Strong density results in trace spaces of maps between manifolds

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Abstract *We deal with strong density results of smooth maps between two manifolds* $\mathcal{X}$ *and* $\mathcal{Y}$ *in the fractional spaces given by the traces of Sobolev maps in* $W^{1,p}$.

1 Introduction

In the last years there has been a growing interest in studying the fractional Sobolev spaces of mappings defined between manifolds, see e.g. [3, 4, 5, 6, 7, 8, 10, 14]. Motivated by these papers, in this note we are concerned with strong density results of smooth maps between two manifolds $\mathcal{X}$ and $\mathcal{Y}$ in the fractional spaces $W^{1-1/p,p}$ given by the traces of Sobolev maps in $W^{1,p}$, for $p>1$. We recall that the analogous strong density problem for Sobolev mappings between manifolds was settled in [2] and [11].

We shall consider smooth, connected, compact Riemannian manifolds $\mathcal{X}$ and $\mathcal{Y}$ without boundary, that are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^N$, respectively. We shall equip $\mathcal{X}$ and $\mathcal{Y}$ with the metric induced by the Euclidean norms on the ambient spaces, and we let $n:=\dim\mathcal{X}$.

Let p be a given exponent, $1<p<\infty$, and denote by $[p]$ the integer part of p. We recall, see e.g. [1], that the fractional Sobolev space $W^{1/p}(\mathcal{X}):=W^{1-1/p,p}(\mathcal{X})$ is the Banach space of L^p-functions $u:\mathcal{X}\to\mathbb{R}$ which have finite $W^{1-1/p,p}$-seminorm

$$|u|^p_{1/p,\mathcal{X}} := \int_{\mathcal{X}}\int_{\mathcal{X}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-1}}\,dx\,dy$$

endowed with the norm

$$\|u\|^p_{1/p,\mathcal{X}} := \|u\|^p_{L^p(\mathcal{X})} + |u|^p_{1/p,\mathcal{X}}\,. \tag{1.1}$$

$W^{1/p}(\mathcal{X},\mathbb{R}^N)$ is the space of vector valued maps $u=(u^1,\dots,u^N)$ such that $u^j\in W^{1/p}(\mathcal{X})$ for every $j=1,\dots,N$. Recall that if $\mathcal{X}=\partial\mathcal{M}$ for some smooth manifold $\mathcal{M}$, e.g., $\mathcal{X}=\mathbb{S}^n$, the unit sphere in $\mathbb{R}^{n+1}$, then $W^{1/p}(\partial\mathcal{M},\mathbb{R}^N)$ can be characterized as the space of functions u that are *traces* of functions U in the Sobolev space $W^{1,p}(\mathcal{M},\mathbb{R}^N)$. More generally, since $\mathcal{X}\subset\mathbb{R}^l$, denoting by $\mathcal{C}^{n+1}$ the cylinder

$$\mathcal{C}^{n+1} := \mathcal{X}\times I\subset\mathbb{R}^l\times\mathbb{R}\,,\qquad I:=]-1,1[\,,$$

$W^{1/p}(\mathcal{X},\mathbb{R}^N)$ can be seen as the space of functions u that are traces of functions U in the Sobolev space $W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$.

If $U\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$, and $\mathcal{H}^k$ is the k-dimensional *Hausdorff measure* in $\mathcal{C}^{n+1}$, we will denote by

$$\mathcal{E}_p(U) := \frac{1}{p^{p/2}}\int_{\mathcal{C}^{n+1}} |Du(z)|^p\,d\mathcal{H}^{n+1}(z)$$

the *p-energy* of u. Moreover, we will write $\mathbf{T}(U)=u$ if $u\in W^{1/p}(\mathcal{X},\mathbb{R}^N)$, $U\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ and $U=u$ on $\mathcal{X}\times\{0\}$. For $u\in W^{1/p}(\mathcal{X},\mathbb{R}^N)$, we shall denote by $\mathrm{Ext}(u)$ a function in $W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ that minimizes the p-energy $\mathcal{E}_p(U)$ among all Sobolev maps $U\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ such that $\mathbf{T}(U)=u$.

Instead of working with the norm (1.1), we shall equip $W^{1/p}(\mathcal{X},\mathbb{R}^N)$ with the equivalent norm given by

$$|||u|||_{1/p,\mathcal{X}} := ||u||_{L^p(\mathcal{X})} + \mathcal{E}_p(\mathrm{Ext}(u))\,.$$

We shall study strong density results for the class

$$W^{1/p}(\mathcal{X},\mathcal{Y}) := \{u\in W^{1/p}(\mathcal{X},\mathbb{R}^N)\mid u(x)\in\mathcal{Y}\quad\text{for }\mathcal{H}^n\text{-a.e. }x\in\mathcal{X}\}\,,$$

for $1<p<\infty$. We will then denote

$$H_S^{1/p}(\mathcal{X},\mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X},\mathcal{Y}) \quad | \quad \text{there exists } \{u_k\} \subset C^\infty(\mathcal{X},\mathcal{Y}) \text{ such that } u_k \to u \text{ strongly in } W^{1/p}\}.$$

It is well-known that

$$H_S^{1/p}(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y}) \qquad \text{if} \quad p \geq n+1\,.$$

This follows from a standard convolution argument if $p > n+1$, compare e.g. [5], and was extended by Bethuel [3] to the critical case $p = n+1$. Therefore, from now on we shall always *assume that* $\mathcal{X}$ *has dimension* $n > p-1$ *or, equivalently,* $n \geq [p]$.

For $n \geq [p]$, we let $R^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ and $R^0_{1/p}(\mathcal{X},\mathcal{Y})$ denote, respectively, the set of all maps $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^r \Sigma_i\,, \qquad r \in \mathbb{N}\,, \tag{1.2}$$

where Σ_i is a smooth $(n-[p])$-dimensional subset of B^n with smooth boundary, if $n \geq [p]+1$, and Σ_i is a point if $n = [p]$.

Using arguments from [7], in Sec. 2 we will first prove the following

Theorem 1.1 *For every* $1<p<n+1$*, where* $n = \dim(\mathcal{X})$*, the class* $R^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ *is dense in* $W^{1/p}(\mathcal{X},\mathcal{Y})$*.*

In the case $p=2$, this density result was proved in [14], compare also [5], in dimension $n=2$, for $\mathcal{X} = \mathbb{S}^2$ and with $\mathcal{Y} = \mathbb{S}^1$, the standard unit circle. For $p=2$, it was extended in [7] to the case $\mathcal{X} = \mathbb{S}^n$ in higher dimension $n \geq 2$ and for general target manifolds $\mathcal{Y}$, see also [9].

Moreover, in [3] it was noticed that if $\pi_{[p]-1}(\mathcal{Y}) \neq 0$, and $n \geq [p]$, in general the strict inclusion

$$H_S^{1/p}(\mathcal{X},\mathcal{Y}) \subsetneqq W^{1/p}(\mathcal{X},\mathcal{Y})$$

holds. More precisely, there exist functions $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ which cannot be approximated in $W^{1/p}$ by sequences of smooth maps in $W^{1/p}(\mathcal{X},\mathcal{Y})$.

If $n \leq p < n+1$, the converse holds true. In fact, we have:

Theorem 1.2 *If* $n \leq p < n+1$*, and* $p>1$*, then* $H_S^{1/p}(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y})$ *if and only if* $\pi_{n-1}(\mathcal{Y}) = 0$*.*

The argument given in [3, Lemma 4] to prove Theorem 1.2 is not clear to us; therefore, in Sec. 2 we shall give a different proof.

In the case of higher dimension $n>p$, i.e., $n \geq [p]+1$, in order to remove the $(n-[p])$-dimensional singular set of mappings in $R^\infty_{1/p}(\mathcal{X},\mathcal{Y})$, following observations by Hang-Lin [11], we shall see that the possibly non-trivial topology of the domain manifold $\mathcal{X}$ plays a role.

For our purposes, it is more convenient to consider "cubeulations" instead of triangulations of $\mathcal{X}$. These ones can be obtained by taking barycentric subdivisions of the n-simplices of any triangulation.

We let X^k denote the k-skeleton of some finite cubeulation X of $\mathcal{X}$. If $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$, possibly slightly moving the faces of X we may assume that the restriction of u to F belongs to $W^{1/p}(F,\mathcal{Y})$ for every k-face F of X^k, where $k = [p]-1,\ldots,n$. In this case, we will say that X is in *generic position* with respect to u. Moreover, if $u \in R^0_{1/p}(\mathcal{X},\mathcal{Y})$, and $\Sigma(u)$ is the $(n-[p])$-dimensional singular set of u, compare (1.2), we say that X is in *dual position* with respect to u if X is in generic position with respect to u and $X^{[p]-1} \cap \Sigma(u) = \emptyset$. Possibly slightly moving the faces of $X^{[p]-1}$, it turns out that the cubeulation X is in dual position with respect to u.

