Optimal transport with convex obstacle

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Abstract

We consider the Monge transportation problem when the cost is the squared geodesic distance around a convex obstacle. We show that there exists at least one—and in general infinitely many—optimal transport maps.

Key-words :   Monge transport problem. Dynamic formulation. Convex obstacle.

1   Introduction

The Monge-Kantorovich has given birth to a huge literature through the last decades. Given two fixed probability measures f^+ and f^-, it consists in finding a *transport map T* which pushes forward f^+ to f^-, and minimizes the following energy:

$$(MK) \quad \int F(x, Tx) \, df^+(x).$$

The quantity $F(x, y)$ denotes the cost of moving a mass 1 from a point x to a point y. The relaxed version introduced by Kantorovich [21] is the following:

$$(\overline{MK}) \quad \inf_{\pi} \int F(x, y) \, d\pi(x, y)$$

where the infimum is taken over probability measures π with fixed marginals f^+ and f^-. It is easily seen that as soon as F is l.s.c. and finite, this last formulation has a solution μ called *optimal transport plan*. Moreover this solution μ happens to be associated to an optimal transport map T if $Id \otimes T$ pushes forward f^+ to μ.

In the initial formulation of Monge [23], F is a distance. The Monge problem is actually very difficult and has been solved in the case of the Euclidian distance (see [15], [13], [24], [2]) in the end of the 90'. The case of a general distance in $\mathbb{R}^N$ is treated in a recent articles by Champion and De Pascale [12] (for a strictly convex norm see also Caravenna

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[10] and Champion and De Pascale [11]).

Motivated by some crowd models introduced in [9], we focus here on the case where f^+ and f^- are supported on a subdomain of $\mathbb{R}^N$ with a convex obstacle $\mathbf{C}$ and where F is the squared geodesic distance: $F(x, y) = d^2(x, y)$, d being the geodesic distance on the domain $\mathbb{R}^N \backslash \mathbf{C}$. In the unconstrained case, i.e., when $F(x, y) = |y - x|^2$, existence and characterization of the optimal transport map have been established in the pioneering work by Brenier [7]. The obstacle case itself has been studied by Feldman and McCann [18], for the linear cost $F = d$, on a manifold with a geodesically strictly convex obstacle and when f^- is absolutely continuous. Here we consider what happens for the squared geodesic distance and when the mass f^- is not necessarily absolutely continuous.

Let us first briefly recall Brenier's approach of the problem. The first idea is to use the dual formulation of the Monge-Kantorovich problem introduced by Kantorovich [21] (see also [25]):

$$\sup_{u_1(y) - u_0(x) \leq \frac{|y-x|^2}{2}} \left\{ \int_\Omega u_1(y) \, df^-(x) \; - \; \int_\Omega u_0(x) \, df^+(x) \right\}.$$

If (u_0, u_1) is a pair of optimal Lipschitz functions for the above dual problem, the solution μ of the relaxed Monge-Kantorovich problem must satisfy the following primal-dual optimality condition (necessary and sufficient):

$$u_1(y) - u_0(x) = \frac{|x - y|^2}{2} \quad \mu - \text{a.e. } (x, y).$$

Then deriving formally this relation with respect to x leads to

$$y = x + Du_0(x) \quad \mu - \text{a.e. } (x, y).$$

Hence μ is equal to $(Id \otimes (Id + Du_0)) \sharp f^+$ and the existence of a solution for the Monge-Kantorovich problem characterized by $Tx = x + Du_0(x)$.

In our case, the dual formulation and the primal-dual optimality condition are similar:

$$\sup_{u_1(y) - u_0(x) \leq \frac{d(x,y)^2}{2}} \left\{ \int_\Omega u_1(y) \, df^-(x) \; - \; \int_\Omega u_0(x) \, df^+(x) \right\}.$$

$$u_1(y) - u_0(x) = \frac{d(x, y)^2}{2} \quad \mu - \text{a.e. } (x, y).$$

The derivation of this last equality gives:

$$Du_0(x) = \gamma'_{x,y}(0) \quad \mu - \text{a.e. } (x, y) \,, \tag{1}$$

where $\gamma_{x,y}$ is the constant speed geodesic joining x to y for μ-almost every (x, y). To conclude that μ is associated to a transport map we need to know that x is joined to a unique y such that (x, y) is in the support of μ for f^+−a.e. x. Unfortunately, the knowledge of x and $\gamma'_{x,y}(0)$ does not determine y (by far). The following example extracted from [9] enlightens the difference between the classical quadratic case and the case with a convex obstacle. In particular it shows that no uniqueness holds for the obstacle problem even for the squared distance.

