Lie algebraic obstructions to Γ-convergence of optimal control problems.

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1 Introduction.

One of the developments of Γ-convergence theory concerns finite-dimensional optimal control problems, that is, minimum problems where the unknown is a map from a time–interval $[0,T]$ into $\mathbb{R}^N$ subject to a differential constraint, namely an ordinary differential equation containing a *control* $u : [0,T] \to \mathbb{R}^N$. In particular, some authors (see e.g. [BC89],[BF93],[Fr98]) studied the Γ-*limit* —see e.g. the Appendix— of a sequence of control problems having the form:

$$\begin{cases} \dot{x} = g_0(t,x) + \displaystyle\sum_{i=1}^{M} g_i b_{n_i}(t) u_i(t) \quad , \quad x(0) = x_0 \\ \displaystyle\min_{u \in L^2(0,T;\mathbb{R}^M)} \left\{ J(x,u) = \int_0^T (l(t,x) + \sum_{i=1}^{M} h_i^2 \, u_i^2(t)) dt \right\} \end{cases} \tag{1.1}$$

where the $g_1, \dots, g_M$ and the $h_1, \dots, h_M$ are *constant* vectors of $\mathbb{R}^N$ and $\mathbb{R}$, respectively. In [BF93],[Fr98] one studies the limit of these problems when the L^2 structural parameters $b_n(\cdot) = (b_{n_1}, \cdots, b_{n_M})(\cdot)$ converge, say weakly, to a L^2 map $b(\cdot) = (b_1, \cdots, b_M)(\cdot)$. It turns out that in order to single out the limit problem one needs to know the (weak) limit $\mu = (\mu_1, \cdots, \mu_M)(\cdot)$ of the maps $b_n^2 = (b_{n_1}^2, \dots, b_{n_M}^2)(\cdot)$ as well. Let us recall that this limit, when it exists, can well be different from $b^2(\cdot)$ (actually, one has $\mu \geq b^2$). Moreover, in general it is not a L^1 function. Rather it is a measure on $[0,T]$. The main point established in the quoted papers consists in the fact that the pair (b, μ) does single out the limit problem. We just recall that, unless $\mu = b^2$, such limit problem does not display the standard form of an optimal control problem.

These results share an essential assumption: the "fields" g_i and the maps h_i are independent of x, that is, they are constant. This fact is crucial, in that it allows one to regard the limit equation and the limit payoff as relations *in measure*. On the contrary, as soon as the g_i actually depend on x —and a certain commutativity assumption (see below) is not verified—, the measure-theoretical approach does not work, as shown by the simple example in Section 3.

In this paper we to study the Γ-limit of problems of the form

$$(\mathcal{P})_{(b_n)} \qquad \begin{cases} \dot{x} = g_0(t,x) + \sum_{i=1}^{M} g_i(x) b_{n_i}(t) u_i(t) \quad , \quad x(0) = x_0 \\ \min_u \left\{ J(x,u) = \int_0^T \left(l(t,x) + \sum_{i=1}^{M} k_i(t,x) b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2(x) u_i^2(t) \right) dt \right\} \end{cases}$$

when

$$\begin{array}{l} \lim_{n\to\infty} b_{n_i}(\cdot) = b_i(\cdot) \quad \text{weakly } L^2(0,T) \\ \lim_{n\to\infty} b_{n_i}^2(\cdot) = \mu_i(\cdot) \quad \text{weakly}^* \mathcal{M}([0,T]) \end{array} \tag{1.2}$$

The main difference between (1.1) and $(\mathcal{P})_{(b_n)}$ consists in the fact that *the g_i and the h_i can depend on x*. Moreover, terms which are linear in u are also allowed in the Lagrangian.

Our aim is threefold. To begin with, in Section 2 we assume a *commutativity* hypothesis, which generalizes the case where the g_i are constant. Namely, we assume that $[g_i, g_j] = 0$ for all $i, j = 1, \ldots, M$, where $[g_i, g_j]$ denotes the Lie bracket of the fields g_i and g_j. It is remarkable that, under this assumption, one can prove the same result as in the case where the g_i are x-independent. In other words, the limit problem of the $(\mathcal{P})_{(b_n)}$ is still singled out by the limit (b, μ) of the pairs (b_n, b_n^2). This limit is denoted by $\Phi^{-1}(Q_{(b,\mu)})$, for it is the preimage of a simple problem $(\mathcal{Q})_{(b,\mu)}$ via a diffeomorphism Φ, which in turn is determined by the (commutative) fields g_i. The result in this section provide a geometrical insight to the results in [BC89],[BF93],[Fr98] as well, for, while the property of "being independent of x" is not chart-invariant, commutativity has an intrinsic meaning.

Secondly, in Section 3 we present an example that reveals the crucial difference between the case with vanisching Lie brackets and the general case. Actually, in this example (the Lie brackets do not vanish and) two sequences $((b_n, b_n^2))_{n\in N}$ and $((\tilde{b}_n, \tilde{b}_n^2))_{n\in N}$ converge to the *same* pair (b, μ), while the corresponding problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ converge to *different* limit problems. Hence, provided a limit problem exists (in some possibly extended sense), in order to characterize it one needs some "extra information" besides the one contained in the assignment of the pair (b, μ).

The construction of an extended setting for problems with no commutativity assumptions is, in fact, the third aim of the paper. We pursue this objective in Section 4, by redefining the minimum problems in the space of the graphs. Whithin this extended setting every minimum problem is identified by a triple of functions (B_0, B, γ) defined on $[0, 1]$, this triple replacing the role of the pair (b, μ). The map B_0, whose square root is the derivative of time t with respect to the parameter s in the interval $[0, 1]$, assumes values greater than or equal to zero. A particular case is represented by the original problems $(\mathcal{P})_{(b_n)}$, which are identified with problems corresponding to triples of the form (B_{0_n}, B_n, B_n^2), with B_{0_n} *strictly greater than zero a.e. in* $[0, 1]$ and $B_n \doteq b_n B_{0_n}$. On the other hand the extra information needed in order to single out the limit problem is provided by the restriction of γ to the subintervals of $[0, 1]$ where B_0 is equal to zero.

Lastly, in Section 5 we prove some statements aiming to compose the (apparent) discrepancy between the case with vanishing Lie brackets —which is treated in Section 2 in terms of the original time t— and the general case —which is addressed in Section 4 in an extended framework. The key points consist in a projection of the set of triples (B_0, B, γ) onto the set of the pairs (b, μ) and in the consequent partition of the set of triples. Roughly speaking, when the commutativity hypothesis holds all extended problems in a class of this partition correspond to a unique problem, namely the one singled out by the (unique) projection (b, μ) of the triples in the class.

For the sake of self-consistency we conclude the paper with an Appendix where some basic facts from the general theory of Γ-convergence are briefly recalled.

Let us point out that a reader interested only in the case with vanishing Lie brackets may read just Section 2. On the other hand, the construction of the extended setting for the general case, which is performed in Section 4, is self-contained and independent of the antecedent material of the paper.

Notations. We will write $L^p(0,T;\mathbb{R}^M)$ to denote the space of p-integrable functions from $[0,T]$ into $\mathbb{R}^M$ endowed with the usual norm $\|\cdot\|_p$. Moreover, $\mathcal{M}([0,T];\mathbb{R}^M)$ and $BV([0,T];\mathbb{R}^M)$ will denote the space of $\mathbb{R}^M$-valued Borel measure on $[0,T]$ and the space of $\mathbb{R}^M$-valued functions with bounded variation on $[0,T]$, respectively. If $M=1$ we write $L^p(0,T)$, $\mathcal{M}([0,T])$, $BV([0,T])$ instead of $L^p(0,T;\mathbb{R})$, $\mathcal{M}([0,T];\mathbb{R})$, $BV([0,T];\mathbb{R})$, respectively.

If $\mu \in \mathcal{M}([0,T])$, μ^a and μ^s stand for the absolutely continuous and the singular part of μ with respect to the Lebesgue measure dt, respectively. If μ_1 and μ_2 are a vector measure and a scalar measure on $[0,T]$, respectively, we write $\mu_1 << \mu_2$ to mean that μ_1 is absolutely continuous with respect to μ_2. Moreover we denote the derivative of μ_1 with respect to μ_2 (in the sense of Radon-Nikodym's theorem) by $\frac{d\mu_1}{d\mu_2}$. Finally, by $supp\mu$ we mean the support of the measure μ.

2 Null Lie brackets.

In this section we consider sequences of minimum problems of the form

$$(\mathcal{P})_{(b_n)} \qquad \begin{cases} \dot{x} = g_0(t,x) + \displaystyle\sum_{i=1}^{M} g_i(x) b_{n_i}(t) u_i(t) \quad , \quad x(0) = x_0 \\ \displaystyle\min_u \left\{ J_n(x,u) = \int_0^T \left(l(t,x) + \sum_{i=1}^{M} k_i(t,x) b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2(x) u_i^2(t) \right) dt \right\} \end{cases}$$

Let us notice that we allow the maps multiplying the controls to depend on the state as well, both in the dynamics and in the Lagrangian. We assume here the *commutativity condition* (HC) below, which states that all Lie brackets $[g_i, g_j]$, $i,j = 1,\dots,M$ are identically equal to zero. This allows us to prove a result of Γ-convergence (see Appendix for the definition of Γ-limit) analogous to the one proved in [Fr98] where the vectors multiplying the control were assumed x-independent. This gives a geometrical insight to the question, for the case with constant g_i is nothing but a particular occurrence of the commutativity condition. We will see in the next sections that such a result does not hold when the commutativity assumption is not assumed.

Let us posit precisely the assumptions on the data and the controls. Our main hypothesis is the following *commutativity condition*:

(HC) For every $i, j = 1, \dots, M$ the Lie bracket

$$[g_i, g_j](x) = Dg_j(x)g_i(x) - Dg_i(x)g_j(x)$$

(where $Dg(x)$ denotes the derivative of g at x) is identically equal to zero.

Moreover we assume the following set of hypotheses.

(Hg0) The function $g_0 : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Moreover, for every compact subset $Q \subset \mathbb{R}^N$ there exists a continuous function $\gamma_0(t)$ such that, for every $t \in [0,T]$ and for every $x, y \in Q$ one has

$$|g_0(t,x) - g_0(t,y)| \leq \gamma_0(t)|x-y|$$

(Hg1) For each $i = 1, ..., M$ the vector fields g_i from $\mathbb{R}^N$ into $\mathbb{R}^N$ are of class C^2 and the trajectories of the equations $\dot{x} = g_i(x)$ exist globally.