Using arguments from [3], that go back to [16], in Sec. 3 we will prove the following $([p]-1)$-*homotopy type* property for the class of maps in $R^0_{1/p}(\mathcal{X},\mathcal{Y})$:

Proposition 1.3 *Let* $n+1>p\geq 2$*. Let* $u_\infty \in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ *and* X *be a finite cubeulation of* $\mathcal{X}$ *in dual position with respect to* u_∞*. Let* $\{u_i\} \subset W^{1/p}(X^{[p]-1},\mathcal{Y}) \cap C^\infty$ *be a sequence of smooth maps strongly converging in* $W^{1/p}$ *to the restriction* $u_{\infty|X^{[p]-1}}$ *of* u_∞ *to* $X^{[p]-1}$*. Then, we find* $k_0 \in \mathbb{N}^+$ *such that for every* $i \geq k_0$ *the maps* u_i *and* $u_{\infty|X^{[p]-1}}$ *are homotopic as maps from* $X^{[p]-1}$ *to* $\mathcal{Y}$*.*

As a consequence, in Sec. 4 we shall provide a characterization of strongly approximable $R^0_{1/p}$-maps:

Theorem 1.4 *Let $n+1>p>1$. Let $u\in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ and let X be a cubeulation of $\mathcal{X}$ in dual position with respect to u. Then, u belongs to $H^{1/p}_S(\mathcal{X},\mathcal{Y})$, i.e., u is the strong $W^{1/p}$-limit of a sequence of smooth maps in $C^\infty(\mathcal{X},\mathcal{Y})$, if and only if the restriction $u_{|X^{[p]-1}}$ of u to $X^{[p]-1}$ can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$.*

Following Hang-Lin [11], we now recall that $\mathcal{X}$ is said to satisfy the *k-extension property with respect to $\mathcal{Y}$*, where $k\in\mathbb{N}$, if for any given CW-complex X on $\mathcal{X}$, denoting by X^k its k-dimensional skeleton, any continuous map $f:X^{k+1}\to\mathcal{Y}$ is such that its restriction to X^k can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. We recall that the k-extension property does not depend on the choice of the CW-complex structure on $\mathcal{X}$, compare [11, Sec. 2.2]. Moreover, we refer to [11, Sec. 5] for examples of manifolds $\mathcal{X}$ and $\mathcal{Y}$ such that the k-extension property fails to hold.

As an application of the previous facts, in Sec. 4 we shall then prove the following characterization:

Theorem 1.5 *If $n>p>1$, smooth maps in $C^\infty(\mathcal{X},\mathcal{Y})$ are sequentially dense in $W^{1/p}(\mathcal{X},\mathcal{Y})$, i.e., $H^{1/p}_S(\mathcal{X},\mathcal{Y})=W^{1/p}(\mathcal{X},\mathcal{Y})$, if and only if we have $\pi_{[p]-1}(\mathcal{Y})=0$ and $\mathcal{X}$ satisfies the $([p]-1)$-extension property with respect to $\mathcal{Y}$.*

We remark that in the case $n\le p<n+1$ and $p>1$, Theorem 1.5 is equivalent to Theorem 1.2, as the $(n-1)$-extension property is automatically satisfied if $\pi_{n-1}(\mathcal{Y})=0$.

In particular, from Theorem 1.5 we deduce:

Corollary 1.6 *If $n>p>1$ and $\pi_k(\mathcal{Y})=0$ for every integer $k=[p]-1,\dots,n-1$, then $H^{1/p}_S(\mathcal{X},\mathcal{Y})=W^{1/p}(\mathcal{X},\mathcal{Y})$.*

Corollary 1.7 *Let $n>p\ge 2$ and k be an integer, with $k=1,\dots,[p]-1$. If $\pi_i(\mathcal{X})=0$ for every $i=0,\dots,k-1$ and $\pi_j(\mathcal{Y})=0$ for every $j=k,\dots,[p]-1$, then $H^{1/p}_S(\mathcal{X},\mathcal{Y})=W^{1/p}(\mathcal{X},\mathcal{Y})$.*

In the model case $\mathcal{X}=\mathbb{S}^n$, since $\mathbb{S}^n$ is $(n-1)$-connected, i.e., $\pi_i(\mathbb{S}^n)=0$ for $i=0,\dots,n-1$, taking $k=[p]-1$ in Corollary 1.7, on account of Theorem 1.2 we immediately obtain:

Corollary 1.8 *If $n+1>p>1$, smooth maps in $C^\infty(\mathbb{S}^n,\mathcal{Y})$ are sequentially dense in $W^{1/p}(\mathcal{X},\mathcal{Y})$, i.e., $H^{1/p}_S(\mathbb{S}^n,\mathcal{Y})=W^{1/p}(\mathbb{S}^n,\mathcal{Y})$, if and only if $\pi_{[p]-1}(\mathcal{Y})=0$.*

Finally, we remark that the case of domain manifolds $\mathcal{X}$ with non zero smooth boundary can be treated in a similar way, giving analogous density results, possibly with prescribed Dirichlet conditions, compare [12] for the case of Sobolev mappings between manifolds.

2 Density results for $W^{1/p}$-maps

In this section we shall prove the theorems 1.1 and 1.2. We shall essentially use arguments given in [7] for the case $p=2$. However, we prefer to give a complete proof.

Since the approximation argument is *local*, by using of a standard approach based on local coordinate charts, we deduce that it suffices to prove Theorem 1.1 in the case of maps defined in the unit n-ball B^n. Moreover, since B^n is bilipschitz homeomorphic to the unit open n-cube

$$\mathcal{Q}^n :=]0,1[^n\,,$$

it suffices to prove Theorem 1.1 in the case of maps defined in $\mathcal{Q}^n$. Therefore, in the sequel of this section we will denote

$$z=(x,t)=(x_1,\dots,x_n,t)\in\mathbb{R}^n\times\mathbb{R}$$

a point in the cylinder $\mathcal{Q}^n\times I$, where $I=]-1,1[$. If $U\in W^{1,p}(\mathcal{Q}^n\times I,\mathbb{R}^N)$ and A is a "smooth" $\mathcal{H}^k$-measurable k-dimensional subset of $\mathcal{Q}^n\times I$, we denote

$$\mathcal{E}_p(U,A):=\frac{1}{p^{p/2}}\int_A |DU_{|A}|^p\,d\mathcal{H}^k\,,\qquad \mathcal{E}_p(U):=\mathcal{E}_p(U,\mathcal{Q}^n\times I)\,,$$

the k-dimensional p-energy integral of the restriction $U_{|A}$ of U to A. As in the introduction, we will write $\mathbf{T}(U) = u$ if $u \in W^{1/p}(\mathcal{Q}^n, \mathbb{R}^N)$ is the *trace* of U on $\mathcal{Q}^n \times \{0\}$. If $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$, we set

$$\|v\|_k := \max_{1 \le i \le k} |v_i| \,.$$

Also, for $i = 1, \ldots, n+1$ and $\lambda \in \mathbb{R}$, we denote by $P(\lambda, i)$ the restriction to $\mathcal{Q}^n \times I$ of the hyperplane of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ containing the point $\lambda\, e_i$ and orthogonal to e_i, where $\{e_1, \ldots, e_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+1}$, i.e.,

$$P(\lambda, i) := \{z \in \mathcal{Q}^n \times I \mid (z - \lambda\, e_i \mid e_i)_{\mathbb{R}^{n+1}} = 0\} \,.$$

For $m \in \mathbb{N}^+$ and $a = (a_1, \ldots, a_n) \in [1/(4m), 3/(4m)]^n$ we denote by $\mathcal{L}_m$ the grid of $\mathbb{R}^n \times \mathbb{R}$

$$\mathcal{L}_m := \bigcup_{i=1}^{n} \bigcup_{j=0}^{m-1} P(a_i + j/m, i) \tag{2.1}$$

and by $C_m^{(k)}$ the k-skeleton of the grid of $\mathcal{Q}^n$ given by the intersection of $\mathcal{L}_m$ with the n-space $\mathbb{R}^n \times \{0\}$. Moreover, we denote

$$\begin{aligned} \mathcal{Q}_m^n &:= a + [0, (m-1)/m]^n \\ \Sigma_m^{(k)} &:= C_m^{(k)} \cap \mathcal{Q}_m^n \,, \qquad k = 0, \ldots, n \end{aligned} \tag{2.2}$$

the closed n-cube of side $(m-1)/m$ inside $\mathcal{Q}^n$ and the part of the k-skeleton $C_m^{(k)}$ that is contained in $\mathcal{Q}_m^n$.

Remark 2.1 For future use, we set

$$\mathcal{Y}_\varepsilon := \overline{U_\varepsilon(\mathcal{Y})} \,,$$

where $U_\varepsilon(A) := \{y \in \mathbb{R}^N \mid \operatorname{dist}(y, A) < \varepsilon\}$ is the ε-neighborhood of $A \subset \mathbb{R}^N$, and we observe that, since $\mathcal{Y}$ is smooth and compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $\operatorname{Lip}(\Pi_\varepsilon) \le (1 + c\,\varepsilon) \to 1^+$ as $\varepsilon \to 0^+$. Notice that for $0 < \varepsilon \le \varepsilon_0$, the ε-neighborhood $\mathcal{Y}_\varepsilon$ is equivalent to $\mathcal{Y}$ in the sense of the algebraic topology.

Let $u \in W^{1/p}(\mathcal{Q}^n, \mathcal{Y})$ and $U : \mathcal{Q}^n \times I \to \mathbb{R}^N$ be the extension $\operatorname{Ext}(u)$ of u, so that $U \in W^{1,p}(\mathcal{Q}^n \times I, \mathbb{R}^N)$ and $\mathbf{T}(U) = u$. Notice that U is continuous outside $\mathcal{Q}^n \times \{0\}$. Moreover, we denote

$$U^{(m)} := U_{|C_m^{(d-1)} \times I} \tag{2.3}$$

the restriction of U to the d-skeleton $C_m^{(d-1)} \times I$, where $d = [p]$.

In order to prove Theorem 1.1, we first make use of the argument of [3, 2.1], that goes back to [15], and show that if the restriction $U^{(m)}$ belongs to $W^{1,p}(C_m^{(d-1)} \times I, \mathbb{R}^N)$, then it can be approximated by continuous maps $U_h^{(m)}$ such that their traces take values in the neighborhood $\mathcal{Y}_{\varepsilon_0}$ of $\mathcal{Y}$, Proposition 2.2. Secondly, we will suitably modify the extension U in such a way that it agrees with $U_h^{(m)}$ on the d-skeleton $C_m^{(d-1)} \times I$, Proposition 2.4.