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Example 1.1 *In dimension 2, we assume that the obstacle is a disc K, $f^+ = dx \llcorner S$ and $f^- = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$ as in the figure; S is a disc of area 1 and A, B are two points such that $d(A, P) = d(B, P)$ and such that any geodesic connecting a point of S to A or B passes through P. Then any admissible transport plan π with fixed marginals f^+ and f^- is optimal, contrary to the usual quadratic case where the unique optimal transport plan is the unique optimal transport map. Note also that, for any potential pair (u_0, u_1) and*
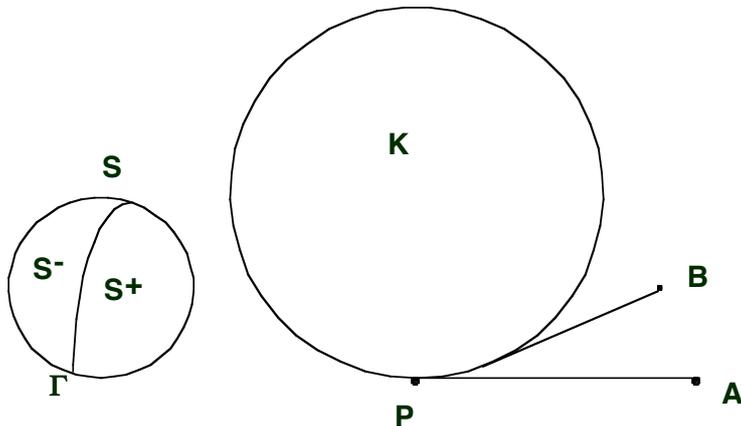
any (x, y) belonging to the support of an optimal transport plan μ, condition (1) gives no hint whether $y = A$ or $y = B$.

As the above example show, there is no way to use Brenier's approach in our framework. In order to overcome this issue we use the strategy of proof initiated by Feldman and McCann [18] for the obstacle problem with linear cost and an absolutely continuous measure f^-: the idea is to project the measures f^+ and f^- onto $\partial\mathbf{C}$. For the linear cost and for absolutely continuous measure f^-, there is a natural and intrinsic way to define this projection: it is just a consequence of the geometry of the "transport rays" associated to the dual problem. Then the construction of the optimal transport map between f^+ and f^- reduces to the construction of an optimal transport map between these projected measures $\hat{f}^+_\pi$ and $\hat{f}^-_\pi$.

In our framework, with a cost which is a squared distance and when f^- is any measure, there is no "transport rays" and therefore no natural way to associate to our obstacle transport problem a transport problem on $\partial\mathbf{C}$. So we have first to transform our obstacle problem into an obstacle problem with a "linear-like" distance in space-time. For this we use Brenier's dynamic formulation [8] of the Monge-Kantorovich problem:

$$\inf_\omega \int c((x,s),(y,t))d\omega((x,s),(y,t))$$

where the infimum is taken over every probability ω with fixed marginals $f^+ \otimes \delta_0$, $f^- \otimes \delta_1$ and c is a space-time cost given by $c((x,s),(y,t)) = \frac{d^2(x,y)}{t-s}$ whenever $t > s$. The interest of this setting is that the cost c behave almost like a distance, and in particular satisfies some triangle inequality. Thanks to this property, we can define in the same spirit of

[18] some "projected measures" $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ of f^+ and f^- onto $\partial\mathbf{C} \times [0,1]$. Note however that—because f^- is not absolutely continuous—these measures are not intrinsically defined, but depend on the choice of the optimal transport plan μ and of the optimal potential $\hat{u}$.