(Hb) For each $n \in N$, $b_n(t) = (b_{n_1}(t), ..., b_{n_M}(t)) \in L^2(0,T;\mathbb{R}^M)$.

(Hu) The controls $u(t) = (u_1(t), ..., u_M(t))$ belong to $L^2(0,T;\mathbb{R}^M)$.

(Hl) The function $l : [0,T] \times \mathbb{R}^N \to [0,\infty]$ is a Borel function. For every compact subset $Q \subset \mathbb{R}^N$ there exists a L^1 function $\eta(t)$ such that, for every $t \in [0,T]$ and for every $x, y \in Q$,

$$|l(t,x) - l(t,y)| \leq \eta(t)\,|x-y|$$

Moreover, the sequence $l(t,0)$ is bounded in $L^1(0,T)$.

(Hk) For each $i = 1, ..., M$, $k_i : [0,T] \times \mathbb{R}^N \to [0,\infty]$ is a continuous function. There exists a constant $C > 0$ such that, for each $t \in [0,T]$ and for each $y \in \mathbb{R}^N$

$$|k_i(t,y)| \leq C \qquad i = 1, ..., M \tag{2.1}$$

Moreover, there exists a constant $L_k > 0$ such that, for each $t \in [0,T]$ and for each $y, z \in \mathbb{R}^N$,

$$|k_i(t,y) - k_i(t,z)| \leq L_k|y-z| \qquad i = 1, ..., M \tag{2.2}$$

(Hh) For each $i = 1, ..., M$, $h_i : \mathbb{R}^N \to [0,\infty]$ is a Borel function and for every compact subset $Q \subset \mathbb{R}^N$ there exists a constant L_h such that

$$|h_i(x) - h_i(y)| \leq L_h\,|x-y|\,,\ i = 1, ..., M$$

for every $x, y \in Q$. Moreover, we assume the following *coercivity* hypothesis: there exists a constant $K > 0$ such that, for every $x \in \mathbb{R}^N$,

$$\sum_{i=1}^{M} h_i^2(x)u_i^2 \geq K\,|u|^2$$

for every $u \in L^2(0,T)$, for every $x \in \mathbb{R}^N$.

Remark 2.1 Some of these hypotheses can be weakened further. For example, in view of Section 5, the constants C and L_k in (2.1), (2.2) may be replaced by two functions in $L^1(0,T)$. Moreover, at the cost of some technical complications in the computation of the Γ-limit in Definition 2.4 below, the maps l, h_i, and k_i may be allowed to depend on n as well.

Statement of the problem.
We shall investigate the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ under the following convergence assumptions:

$$\begin{array}{l} \lim_{n\to\infty} b_{n_i}(\cdot) = b_i(\cdot) \quad \text{weakly } L^2(0,T) \\ \lim_{n\to\infty} b^2_{n_i}(\cdot) = \mu_i(\cdot) \quad \text{weakly}^* \mathcal{M}([0,T]) \end{array} \tag{2.3}$$

In order to define the Γ-limit, we introduce a suitable coordinate transformation which is induced by the fields $g_1, ..., g_M$. This transformation is made possible by the crucial commutativity assumption (HC). Let us begin by adding the auxiliary equations $z_i(t) = \int_0^t b_{n_i}(s)u_i(s)ds$, $i = 1, ..., M$. Then the state equation of $(\mathcal{P})_{(b_n)}$ reads

$$\begin{pmatrix} \dot{z} \\ \dot{x} \end{pmatrix} = \tilde{g}_0(t,x) + \sum_{i=1}^{M} \tilde{g}_i(x) b_{n_i}(t) u_i(t)$$

where

$$\begin{array}{rccc} \tilde{g}_0 : & [0,T]\times \mathbb{R}^N & \to & \mathbb{R}^M \times \mathbb{R}^N \\ & (t,x) & \mapsto & \begin{pmatrix} 0_M \\ g_0^1(t,x) \\ \cdot \\ \cdot \\ g_0^N(t,x) \end{pmatrix} \end{array}$$

and, for every $i = 1, ..., M$,

$$\begin{array}{rccc} \tilde{g}_i : & \mathbb{R}^N & \to & \mathbb{R}^M \times \mathbb{R}^N \\ & x & \mapsto & \begin{pmatrix} e_i \\ g_i^1(x) \\ \cdot \\ \cdot \\ g_i^N(x) \end{pmatrix} \end{array}$$

0_M and e_i being the zero vector and the $i-$th (column) vector of the canonical basis in $\mathbb{R}^M$, respectively.

In the extended state space $\mathbb{R}^M \times \mathbb{R}^N$ problem $(\mathcal{P})_{(b_n)}$ is now formulated as

$$(\mathcal{P})_{(b_n)} \quad \begin{cases} \begin{pmatrix} \dot z \\ \dot x \end{pmatrix} = \tilde g_0(t,x) + \sum_{i=1}^M \tilde g_i(x) b_{n_i}(t) u_i(t) \quad , \quad (z(0), x(0)) = (0, x_0) \\ \min_u \left\{ J_n((z,x),u) = \int_0^T \left(l(t,x) + \sum_{i=1}^M k_i(t,x) b_{n_i}(t) u_i(t) + \sum_{i=1}^M h_i^2(x) u_i^2(t) \right) dt \right\} \end{cases}$$

(Notice that we use the same notation, namely $(\mathcal{P})_{(b_n)}$, to mean both the problem in $\mathbb{R}^N$ and the corresponding one in $\mathbb{R}^M \times \mathbb{R}^N$.)
Let us set

$$\begin{aligned} \Phi_1(z,x) &= z \\ \Phi_2(z,x) &= exp(-z_M g_M) \circ ... \circ exp(-z_1 g_1) x \end{aligned}$$

(where $exp(sg)x$ stands for the value at time s of the solution of the Cauchy problem $\dot y(s) = g(y(s))$, $y(0) = x$) and let us consider the map Φ defined by

$$\begin{pmatrix} z \\ y \end{pmatrix} = \Phi(z,x) \doteq \begin{pmatrix} \Phi_1(z,x) \\ \Phi_2(z,x) \end{pmatrix}$$

We shall also use the notations $(z, x(z,y))$ and $(z, y(z,x))$ instead of $\Phi^{-1}(z,y)$ and $\Phi(z,x)$, respectively. Notice that, since the maps g_i are of class C^2, Φ is a local diffeomorphism. Actually, Φ is a global diffeomorphism.
Let us define the vector fields $\breve g_0 : (0,T) \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$ and $\breve g_i : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $i = 1, ..., M$, by setting

$$\begin{aligned} \breve g_0(t,z,y) &\doteq D\Phi(z,x)\, \tilde g_0(t,x) \\ \breve g_i(z,y) &\doteq D\Phi(z,x)\, \tilde g_i(x) \end{aligned}$$

where $(z,x) = \Phi^{-1}(z,y)$. Notice that $\breve g_0$ and $\breve g_i$ are the expressions of $\tilde g_0$ and $\tilde g_i$, respectively, in the new coordinate (z,y).

Proposition 2.2 *The first components of the vector field $\breve g_0 : (0,T) \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$ are equal to zero, that is*

$$\breve g_0(t,z,y) = \begin{pmatrix} 0_M \\ g_0^\sharp(t,z,y) \end{pmatrix}$$

where the (column) vector field $g_0^\sharp(t,z,y)$ is given by $g_0^\sharp(t,z,y) = D_x\Phi_2(z,x)\tilde g_0(t,x)$, with $(z,x) = \Phi^{-1}(z,y)$. In particular $\breve g_0$ verifies (Hg0) *(with N replaced by $M+N$).*
Moreover one has, for $i = 1, ..., M$,

$$\breve g_i(z,y) = \begin{pmatrix} e_i \\ 0_N \end{pmatrix}$$

where 0_N stands for the (column) zero vector of $\mathbb{R}^N$.

A proof of this proposition can be found in [BR91].

By means of this coordinate change, problem $(\mathcal{P})_{(b_n)}$ is transformed into the problem

$$
(\mathcal{Q})_{(b_n)} \qquad \begin{cases} \begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = \check{g}_0(t,z,y) + \sum_{i=1}^{M} \check{g}_i(z,y) b_{n_i}(t) u_i(t) \quad , \quad (z(0), y(0)) = (0, x_0) \\ \min_u \left\{ \check{J}_n((z,y),u) \right\} \end{cases}
$$

$$
\check{J}_n((z,y),u) = \int_0^T \left(l(t, x(z,y)) + \sum_{i=1}^{M} k_i(t, x(z,y)) b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2(x(z,y)) u_i^2(t) \right) dt
$$

which thanks to Proposition 2.2, display the following, particularly simple, form:

$$
(\mathcal{Q})_{(b_n)} \qquad \begin{cases} \dot{z}_1(t) = b_{n_1}(t) u_1(t) \\ \cdot \\ \cdot \\ \dot{z}_M(t) = b_{n_M}(t) u_M(t) \\ \\ \dot{y}(t) = g_0^\sharp(t,z,y) \\ \\ \min_u \left\{ \check{J}_n((z,y),u) \right\} \end{cases}
$$

Remark 2.3 By saying that "$(\mathcal{P})_{(b_n)}$ is transformed into $(\mathcal{Q})_{(b_n)}$" we mean that:
(i) a trajectory-control pair $((z,y),u)$ is admissible for the problem $(\mathcal{Q})_{(b_n)}$ if and only if the trajectory-control pair $((z,x),u) \doteq (\Phi^{-1}(z,y),u)$ is admissible for $(\mathcal{P})_{(b_n)}$.
(ii) For each trajectory-control pair $((z,y),u)$, if $((z,x),u) \doteq (\Phi^{-1}(z,y),u)$ then

$$
J_n((z,x),u) = \check{J}_n(\Phi(z,x),u)
$$

for every $u \in L^2(0,T;\mathbb{R}^M)$.
In particular a trajectory-control pair $((z^\sharp, x^\sharp), u^\sharp)$ is optimal for $(\mathcal{P})_{(b_n)}$ if and only if $((z^\sharp, y^\sharp), u^\sharp)$ is optimal for $(\mathcal{Q})_{(b_n)}$ where $y^\sharp = \Phi_2(z^\sharp, x^\sharp)$.