Proposition 2.2 *Let $n + 1 > p \ge 2$ and $d = [p]$. Assume that $U^{(m)} \in W^{1,p}(C_m^{(d-1)} \times I, \mathbb{R}^N)$. There exists a sequence of continuous maps $\{U_h^{(m)}\}_h$ in $W^{1,p}(\Sigma_m^{(d-1)} \times I, \mathbb{R}^N)$ such that $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}(\Sigma_m^{(d-1)} \times I, \mathbb{R}^N)$ and the traces $\mathbf{T}(U_h^{(m)}) \in W^{1/p}(\Sigma_m^{(d-1)}, \mathcal{Y}_{\varepsilon_0})$ for every h.*

Remark 2.3 If $1 < p < 2$, since $d = [p] = 1$, Proposition 2.2 holds true by taking $U_h^{(m)} = U^{(m)}$, see (2.3).

Proof of Proposition 2.2: If $z = (x, t) \in \Sigma_m^{(d-1)} \times I$ and $0 < h < 1/(4m)$ we denote by

$$C(z, h) := \overline{B}^n(x, h/2) \times [t - h/2, t + h/2]$$

the cylinder centered at z over the ball of diameter h and height h, and by

$$\Sigma(z, h) := C(z, h) \cap (C_m^{(d-1)} \times I)$$

the intersection of the cylinder with the d-skeleton $C_m^{(d-1)}\times I$. Setting then, for $z\in\Sigma_m^{(d-1)}\times I$,

$$U_h^{(m)}(z):=\fint_{\Sigma(z,h)}U^{(m)}(y)\,d\mathcal{H}^d(y):=\frac{1}{\mathcal{H}^d\big(\Sigma(z,h)\big)}\int_{\Sigma(z,h)}U^{(m)}(y)\,d\mathcal{H}^d(y)\,,$$

it is not difficult to show that $U_h^{(m)}\in W^{1,p}(\Sigma_m^{(d-1)}\times I,\mathbb{R}^N)$ is continuous and that $U_h^{(m)}\to U^{(m)}$ strongly in $W^{1,p}$ as $h\to 0^+$.

It remains to show that if $u_h^{(m)}:=\mathbf{T}(U_h^{(m)})$, possibly passing to a subsequence $u_h^{(m)}(\Sigma_m^{(d-1)})\subset\mathcal{Y}_{\varepsilon_0}$ for every h. To this aim, for $\varepsilon>0$ to be determined later, choose $h_\varepsilon>0$ small so that for any $0<h\le h_\varepsilon$

$$\int_{\Sigma(z,h)}|DU^{(m)}(y)|^p\,d\mathcal{H}^d(y)\le\varepsilon\qquad\forall\, z\in\Sigma_m^{(d-1)}\times I\,. \tag{2.4}$$

For fixed $P_0\in\Sigma_m^{(d-1)}\times\{0\}$, we observe that the connected set $\Sigma(P_0,h)$ always contains a d-cube R_1 of side h. More precisely, assume for example $P_0=(x_0^1,\dots,x_0^n,0)$, where $x_0^l\in a_l+[0,(m-1)/m]$, for $l=1,\dots,d-1$, and $x_0^i=a_i+j_i/m$, for $i=d,\dots,n$. Then we have

$$\Sigma(P_0,h)=R_1\cup\bigcup_{i=2}^K R_i\,,\qquad K=\binom{n}{d-1}\,,$$

where R_1 is the d-cube

$$R_1:=\left(\prod_{l=1}^{d-1}[x_0^l-h/2,x_0^l+h/2]\right)\times\{(x_0^d,\dots,x_0^n)\}\times[-h/2,h/2]$$

and $R_i:=\widetilde{R}_i\times[-h/2,h/2]$ for $i=2,\dots,K$, where $\widetilde{R}_i$ is a possibly degenerate $(d-1)$-parallelepiped of diameter lower than $\sqrt{d-1}\,h$, and edges parallel to the coordinate axes. In particular, we have

$$h^d\le\mathcal{H}^d(\Sigma(P_0,h))\le c\,h^d$$

for some dimensional constant $c>0$, not depending on P_0.

Slicing the d-cube R_1 with hyperplanes orthogonal to the direction e_1, and setting $c_p:=2p^{-p/2}$, for every $h\le h_\varepsilon$ we find $h_1\in[x_0^1-h/2,x_0^1+h/2]$ such that

$$\begin{aligned}\mathcal{E}_p(U^{(m)},R_1\cap P(h_1,1)) &\le \frac{2}{h}\mathcal{E}_p(U^{(m)},R_1)\\ &\le \frac{c_p}{h}\int_{\Sigma(P_0,h)}|DU^{(m)}(y)|^p\,d\mathcal{H}^d\le c_p\,\frac{\varepsilon}{h}\,.\end{aligned}$$

We now choose $z_0\in R_1\cap P(h_1,1)\cap(\Sigma_m^{(d-1)}\times\{0\})$ and set $y_h^{(m)}:=U^{(m)}(z_0)$ in such a way that $y_h^{(m)}\in\mathcal{Y}$. Applying the Sobolev embedding theorem, since $R_1\cap P(h_1,1)$ is a $(d-1)$-cube of side h, and $d=[p]$, it follows that

$$\max_{z\in R_1\cap P(h_1,1)}|U^{(m)}(z)-y_h^{(m)}|\le c\,h^{1-d/p}\,\varepsilon^{1/p}\le c\,\varepsilon^{1/p}\,. \tag{2.5}$$

Moreover, we note that

$$|U_h^{(m)}(P_0)-y_h^{(m)}|\le\fint_{\Sigma(P_0,h)}|U^{(m)}(y)-y_h^{(m)}|\,d\mathcal{H}^d(y)\,. \tag{2.6}$$

Let η be a positive number to be determined later. We slice the d-dimensional set $\Sigma(P_0,h)$ with hyperplanes orthogonal to the "vertical" direction e_{n+1}, and denote

$$\Omega_{h'}:=\Sigma(P_0,h)\cap P(h',n+1)\,,\qquad h'\in[-h/2,h/2]\,.$$

Setting

$$A_h:=\{h'\in[-h/2,h/2]\,:\,p^{p/2}\,\mathcal{E}_p(U^{(m)},\Omega_{h'})\le\varepsilon\,\eta/h\}$$

and $B_h := [-h/2, h/2] \setminus A_h$, by (2.4) we have $\mathcal{L}^1(B_h) \le h/\eta$. Moreover, for every h' the set $\Omega_{h'}$ is given by the connected union of $K = \binom{n}{d-1}$ parallelepipeds of dimension not greater than $d-1$ and diameter lower than $\sqrt{d-1}\,h$. Since $h^{1-d/p} \le 1$, by the Sobolev theorem we obtain that for every $h' \in A_h$

$$\max_{z,y\in\Omega_{h'}} |U^{(m)}(z) - U^{(m)}(y)| \le c\,\eta^{1/p}\,\varepsilon^{1/p}\,.$$

Note that $\Omega_{h'}$ intersects $R_1 \cap P(h_1, 1)$ for every h'. Therefore, combining with (2.5) we obtain

$$\max_{y\in\Omega_{h'}} |U^{(m)}(y) - y_h^{(m)}| \le c\,(\eta^{1/p} + 1)\,\varepsilon^{1/p} \qquad \forall\, h' \in A_h\,. \tag{2.7}$$

By Fubini theorem we write

$$\begin{aligned} \int_{\Sigma(P_0,h)} |U^{(m)}(y) - y_h^{(m)}|\, d\mathcal{H}^d(y) &= \int_{B_h}\int_{\Omega_{h'}} |U^{(m)}(y) - y_h^{(m)}|\, d\mathcal{H}^{d-1}\, dh' \\ &+ \int_{A_h}\int_{\Omega_{h'}} |U^{(m)}(y) - y_h^{(m)}|\, d\mathcal{H}^{d-1}\, dh'\,. \end{aligned}$$

Since $\|U^{(m)}\|_\infty \le K_\infty < \infty$ by the compactness of $\mathcal{Y}$, whereas $\mathcal{L}^1(B_h) \le h/\eta$ and $h^d \le \mathcal{H}^d(\Sigma(P_0,h)) \le c\,h^d$, using (2.6) and (2.7) we get

$$|U_h^{(m)}(P_0) - y_h^{(m)}| \le c_1\,\frac{K_\infty}{\eta} + c_2\,(\eta^{1/p} + 1)\,\varepsilon^{1/p}\,. \tag{2.8}$$

Finally, taking first η large so that $c_1 K_\infty/\eta < \varepsilon_0/2$, and then ε small so that $c_2\,(\eta^{1/p}+1)\,\varepsilon^{1/p} < \varepsilon_0/2$, by the arbitrariness of P_0 in $\Sigma_m^{(d-1)} \times \{0\}$ we conclude that

$$\mathrm{dist}\big(u_h^{(m)}(x), \mathcal{Y}\big) < \varepsilon_0 \qquad \forall\, x \in \Sigma_m^{(d-1)}$$

for every $h \le h_\varepsilon$, which clearly yields the assertion. □

Proposition 2.4 *Let $n+1 > p > 1$ and $d = [p]$. Assume that $U^{(m)} \in W^{1,p}(C_m^{(d-1)} \times I, \mathbb{R}^N)$. Then there exists a sequence of maps $\{V_h^{(m)}\}_h$ in $W^{1,p}(\mathcal{Q}_m^n \times I, \mathbb{R}^N)$, continuous out of $\mathcal{Q}_m^n \times \{0\}$, such that $V_h^{(m)} \to U_{|\mathcal{Q}_m^n \times I}$ strongly in $W^{1,p}(\mathcal{Q}_m^n \times I, \mathbb{R}^N)$, with ${V_h^{(m)}}_{|\Sigma_m^{(d-1)} \times I} = U_h^{(m)}$, see Proposition 2.2. In particular we have*

$$\mathbf{T}(V_h^{(m)})_{|\Sigma_m^{(d-1)}} \in W^{1/p}(\Sigma_m^{(d-1)}, \mathcal{Y}_{\varepsilon_0}) \qquad \forall\, h\,.$$

Proof: We first consider the case $n = d = [p]$.