Once $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ are defined, it remains to build an optimal transport plan between these measures. For this it is mandatory to know that $\hat{f}_\pi^+$ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ on $\partial\mathbf{C} \times [0,1]$ (or at least that it does not charge "small sets", see [25]). By construction of $\hat{f}_\pi^+$, this property is strongly related with the semi-concavity property of the potential $\hat{u}$ in a neighborhood of $\partial\mathbf{C}$, or, equivalently, with the "Lipschitz continuity of $D\hat{u}$ inside the rays" (see section 3 of [19]). Unfortunately $\hat{u}$ fails to be semi-concave in a neighborhood of $\partial\mathbf{C} \times (0,1)$ in general: its second order normal derivative usually blows up (see an example in [14]). However $\hat{u}$ enjoys a kind of "tangential semi-concavity", which is enough to conclude to the absolutely continuity of $\hat{f}_\pi^+$. The construction of the optimal transport on $\partial\mathbf{C} \times (0,1)$ then follows from classical arguments (we use here the Champion and De Pascale approach [11]).

Let us finally discuss the uniqueness of this transport map: as explained in the above example, there is basically no hope to have some uniqueness in our problem. However it is interesting to note that, in the particular case where f^- is absolutely continuous, this lack of uniqueness is completely due to the lack of uniqueness of the optimal transport between the projected measures $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$, for the "linear" space-time cost c. Indeed, in this case, the projected measures are actually independent of the choice of the optimal transport plan and of the potentials. Then there is a one-to-one correspondence between optimal transport plans between f^+ and f^- and the optimal transport plans—for the space-time problem—between $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$. In this sense the introduction of Brenier's dynamic formulation in our context not only allows to solve the problem, but also gives a precise description of failure of uniqueness. Note that, as in the classical Monge problem, there is therefore a canonical way to select a good transport map (the one which is increasing along the rays). To conclude on this point, let us finally underline that this nice picture breaks down when f^- is not absolutely continuous. Indeed in this case the projected measure $\hat{f}_\pi^-$ is no longer intrinsically defined: as we show through an example, it depends on the choice of the optimal transport plan, and there does not seem to be a canonical way to choose the optimal transport map.

Let us briefly explain how this paper is organized: in section 2 we recall some basic facts on Brenier's formulation and introduce the main notations used in the paper. Section 3 is dedicated to the proof of the absolutely continuity of $\hat{f}_\pi^+$. The construction of the optimal transport map between the projected measures is carried out in Section 4. Finally the initial problem is solved in section 5, where we also discuss the uniqueness issue.

General notations: Throughout the paper, $|\cdot|$ denotes the Euclidean norm of the ambient space ($\mathbb{R}^N$ or $\mathbb{R}^{N+1}$), $B(\xi, r)$ the ball centered at ξ and of radius r (again in $\mathbb{R}^N$ if $\xi \in \mathbb{R}^N$ or in $\mathbb{R}^{N+1}$ if $\xi \in \mathbb{R}^{N+1}$). Given a function $v = v(x,t)$, $\partial_t v$ denotes its partial

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derivative with respect to the t variable, Dv its partial derivative with respect to the x variable and $D_{x,t}v$ its differential. When we consider the restriction of v to $\partial \mathbf{C} \times (0,1)$, we denote by $D^\tau v(x,t)$ and $D^\tau_{x,t}v(x,t)$ the tangential space derivative and the tangential differential respectively.

