} \in \mathrm{CX}_{n-1}(\mathrm{WF}(U))$, parallel to $\{x_{i_h} = 0\}$ and for which (59) holds, then (61) holds, with $\delta = 2^{-l}$, for $I = F$ and for each $I \in \mathrm{CX}_{n-j}(\mathrm{WF}(U))$, $I \subset \partial F$, $j \in \{l+2, \ldots, n\}$, where $\xi_{i_h} \in \mathbf{R}$ is chosen at ${i_h}^{th}$ step in correspondence to the $(n-1)$-face K_h.

Remark 7 Actually, since $n \geq 3$, if $k \in \{n-1, n\}$ and $l = n-2$, for each $F \in \mathrm{CX}_1(\mathrm{WF}(U))$ as before, it suffices to require finiteness of the 1-dimensional integrals in (61).

By means of suitable intersections of the previously defined $(n-1)$-faces $\Omega_K \cap \{x_k = \xi_k\}$, $k \in \{1, \ldots, n\}$, $K \in \mathrm{CX}_{n-1}(\mathrm{WF}(U))$, it is possible to define the partition of the open subset $U \subset \Omega$ as described in Proposition 2. In particular, one infers that (57) follows by (60) and by iterated applications of (61), where $\delta = 2^{-l}$. □

Remark 8 For future purposes, we finally note that the polyhedral regions Q of the partition in Proposition 2 have the following size upper bound. If $Q \in \mathcal{F}_k$ and $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U))$ is the corresponding k-face, with level$(\widetilde{Q}) = l$, then the diameter of Q is lower than $\sqrt{n}\, 2^{-l+1}$. Moreover, Q is strictly starshaped w.r.t. some point $x_0 \in \mathrm{int}(Q)$ so that, if $\lambda_v > 0$ corresponds to $v \in \mathbf{R}^n$, $|v| = 1$, see Proposition 2, then for some absolute constant $c > 0$

$$\sup\{\lambda_v \mid v \in \mathbf{R}^n, |v| = 1\} \leq c\, 2^{-l}.$$

4. Proof of the Main Result

To show the "if" part, we recall by [12] that every continuous map in $\mathrm{cart}^1(\Omega,\mathbf{R}^N)$ can be approximated by smooth graphs. We therefore reduce to prove that if $u\in\mathrm{cart}^1(\Omega,\mathbf{R}^N)$ is such that (2) holds for $\mathcal{H}^{n-2}$-a.e. $x\in(E_2)^\perp$, with E_2 any coordinate plane of $\mathbf{R}^n$, then we can define, for each small $\varepsilon>0$, a continuous map $u_\varepsilon:\Omega\to\mathbf{R}^N$ in $\mathrm{cart}^1(\Omega,\mathbf{R}^N)$ such that u_ε is equal to u in $\Omega\setminus U_\varepsilon$, with U_ε bounded open set of arbitrary small measure, and such that, by absolute continuity of the area integral (1)

$$\mathrm{A}(u_\varepsilon,U_\varepsilon)\le\mathrm{A}(u,U_\varepsilon)+c\,\varepsilon$$

where $c>0$ is an absolute constant.

Recall to this aim that by Liu's theorem [9], if $u\in\mathrm{cart}^1(\Omega,\mathbf{R}^N)$, for each small $\varepsilon>0$ there is a closed set $C_\varepsilon\subset\Omega$ such that $|\Omega\setminus C_\varepsilon|\le\varepsilon$ and u is of class C^1 in C_ε. Consider then the Whitney family $\mathrm{WF}(U_\varepsilon)$ of cubes associated to the open set

$$U_\varepsilon:=\Omega\setminus C_\varepsilon$$

and partition the singular set U_ε in a countable family of polyhedral regions as in Proposition 2 of Sec. 3. In particular, if Q is any region of the partition, with $Q\in\mathcal{F}_k$, $k\in\{0,\dots,n\}$, $Q_{(n-j)}$ denotes its $(n-j)$-skeleton and $\widetilde{Q}\in\mathrm{CX}_k(\mathrm{WF}(U_\varepsilon))$ corresponds to Q, then (57) holds for each $j\in\{1,\dots,n-2\}$, where $u_K:=u_{|K}:K\to\mathbf{R}^N$ is the restriction of u to the $(n-j)$-face K of $Q_{(n-j)}$, $\Omega_{\widetilde{Q}}$ is a small neighbor of the k-face $\widetilde{Q}$, $c>0$ is an absolute constant and $l=\mathrm{level}(\widetilde{Q})$. In particular, u_K is in $\mathrm{cart}^1(K,\mathbf{R}^N)$ for any such $(n-j)$-face K. Moreover, by the generic position of Q, since (2) holds for $\mathcal{H}^{n-2}$-a.e. 2-plane, by an obvious generalization of Lemma 1, we infer that i) and ii) in Sec. 2, in particular (16), (17) and (18), hold for the restriction $u_F:=u_{|F}:F\to\mathbf{R}^N$ of u to any 2-face F of $Q_{(2)}$.