The case $n = d = [p]$. Let $\mathcal{C}_m$ denote the family of all n-cubes Q of side $1/m$ with boundary contained in the $(n-1)$-grid $\Sigma_m^{(n-1)}$, i.e. $\partial Q \subset \Sigma_m^{(n-1)}$, so that

$$\cup\,\mathcal{C}_m = \mathcal{Q}_m^n\,.$$

Let $0 < \varepsilon < 1/2$ to be fixed later. If $Q \in \mathcal{C}_m$, we define $V_h^{(Q)} : Q \times I \to \mathbb{R}^N$ by setting for every $(x,t) \in Q \times I$

$$V_h^{(Q)} := \begin{cases} U\Big(q + \dfrac{x-q}{1-\varepsilon}, t\Big) & \text{if } \rho \le \dfrac{1-\varepsilon}{2m} \\ S(\rho)\,U_h^{(m)}(y,t) + (1 - S(\rho))\,U(y,t) & \text{if } \dfrac{1-\varepsilon}{2m} \le \rho \le \dfrac{1}{2m}\,. \end{cases} \tag{2.9}$$

Here $\rho = \rho(x) := \|x - q\|_n$, where q is the center of Q, so that $\rho(x) = 1/(2m)$ if $x \in \partial Q$; moreover

$$y = y(x) := q + \frac{1}{2m}\,\frac{x-q}{\rho(x)}$$

and finally

$$S(\rho) := \frac{2m}{\varepsilon}\,\rho + \frac{\varepsilon - 1}{\varepsilon}\,, \tag{2.10}$$

so that $S\big(1/(2m)\big) = 1$ and $S\big((1-\varepsilon)/(2m)\big) = 0$. Trivially $V_h^{(Q)}$ is a function in $W^{1,p}(Q \times I, \mathbb{R}^N)$, continuous out of $Q \times \{0\}$. Moreover, it is readily checked that

$$\int_{\{\rho(x)\le(1-\varepsilon)/(2m)\}\times I} |DV_h^{(Q)}|^p\,dx\,dt \le (1-\varepsilon)^{n-p}\,p^{p/2}\,\mathcal{E}_p(U, Q\times I)$$

and

$$\int_{\{(1-\varepsilon)/(2m)\le\rho(x)\le 1/(2m)\}\times I} |DV_h^{(Q)}|^p\,dx\,dt \le c\,(m,p)\,\frac{1}{\varepsilon}\int_{\partial Q\times I} |U - U_h^{(m)}|^p\,d\mathcal{H}^n$$
$$+\,c\,(m,p)\,\varepsilon \int_{\partial Q\times I} \big(|D_\tau U|^p + |D_\tau U_h^{(m)}|^p\big)\,d\mathcal{H}^n\,,$$

where τ is an orthonormal frame to $\Sigma_m^{(n-1)} \times I$ and $c\,(m,p) > 0$ only depends on m and p. Define now $V_h^{(m)} : \mathcal{Q}_m^n \times I \to \mathbb{R}^N$ by $V_h^{(m)}(x,t) := V_h^{(Q)}(x,t)$ if $x \in Q$ for some $Q \in \mathcal{C}_m$. Then $\{V_h^{(m)}\}_h$ is a sequence in $W^{1,p}(\mathcal{Q}_m^n \times I, \mathbb{R}^N)$, continuous out of $\mathcal{Q}_m^n \times \{0\}$, such that

$$\begin{aligned}
\mathcal{E}_p(V_h^{(m)}, \mathcal{Q}_m^n \times I) \;&\le\; (1-\varepsilon)^{n-p}\,\mathcal{E}_p(U, \mathcal{Q}_m^n\times I)\\
&+\; c_1\,(m,p)\,\frac{1}{\varepsilon}\int_{\Sigma_m^{(n-1)}\times I} |U^{(m)} - U_h^{(m)}|^p\,d\mathcal{H}^n\\
&+\; c_2\,(m,p)\,\varepsilon\int_{\Sigma_m^{(n-1)}\times I} \big(|D_\tau U^{(m)}|^p + |D_\tau U_h^{(m)}|^p\big)\,d\mathcal{H}^n\,,
\end{aligned}$$

see (2.3). Moreover, since $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, see Proposition 2.2, there exists $\overline{h} \in \mathbb{N}$ such that for every $h \ge \overline{h}$

$$\int_{\Sigma_m^{(n-1)}\times I} |D_\tau U_h^{(m)}|^p\,d\mathcal{H}^n \le 2\int_{\Sigma_m^{(n-1)}\times I} |D_\tau U^{(m)}|^p\,d\mathcal{H}^n\,.$$

Now, for every $j \in \mathbb{N}^+$ we first choose $\varepsilon = \varepsilon_j \in (0, 1/2)$ small so that $\varepsilon_j \searrow 0$,

$$(1-\varepsilon_j)^{n-p}\,\mathcal{E}_p(U, \mathcal{Q}_m^n \times I) \le \mathcal{E}_p(U, \mathcal{Q}_m^n \times I) + \frac{1}{j}$$

and

$$3\,c_2\,(m,p)\,\varepsilon_j \int_{\Sigma_m^{(n-1)}\times I} |D_\tau U^{(m)}|^p\,d\mathcal{H}^n \le \frac{1}{j}\,.$$

Secondly, by the L^p-convergence of $U_h^{(m)}$ to $U^{(m)}$, we take $h = h_j \ge \overline{h}$ large so that $h_{j+1} > h_j$ and

$$c_1\,(m,p)\,\frac{1}{\varepsilon_j}\int_{\Sigma_m^{(n-1)}\times I} |U^{(m)} - U_{h_j}^{(m)}|^p\,d\mathcal{H}^n \le \frac{1}{j} \qquad \forall\, j\,.$$

Finally, since by the previous estimates

$$\mathcal{E}_p(V_{h_j}^{(m)}, \mathcal{Q}_m^n \times I) \le \mathcal{E}_p(U, \mathcal{Q}_m^n \times I) + \frac{3}{j}\,,$$

we relabel $\{V_j^{(m)}\}$ the subsequence $\{V_{h_j}^{(m)}\}$, where $\varepsilon = \varepsilon_j$ in (2.9). Using again the strong convergence of $U_h^{(m)}$ to $U^{(m)}$ in $W^{1,p}(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, the Poincaré inequality yields the strong L^p-convergence of $V_j^{(m)}$ to U, and hence the assertion, by uniform convexity.

The case $n-1 \geq d = [p]$. We first set $V_h^{(m)} = U_h^{(m)}$ on $\Sigma_m^{(d-1)} \times I$, according to Proposition 2.2. Arguing by induction on the dimension $k = d, \ldots, n$, by the inductive hypothesis we have already defined $V_h^{(m)} : \Sigma_m^{(k-1)} \times I \to \mathbb{R}^N$ in such a way that $V_h^{(m)} \to U_{|\Sigma_m^{(k-1)} \times I}$ strongly in $W^{1,p}(\Sigma_m^{(k-1)} \times I, \mathbb{R}^N)$.

We now extend $\{V_h^{(m)}\}$ to $\Sigma_m^{(k)} \times I$ as follows. Let F be a k-face of side $1/m$ of $\Sigma_m^{(k)}$, and hence with boundary contained in $\Sigma_m^{(k-1)}$. Without loss of generality, we suppose F oriented by $e_1 \wedge \cdots \wedge e_k$, and we set

$$x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^k \times \mathbb{R}^{n-k} .$$

Similarly to (2.9), we define $V_h^{(F)} : F \times I \to \mathbb{R}^N$ by setting for $(x,t) \in F \times I$

$$V_h^{(F)} := \begin{cases} U\Big(\widetilde{q} + \dfrac{\widetilde{x} - \widetilde{q}}{1-\varepsilon}, \widehat{q}, t\Big) & \text{if } \rho \leq \dfrac{1-\varepsilon}{2m} \\ S(\rho)\, V_h^{(m)}\big(y, \widehat{q}, t\big) + \big(1 - S(\rho)\big)\, U\big(y, \widehat{q}, t\big) & \text{if } \dfrac{1-\varepsilon}{2m} \leq \rho \leq \dfrac{1}{2m} . \end{cases}$$

Here $\rho = \rho(\widetilde{x}) := \|\widetilde{x} - \widetilde{q}\|_k$, where $(\widetilde{q}, \widehat{q}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is the center of F; moreover

$$y = y(\widetilde{x}) := \widetilde{q} + \frac{1}{2m}\, \frac{\widetilde{x} - \widetilde{q}}{\rho(\widetilde{x})}$$

and $S(\rho)$ is given by (2.10).