In order to provide a representation of the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ we shall be concerned with the following set of *data pairs*

$$
A = \left\{ (b,\mu) \in L^2(0,T;\mathbb{R}^M) \times \mathcal{M}([0,T];\mathbb{R}^M) \ : \ \mu \geq b^2 \right\}
$$

where, $\mu = (\mu_1, ..., \mu_M)$, $b = (b_1, ..., b_M)$, and the inequality has to be interpreted as $\mu_i \geq b_i^2$, $\forall i = 1, ..., M$, (in the measure-theoretical sense). In particular, we shall consider the subset $A_s \subset A$ defined by

$$
A_s = \left\{ (b,\mu) \in A \ : \ \mu = b^2 \right\}
$$

which we call the subset of *simple data pairs* of A. (We recall that b^2 denotes the vector $(b_1^2, ..., b_M^2)$.)

Definition 2.4 *Let* $(b,\mu)\in A$*, and let us set* $\sigma=\sum_{i=1}^{M}\mu_i^s$*. We consider the variational problem*

$$(\mathcal{Q})_{(b,\mu)} \qquad \min_{((z,y),u)}\left\{\check{J}\left((z,y),u\right) \quad : \quad \dot{z} << dt+\sigma \quad , \quad \dot{y}=g_0^{\sharp}(t,z,y)\right\}$$

where the minimum is searched over the trajectory-control pairs $((z,y),u)$ *in* $BV([0,T];\mathbb{R}^M\times\mathbb{R}^N)\times L^2(0,T;\mathbb{R}^M)$ *and the cost functional* $\check{J}$ *is defined by*

$$\begin{aligned}
&\check{J}\left((z,y),u\right)\doteq\\
&\doteq\int_0^T\left[l(t,w)+\sum_{i=1}^{M}(k_i(t,w)z_i^a(t)+h_i^2(w)u_i^2(t)+h_i^2(w)\frac{(b_i(t)u_i(t)-z_i^a(t))^2}{(\mu_i^a(t)-b_i^2(t))})\right]dt+\\
&+\int_{\Omega_s\setminus\{0,T\}}\sum_{i=1}^{M}\left(h_i^2(w)\left|\frac{d\dot{z}_i}{d\sigma}\right|^2+k_i(t,w)\left|\frac{d\dot{z}_i}{d\sigma}\right|\right)d\sigma+\\
&+\sum_{i=1}^{M}\left(h_i^2(w)\left|\frac{z_i(0^+)-z_i(0^-)}{\sigma(\{0\})}\right|^2+k_i(0,w)\left|\frac{z_i(0^+)-z_i(0^-)}{\sigma(\{0\})}\right|\right)+\\
&+\sum_{i=1}^{M}\left(h_i^2(w)\left|\frac{z_i(T^+)-z_i(T^-)}{\sigma(\{T\})}\right|^2+k_i(T,w)\left|\frac{z_i(T^+)-z_i(T^-)}{\sigma(\{T\})}\right|\right)
\end{aligned}$$

where we have set $w=x(z,y)$ *and* $\Omega_s=supp\sigma$*.*

Remark 2.5 We adopt here the convention (already used in [BC89], [BF93], [Fr98]) according to which the fractions appearing in the definition of $\check{J}$ are zero as soon as their denominator is zero.

Remark 2.6 If one has $b_i^2=\mu_i$ for $i=1,...,M$, then the limit problem $(\mathcal{Q})_{(b,\mu)}$ reduces to the standard form

$$\begin{cases}\begin{pmatrix}\dot{z}\\ \dot{y}\end{pmatrix}=\check{g}_0(t,z,y)+\sum_{i=1}^{M}\check{g}_i(z,y)b_i(t)u_i(t) \quad , \quad (z(0),y(0))=(0,x_0)\\ \min_u\left\{\check{J}((z,y),u)\right\}\end{cases}$$

$$\check{J}((z,y),u)=\int_0^T\left(l(t,x(z,y))+\sum_{i=1}^{M}k_i(t,x(z,y))b_i(t)u_i(t)+\sum_{i=1}^{M}h_i^2(x(z,y))u_i^2(t)\right)dt$$

Definition 2.7 *Let us rewrite problem problem* $(\mathcal{Q})_{(b,\mu)}$ *in the form*

$$(\mathcal{Q})_{(b,\mu)} \qquad \min\left\{\check{F}((z,y),u):(z,y)\in BV([0,T];\mathbb{R}^M\times\mathbb{R}^N)\ ,\ u\in L^2(0,T;\mathbb{R}^M)\right\}$$

where $\check{F}((z,y),u)\doteq\check{J}\left((z,y),u\right)+\chi_{\{\dot{z}<<dt+\sigma\ ,\ \dot{y}=g_0^{\sharp}(t,w)\}}$*. We define* problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$ *as follows:*

$$\Phi^{-1}((\mathcal{Q})_{(b,\mu)}) \quad \min\left\{\check{F}(\Phi((z,x)),u):(z,x)\in BV([0,T];\mathbb{R}^M\times\mathbb{R}^N),u\in L^2(0,T;\mathbb{R}^M)\right\}$$

The next result states that problems $(\mathcal{P})_{(b_n)}$ converge to the variational problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. For the basic facts concerning the Γ-convergence see Appendix and the reference therein.

Theorem 2.8 *If the (b_n, b_n^2) converge to (b,μ) as in (2.3), then the problems $(\mathcal{P})_{(b_n)}$ Γ-converge to $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.*

Proof. In view of Lemma 2.9 below we only have to prove that the $(\mathcal{Q})_{(b_n)}$ Γ-converge to $(\mathcal{Q})_{(b,\mu)}$. Now, the optimal control problems $(\mathcal{Q})_{(b_n)}$ verify hypotheses (7.1)-(7.5) in [Fr98]. Moreover assumption (2.3) here implies (7.17) and (7.18) therein. Hence, in view of the results in [Fr98], problems $((\mathcal{Q})_{(b_n)})_{n\in N}$ Γ-converge to the problem $(\mathcal{Q})_{(b,\mu)}$ introduced in Definition 2.4. $\diamond$

Lemma 2.9 *If the sequence of problems $((\mathcal{Q})_{(b_n)})_{n\in N}$ Γ-converges to $(\mathcal{Q})_{(b,\mu)}$ then the sequence $((\mathcal{P})_{(b_n)})_{n\in N}$ Γ-converges to $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.*

Proof. To begin with, for each $n \in N$ we set

$$\check{F}_n((z,y),u) \doteq \check{J}_n((z,y),u) + \chi_{\check{C}_n}((z,y),u)$$

where $\check{C}_n$ is the set of admissible trajectory-control pairs for $(\mathcal{Q})_{(b_n)}$ (see Appendix). By assumption we have (see Definition 6.3 in Appendix)

$$\Gamma(N, U^-, Y^-) \lim_{n\to\infty} \check{F}_n((z,y),u) = \check{F}((z,y),u) \tag{2.4}$$

Now, (see Remark 2.3)

$$F_n((z,x),u) \doteq J_n((z,x),u) + \chi_{C_n}((z,x),u) =$$

$$= \check{J}_n(\Phi(z,x),u) + \chi_{\check{C}_n}(\Phi(z,x),u)) = \check{F}_n(\Phi(z,x),u)$$

where C_n is the set of admissible trajectory-control pairs for $(\mathcal{P})_{(b_n)}$. Hence, by (2.4),

$$\Gamma(N, U^-, Y^-) \lim_{n\to\infty} F_n((z,x),u) =$$

$$= \Gamma(N, U^-, Y^-) \lim_{n\to\infty} \check{F}_n(\Phi(z,x),u) = \check{F}(\Phi(z,x),u)$$

which proves the Lemma. $\diamond$

Theorem 2.8 says that the Γ-limit of a sequence of problems $(\mathcal{P})_{(b_n)}$ has the form $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. Conversely, we have:

Theorem 2.10 *For each problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$ with $(b,\mu) \in A$, there exists a sequence of problems $((\mathcal{P})_{(b_n)})_{n\in N}$ which Γ-converges to $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.*

In order to prove this theorem, let us demonstrate the following result.

Lemma 2.11 *For each* $(b,\mu) \in A$ *(with* $M = 1$*) there exists a sequence* $(b_n)_{n\in N} \in L^2(0,T)$ *such that* $b_n \to b$ *weakly in* $L^2(0,T)$ *and* $b_n^2 \to \mu$ *weakly*$^*\mathcal{M}([0,T])$.

In the case where μ is a L^∞-function we can sharpen the above result as follows.

Lemma 2.12 *If* $(b,\mu) \in A$ *(with* $M = 1$*) and* $\mu \in L^\infty(0,T)$ *then there exists a sequence* $(b_n)_{n\in N} \in L^2(0,T)$ *such that* $b_n \to b$ *weakly in* $L^2(0,T)$ *and* $b_n^2 \to \mu$ *weakly*$^* L^\infty(0,T)$.

Proof of Lemma 2.11. This result is mostly based on some arguments of the proof of Theorem 3.2 in [BR93]. However, for the sake of self consistency, let us provide a detailed proof. Let us set

$$p(t) = \int_0^t b^2(s)ds$$

and

$$w(t) = \begin{cases} 0 & t = 0 \\ \int_{[0,t]} d\mu & 0 < t \leq T \end{cases}$$

$$\psi(t) = w(t) - p(t)$$

and let us observe that p, w, ψ are nondecreasing functions, $p(0) = \psi(0) = 0$, $\psi(T) \leq w(T) \leq K$. Since ψ has bounded variation, it can have at most countably many points of discontinuity, say $\{\tau_i\ ,\ i \geq 1\}$. Let $\{\xi_i\ ,\ i \geq 1\}$ be any sequence of points dense on $[0,T]$. For every $n \geq 1$, let us consider the set

$$S_n = \{0, T\} \cup \{\tau_1, ...\tau_n\} \cup \{\xi_1, ..., \xi_n\}$$

which we arrange in increasing order, that is

$$S_n = \{0 = t_{n,0} < t_{n,1} < ... < t_{n,\nu_n} = T\} \quad (\nu_n \leq 2n+1)$$

For each $i \in \{1, ..., \nu_n\}$, let us set

$$\bar{t}_{n,i} = t_{n,i} - \frac{t_{n,i} - t_{n,i-1}}{n} \qquad \omega_{n,i} = \sqrt{n\frac{\psi(t_{n,i}) - \psi(t_{n,i-1})}{t_{n,i} - t_{n,i-1}}}$$

and let us define the functions

$$v_n(t) = \begin{cases} 0 & \text{if } t \in [\bar{t}_{n,i}, t_{n,i}] \text{ for some } i \\ b(t) & \text{otherwise} \end{cases}$$

$$\psi_n(t) = \begin{cases} \omega_{n,i} & \text{if } t \in [\bar{t}_{n,i}, t_{n,i}] \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

$$w_n(t) = \int_0^t [v_n(s) + \psi_n(s)]^2 ds$$

Observe that the above definitions imply

$$\int_{t_{n,i-1}}^{t_{n,i}} \psi_n^2(s)ds = \psi(t_{n,i}) - \psi(t_{n,i-1}) \quad \forall i \in \{1, ..., \nu_n\}$$

If we can show that

$$v_n + \psi_n \to b \text{ weakly } L^2(0,T) \tag{2.5}$$

$$w_n(t) \to w(t) \ \forall t \ \in [0,T] \tag{2.6}$$

then, setting $b_n(t) \doteq v_n(t) + \psi_n(t)$ we have $b_n \to b$ weakly $L^2(0,T)$ and —in view of the fact that the w_n are increasing maps— (see Proposition 7.19 in [Fo84])

$$\dot{w}_n \to \dot{w} \text{ weakly } {}^*\mathcal{M}([0,T])$$

Since by construction $\dot{w}_n = b_n^2(t)$ and $\dot{w} = \mu$, we can conclude.