2 Setting of the Problem and basic results

Let X be a compact convex subset of $\mathbb{R}^N$, $\mathbf{C} \subset \text{Int}(X)$ a closed convex subset of X with a $\mathcal{C}^3$ boundary. We set $\Omega = X \backslash \mathbf{C}$. We denote by d the geodesic distance in Ω:

$$d(x,y) = \inf_{\xi \text{ Lipschitz}} \left\{ \int_0^1 |\xi'(s)|ds \ ; \ \xi(0) = x, \ \xi(1) = y, \ \xi(s) \in \overline{\Omega} \ \forall s \in [0,1] \right\}.$$

We also introduce $\delta : X \to \mathbb{R}^+$ the signed distance to the boundary $\partial \mathbf{C}$ of $\mathbf{C}$ (negative inside $\mathbf{C}$). Let us recall that δ is of class $\mathcal{C}^3$ in a neighborhood of $\partial \mathbf{C}$. Given f^+ and f^- two probability measures, we investigate the problem

$$(MK2) \qquad I = \min_{T:\overline{\Omega} \to \overline{\Omega}} \int_\Omega \frac{d^2(x,T(x))}{2} \, df^+(x)$$

where the minimum is taken over all the Borel measurable maps $T : \Omega \to \Omega$ such that T pushes forward f^+ to f^- (denoted by $T \sharp f^+ = f^-$). We assume f^+ is absolutely continuous with respect to the Lebesgue measure and will also write f^+ for its density.

The relaxed and dual formulations of problem $(MK2)$ are (see for instance [25]):

$$(\overline{MK2}) \qquad \min_{\mu \in \Pi(f^+, f^-)} \left\{ \int_{\Omega^2} \frac{d^2(x,y)}{2} \, d\mu(x,y) \right\},$$

where $\Pi(f^+, f^-)$ are the probability measures with marginals f^+ and f^-,

$$(D) \qquad \max_{v_1,v_0 \in C_b(\overline{\Omega})} \left\{ \int_\Omega v_1(y) \, df^-(x) \ - \ \int_\Omega v_0(x) \, df^+(x) \right\}$$

where the maximum is taken under the constraints $v_1(y) - v_0(x) \leq \frac{d^2(x,y)}{2} \ \forall (x,y) \in \overline{\Omega}^2$. Then the *primal-dual optimality condition* take the form (see [25]):

$$(PD) \qquad (\mu, u_0, u_1) \text{ are optimal in } (\overline{MK2}) \text{ and } (D) \Leftrightarrow u_1(y) - u_0(x) = \frac{d^2(x,y)}{2} \ \mu-\text{a.e.}(x,y).$$

Recall also that solving $(MK2)$ is equivalent to building a solution μ of $(\overline{MK2})$ which is concentrated on a graph of a Borel measurable map $T : \overline{\Omega} \to \overline{\Omega}$ (see [25]).

Let (u_0, u_1) be an optimal pair for (D). We call it a pair of *optimal potentials*. Without loss of generality (see [25]) we can assume that

$$u_0(x) = \sup_{y \in \overline{\Omega}} \{ u_1(y) - \frac{1}{2} d^2(x,y) \} \qquad \text{while} \qquad u_1(y) = \inf_{x \in \overline{\Omega}} \{ u_0(x) + \frac{1}{2} d^2(x,y) \}$$

for any $x, y \in \overline{\Omega}$. Note that u_0 and u_1 are locally Lipschitz continuous in $\overline{\Omega}$. Following [6] and [22] we set

$$c((x,s),(y,t)) = \begin{cases} \dfrac{d^2(x,y)}{2(t-s)} & \text{if } (x,s), (y,t) \in \overline{\Omega} \times [0,1] \text{ with } s < t, \\ 0 & \text{if } (x,s) = (y,t), \\ +\infty & \text{otherwise.} \end{cases}$$

Then c looks very much like a "distance" since

$$c((x,s),(y,t)) = \inf\left\{ \frac{1}{2}\int_s^t |\gamma'(\tau)|^2 d\tau \; ; \; \gamma : [s,t] \to \overline{\Omega} \text{ is Lipschitz continuous and } \gamma(s) = x, \; \gamma(t) = y \right\}.$$

In particular, we have the triangle inequality

$$c((x,s),(y,t)) \leq c((x,s),(z,\tau)) + c((z,\tau),(y,t))$$

for any $(x,s), (y,t), (z,\tau) \in \overline{\Omega} \times (0,1)$ with $s < \tau < t$. Following [8], [6] let us introduce the maps

$$\hat{u}(y,t) = \inf_{x \in \Omega}\left\{ u_0(x) + c((x,0),(y,t)) \right\} \qquad \text{and} \qquad \check{u}(x,s) = \sup_{y \in \Omega}\left\{ u_1(y) - c((x,s),(y,1)) \right\}$$