We then extend $V_h^{(m)} : \Sigma_m^{(k)} \times I \to \mathbb{R}^N$ by setting $V_h^{(m)}(x,t) := V_h^{(F)}(x,t)$ if $x \in F$ for some k-face F as above. Similarly to the case $n = d = [p]$, using that $V_h^{(m)} \to U_{|\Sigma_m^{(k-1)} \times I}$ strongly in $W^{1,p}(\Sigma_m^{(k-1)} \times I, \mathbb{R}^N)$, and by suitably choosing $\varepsilon = \varepsilon_j \searrow 0$, we infer that $\{V_h^{(m)}\}_h$ is a sequence in $W^{1,p}(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$, continuous out of $\Sigma_m^{(k)} \times \{0\}$, such that, possibly passing to a subsequence, $V_{h_j}^{(m)} \to U_{|\Sigma_m^{(k)} \times I}$ strongly in $W^{1,p}(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$. The proof of Proposition 2.4 is complete. □

Proof of Theorem 1.1: Let $u \in W^{1/p}(\mathcal{Q}^n, \mathcal{Y})$ and $U : \mathcal{Q}^n \times I \to \mathbb{R}^N$ be the extension $\mathrm{Ext}(u)$ of u, so that $U \in W^{1,p}(\mathcal{Q}^n \times I, \mathbb{R}^N)$ and $\mathbf{T}(U) = u$. We proceed along the lines of [3, Lemma 5], and we first consider the case $n = [p]$.

The case $n = d = [p]$. Let $m \in \mathbb{N}^+$. Since for $i = 1, \ldots, n$ we have

$$\begin{aligned} \int_{1/(4m)}^{3/(4m)} \sum_{j=0}^{m-1} \mathcal{E}_p(U, P(t + j/m, i))\, dt \quad &\leq \quad \sum_{j=0}^{m-1} \mathcal{E}_p(U, \{j/m \leq x_i \leq (j+1)/m\}) \\ &= \quad \mathcal{E}_p(U, \mathcal{Q}^n \times I) , \end{aligned}$$

we find a vector $a = a(m) \in [1/(4m), 3/(4m)]^n$ such that

$$U_{|P(a_i + j/m, i)} \in W^{1,p}(P(a_i + j/m, i), \mathbb{R}^N)$$

for every $i = 1, \ldots, n$ and $j = 0, \ldots, m-1$, and

$$\mathcal{E}_p(U, C_m^{(n-1)} \times I) \leq c\, m\, \mathcal{E}_p(U, \mathcal{Q}_m^n \times I) . \tag{2.11}$$

We now apply Propositions 2.2 and 2.4 with $a = a(m)$. Slicing the cylinder $\mathcal{Q}_m^n \times I$ with hyperplanes $P(t, n+1)$ orthogonal to the "vertical" direction e_{n+1}, since $\{V_h^{(m)}\}$ converges to $U_{|\mathcal{Q}_m^n \times I}$ strongly in $W^{1,p}(\mathcal{Q}_m^n \times I, \mathbb{R}^N)$, Proposition 2.4, we may and do choose $a_{n+1} \in [1/(4m), 3/(4m)]$ so that

$$V^{(m)}_{h|P(a_{n+1} - j/m, n+1)} \in W^{1,p}(P(a_{n+1} - j/m, n+1), \mathbb{R}^N)$$

for every h and for $j = 0, 1$, with

$$\sum_{j=0,1} \mathcal{E}_p(V_h^{(m)}, P(a_{n+1} - j/m, n+1)) \leq c\, m\, \mathcal{E}_p(U, \mathcal{Q}_m^n \times I) \tag{2.12}$$

for every h. Let $\mathcal{F}_m$ denote the family of $(n+1)$-cubes of $\mathcal{Q}^n_m \times I$, of side $1/m$, whose boundary lies in the n-skeleton

$$\mathcal{L}_m \cup \bigcup_{j=0,1} P(a_{n+1} - j/m, n+1)\,,$$

compare (2.1), and let $\{C_l\}_{l=1}^{(m-1)^n}$ be a list of the $(n+1)$-cubes in $\mathcal{F}_m$. Notice that each C_l intersects the n-cube $\mathcal{Q}^n \times \{0\}$.

Recall that $V_h^{(m)}{}_{|\Sigma_m^{(n-1)}\times I} = U_h^{(m)}$, where $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, see Propositions 2.2 and 2.4. Then, as in [3, Lemma 5], by refining the slicing arguments in (2.11) and (2.12) we in fact may and do choose $(a_1, \dots, a_{n+1}) \in [1/(4m), 3/(4m)]^{n+1}$ in such a way that

$$\sum_{l=1}^{(m-1)^n} \mathcal{E}_p(V_h^{(m)}, \partial C_l) \le c\, m\, \mathcal{E}_p(U, G_m) \qquad \forall\, h \ge \overline{h}\,, \tag{2.13}$$

where

$$G_m := \mathcal{Q}^n \times]-10m^{-1}, 10m^{-1}[\,.$$

For every l let f_l be a bilipschitz homeomorphism between C_l and $[-1/(2m), 1/(2m)]^{n+1}$ such that

$$\begin{array}{rcl} f_l(C_l \cap (\mathcal{Q}^n \times \{0\})) & = & [-1/(2m), 1/(2m)]^n \times \{0\} \\ f_l(\partial C_l \cap (\mathcal{Q}^n \times \{0\})) & = & \partial[-1/(2m), 1/(2m)]^n \times \{0\} \end{array}$$

and $\|Df_l\|_\infty \le K$, $\|Df_l^{-1}\|_\infty \le K$. We then define $W_h^{(m)}$ on C_l by

$$W_h^{(m)}(z) := V_h^{(m)}\Big[f_l^{-1}\Big(\frac{f_l(z)}{2m\|f_l(z)\|_{n+1}}\Big)\Big]\,, \tag{2.14}$$

so that

$$\mathcal{E}_p(W_h^{(m)}, C_l) \le \frac{c}{m}\, \mathcal{E}_p(V_h^{(m)}, \partial C_l)$$

for every l and hence, by (2.13),

$$\mathcal{E}_p(W_h^{(m)}, \cup\mathcal{F}_m) \le C\, \mathcal{E}_p(U, G_m)\,. \tag{2.15}$$

Setting

$$W_h^{(m)}(z) = V_h^{(m)}(z) \qquad \forall\, z \in (\mathcal{Q}^n_m \times I) \setminus \cup\mathcal{F}_m\,,$$

the function $W_h^{(m)}$ is continuous on $\mathcal{Q}^n_m \times I$ except at one singular point on each C_l, which lies on $\mathcal{Q}^n_m \times \{0\}$. Moreover, $\{W_h^{(m)}\}$ is a sequence in $W^{1,p}(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$ such that for h large enough

$$\mathcal{E}_p(W_h^{(m)} - V_h^{(m)}, \mathcal{Q}^n_m \times I) \le C\, \mathcal{E}_p(U, G_m)$$

and therefore, by Proposition 2.4,

$$\limsup_{h\to\infty} \mathcal{E}_p(W_h^{(m)}, \mathcal{Q}^n_m \times I) \le \mathcal{E}_p(U, \mathcal{Q}^n_m \times I) + C\, \mathcal{E}_p(U, G_m)\,.$$

Remark 2.5 For every $(n+1)$-cube C_l in $\mathcal{F}_m$ we have that $W_h^{(m)}{}_{|\partial C_l} = V_h^{(m)}{}_{|\partial C_l}$, where the traces $\mathbf{T}(V_h^{(m)})_{|\Sigma_m^{(n-1)}}$ belong to $W^{1/p}(\Sigma_m^{(n-1)}, \mathcal{Y}_{\varepsilon_0})$, see Proposition 2.4. As a consequence, by the definition (2.14) we infer that the traces $\mathbf{T}(W_h^{(m)})$ are functions in $W^{1/p}(\mathcal{Q}^n_m, \mathcal{Y}_{\varepsilon_0})$ for every h.

Now, let $\psi_m : \mathcal{Q}^n \to \mathcal{Q}^n_m$ be an affine bijective function such that $\mathrm{Lip}\,\psi_m = (m-1)/m$ and $\psi_m \to Id_{\mathcal{Q}^n}$ uniformly as $m \to \infty$. Setting $U_m(x,t) := W_{h_m}^{(m)}(\psi_m(x), t)$ for some increasing sequence $h_m \nearrow \infty$, since $\mathrm{meas}(G_m) \to 0$ as $m \to \infty$ we easily infer that $\{U_m\}_m$ is a sequence of maps in $W^{1,p}(\mathcal{Q}^n \times I, \mathbb{R}^N)$, continuous out of a finite number of points, such that $U_m \to U$ strongly in $W^{1,p}$. Moreover by Remark 2.5 it follows that the traces $\mathbf{T}(U_m) \in W^{1/p}(\mathcal{Q}^n, \mathcal{Y}_{\varepsilon_0})$ for every m. Therefore, taking $u_m(x) := \Pi_{\varepsilon_0} \circ \mathbf{T}(U_m)(x)$, compare Remark 2.1, clearly $\{u_m\} \subset W^{1/p}(\mathcal{Q}^n, \mathcal{Y})$ is continuous out of a discrete set of points and $u_m \to u$

in $W^{1/p}$. Finally, e.g. as in [2, Appendix], every function u_m can be approximated by maps in $R^\infty_{1/p}(\mathcal{Q}^n,\mathcal{Y})$.

The case $n-1\geq d=[p]$. By applying iteratively the Fubini theorem, we fist observe that for a.e. $a=a(m)$ as above, the restriction of U to each k-face F of $C^{(k)}_m$ belongs to $W^{1,p}(F,\mathbb{R}^N)$, for every $k=d-1,\dots,n$. We then may and do apply Propositions 2.2 and 2.4 with $a=a(m)$.