We claim that $\psi_n \to 0$ weakly in $L^2(0,T)$, which implies (2.5). Indeed, let us fix $\phi \in L^2(0,T)$ and for any $\varepsilon > 0$, let us choose $\phi' \in C([0,T])$ such that $\| \phi - \phi' \|_2 < \varepsilon$. Then

$$\limsup_{n\to\infty} \int_0^T \psi_n \phi dt \le \limsup_{n\to\infty} \int_0^T \psi_n \phi' dt + \limsup_{n\to\infty} \int_0^T \psi_n(\phi - \phi')dt \le$$
$$\le \limsup_{n\to\infty} \| \psi_n \|_1 \| \phi' \|_\infty + \limsup_{n\to\infty} \| \psi_n \|_2 \| \phi' - \phi \|_2 \le 0 + \sqrt{K}\varepsilon$$

which proves the claim.

To prove (2.6), consider first a point t where w is discontinuous. Then $t = \tau_{m,k}$ for some m and some $k \le \nu_m$. Hence, for every $n \ge \nu$ we have

$$\int_0^t \psi_n^2(s)ds = \psi(t)$$

while

$$\lim_{n\to\infty} \int_0^t v_n^2(s)ds = \int_0^t b^2(s)ds$$

Hence, (2.6) holds. Of course, the same argument applies when $t = T$.

Next, consider the case where w is continuous at t. For any $\varepsilon > 0$, choose $\xi_\mu < t < \xi_\nu$ such that

$$w(t) - \varepsilon \le w(\xi_\mu) \le w(\xi_\nu) \le w(t) + \varepsilon$$

If $n \ge \max\{\mu, \nu\}$, then

$$\int_0^{\xi_\mu} \psi_n^2(s)ds = \int_0^{\xi_\mu} \psi^2(s)ds, \qquad \int_0^{\xi_\nu} \psi_n^2(s)ds = \int_0^{\xi_\nu} \psi^2(s)ds$$

Hence

$$w(t) - \varepsilon \le w(\xi_\mu) = \lim_{n\to\infty} w_n(\xi_\mu) \le \liminf_{n\to\infty} w_n(t) \le$$
$$\le \limsup_{n\to\infty} w_n(t) \le \lim_{n\to\infty} w_n(\xi_\nu) = w(\xi_\nu) \le w(t) + \varepsilon$$

Since ε was arbitrary, (2.6) is proved in this case as well. $\diamond$

Proof of Lemma 2.12. Let us begin by noting that there are no points τ_i of discontinuity of ψ. Since $w(\cdot)$ is Lipschitz continuous, for every $\varepsilon > 0$ we have $|w(t_1) - w(t_2)| < \varepsilon$ as soon as $|t_1 - t_2| < \frac{\varepsilon}{\|\mu\|_\infty}$. Let us fix ε and let n be such that $\frac{T}{n} < \frac{\varepsilon}{\|\mu\|_\infty}$. Correspondingly, let us consider the family of points $\tilde{S}_n = \{t_{n,i} = \dfrac{iT}{n},\ i = 0, ..., n\}$, and let $\bar{t}_{n,i}, \omega_{n,i}, v_n, \psi_n, w_n$ be defined as in the previous proof. Still we have $w(t_{n,i}) = w_n(t_{n,i})$ for every n and every $i = 0, ..., n$. Hence, if $t \in [t_{n,i-1}, t_{n,i}]$ one has

$$w(t) - \varepsilon \leq w(t_{n,i-1}) = w_n(t_{n,i-1}) \leq w_n(t) \leq w(t_{n,i}) = w_n(t_{n,i}) \leq w(t) + \varepsilon$$

which implies the thesis. ◇

Proof of Theorem 2.10. In view of Lemma 2.11, for each $(b, \mu) \in L^2(0, T; \mathbb{R}^M) \times \mathcal{M}([0, T]; \mathbb{R}^M)$ such that $(b, \mu) \in A$ there exist sequences $(b_{n_i})_{n \in N}$ in $L^2(0, T)$ such that $b_{n_i} \to b_i$ weakly $L^2(0, T)$ and $b^2_{n_i} \to \mu_i$ weakly$^*\mathcal{M}([0, T])$ for $i = 1, ..., M$. Hence in view of Theorem 2.8 the sequence of problems $((\mathcal{P})_{(b_n)})_{n \in N}$ Γ-converges to $\Phi^{-1}((\mathcal{P})_{(b,\mu)})$. ◇

3 Non vanishing Lie brackets: an example.

In the previous section it has been shown that whenever the vector fields commute the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ for (b_n, b^2_n) converging to (b, μ) *does exist.* However this is no longer true whenever some Lie bracket is not vanishing, as shown in the example below. In the next sections we will provide a theoretical framework from which it will be clear that in general there exist infinitely many limit problems corresponding to the pair (b, μ).

In order to make rid of the suspicion that having a state's dimension larger than the control's dimension might matter with the convergence question, the state in this example is one-dimensional.

Let $N = 1$, $M = 2$ and consider the state equation

$$\begin{cases} \dot{x}(t) = b_{n_1}(t)u_1(t) + x(t)b_{n_2}(t)u_2(t) \\ x(0) = 0 \end{cases}$$

Hence $g_1(t, x)$ coincides with the constant 1, and $g_2(x) = x$. Note that $[g_1, g_2] = -1$ so the commutativity condition (HC) is not fulfilled. Let us assume (Hb), (Hu), $T = 1$ and let us consider the cost functional

$$J_{b_n}(x, u) = \int_0^1 \left(|u(t)|^2 + b_{n_1}(t)u_1(t) + x(t)b_{n_2}(t)u_2(t)\right) dt \left(= \int_0^1 |u(t)|^2\, dt + x(1)\right)$$

If we set $h_1(x) = 1$, $h_2(x) = 1$, $k_1(t, x) = 1$, $k_2(t, x) = x$ and $l(t, x) = 0$ the hypotheses in Section 2 turns out to be satisfied (see also Remark 2.1).

Let us consider the two sequences of coefficients

$$(b_{n_1}(t), b_{n_2}(t)) \doteq \left(\sqrt{n}, \sqrt{n}\right)\ \mathrm{I}_{[1-\frac{1}{n},1]}(t)$$

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) \doteq \left(0, \sqrt{2n}\right) \mathrm{I}_{[1-\frac{1}{n},1-\frac{1}{2n}]}(t) + \left(\sqrt{2n}, 0\right) \mathrm{I}_{[1-\frac{1}{2n},1]}(t)$$

where $\mathrm{I}_{[a,b]} = 1$ if $t \in [a, b]$ and $\mathrm{I}_{[a,b]} = 0$ if $t \notin [a, b]$. Let us observe that

$$(b_{n_1}(t), b_{n_2}(t)) \to (0, 0) \quad \text{weakly } L^2(0, T)$$

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) \to (0, 0) \quad \text{weakly } L^2(0, T)$$

$$\left(b^2_{n_1}(t), b^2_{n_2}(t)\right) \to (\delta_1, \delta_1) \quad \text{weakly}^* \mathcal{M}([0,T])$$

$$\left(\tilde{b}^2_{n_1}(t), \tilde{b}^2_{n_2}(t)\right) \to (\delta_1, \delta_1) \quad \text{weakly}^* \mathcal{M}([0,T])$$

where δ_1 denotes the Dirac measure at $T = 1$. Hence the two sequences fulfill the convergence assumption (2.3) with the *same* limit $(b_1, b_2) = (0, 0)$ and $(\mu_1, \mu_2) = (\delta_1, \delta_1)$. Yet, the corresponding sequences $((\mathcal{P})_{(b_n)})_{n\in N}$ and $((\mathcal{P})_{(\tilde{b}_n)})_{n\in N}$ *cannot* converge to the same Γ-limit. Indeed, if we implement the control

$$(u^n_1(t), u^n_2(t)) = \left(-\frac{1}{2}\sqrt{n}\, \mathrm{I}_{[1-\frac{1}{n},1]}(t), 0\right)$$

in the system driven by the (b_n) we obtain a trajectory x_n verifying $x_n(1) = -\frac{1}{2}$. Thus,

$$J_{b_n}(x_n, u_n) = -\frac{1}{4}$$

On the contrary, if we consider $(\mathcal{P})_{(\tilde{b}_n)}$ by the Pointryagin Maximum Principle we obtain that

$$(\hat{u}^n_1(t), \hat{u}^n_2(t)) = \left(-\frac{1}{2}\, \tilde{b}_{n_1}, 0\right)$$

is an optimal control. The corresponding optimal trajectory $\hat{x}_n$ solves

$$\dot{\hat{x}}_n(t) = -\frac{1}{2}\, \tilde{b}^2_{n_1}, \qquad x_n(0) = x_0$$

Hence,

$$\frac{1}{2} = J_{\tilde{b}_n}(\hat{x}_n, \hat{u}_n) = \min_u \{J_{\tilde{b}_n}(x, u)\}$$

In particular one has

$$\liminf_{n\to\infty} \left(\inf_u \{J_{b_n}(x, u)\}\right) \leq -\frac{1}{4} < \frac{1}{2} = \liminf_{n\to\infty} \left(\min_u \{J_{\tilde{b}_n}(x, u)\}\right)$$

Hence, although the (b_{n_1}, b_{n_2}) and $(\tilde{b}_{n_1}, \tilde{b}_{n_2})$ converge to the same (b, μ) in the sense of (2.3), the Γ-limit of the $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ are necessarily different. (See Theorem 6.2 in the Appendix.)