Then, by an adaptation of the local approximation argument in Sec. 2, Proposition 1 holds for any polyhedral region Q of the partition of U_ε. More precisely, if Q is strictly starshaped w.r.t. $x_0\in\mathrm{int}(Q)$, and up to a translation $x_0=0_{\mathbf{R}^n}$, if $K\in Q_{(k-1)}$ is a $(k-1)$-face of Q, for example $K=\widetilde{K}\times\{\widehat{\delta}\}$, where $\widetilde{K}$ is a $(k-1)$-dimensional polyhedral region of $\mathbf{R}^{k-1}$ and $\widehat{\delta}\in\mathbf{R}^{n-k+1}$ is fixed, we modify k^{th} step by setting

$$\|\widetilde{x}\|_{k-1}:=\inf\{\alpha>0\mid\frac{\delta}{\alpha}\,\widetilde{x}\in\widetilde{K}\}\,,\qquad\widetilde{x}\in\mathbf{R}^{k-1}\,,\tag{63}$$

$$\Delta^1_K:=\{(\widetilde{x},\frac{t}{\delta}\,\widehat{\delta})\mid\delta\,(\frac{\lambda+1}{2})<t\le\delta\,,\quad\|\widetilde{x}\|_{k-1}\le\delta\,f(t)\}$$

and

$$\begin{aligned}G^{(1)}&:]\delta\,(\tfrac{\lambda+1}{2}),\delta]\times\widetilde{K}\longrightarrow\Delta^1_K\times\mathbf{R}^N\\ G^{(1)}(t,\widetilde{x})&:=((f(t)\,\widetilde{x},\,\tfrac{t}{\delta}\,\widehat{\delta}),v_F(\,\widetilde{x},\widehat{\delta}\,))\end{aligned}$$

for example, where $\delta=2^{-l}$, $l=\mathrm{level}(\widetilde{Q})$, $\widetilde{Q}\in\mathrm{CX}(\mathrm{WF}(U_\varepsilon))$ corresponding to the given region Q as in Proposition 2. Moreover, Lemma 3 easily extends to any k-face L of $Q_{(k)}$, with $k\ge3$, whereas, taking account of Remark 8, (41) and (56) hold again (with $j\in\{1,\dots,k\}$, L for Q, $\|\cdot\|_k$ given as in (63) w.r.t. L) and so on.

Therefore, for each region Q, for a small $\sigma > 0$ to be fixed, there exists a map $v_Q : Q \to \mathbf{R}^N$ in $\mathrm{cart}^1(Q, \mathbf{R}^N)$, continuous up to the closure of Q, such that the restriction of v_Q to each $(n-j)$-face K of $Q_{(n-j)}$, $j \in \{1, \dots, n\}$, is a map in $\mathrm{cart}^1(K, \mathbf{R}^N)$ (for $j < n$) which only depends on the restriction u_K of u to K and finally

$$\begin{aligned} \mathbf{M}(G_{v_Q} \llcorner Q \times \mathbf{R}^N) &\le \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + \sigma \\ &+ \widetilde{c} \sum_{j=1}^{n-2} (2^{-l})^j \sum_{K \in Q_{(n-j)}} \int_K |M(Du_K)| \, d\mathcal{H}^{n-j} \end{aligned} \tag{64}$$

where $\widetilde{c} > 0$ is an absolute constant, $\sigma > 0$ is chosen small and $l = \mathrm{level}(\widetilde{Q})$, with $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U_\varepsilon))$ corresponding to $Q \in \mathcal{F}_k$.

Let now m_l, $l \in \mathbf{Z}$, denote the number of n-cubes of level l of $\mathrm{WF}(U_\varepsilon)$. Since $|U_\varepsilon| < \varepsilon$, clearly $m_l = 0$ if $l < l_0 := [\log_2(\varepsilon^{-1/n})] + 1$ whereas, for $l \ge l_0$, $m_l \le \varepsilon\, (2^l)^n$. Moreover, since each n-cube has $2^{n-k} \binom{n}{k}$ k-faces, the number of k-faces of level l of $\mathrm{CX}_k(\mathrm{WF}(U))$ is lower than $2^{n-k} \binom{n}{k} m_l$. Then, by summation w.r.t. $Q \in \mathcal{F}_k$, taking $\sigma = [2^{n-k} \binom{n}{k} m_l]^{-1} \varepsilon$ in (64) if $Q \in \mathcal{F}_k$ and $l = \mathrm{level}(\widetilde{Q})$, with $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U_\varepsilon))$ corresponding to Q, by (64) and (57) one infers, for each $k \in \{0, \dots, n\}$,