Let $\mathcal{F}^{(k)}_m$ be the k-dimensional skeleton of $\mathcal{F}_m$, i.e. the union of the k-faces of the $(n+1)$-cubes C_l of $\mathcal{F}_m$. Since $V^{(m)}_h\to U$ in $W^{1,p}(\mathcal{Q}^n_m\times I,\mathbb{R}^N)$, by using a more refined slicing argument as e.g. in [13, Sec. 4], we may and do choose $(a_1,\dots,a_{n+1})\in[1/(4m),3/(4m)]^{n+1}$ so that for every h sufficiently large the following holds:

(i) for every $k=d,\dots,n$ the restriction of $V^{(m)}_h$ to any k-face Q of $\mathcal{F}^{(k)}_m$ is a function in $W^{1,p}(Q,\mathbb{R}^N)$;

(ii) there exists some absolute constant $c>0$, not depending on h, such that for every $k=d,\dots,n$

$$\mathcal{E}_p(V^{(m)}_h,\mathcal{F}^{(k)}_m)\leq c\,m^{n+1-k}\,\mathcal{E}_p(U,G_m)\,. \tag{2.16}$$

First we let $W^{(m)}_h\equiv V^{(m)}_h$ on $\mathcal{F}^{(d)}_m$. Arguing by induction on $k=d,\dots,n$, we now extend $W^{(m)}_h$ to $\mathcal{F}^{(k+1)}_m$. To this aim, for every $(k+1)$-face Q in $\mathcal{F}^{(k+1)}_m$ we distinguish two cases.

If Q is "horizontal", i.e. the direction e_{n+1} is not spanned by the vector space underlying Q, we let

$$W^{(m)}_h\equiv V^{(m)}_h\qquad\text{on }\,Q\,. \tag{2.17}$$

If Q is not "horizontal", as in the case $n=d=[p]$ we let f_Q be a bilipschitz homeomorphism between Q and $[-1/(2m),1/(2m)]^{k+1}$ such that

$$\begin{aligned}f_Q(Q\cap(\mathcal{Q}^n\times\{0\}))&=[-1/(2m),1/(2m)]^k\times\{0\}\\ f_Q(\partial Q\cap(\mathcal{Q}^n\times\{0\}))&=\partial[-1/(2m),1/(2m)]^k\times\{0\}\end{aligned}$$

and $\|Df_Q\|_\infty\leq K$, $\|Df_Q^{-1}\|_\infty\leq K$. Since we have already defined $W^{(m)}_h$ on ∂Q, we extend $W^{(m)}_h$ to Q by setting

$$W^{(m)}_h(z)=W^{(m)}_h\Big[f_Q^{-1}\Big(\frac{f_Q(z)}{2m\|f_Q(z)\|_{k+1}}\Big)\Big]\,, \tag{2.18}$$

so that

$$\mathcal{E}_p(W^{(m)}_h,Q)\leq\frac{c}{m}\,\mathcal{E}_p(W^{(m)}_h,\partial Q)\,. \tag{2.19}$$

Repeating the argument for $k=d,\dots,n$, we then easily estimate

$$\mathcal{E}_p(W^{(m)}_h,\cup\mathcal{F}_m)\leq C(n,p)\sum_{k=d}^n\frac{1}{m^{n+1-k}}\mathcal{E}_p(V^{(m)}_h,\mathcal{F}^{(k)}_m) \tag{2.20}$$

and hence, by (2.16), we obtain again (2.15). Setting then $W^{(m)}_h(z)=V^{(m)}_h(z)$ for every $z\in(\mathcal{Q}^n_m\times I)\setminus\cup\mathcal{F}_m$, this way $W^{(m)}_h$ is continuous on $\mathcal{Q}^n_m\times I$ outside an $(n-d)$-dimensional singular set, which lies on $\mathcal{Q}^n_m\times\{0\}$, given by the union of a finite number (depending on n, d, and m) of smooth subsets of affine $(n-d)$-planes parallel to the coordinate directions in $\mathbb{R}^n\times\{0\}$. Moreover, by the construction we infer that the traces $\mathbf{T}(W^{(m)}_h)\in W^{1/p}(\mathcal{Q}^n_m,\mathcal{Y}_{\varepsilon_0})$ for every m. The rest of the proof follows as in the case $n=d=[p]$. $\square$

Proof of Theorem 1.2: We have to show that $H^{1/p}_S(\mathcal{X},\mathcal{Y})=W^{1/p}(\mathcal{X},\mathcal{Y})$ provided that $\pi_{n-1}(\mathcal{Y})=0$. On account of Theorem 1.1, it suffices to prove that $R^\infty_{1/p}(\mathcal{X},\mathcal{Y})\subset H^{1/p}_S(\mathcal{X},\mathcal{Y})$. Moreover, since the argument is local, without loss of generality we assume that $\mathcal{X}=\mathcal{Q}^n$ and $u\in R^\infty_{1/p}(\mathcal{Q}^n,\mathcal{Y})$ is smooth outside the origin. For $0<r<1$ we denote

$$Q_r:=[-r,r]^{n+1}\,,\qquad F_r:=Q_r\cap(\mathbb{R}^n\times\{0\})\,.$$

Let $U=\mathrm{Ext}(u)\in W^{1,p}(\mathcal{Q}^n\times I,\mathbb{R}^N)$ be the extension of u. For every fixed $\varepsilon>0$ let $0<R=R(\varepsilon)\ll 1$ be such that $\mathcal{E}_p(U,Q_R)\leq\varepsilon$. Since

$$\mathcal{E}_p(U,Q_R\setminus Q_{R/2})=\frac{1}{p^{p/2}}\int_{R/2}^R dr\int_{\partial Q_r}|DU|^p\,d\mathcal{H}^n\,,$$

there exists $r=r(\varepsilon)\in[R/2,R]$ such that

$$\mathcal{E}_p(U,\partial Q_r):=\frac{1}{p^{p/2}}\int_{\partial Q_r}|DU|^p\,d\mathcal{H}^n\leq\frac{2}{R}\,\mathcal{E}_p(U,Q_R\setminus Q_{R/2})\leq\frac{2\,\varepsilon}{R}\,. \tag{2.21}$$

Since $u_{|\partial F_r}:\partial F_r\to\mathcal{Y}$ is a smooth map in $W^{1/p}(\partial F_r,\mathcal{Y})$ and $\pi_{n-1}(\mathcal{Y})=0$, there exists a smooth extension $u_r:F_r\to\mathcal{Y}$ of u with finite $W^{1,p}$-norm.

Let now $Q_r^\pm:=\{z=(x,t)\in Q_r\mid\pm t\geq 0\}$ be the upper and lower half $(n+1)$-cubes of Q_r. Moreover, let $V_r^\pm:Q_r^\pm\to\mathbb{R}^N$ be a function that minimizes the p-energy on $Q_r^\pm$ among all maps in $W^{1,p}(Q_r^\pm,\mathbb{R}^N)$ satisfying the boundary condition

$$\begin{cases} V_r^\pm=U & \text{on}\quad \partial Q_r^\pm\cap\{(x,t)\mid\pm t>0\}\\ V_r^\pm=u_r & \text{on}\quad F_r\end{cases}$$

and let $V_r:Q_r\to\mathbb{R}^N$ be given by $V_r(z)=V_r^\pm(z)$ if $z\in Q_r^\pm$. Define then $W_r:\mathcal{Q}^n\times I\to\mathbb{R}^N$ by

$$W_r(z):=\begin{cases} V_r\Big(\dfrac{r}{\delta}z\Big) & \text{if}\quad \|z\|_{n+1}\leq\delta\\ U\Big(\dfrac{r\,z}{\|z\|_{n+1}}\Big) & \text{if}\quad \delta\leq\|z\|_{n+1}\leq r\\ U(z) & \text{if}\quad \|z\|_{n+1}\geq r\end{cases}$$

for a suitable $0<\delta<r$. Since $V_r^\pm$ is continuous, $W_r\in W^{1,p}(\mathcal{Q}^n\times I,\mathbb{R}^N)$ is continuous and with trace $\mathbf{T}(W_r)\in W^{1/p}(\mathcal{Q}^n,\mathcal{Y})$. We easily estimate

$$\mathcal{E}_p(W_r,\mathcal{Q}^n\times I)\leq\mathcal{E}_p(U,\mathcal{Q}^n\times I)+c\,r\,\mathcal{E}_p(U,\partial Q_r)+\Big(\frac{\delta}{r}\Big)^{n+1-p}\mathcal{E}_p(V_r,Q_r)$$

for some absolute constant $c>0$, depending on n and p, so that by (2.21), and since $r<R$,

$$\begin{aligned}\mathcal{E}_p(W_r,\mathcal{Q}^n\times I)&\leq\mathcal{E}_p(U,\mathcal{Q}^n\times I)+2\,c\,\varepsilon+\Big(\frac{\delta}{r}\Big)^{n+1-p}\mathcal{E}_p(V_r,Q_r)\\&\leq\mathcal{E}_p(U,\mathcal{Q}^n\times I)+(2c+1)\,\varepsilon\,,\end{aligned}$$

taking $\delta=\delta(r,\varepsilon)$ sufficiently small. Letting $\varepsilon\to 0$ we infer that $W_{r(\varepsilon)}\to U$ in $W^{1,p}(\mathcal{Q}^n\times I,\mathbb{R}^N)$ and hence that $\mathbf{T}(W_{r(\varepsilon)})\to u$ in $W^{1/p}(\mathcal{Q}^n,\mathcal{Y})$. Since the trace $\mathbf{T}(W_r)\in W^{1/p}(\mathcal{Q}^n,\mathcal{Y})$ is continuous, then in a standard way it can be approximated by smooth maps, as required. □

3 Homotopy type of $W^{1/p}$-maps

In this section we let $n+1>p\geq 2$ and $d=[p]$. We shall prove the following

Proposition 3.1 *Let $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ and X be a finite cubeulation of $\mathcal{X}$ in generic position with respect to u. For any smooth sequence $\{u_i\}\subset W^{1/p}(X^{d-1},\mathcal{Y})\cap C^\infty$ strongly converging to $u_{|X^{d-1}}$ in $W^{1/p}$, we find $k_0\in\mathbb{N}^+$ such that for every $i,j\geq k_0$ the maps u_i and u_j are homotopic as maps from X^{d-1} to $\mathcal{Y}$.*

By the argument of Proposition 3.1 we shall then obtain Proposition 1.3.