4 Non vanishing Lie brackets: an extended setting.

In this section we still assume hypotheses (Hg0), (Hg1), (Hl), (Hk), (Hh) but we *do not* assume the commutativity hypothesis (HC). The previous example shows that in order to determine the limit problem *it is not enough* to assume that $b_n \to b$ weakly in $L^2(0, T; \mathbb{R}^M)$ and $b^2_n \to \mu$ weakly* in $\mathcal{M}([0, T]; \mathbb{R}^M)$. In fact, due to the noncommutativity of the vector fields g_i $(i = 1, ..., M)$ some extra information —related to the choice of the particular sequence (b_n, b^2_n) approximating (b, μ)— is needed. It turns out that this extra information can be neatly represented by first embedding the problem in the (t, x)-space and then reparametrizing time with a nondecreasing map whose derivative is zero for those values of t where μ is concentrated. In particular this embedding allows one to keep track of the particular sequence (b_n, b^2_n) approximating (b, μ).

Let us begin with some definitions.

Definition 4.1 *The set of* data triples *is defined as*

$$\mathcal{A} \doteq \Big\{(B_0, B, \gamma) : \ B_0 : [0,1] \to \mathbb{R}^+ \cup \{0\} \ , \ B : [0,1] \to \mathbb{R}^M \ , \ \gamma : [0,1] \to (\mathbb{R}^+ \cup \{0\})^M$$
$$\text{are Borel functions in } L^\infty(0,T;\mathbb{R}^M) : \gamma_i \geq B_i^2 \ \forall i = 1, ..., M \text{ and } \int_0^1 B_0^2(s)ds = T\Big\}$$

The subset A_{NI} *of* nonimpulsive data triples *is defined as*

$$\mathcal{A}_{NI} \doteq \{(B_0, B, \gamma) \in \mathcal{A} \ : \ B_0 > 0 \ a.e. \ on \ [0,1]\}$$

The subset of simple data triples *is defined as*

$$\mathcal{A}_s \doteq \Big\{(B_0, B, \gamma) \in \mathcal{A} \ : \ \gamma_i = B_i^2 \ \ a.e. \ on \ [0,1] \ , \ i = 1, ..., M \ \Big\}$$

We will denote the vector $(B_1^2, ..., B_M^2)$ by B^2.

For each triple $(B_0, B, \gamma) \in \mathcal{A}$ let us consider the *space-time* optimal control problem

$$(\mathcal{P})_{(B_0,B,\gamma)} \qquad \begin{cases} y'(s) = g_0(s,y)B_0^2(s) + \sum_{i=1}^M g_i(y)V_i(s) \ , \ y(0) = y_0 \\ \min_{U,V} \Big\{\hat{J}(y,U,V)\Big\} \end{cases}$$

$$\begin{aligned} \hat{J}(y,U,V) \ &= \ \int_0^1 \Bigg(l(s,y)B_0^2(s) + \sum_{i=1}^M k_i(s,y)V_i(s) + \sum_{i=1}^M h_i^2(y)U_i^2(s) + \\ &\quad + \sum_{i=1}^M h_i^2(y)\frac{(B_i(s)U_i(s) - V_i(s))^2}{(\gamma_i(s) - B_i^2(s))}\Bigg) \, ds \end{aligned}$$

where $U \in L^2(0,1;\mathbb{R}^M)$ and $V \in L^2(0,1;\mathbb{R}^M)$.

Remark 4.2 When $(B_0, B, \gamma) \in \mathcal{A}_s$, that is $\gamma_i = B_i^2$, $i = 1, ..., M$, the optimal control problem $(\mathcal{P})_{(B_0,B,\gamma)}$ reduces to the following standard form

$$(\mathcal{P})_{(B_0,B,B^2)} \qquad \begin{cases} y'(s) = g_0(s,y)B_0^2(s) + \sum_{i=1}^M g_i(y)B_i(s)U_i(s) \ , \ y(0) = y_0 \\ \min_{U} \Big\{\hat{J}(y,U)\Big\} \end{cases}$$

$$\hat{J}(y,U) = \int_0^1 \Bigg(l(s,y)B_0^2(s) + \sum_{i=1}^M k_i(s,y)B_i(s)U_i(s) + \sum_{i=1}^M h_i^2(y)U_i^2(s)\Bigg) \, ds$$

We shall show that the class of problems $(\mathcal{P})_{(b)}$ —where $(\mathcal{P})_{(b)}$ stands for a problem like $(\mathcal{P})_{(b_n)}$ when b_n is replaced by b—, can be put into one-to-one correspondence with the class of space-time problems $\{(\mathcal{P})_{(B_0,B,B^2)} \ : \ (B_0, B, B^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}\}$. Then we shall give sufficient conditions for the Γ-convergence of a sequence of problems $(\mathcal{P})_{(B_{0n},B_n,B_n^2)}$ to a problem $(\mathcal{P})_{(B_0,B,\gamma)}$. Lastly we shall see that *every* such problem is the Γ-limit of a suitable sequence of problems $(\mathcal{P})_{(B_{0n},B_n,B_n^2)}$.

Definition 4.3 *Given* $(B_0, B, \gamma) \in \mathcal{A}$ *let us define* $\alpha(B_0, B, \gamma) \doteq (b, \mu)$ *by setting:*

i)

$$t(s) \doteq \int_0^s B_0^2(u)du$$

and, whenever there exists $\delta > 0$ *such that* $B_0 > 0$ *a.e. on* $[s-\delta, s+\delta] \cap [0,1]$,

$$b_i(t(s)) \doteq \frac{B_i(s)}{B_0(s)} \qquad (i = 1, ..., M)$$

ii) *For each Borel subset* $E \subseteq [0, T]$

$$\mu_i(E) \doteq \int_I \gamma_i(s)ds \qquad (i = 1, ..., M)$$

when $E = t(I)$.

Remark 4.4

a) The function $b(\cdot)$ is well defined. Indeed, the set of values of t such that t^{-1} is not a singleton is at most countable.

b) For each $(B_0, B, \gamma) \in \mathcal{A}$ the pair $(b, \mu) = \alpha(B_0, B, \gamma)$ is in A; in particular if $(B_0, B, B^2) \in \mathcal{A}_{NI} \cap \mathcal{A}_s$ then $\alpha(B_0, B, B^2) = (b, b^2) \in A_s$.

c) The definition of μ is equivalent to

$$\int_{[0,T]} \phi(t)d\mu = \int_0^1 < \gamma_i(s), \phi(t(s)) > ds \quad \forall \phi \in C([0,T]; \mathbb{R}^M) \tag{4.1}$$

where $< \cdot, \cdot >$ denotes the scalar product in $\mathbb{R}^M$.

d) The map α is not injective, unless it is restricted to $\mathcal{A}_{NI} \cap \mathcal{A}_s$. Let us show that it is surjective. Indeed for every (b, μ) let us set

$$s(t) = \begin{cases} 0 & t = 0 \\ \dfrac{t + \int_{]0,t]} d\mu}{T + \int_{]0,T]} d\mu} & 0 < t < T \\ 1 & t = T \end{cases}$$

and let us define $t(s)$ as the unique nondecreasing continuous map such that $t \circ s(\tau) = \tau$ for all $\tau \in [0, T]$. Correspondingly let us set $B_0^2(s) = t'(s)$, $B_i(s) = b_i(t(s))B_0(s)$, $s \in [0,1]$, $i = 1, ..., M$. Finally let us choose $\gamma_i(s)$ such that

$$\int_{s_1}^{s_2} \gamma_i(s)ds = \int_{t(s_1,s_2)} d\mu_i, \quad i = 1, ..., M$$

for each subinterval (s_1, s_2) of $[0, 1]$. Then, $\alpha(B_0, B, \gamma) = (b, \mu)$. We will call this data triple *the canonical preimage* of (b, μ).

Let us notice that for every (b,μ), $\alpha^{-1}(b,\mu)$ turns out to be the class of data triples in $\mathcal{A}$ such that

$$(B_0,B,\gamma),(\tilde{B}_0,\tilde{B},\tilde{\gamma})\in\alpha^{-1}(b,\mu) \quad \Leftrightarrow \quad \begin{cases} B_0=\tilde{B}_0 \ a.e. \ , \\ B=\tilde{B} \ a.e. \ , \\ \gamma_i(s)=\tilde{\gamma}_i(s) \text{ for } a.e. \ s\in[0,1]\setminus\cup I_j \\ \text{ and } \displaystyle\int_{I_j}\gamma_i(s)ds=\int_{I_j}\tilde{\gamma}_i(s)ds \ \ \forall j \end{cases}$$

where $\{I_j\}$ is the (countable) family of (disjoint) subintervals of $[0,1]$ such that $B_0=\tilde{B}_0=0$ on each I_j.

In the following two theorems we establish a one-to-one correspondence between the class of problems $(\mathcal{P})_{(b)}$, $b\in L^2(0,T;\mathbb{R}^M)$ and the class of problems $(\mathcal{P})_{(B_0,B,\gamma)}$, $(B_0,B,\gamma)\in\mathcal{A}_{NI}\cap\mathcal{A}_s$ (i.e. $B^2=\gamma$ and $B_0>0$). Before stating these results let us notice that α is one-to-one from $\mathcal{A}_{NI}\cap\mathcal{A}_s$ onto A_s.

Theorem 4.5 *Let b and u satisfy* (Hb) *and* (Hu) *and let $x(\cdot)$ be the corresponding solution of the state equation of $(\mathcal{P})_{(b)}$*

$$\begin{cases} \dot{x}=g_0(t,x)+\displaystyle\sum_{i=1}^{M}g_i(x)b_i(t)u_i(t) \\ x(0)=x_0 \end{cases} \tag{4.2}$$

Let $(B_0,B,B^2)=\alpha^{-1}(b,b^2)$ and set $U(s)\doteq[u\circ t(s)]B_0(s)$. Let y be the solution of the state equation of $(\mathcal{P})_{(B_0,B,B^2)}$

$$\begin{cases} y'(s)=g_0(s,y)B_0^2(s)+\displaystyle\sum_{i=1}^{M}g_i(y)B_i(s)U_i(s) \\ y(0)=y_0 \end{cases} \tag{4.3}$$

Then

$$y(s)=x(t(s)) \quad \forall s\in[0,1]$$

Conversely, let $(B_0,B,B^2)\in\mathcal{A}_s\cap\mathcal{A}_{NI}$, $U\in L^2(0,1;\mathbb{R}^M)$ and let $y(\cdot)$ be the corresponding solution of (4.3). *Setting $(b,b^2)=\alpha(B_0,B,B^2)$ let us define*

$$u_i(t)\doteq\frac{U_i(s(t))}{B_0(s(t))} \quad i=1,...,M$$

If $x(\cdot)$ is the solution of (4.2) corresponding to these b_i and u_i, then

$$x(t)=y(s(t)) \quad \forall t\in[0,T]$$

Proof. The proof of this theorem relies essentially on the uniqueness properties of the equations (4.2) and (4.3). For this reason we omit it. $\diamond$

An analogous result holds for the payoffs J and $\hat{J}$:

Theorem 4.6 *Consider b, u, x, B_0, B, U, y as in the first part of Theorem 4.5, and set*

$$J(x,u) = \int_0^T \left(l(t,x) + \sum_{i=1}^M k_i(t,x) b_i(t) u_i(t) + \sum_{i=1}^M h_i^2(x) u_i^2(t) \right) dt$$

$$\hat{J}(y,U) = \int_0^1 \left(l(s,y) B_0(s) + \sum_{i=1}^M k_i(s,y) B_i(s) U_i(s) + \sum_{i=1}^M h_i^2(y) U_i^2(s) \right) ds$$

Then $\hat{J}(y,U) = J(x,u)$.
Conversely, if B_0, B, U, y and b, u, x are as in the second part of Theorem 4.5, then $J(x,u) = \hat{J}(y,U)$.