$$\begin{aligned} \sum_{Q \in \mathcal{F}_k} \mathbf{M}(G_{v_Q} \llcorner Q \times \mathbf{R}^N) &\le \sum_{Q \in \mathcal{F}_k} \mathbf{M}(G_u \llcorner Q \times \mathbf{R}^N) + \varepsilon \\ &+ c\, \widetilde{c}\, (n-2) \sum_{Q \in \mathcal{F}_k} \int_{\Omega_{\widetilde{Q}}} |M(Du)| \, dx . \end{aligned}$$

Finally, for fixed k, since the domains $\Omega_{\widetilde{Q}}$, for $\widetilde{Q} \in \mathrm{CX}_k(\mathrm{WF}(U_\varepsilon))$, are contained in U_ε and are pairwise disjoint if $k = 0$ or $k = n$ whereas, if $0 < k < n$, different $\Omega_{\widetilde{Q}}$'s intersect if and only if the corresponding k-faces $\widetilde{Q}$ do intersect (see Proposition 2), one has

$$\sum_{Q \in \mathcal{F}_k} \mathbf{M}(G_{v_Q} \llcorner Q \times \mathbf{R}^N) \le [1 + c\, \widetilde{c}\, (n-2)\, \widehat{c}\,]\, \mathbf{M}(G_u \llcorner U_\varepsilon \times \mathbf{R}^N) + \varepsilon \tag{65}$$

where $\widehat{c} = \widehat{c}\,(n,k)$ is an absolute constant. Then, taking $u_\varepsilon \equiv u$ in $\Omega \setminus U_\varepsilon$ and $u_\varepsilon = v_Q$ in Q, for each domain $Q \in \mathcal{F}_k$, $k \in \{0, \dots, n\}$, the mapping $u_\varepsilon : \Omega \to \mathbf{R}^N$, for $\{\varepsilon_j\} \downarrow 0$, defines the approximating sequence.

In fact, by summation on $k \in \{0, \dots, n\}$ in (65) one infers

$$\mathbf{M}(G_{u_\varepsilon} \llcorner U_\varepsilon \times \mathbf{R}^N) \le C\, \mathbf{M}(G_u \llcorner U_\varepsilon \times \mathbf{R}^N) + (n+1)\, \varepsilon \tag{66}$$

where $C := (n+1) + c\, \widetilde{c}\, (n-2) \sum_{k=0}^n \widehat{c}\,(n,k)$ is an absolute constant. Also, since u_ε is continuous in Q up to the closure, for each region Q of the partition of U_ε, with $u_{\varepsilon|K}$ depending on $u_{|K}$, for each $(n-j)$-face K of $Q_{(n-j)}$, $j \in \{1, \dots, n\}$, then u_ε is continuous on U_ε. Moreover, u is Lipschitz-continuous in $\Omega \setminus U_\varepsilon$ and u_ε coincides with u on the 1-skeleton of each region Q, see Proposition 1, with Q smaller and smaller w.r.t. its distance from $\Omega \setminus U_\varepsilon$, see Sec. 3. We then conclude that u_ε is continuous in Ω. Also, $u_{\varepsilon|Q} \in \mathrm{cart}^1(Q, \mathbf{R}^N)$ for each region Q whereas, if F is a common $(n-1)$-face between two regions Q_1, Q_2, by Sec. 2 we have

$$\partial(G_{u_\varepsilon} \llcorner Q_1 \times \mathbf{R}^N) \llcorner F \times \mathbf{R}^N = -\partial(G_{u_\varepsilon} \llcorner Q_2 \times \mathbf{R}^N) \llcorner F \times \mathbf{R}^N .$$

Therefore, since there is no creation of boundary when defining u_ε on U_ε, whereas $u_\varepsilon = u$ is Lipschitz in the closed set $\Omega \setminus U_\varepsilon$, we thus infer

$$\partial G_{u_\varepsilon} \llcorner \Omega \times \mathbf{R}^N = 0$$

and then $u_\varepsilon \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$. Finally, since $|U_\varepsilon| < \varepsilon$, by absolute continuity of the area integral and (66), one infers that $\{u_\varepsilon\}$ converges weakly as currents and in area to u, as $\varepsilon \to 0^+$. Since by [12] each continuous Cartesian map u_ε can be approximated by smooth maps, the claim is proved.

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