Proof of Proposition 3.1: Following the notation from Sec. 2, we shall give the proof in the case $\mathcal{X}=\mathcal{Q}^n$ and $X^k:=\Sigma_m^{(k)}$, see (2.2), making use of the argument from [3, Lemma 1], that goes back to [16].

The case of general X is obtained by means of an easy adaptation of the argument below. In fact, the manifold $\mathcal{X}$ being smooth and compact, for any given finite cubeulation X, taking local coordinate charts, we find a bilipschitz homeomorphism ψ, with Lipschitz constants $\operatorname{Lip}\psi$ and $\operatorname{Lip}\psi^{-1}$ bounded by a constant not depending on the local chart, and a number $m \in \mathbb{N}^+$, such that in each coordinate chart $\psi(X^k) = \Sigma_m^{(k)}$ for every dimension k.

We now let $\mathcal{X} = \mathcal{Q}^n$ and $X^k := \Sigma_m^{(k)}$. Let $U_i \in W^{1,p}(\Sigma_m^{(d-1)} \times I, \mathbb{R}^N)$ be such that $\mathbf{T}(U_i) = u_i$ and U_i minimizes the p-energy among all maps $V \in W^{1,p}(\Sigma_m^{(d-1)} \times I, \mathbb{R}^N)$ such that $\mathbf{T}(V) = u_i$. Let $\sigma_0 > 0$ to be chosen. By the strong convergence of u_i to u, we can find $k_0 \in \mathbb{N}$ such that

$$\|U_i - U_j\|_{W^{1,p}(\Sigma_m^{(d-1)} \times I)} < \sigma_0 \qquad \forall\, i, j \geq k_0 \,. \tag{3.1}$$

If $z = (x, t) \in \Sigma_m^{(d-1)} \times I$ and $0 < h < 1/(4m)$, we let

$$U_i(h, z) := \fint_{\Sigma(z,h)} U_i(y)\, d\mathcal{H}^d(y) := \frac{1}{\mathcal{H}^d\big(\Sigma(z,h)\big)} \int_{\Sigma(z,h)} U_i(y)\, d\mathcal{H}^d(y) \,,$$

where the d-dimensional set $\Sigma(z, h)$ is defined as in the proof of Proposition 2.2 from Sec. 2. Moreover, let $u_i(h, \cdot) := \mathbf{T}(U_i(h, \cdot)) \in W^{1/p}(\Sigma_m^{(d-1)}, \mathbb{R}^N)$. For every i, we infer that $U_i(h, z)$ is continuous, whereas $U_i(h, \cdot)$ tends to U_i and $u_i(h, \cdot)$ tends to u_i uniformly as $h \to 0$. Let $\varepsilon_1 > 0$ to be chosen. By the strong convergence of u_i to u, we also may and do fix a positive number $h_0 < 1/(4m)$ such that for every $z \in \Sigma_m^{(d-1)} \times I$, and for any $0 < h \leq h_0$, we have

$$\mathcal{E}_p(U_i, \Sigma(z, h)) \leq \varepsilon_1 \qquad \forall\, i \,. \tag{3.2}$$

If $\xi := (x, 0) \in \Sigma_m^{(d-1)} \times \{0\}$, for $i \neq j$ we estimate

$$\begin{aligned}
|u_i(h_0, x) - u_j(h_0, x)| &= |U_i(h_0, \xi) - U_j(h_0, \xi)| \\
&= \Big(\fint_{\Sigma(\xi, h_0)} |U_i(h_0, \xi)) - U_j(h_0, \xi)|^p \, d\mathcal{H}^d(y) \Big)^{1/p} \\
&\leq \Big(\fint_{\Sigma(\xi, h_0)} |U_i(h_0, \xi)) - U_i(y)|^p \, d\mathcal{H}^d(y) \Big)^{1/p} \\
&+ \Big(\fint_{\Sigma(\xi, h_0)} |U_j(h_0, \xi)) - U_j(y)|^p \, d\mathcal{H}^d(y) \Big)^{1/p} \\
&+ \Big(\fint_{\Sigma(\xi, h_0)} |U_i(y) - U_j(y)|^p \, d\mathcal{H}^d(y) \Big)^{1/p} \\
&=: I_1 + I_2 + I_3 \,.
\end{aligned}$$

Using (3.2) and the Poincaré inequality, we have

$$I_1 + I_2 \leq c\, h_0^{(p-d)/p} \, {\varepsilon_1}^{1/p} \,,$$

whereas by (3.1), using that $\mathcal{H}^d(\Sigma(\xi, h_0)) \geq h_0^d$, we infer that if $i, j \geq k_0$

$$I_3 \leq h_0^{-d/p} \, \|U_i - U_j\|_{W^{1,p}(\Sigma_m^{(d-1)} \times I)} \leq C\, h_0^{-d/p} \, \sigma_0 \,.$$

Since $d \leq p$ and $h_0 < 1$, we then obtain for every $x \in \Sigma_m^{(d-1)}$, and for $i, j \geq k_0$,

$$|u_i(h_0, x) - u_j(h_0, x)| \leq c_3 \, {\varepsilon_1}^{1/p} + c_4 \, h_0^{-d/p} \, \sigma_0 \,. \tag{3.3}$$

Let now $\varepsilon_0 > 0$ be given by Remark 2.1. As in the proof of Proposition 2.2, see (2.8), using that $d = [p]$, taking first η large so that $c_1 \, K_\infty / \eta < \varepsilon_0 / 2$, we infer that if ε_1 satisfies

$$c_2 \, (\eta^{1/p} + 1) \, {\varepsilon_1}^{1/p} < \varepsilon_0 / 2 \,, \tag{3.4}$$

by (3.2) we obtain that for $0 < h \le h_0$ and for every i

$$\mathrm{dist}(u_i(h,x),\mathcal{Y}) < \varepsilon_0 \qquad \forall\, x\in \Sigma_m^{(d-1)}\,. \tag{3.5}$$

We then fix ε_1 so that both (3.4) and $c_3\,{\varepsilon_1}^{1/p} \le \varepsilon_0/2$ hold true, and determine h_0 by condition (3.2). We then choose $\sigma_0 > 0$ small in such a way that $c_4\, h_0^{-d/p}\,\sigma_0 \le \varepsilon_0/2$, and select k_0. By (3.3) we obtain

$$|u_i(h_0,x) - u_j(h_0,x)| < \varepsilon_0 \qquad \forall\, x\in \Sigma_m^{(d-1)}\,, \quad \forall\, i,j \ge k_0\,. \tag{3.6}$$

Setting $u_i(\cdot,0) = u_i$, on account of (3.5) for every $i \ge k_0$ the homotopy maps

$$H_i : [0,h_0]\times \Sigma_m^{(d-1)} \to \mathcal{Y}_{\varepsilon_0}\,, \qquad H_i(h,x) := u_i(h,x)$$

are well defined. Therefore, the functions u_i and $u_i(h_0,\cdot)$ are homotopic, as maps from $\Sigma_m^{(d-1)}$ into $\mathcal{Y}_{\varepsilon_0}$. Moreover, (3.6) says that $u_i(h_0,\cdot)$ and $u_j(h_0,\cdot)$ are homotopic in the same sense, for $i,j\ge k_0$. This yields that u_i and u_j are homotopic, too, for $i,j \ge k_0$, and hence the assertion, by projecting $\mathcal{Y}_{\varepsilon_0}$ onto $\mathcal{Y}$. $\square$

Proof of Proposition 1.3: Let U_∞, $U_\infty(h,z)$ and $u_\infty(h,\cdot)$ be defined as in the proof of Proposition 3.1, but for $u = u_\infty$. With our hypotheses, it turns out that $U_\infty(h,z)$ is continuous, whereas $U_\infty(h,\cdot)$ tends to U_∞ and $u_\infty(h,\cdot)$ tends to u_∞ uniformly as $h\to 0$. Moreover, we can assume that both (3.1) and (3.2) hold true also for $i=\infty$. The assertion readily follows. $\square$

4 A characterization of approximable $W^{1/p}$-maps

In this section we shall prove Theorem 1.4 and its consequences, Theorem 1.5 and Corollaries 1.6 and 1.7.

Proof of Theorem 1.4: Let $d = [p]$. Assume that $u \in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ is the strong $W^{1/p}$-limit of a sequence of smooth maps $\{u_i\}$ in $C^\infty(\mathcal{X},\mathcal{Y})$. Let X be a finite cubeulation of $\mathcal{X}$ in dual position with respect to u. Then, denoting by $\{\widetilde u_i\} \subset W^{1/p}(X^{d-1},\mathcal{Y})$ the restriction of u_i to X^{d-1}, possibly slightly moving the faces of X, by Fubini theorem we have that $\widetilde u_i$ strongly converges to $\widetilde u := u_{|X^{(d-1)}}$ in $W^{1/p}$. If $d\ge 2$, by Proposition 1.3 we infer that for i sufficiently large $\widetilde u_i$ is homotopically equivalent to $\widetilde u$, as maps from X^{d-1} to $\mathcal{Y}$. Since each $\widetilde u_i$ is the restriction of a smooth map from $\mathcal{X}$ to $\mathcal{Y}$, and $(\mathcal{X}, X^{d-1})$ satisfies the so called *homotopy extension property*, see e.g. [11, Prop. 2.1], this yields that $\widetilde u$ can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. The same conclusion holds for any cubeulation X in dual position with respect to u. Finally, if $d=1$ the conclusion trivially follows.