Proof. In view of Theorem 4.5 the proof of this theorem is straightforward. $\diamond$

When the problems $(\mathcal{P})_{(b)}$ and $(\mathcal{P})_{(B_0,B,B^2)}$ are related as in the previous result we say that they are *isomorphic*. In view of Theorem 4.5 and Theorem 4.6 the map $(\mathcal{P})_{(b)} \mapsto (\mathcal{P})_{(B_0,B,B^2)}$ with $\alpha(B_0,B,B^2) = (b,b^2)$, establishes a *one-to-one correspondence between the class of problems $\{(\mathcal{P})_{(b)}\ ,\ b \in L^2(0,T;\mathbb{R}^M)\}$ and the subset $\{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0,B,\gamma) \in \mathcal{A}_s \cap \mathcal{A}_{NI}\} \subset \{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0,B,\gamma)\ \in \mathcal{A}\}$* . This one-to-one correspondence can be regarded as an embedding of the original class of problems $\{(\mathcal{P})_{(b)}\ :\ b \in L^2(0,T;\mathbb{R}^M)\}$ in the larger class $\{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0,B,\gamma)\ \in \mathcal{A}\}$. In this extended setting we are now able to provide a convergence result, so giving an answer to the question raised with the example in Section 3. In other words, we are going to replace the assumptions on the sequence $((\mathcal{P})_{(b_n)})_{n\in N}$ with hypotheses on the sequence of isomorphic problems $(\mathcal{P}_{(B_{0_n},B_n,B_n^2)})_{n\in N}$. And assigning the limit of the triples (B_{0_n}, B_n, B_n^2) we actually provide the extra informations whose lackness was revealed by the example of Section 3.

Here is the main result.

Theorem 4.7 *Let $(B_{0_n}, B_n, B_n^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}$ and $(B_0, B, \gamma) \in \mathcal{A}$ verify*

$$\lim_{n\to\infty} B_{0_n}(\cdot) = B_0(\cdot) \ \textit{a.e. on } [0,1] \tag{4.4}$$

$$\lim_{n\to\infty} B_n(\cdot) = B(\cdot) \ \textit{weakly } L^1(0,1;\mathbb{R}^M) \tag{4.5}$$

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \ \textit{weakly } L^1(0,1;\mathbb{R}^M) \tag{4.6}$$

Then problems $\mathcal{P}_{(B_{0_n},B_n,B_n^2)}$ Γ-converge to problem $(\mathcal{P})_{(B_0,B,\gamma)}$.

Remark 4.8 In the previous statement one possibly has $\alpha(B_{0_n}, B_n, B_n^2) = (b_n, b_n^2)$, with $b_n \in L^2(0,T;\mathbb{R}^M)$. So, in particular, Theorem 4.7 can be regarded as a convergence result concerning the original problems $(\mathcal{P})_{(b_n)}$.

Proof of Theorem 4.7. Thanks to the performed rescaling of the problem we can exploit the general results proved by G. Buttazzo and E. Cavazzuti in [BC89]. Actually hypotheses (Hg0), (Hg1), (Hl), (Hk), (Hh) and (4.4)-(4.6) imply (3.6) – (3.10) and (3.12) – (3.15) in [BC89], respectively. Hence Propositions 3.2 and 3.3 of [BC89] state that $(\mathcal{P})_{(B_0,B,\gamma)}$ is the Γ-limit of the $\mathcal{P}_{(B_{0_n},B_n,B_n^2)}$. $\diamond$

Similarly to what has been done in the case where the Lie brackets vanish, we now prove that *each* problem $(\mathcal{P})_{(B_0,B,\gamma)}$, with $(B_0,B,\gamma)\in\mathcal{A}$, is indeed the Γ-limit of a sequence of problems of the form $\mathcal{P}_{(B_{0_n},B_n,B_n^2)}$, with $B_{0_n}>0$. (Which, up to the introduced one-to-one correspondence, means that $(\mathcal{P})_{(B_0,B,\gamma)}$ is the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ with $(b_n,b_n^2)=\alpha(B_{0_n},B_n,B_n^2)$).

Theorem 4.9 *For every $(B_0,B,\gamma)\in\mathcal{A}$, the problem $(\mathcal{P})_{(B_0,B,\gamma)}$ is the Γ-limit of a suitable sequence $(\mathcal{P}_{(B_{0_n},B_n,B_n^2)})_{n\in N}$, with $(B_{0_n},B_n,B_n^2)\in\mathcal{A}_s\cap\mathcal{A}_{NI}$.*

Proof. By Lemma 2.12 for each $(B_0,B,\gamma)\in A$ there is a sequence $((B_0,B_n,B_n^2))_{n\in N}(\in\mathcal{A}_s)$ such that $B_{n_i}\to B_i$ weakly in $L^2(0,1)$ and $B^2_{n_i}\to\gamma_i$ weakly in $L^1(0,1)$ for $i=1,...,M$. Moreover, by setting

$$B_{0_n}(s)=\sqrt{\frac{T}{T+\frac{1}{n}}\left(B_0^2(s)+\frac{1}{n}\right)}$$

we find that the triples (B_{0_n},B_n,B_n^2) (belong to $\mathcal{A}_s\cap\mathcal{A}_{NI}$ and) verify (4.4)-(4.6). Hence one concludes by Theorem 4.7. $\diamond$

5 Revisiting Section 2 and Section 3 in the light of the extended setting.

On one hand, Section 2 and Section 3 reveal a crucial discrepancy between the case when all the brackets $[g_i,g_j]$ vanish identically and the general case. On the other hand, in Section 4 we have introduced an extended setting in order to state a convergence result in the general case. In this section we are going to revisit both the positive result of Section 2 and the contrexample of Section 3 in the light of the theory developed in Section 4. Let us recall that the map $\alpha:\mathcal{A}\to A$ induces a one-to-one correspondence between the subset $\mathcal{A}_{NI}\cap\mathcal{A}_s\subset\mathcal{A}$ and $A_s\subset A$.

Null Lie brackets.

In Theorem 5.1 below we show —under the commutativity hypothesis (HC) in Section 2— that when problems $(\mathcal{P})_{(B_{0_n},B_n,B_n^2)}$ Γ-converge to a problem $(\mathcal{P})_{(B_0,B,\gamma)}$, then the space-projected problems $(\mathcal{P})_{(b_n)}$ with $(b_n,b_n^2)=\alpha(B_{0_n},B_n,B_n^2)$, Γ-converge to the projected limit $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$, where $(b,\mu)=\alpha(B_0,B,\gamma)$. The most significative point of this theorem consists in the fact that two sequences $(B_{0_n},B_n,B_n^2),(\tilde{B}_{0_n},\tilde{B}_n,\tilde{B}_n^2)$ converging to two *different* triples $(B_0,B,\gamma),(\tilde{B}_0,\tilde{B},\tilde{\gamma})$ such that $(b,\mu)=\alpha(B_0,B,\gamma)=\alpha(\tilde{B}_0,\tilde{B},\tilde{\gamma})$ give rise to problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ Γ-converging to the *same* limit problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. In particular this explains why as soon as all brackets vanish there is in fact no need of the extended setting.

Theorem 5.1 *Let us assume the hypotheses of Section 2 (in particular the commutativity hypothesis* (HC)*). Given* $(B_0, B, \gamma) \in \mathcal{A}$*, let us consider* any *sequence* $(B_{0_n}, B_n, B_n^2) \in \mathcal{A}_{NI} \cap \mathcal{A}_s$ *such that the* B_{0_n} *are equibounded and*

$$\lim_{n\to\infty} B_{0_n}(\cdot) = B_0(\cdot) \ \textit{a.e. on}\ [0,1] \tag{5.1}$$

$$\lim_{n\to\infty} B_n(\cdot) = B(\cdot) \ \textit{weakly}\ L^1(0,1;\mathbb{R}^M) \tag{5.2}$$

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \ \textit{weakly}\ L^1(0,1;\mathbb{R}^M) \tag{5.3}$$

(So that, by Theorem 4.7, $(\mathcal{P})_{(B_{0_n}, B_n, B_n^2)}$ Γ*-converges to* $(\mathcal{P})_{(B_0,B,\gamma)}$*).*
Then, setting $(b_n, b_n^2) = \alpha(B_{0_n}, B_n, B_n^2)$ *and* $(b,\mu) = \alpha(B_0, B, \gamma)$ *one has that the problems* $(\mathcal{P})_{(b_n)}$ Γ*-converge to* $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$*.*

Proof. In view of Theorem 2.8 we only have to prove that

$$b_n \to b \text{ weakly } L^2(0,T;\mathbb{R}^M) \tag{5.4}$$

$$b_n^2 \to \mu \text{ weakly}^* \mathcal{M}([0,T];\mathbb{R}^M) \tag{5.5}$$

We begin by observing that hypotheses (5.1)-(5.3) imply

$$\lim_{n\to\infty} B_{0_n}^2(\cdot) = B_0^2(\cdot) \text{ a.e. on } (0,1)$$

$$\lim_{n\to\infty} B_n(\cdot)B_{0_n}(\cdot) = B(\cdot)B_0(\cdot) \text{ weakly } L^1(0,1;\mathbb{R}^M)$$

and

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \text{ weakly } L^1(0,T;\mathbb{R}^M)$$

Set

$$t_n(s) = \int_0^s B_{0_n}^2(u)du \text{ and } t(s) = \int_0^s B_0^2(u)du$$

Since the t_n tend to t pointwise and each t_n is increasing, the t_n tend to t uniformly on [0,1]. Hence, for each $\varphi \in C([0,T];\mathbb{R}^M)$

$$\lim_{n\to\infty} \left| \int_0^T < b_n^2(t), \varphi(t) > dt - \int_{[0,T]} \varphi(t) d\mu \right| =$$

$$= \lim_{n\to\infty} \left| \int_0^1 < B_n^2(s), \varphi(t_n(s)) > ds - \int_0^1 < \gamma(s), \varphi(t(s)) > ds \right| = 0$$

which proves (5.5).

In order to prove (5.4), let us observe that for any function $\varphi \in C_c^\infty(0,T;\mathbb{R}^M)$

$$\lim_{n\to\infty} \left| \int_0^T < b_n(t), \varphi(t) > dt - \int_0^T < b(t), \varphi(t) > dt \right| =$$

$$\lim_{n\to\infty}\left|\int_0^1 < B_n(s)B_{0_n}(s),\varphi(s) > ds - \int_0^1 < B(s)B_0(s),\varphi(s) > ds\right| = 0$$

Since the B_n are uniformly bounded in $L^2(0,T;\mathbb{R}^M)$, (5.4) follows by the density of $C_c^\infty(0,T;\mathbb{R}^M)$ in $L^2(0,T;\mathbb{R}^M)$. $\diamond$

A compactness result.

We now examine the converse situation, where a sequence $(b_n)_{n\in N}$ is given such that $((b_n,b_n^2))_{n\in N}$ converges —with some regularity— to $(b,\mu)\in A$. We do not assume here that the Lie brackets vanish. It turns out that a subsequence of the corresponding triples (B_{0_n},B_n,B_n^2) converges to an element $(B_0,B,\gamma)\in\alpha^{-1}(b,\mu)$.

Theorem 5.2 *Assume the hypotheses of Section 2, with the exclusion of the commutativity hypothesis* (HC). *Let* $\mathcal{T}=\{t_i, i\in N\}$ *a (countable) subset of* $[0,T]$, *and let* $(b,\mu)\in A$ *and* $(b_n)_{n\in N}$ *be given such that*

(i) $\mu = b^2+\mu^\tau$ *with* μ^τ *a (positive) measure concentrated in* $\mathcal{T}$*;*

(ii) for each $n\in N$, $b_n\in C([0,T]\setminus\mathcal{T};\mathbb{R}^M)$, *and*

$$b_n(\cdot)\to b(\cdot) \text{ uniformly on the compact subsets of } [0,T]\setminus\mathcal{T} \tag{5.6}$$

$$b_n\to b \text{ weakly } L^2(0,T;\mathbb{R}^M)$$

$$b_n^2\to\mu \text{ weakly}^*\mathcal{M}([0,T];\mathbb{R}^M) \tag{5.7}$$

(So, if the commutative hypothesis (HC) *is in force,* $(\mathcal{P})_{(b_n)}\Gamma$*-converges to* $(\mathcal{P})_{(b,\mu)}$*.)*
Then, setting $(B_{0_n},B_n,B_n^2)\doteq\alpha^{-1}(b_n,b_n^2)$ *there exists a subsequence* $(\check{B}_{0_n},\check{B}_n,\check{B}_n^2)$, *and a data triple* $(B_0,B,\gamma)\in\alpha^{-1}(b,\mu)$, *such that the problems* $(\mathcal{P})_{(\check{B}_{0_n},\check{B}_n,\check{B}_n^2)}$ Γ*-converge to* $(\mathcal{P})_{(B_0,B,\gamma)}$.

The proof of this Theorem relies essentially on the following Lemma.

Lemma 5.3 *Assume the hypotheses of Theorem 5.2. Let us set*

$$s(t)\doteq\begin{cases} 0 & t=0\\ \dfrac{t+\int_{]0,t]}d\mu}{T+\int_{]0,T]}d\mu} & 0<t<T\\ 1 & t=T\end{cases}$$

$$s_n(t)\doteq\frac{\int_0^t(1+|b_n|^2(s))ds}{\int_0^T(1+|b_n|^2(s))ds}$$

and let us define $t_n(s)$ *and* $t(s)$ *as the inverse of* $s_n(t)$ *and the unique nondecreasing continuous map such that* $t\circ s(\tau)=id_{[0,T]}$, *respectively. Then* $t_n(\cdot)$ *and* $t(\cdot)$ *are equi-Lipschitz continuous and*

$$\lim_{n\to\infty}t_n(s)=t(s) \tag{5.8}$$

uniformly on $[0,1]$. *Moreover, setting* $B_0(s) \doteq \sqrt{t'(s)}$, *one has*

$$B_0(s) = 0 \quad \forall s \in int(t^{-1}(\mathcal{T})) \tag{5.9}$$

and

$$B_{0_n}(s) \to B_0 \ a.e. \ on \ [0,1] \tag{5.10}$$

Proof. Since $t^{-1}(\mathcal{T}) = \cup_{t_i \in \mathcal{T}} t^{-1}(t_i)$ we immediately obtain (5.9). Let us observe that hypothesis (5.7) implies that

$$\lim_{n\to\infty} s_n(t) = s(t) \quad \text{a. e. on } [0,T]$$

(see e.g. Proposition 7.19 in [Fo84]).
Actually, by the continuity of $s(t)$ on $[0,T] \setminus \mathcal{T}$ one has

$$\lim_{n\to\infty} s_n(t) = s(t) \quad \forall t \in [0,T] \setminus \mathcal{T}$$

Moreover, by the monotonicity of the $s_n(\cdot)$ and of $s(\cdot)$ it follows that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} (s_n(t+\varepsilon) - s_n(t-\varepsilon)) = s(t^+) - s(t^-)$$

which yields

$$\lim_{n\to\infty} t_n(s) = t(s) \quad \text{uniformly on } [0,1] \tag{5.11}$$

Since $B_{0_n}^2 = t_n' \geq 0$ for every n, there exists a subsequence, still denoted with $(B_{0_n})_{n\in N}$, such that

$$B_{0_n}^2 = t_n' \to 0 \quad a.e. \text{ on } int(t^{-1}(\mathcal{T}))$$

The convergence of B_{0_n} to B_0 on the set $int(t^{-1}(\mathcal{T}))$ is proved. In order to conclude let us prove this convergence for every $s \in [0,1] \setminus t^{-1}(\mathcal{T})$. Indeed, by (5.6) and (5.11) one has

$$\lim_{n\to\infty} B_{0_n}^2(s) = \lim_{n\to\infty} t_n'(s) = \lim_{n\to\infty} \frac{1}{\dot{s}(t_n(s))} = \lim_{n\to\infty} \frac{\int_0^T (1 + b_n(u)^2) du}{1 + b_n^2(t_n(s))} =$$

$$= \frac{T + \int_{[0,T]} d\mu}{1 + b^2(t(s))} = B_0^2(s)$$

The Lemma is proved. $\diamond$

Proof of Theorem 5.2 By Theorem 4.7 in Section 4 we only have to prove that the there exists a subsequence $(\check{B}_{0_n}, \check{B}_n, \check{B}_n^2)$ of (B_{0_n}, B_n, B_n^2) and a data triple $(B_0, B, \gamma) \in \alpha^{-1}(b, \mu)$ such that

$$\lim_{n\to\infty} \check{B}_{0_n}(\cdot) = B_0(\cdot) \text{ a.e. on } [0,1] \tag{5.12}$$

$$\lim_{n\to\infty} \check{B}_n(\cdot) = B(\cdot) \text{ weakly } L^1(0,1;\mathbb{R}^M) \tag{5.13}$$

$$\lim_{n\to\infty} \check{B}_n^2(\cdot) = \gamma(\cdot) \text{ weakly } L^1(0,1;\mathbb{R}^M) \tag{5.14}$$

Let us define B_0 as in Lemma 5.3, which yields (5.12).
Moreover, since the B_n are equibounded, there is a subsequence $(\tilde{B}_n)_{n\in N}$ of $(B_n)_{n\in N}$ converging to a map B weakly $L^1(0,1;\mathbb{R}^M)$.
By Ascoli-Arzela's Theorem there exists a subsequence $(\phi_n)_{n\in N}$ of $\tilde{\phi}_n(s) \doteq \int_0^s \tilde{B}_n(\sigma)^2 d\sigma$ converging to a Lipschitz continuous map ϕ. Then the subsequence $\check{B}_n^2(s) \doteq \frac{d\phi_n(s)}{ds}$ converges weakly$^*L^\infty(0,T;\mathbb{R}^M)$ to $\gamma(s) \doteq \frac{d\phi(s)}{ds}$, which implies (5.14).

We claim that $(B_0, B, \gamma) \in \alpha^{-1}(b,\mu)$. Indeed $\check{B}_{0_n}(s)$ tends to $B_0(s)$ for every s such that $B_0 > 0$ a.e. on $[s-\delta, s+\delta]$ for a sufficiently small δ. Moreover, thanks to (5.11) and hypothesis (5.6), $b_n(t_n(s))$ converges to $b(t(s))$ for every such point s. Since $\check{B}_n(s)$ tends almost everywhere to $B(s)$, i) in definition of the mapping α (Definition 4.3) turns out to be satisfied. Finally, in view of (5.14) and (4.1), ii) in the definition of α holds true as well. $\diamond$

Revisiting the example of Section 3.

Let us conclude by framing the example of Section 3 in the extended setting. This will clarify that the distinct limiting behavior of problems $(\mathcal{P})_{(b_n)}$ and problems $(\mathcal{P})_{(\tilde{b}_n)}$ arises from the fact that the corresponding triples $(B_{0_n}, B_n, B_n^2), (\tilde{B}_{0_n}, \tilde{B}_n, \tilde{B}_n^2)$ converge to different limits.

Let us recall that the state equation and the cost functional were given by

$$\begin{cases} \dot{x}(t) = b_{n_1}(t)u_1(t) + x(t)b_{n_2}(t)u_2(t) \\ x(0) = 0 \end{cases}$$

and

$$J_n(x,u) = \int_0^1 \Big(|u(t)|^2 + b_{n_1}(t)u_1(t) + x(t)b_{n_2}(t)u_2(t)\Big)\,dt \left(= \int_0^1 |u(t)|^2 dt + x(1)\right)$$

respectively. The problem of minimizing $J_n(x,u)$ over the controls $u \in L^2(0,T)$ were denoted by $(\mathcal{P})_{(b_n)}$ and by $(\mathcal{P})_{(\tilde{b}_n)}$ when the parameters where identified with

$$(b_{n_1}(t), b_{n_2}(t)) = \left(\sqrt{n}, \sqrt{n}\right) \mathrm{I}_{[1-\frac{1}{n},1]}(t)$$

and

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) = \left(0, \sqrt{2n}\right) \mathrm{I}_{[1-\frac{1}{n},1-\frac{1}{2n}]}(t) + \left(\sqrt{2n}, 0\right) \mathrm{I}_{[1-\frac{1}{2n},1]}(t)$$

respectively.