We now prove the converse, and assume that the restriction $\widetilde u := u_{|X^{(d-1)}}$ can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. We distinguish two cases.

The case $n=d=[p]$. The map $u\in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ is continuous outside a discrete set $\Sigma(u)$. Since the argument is local, without loss of generality we assume that $u \in R^0_{1/p}(\mathcal{Q}^n,\mathcal{Y})$ and u is smooth outside the origin. We then argue as in the proof of Theorem 1.2 from Sec. 2. In fact, this time we infer that $u_{|\partial F_r} : \partial F_r \to \mathcal{Y}$ is a continuous map in $W^{1/p}(\partial F_r,\mathcal{Y})$ for which we can find a continuous extension $u_r : F_r \to \mathcal{Y}$ with finite $W^{1,p}$-norm, as required.

The case $n-1\ge d=[p]$. We use a local argument and return to the proof of Theorem 1.1 from Sec. 2. Recall that the singular set of the approximating maps $W_h^{(m)}$ is contained in $\mathcal{Q}_m^n\times\{0\}$ and intersects every not "horizontal" $(k+1)$-cube Q in $\mathcal{F}_m^{(k+1)}$, for $k=d,\dots,n$, on a $(k-d)$-dimensional set obtained by the "homogeneous" extension (2.18) of the restriction of $W_h^{(m)}$ to the boundary of Q. To remove the singular set, working by induction on $k=d,\dots,n$, it then suffices to modify the definition (2.18) to (4.1) below, where $V_Q : Q\to\mathbb{R}^N$ is a suitable smooth extension of the boundary datum.

To this aim, we now recall that ${V_h^{(m)}}_{|\Sigma_m^{(d-1)}\times I} = U_h^{(m)}$, where $\{U_h^{(m)}\}\subset W^{1,p}(\Sigma_m^{(d-1)}\times I,\mathbb{R}^N)\cap C^\infty$ is such that $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}$, see Propositions 2.2 and 2.4, and the traces $\mathbf{T}(U_h^{(m)}) \in$

$W^{1/p}(\Sigma_m^{(d-1)},\mathcal{Y}_{\varepsilon_0})\cap C^0$. Since the cubeulation given by $C_m^{(k)}$ is in dual position with respect to u, by Proposition 1.3, applied this time with $\mathcal{Y}_{\varepsilon_0}$ instead of $\mathcal{Y}$, see Remark 2.1, we infer that for h large enough $\mathbf{T}(U_h^{(m)})$ is homotopically equivalent to the restriction $u_{|\Sigma_m^{(d-1)}}$ of u to $\Sigma_m^{(d-1)}$, as maps from $\Sigma_m^{(d-1)}$ into $\mathcal{Y}_{\varepsilon_0}$. Moreover, by the hypothesis $u_{|\Sigma_m^{(d-1)}}$ can be extended to a continuous map from $\mathcal{Q}^n$ into $\mathcal{Y}$. As a consequence, the trace $\mathbf{T}(U_h^{(m)})_{|\Sigma_m^{(d-1)}}$ of $U_h^{(m)}$ on $\Sigma_m^{(d-1)}$ can be extended to a continuous map $v_h:\mathcal{Q}^n\to\mathcal{Y}_{\varepsilon_0}$ such that the restriction of v_h to every k-face of $\Sigma_m^{(k)}$ has finite $W^{1/p}$-norm, for every $k=d,\dots,n$.

First we let $W_h^{(m)}\equiv V_h^{(m)}$ on $\mathcal{F}_m^{(d)}$. Arguing by induction on $k=d,\dots,n$, we now extend $W_h^{(m)}$ to $\mathcal{F}_m^{(k+1)}$ as follows.

If Q is a "horizontal" $(k+1)$-cube in $\mathcal{F}_m^{(k+1)}$ define $W_h^{(m)}$ as in (2.17).

If Q is not "horizontal", we let

$$F:=Q\cap(\mathbb{R}^n\times\{0\})$$

be the k-face in $\Sigma_m^{(k)}$ given by the intersection of Q with $\mathcal{Q}^n\times\{0\}$, see (2.2). Moreover, let $u_{h,F}:F\to\mathcal{Y}_{\varepsilon_0}$ be given by the restriction of v_h to F, so that $u_{h,F}\in W^{1/p}(F,\mathcal{Y}_{\varepsilon_0})\cap C^0$.

Let $Q^\pm:=\{z=(x,t)\in Q\mid \pm t\geq 0\}$ be the upper and lower half $(k+1)$-cubes of Q. Moreover, let $V_Q^\pm:Q^\pm\to\mathbb{R}^N$ be the function that minimizes the p-energy on $Q^\pm$ among all maps in $W^{1,p}(Q^\pm,\mathbb{R}^N)$ satisfying the boundary condition

$$\begin{cases} V_Q^\pm = W_h^{(m)} & \text{on} \quad \partial Q^\pm\cap\{(x,t)\mid \pm t>0\} \\ V_Q^\pm = u_{h,F} & \text{on} \quad F \end{cases}$$

and let $V_Q:Q\to\mathbb{R}^N$ be given by $V_Q(z)=V_Q^\pm(z)$ if $z\in Q^\pm$. If f_Q is the bilipschitz homeomorphism between Q and $[-1/(2m),1/(2m)]^{k+1}$ defined in the proof of Theorem 1.1, we modify the definition (2.18) of $W_h^{(m)}$ on Q by setting for every $z\in Q$

$$W_h^{(m)}:=\begin{cases} V_Q\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\delta}\Big)\Big] & \text{if} \quad \|f_Q(z)\|_{k+1}\leq\delta \\ W_h^{(m)}\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\|f_Q(z)\|_{k+1}}\Big)\Big] & \text{if} \quad \delta\leq\|f_Q(z)\|_{k+1}\leq\dfrac{1}{2m}\,. \end{cases} \tag{4.1}$$

Similarly to the proof of Theorem 1.2, we easily infer that (2.19) holds again if $0<\delta<1/(2m)$ is sufficiently small, whereas this time $W_h^{(m)}$ is continuous on Q and with trace $\mathbf{T}(U_h^{(m)})$ in $W^{1/p}(F,\mathcal{Y}_{\varepsilon_0})$.

We then obtain again (2.20) and hence, by (2.16), we conclude again with (2.15). The rest of the proof is similar to the one of Theorem 1.1 from Sec. 2. □

Proof of Theorem 1.5: Let $d=[p]$. Similarly to the proof of [11, Thm. 6.3], if $f\in\text{Lip}(X^{d-1},\mathcal{Y})$, for some cubeulation X of $\mathcal{X}$, by means of homogeneous extensions on the k-faces of X^k, for $k=d,\dots,n$, we find a map $u\in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ such that the restriction $u_{|X^{d-1}}$ agrees with f and X is in dual position with respect to u. If smooth maps in $C^\infty(\mathcal{X},\mathcal{Y})$ are sequentially dense in $W^{1/p}(\mathcal{X},\mathcal{Y})$, by Theorem 1.4 we infer that $u_{|X^{d-1}}=f$ has a continuous (and hence Lipschitz) extension to a map from $\mathcal{X}$ to $\mathcal{Y}$. This implies that $\pi_{d-1}(\mathcal{Y})=0$ and that $\mathcal{X}$ has the $(d-1)$-extension property with respect to $\mathcal{Y}$.

Conversely, let $u\in R^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ and X be in dual position with respect to u. Condition $\pi_{d-1}(\mathcal{Y})=0$ yields that the restriction $u_{|X^{d-1}}$ has a continuous extension $g:X^d\to\mathcal{Y}$. Therefore, by the $(d-1)$-extension property, $u_{|X^{d-1}}$ can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. By Theorem 1.4 we then obtain that u is the strong $W^{1/p}$-limit of a smooth sequence in $W^{1/p}(\mathcal{X},\mathcal{Y})\cap C^\infty$. Theorem 1.1 and a diagonal argument yield the assertion. □

Proof of Corollary 1.6. Taking $d=[p]$, the hypotheses on $\mathcal{Y}$ yield that $\pi_{d-1}(\mathcal{Y})=0$ and that $\mathcal{X}$ has the $(d-1)$-extension property with respect to $\mathcal{Y}$. □

Proof of Corollary 1.7: Using the argument from [16, Sec. 6], we recall the following

Lemma 4.1 *Let $i \in \mathbb{N}^+$. If M, N are compact and connected Riemannian manifolds, $\pi_i(N) = 0$, and $g : M \to N$ is a continuous map $(i-1)$-homotopic to a constant map, then g is i-homotopic to a constant map.*

Applying first Lemma 4.1 with $M = N = \mathcal{X}$ and $i = 0, \dots, k-1$, we infer that there exists a continuous map $\phi : \mathcal{X} \to \mathcal{X}$ homotopic to the identity map and such that the restriction $\phi_{|X^{k-1}}$ is constant. Let $d = [p]$ and $f \in C(X^d, \mathcal{Y})$. Then $f \circ \phi$ is homotopic to f and $f \circ \phi_{|X^{k-1}}$ is constant. Applying then Lemma 4.1 with $M = \mathcal{X}$, $N = \mathcal{Y}$, and $i = k, \dots, d-1$, we infer that $f \circ \phi_{|X^{d-1}}$ is homotopic to a constant map. This yields that $f_{|X^{d-1}}$ can be extended to a continuous map. In conclusion, $\mathcal{X}$ has the $(d-1)$-extension property with respect to $\mathcal{Y}$, whereas $\pi_{d-1}(\mathcal{Y}) = 0$ holds true by the hypothesis. □

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