Following the construction performed in Section 4 let us compute the isomorphic problems $\mathcal{P}_{(B_0,B_n,B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0,\tilde{B}_n,\tilde{B}_n^2)}$, with $(B_0, B_n, B_n^2) = \alpha^{-1}(b_n, b_n^2)$ and $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2) = \alpha^{-1}(\tilde{b}_n, \tilde{b}_n^2)$. In both cases the optimal control problem turns out to have the form

$$\begin{cases} y'(s) = B_{n_1}(s)U_1(s) + y(s)B_{n_2}(s)U_2(s)\ ,\ y(0) = 0 \\ \min\limits_U \left\{\hat{J}_n(y,U)\right\} \end{cases}$$

$$\hat{J}_n(y,U) = \int_0^1 \Big(|U(s)|^2 + B_{n_1}(s)U_1(s) + y(s)B_{n_2}(s)U_2(s)\Big)\,ds$$

with the parameters B identified with

$$
\begin{aligned}
B_{0_n}(s) = &\ \sqrt{3}\, \mathrm{I}_{[0,\frac{1}{3}(1-\frac{1}{n})]} + \sqrt{\frac{3}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(1-\frac{1}{n}),1]} \\
B_n(s) = (B_{n_1}(s), B_{n_2}(s)) = &\ \left(\sqrt{\frac{3n}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(1-\frac{1}{n}),1]} \, , \, \sqrt{\frac{3n}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(1-\frac{1}{n}),1]} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{B}_{0_n}(s) = &\ \sqrt{3}\, \mathrm{I}_{[0,\frac{1}{3}(1-\frac{1}{n})]} + \sqrt{\frac{3}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(1-\frac{1}{n}),1]} \\
\tilde{B}_n(s) = (\tilde{B}_{n_1}(s), \tilde{B}_{n_2}(s)) = &\ \left(\sqrt{\frac{6n}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(2-\frac{1}{2n}),1]} \, , \, \sqrt{\frac{6n}{1+2n}}\, \mathrm{I}_{[\frac{1}{3}(1-\frac{1}{n}),\frac{1}{3}(2-\frac{1}{2n})]} \right)
\end{aligned}
$$

respectively. In order to find the Γ-limit of problems $\mathcal{P}_{(B_0,B_n,B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0,\tilde{B}_n,\tilde{B}_n^2)}$, we need to compute the limits appearing in hypotheses (4.4),(4.5) and (4.6).

For the data triples (B_0, B_n, B_n^2), we obtain

$$
\begin{array}{llll}
\lim\limits_{n\to\infty} B_{0_n}(s) = & B_0(s) \doteq & \sqrt{3}\, \mathrm{I}_{[0,1/3]} & \text{a.e. } [0,1] \\
\lim\limits_{n\to\infty} B_{n_1}(s) = & B_1(s) \doteq & \sqrt{\frac{3}{2}}\, \mathrm{I}_{[1/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} B_{n_2}(s) = & B_2(s) \doteq & \sqrt{\frac{3}{2}}\, \mathrm{I}_{[1/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} B_{n_1}^2(s) = & B_1^2(s) = & \frac{3}{2}\, \mathrm{I}_{[1/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} B_{n_2}^2(s) = & B_2^2(s) = & \frac{3}{2}\, \mathrm{I}_{[1/3,1]} & \text{in } L^1(0,1)
\end{array}
$$

while, for the data triples $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2)$, we have

$$
\begin{array}{llll}
\lim\limits_{n\to\infty} \tilde{B}_{0_n}(s) = & \tilde{B}_0(s) \doteq & \sqrt{3}\, \mathrm{I}_{[0,1/3]} & \text{a.e. } [0,1] \\
\lim\limits_{n\to\infty} \tilde{B}_{n_1}(s) = & \tilde{B}_1(s) \doteq & \sqrt{3}\, \mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} \tilde{B}_{n_2}(s) = & \tilde{B}_2(s) \doteq & \sqrt{3}\, \mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} \tilde{B}_{n_1}^2(s) = & \tilde{B}_1^2(s) = & 3\, \mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1) \\
\lim\limits_{n\to\infty} \tilde{B}_{n_2}^2(s) = & \tilde{B}_2^2(s) = & 3\, \mathrm{I}_{[1/3,2/3]} & \text{in } L^1(0,1)
\end{array}
$$

Let us remark that the limit of (B_0, B_n, B_n^2) and $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2)$ are *different*: this explains why problem $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ in the example cannot converge to the same limit problem. More precisely, in view of Theorem 4.7 problems $\mathcal{P}_{(B_0,B_n,B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0,\tilde{B}_n,\tilde{B}_n^2)}$ Γ-converge to the optimal control problems

$$
\left\{ \begin{array}{l} y'(s) = B_1(s)U_1(s) + y(s)B_2(s)U_2(s) \ , \ y(0) = 0 \\ \min\limits_U \left\{ \hat{J}(y,U) \right\} \end{array} \right.
$$

$$
\hat{J}(y,U) = \int_0^1 \Big(|U(s)|^2 + B_1(s)U_1(s) + y(s)B_2(s)U_2(s) \Big)\, ds
$$

and

$$
\left\{ \begin{array}{l} y'(s) = \tilde{B}_1(s)U_1(s) + y(s)\tilde{B}_2(s)U_2(s) \ , \ y(0) = 0 \\ \min\limits_U \left\{ \hat{J}(y,U) \right\} \end{array} \right.
$$

$$\hat{J}(y,U) = \int_0^1 \left(|U(s)|^2 + \tilde{B}_1(s)U_1(s) + y(s)\tilde{B}_2(s)U_2(s)\right) ds$$

respectively.

6 Appendix: basic tools from Γ-convergence

Let us consider two topological spaces Y and $\mathcal{U}$, the space of states and controls, respectively. For each $n \in N$ let $C_n \subseteq Y \times \mathcal{U}$ denote the set of *admissible trajectory-control pairs* defined by

$$C_n \doteq \{(y,u) \in Y \times \mathcal{U} \,:\, A_n(y) = B_n(u)\}$$

where A_n and B_n map Y and $\mathcal{U}$, respectively, in a third topological space V. Let us consider the sequence of optimal control problems

$$(\mathcal{P}_n) \qquad \min\{J_n(y,u) \,:\, (y,u) \in C_n\}$$

where J_n are real operators defined on $Y \times \mathcal{U}$.

Setting $F_n(y,u) = J_n(y,u) + \chi_{C_n}(y,u)$, (where χ_E is 1 on E and $+\infty$ on $(Y \times \mathcal{U}) \setminus E$) we rephrase problems $(\mathcal{P}_n)$ as follows:

$$(\mathcal{P}_n) \qquad \min\{F_n(y,u) \,:\, (y,u) \in Y \times \mathcal{U}\}$$

We will say that (y_n, u_n) is an *optimal pair* for the problem $(\mathcal{P}_n)$ if $F_n(y_n,u_n) = \min\limits_{Y\times\mathcal{U}} F_n$.

Theorem 6.2 below provides a notion of Γ-*limit* problem $(\mathcal{P})$. In order to state this theorem we recall the definition of multiple Γ-limit operator (see [BDM82]). We shall denote the "sup" and the "inf" operators by $Z(+)$ and $Z(-)$, respectively.

Definition 6.1 *Let X and W be two topological spaces and let $F_n : X \times W \to \overline{\mathbb{R}}$ be a sequences of functions. For every $x \in X$, $w \in W$, $\alpha, \beta, \gamma \in \{+,-\}$ let us define the* Γ-limit of the F_n *by setting*

$$\Gamma(N^\alpha, X^\beta, W^\gamma) \lim_{n\to\infty} F_n(w,x) = \underset{(x_n)\in S(x)}{\mathrm{Z}(\beta)} \; \underset{(w_n)\in S(w)}{\mathrm{Z}(\gamma)} \; \underset{k\in N}{\mathrm{Z}(-\alpha)} \, \underset{n\geq k}{\mathrm{Z}(\alpha)} \, F_n(w_n,x_n)$$

where $S(x)$ and $S(w)$ denote the sets of all sequences $x_n \to x$ in X and $w_n \to w$ in W, respectively. When the Γ-limit does not depend on the sign $+$ or $-$, this sign is omitted. For example, if

$$\Gamma(N^+, X^-, W^+) \lim_{n\to\infty} F_n(w,x) = \Gamma(N^+, X^+, W^+) \lim_{n\to\infty} F_n(w,x)$$

their common value will be indicated by $\Gamma(N^+, X, W^+) \lim_{n\to\infty} F_n(w,x)$.

Theorem 6.2 *Let Y and $\mathcal{U}$ be two topological space and let $F_n\colon Y \times \mathcal{U} \to \overline{\mathbb{R}}$ be a sequence of functions. For each $n \in N$, let (y_n,u_n) be a minimum point for F_n or simply a pair such that*

$$\lim_{n\to\infty} F_n(y_n,u_n) = \lim_{n\to\infty} [\inf_{Y\times\mathcal{U}} F_n]$$

Assume that the (y_n, u_n) *converge to* (y, u) *in* $Y \times \mathcal{U}$ *and there exists*

$$F(y,u) \doteq \Gamma(N, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y_n, u_n) \tag{6.1}$$

Then, we have that:

(i) (y, u) *is a minimum point for* F *on* $Y \times \mathcal{U}$*;*

(ii) $\lim_{n\to\infty} [\inf_{Y\times\mathcal{U}} F_n] = \min_{Y\times\mathcal{U}} F(y,u)$.

(For the proof see [BDM82], Proposition 2.1, page 388).

The above theorem allows one to give the following definition of Γ-limit problem

Definition 6.3 *When (6.1) is verified we say that the problem*

$$(\mathcal{P}) \qquad \min\{F(y,u) \;:\; (y,u) \in Y \times \mathcal{U}\}$$

is the Γ-limit of problems $(\mathcal{P}_n)$.

See e.g. [BDM82], [BC89], [BF93], [BF95], and [Fr98], for the explicit calculation of the Γ-limits in various interesting situations.

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