Then $\hat{u}(\cdot, 0) = \check{u}(\cdot, 0) = u_0$ while $\hat{u}(\cdot, 1) = \check{u}(\cdot, 1) = u_1$. Moreover for any $0 \leq s < t \leq 1$,

$$\hat{u}(y,t) = \inf_{x \in \Omega}\left\{ \hat{u}(x,s) + c((x,s),(y,t)) \right\} \text{ and } \check{u}(x,s) = \sup_{y \in \Omega}\left\{ \check{u}(y,t) - c((x,s),(y,t)) \right\}$$

and $\hat{u} \geq \check{u}$. We have the following Proposition, the proof of which is an easy adaptation of Theorem 3.8. of [22]:

Proposition 2.1 *The following equalities hold:*

$$\min(MK2) = \min(MK_t) = \max(D_t)$$

where (MK_t) *and* (D_t) *are the following optimization problems:*

$$(MK_t) \qquad \inf_{\omega \in \Pi(f^+ \otimes \delta_0, f^- \otimes \delta_1)} \left\{ \iint_{(\Omega \times [0,1])^2} c((x,s),(y,t)) \, d\omega((x,s),(y,t)) \right\},$$

$$(D_t) \qquad \sup_{v \in \mathcal{C}_b(\Omega \times [0,1])} \left\{ \int_\Omega v(y,1) \, df^-(y) - \int_\Omega v(x,0) \, df^+(x) : \; v(y,t) - v(x,s) \leq c((x,s),(y,t)) \right\}.$$

Moreover the function $\hat{u}$ defined above is a solution of (D_t), it is Lipschitz continuous and satisfies the following equation:

$$\partial_t v(x,s) + \frac{|Dv(x,s)|^2}{2} = 0 \quad a.e. \; (x,s) \in \Omega \times [0,1]$$

where Dv and $\partial_t v$ denote the partial derivative of v with respect to x and t.

In a symmetric way, $\check{u}$ is also a solution of (D_t). Let us denote by Γ the set of constant speed (but not normalized) geodesics $\gamma : [0,1] \to \overline{\Omega}$ such that

$$u_0(\gamma(0)) + \frac{1}{2}d^2(\gamma(0), \gamma(1)) = u_1(\gamma(1)) .$$

Elements of Γ are often called transport rays. Then we set

$$\mathcal{T} = \{(x,s) \in \overline{\Omega} \times (0,1) \mid \exists \gamma \in \Gamma, \ \gamma(s) = x, \ \gamma(0) \neq \gamma(1)\} .$$

The following standard Lemma collects some results on the structure of the geodesics of $\overline{\Omega}$.

Lemma 2.2 *Let $\gamma : [0,1] \to \overline{\Omega}$ be a constant speed geodesic such that $\gamma(0) \neq \gamma(1)$. Then either γ is a straight line or there are $0 \leq a \leq b \leq 1$ such that the restriction of γ to $[0,a]$ and to $[b,1]$ are straight lines and γ is a geodesic on the manifold $\partial \mathbf{C}$ on $[a,b]$.*

Moreover there is a non-zero adjoint map $p : [0,1] \to \mathbb{R}^N$ such that the pair (γ, p) solves on $[0,1]$

$$\begin{cases} \gamma'(s) = p(s) \\ -p'(s) = \langle D^2\delta(\gamma(s))p(s), p(s)\rangle \mathbf{1}_{\partial \mathbf{C}}(\gamma(s)) \, D\delta(\gamma(s)). \end{cases}$$

In particular, the $\mathcal{C}^{1,1}$ norm of any constant speed geodesic $\gamma : [0,1] \to \overline{\Omega}$ is proportional to $|\gamma'(s)|$ for some (hence any) $s \in [0,1]$, with a coefficient depending only on the $\mathcal{C}^3$ regularity of $\partial \mathbf{C}$.

The following Lemma will be used to prove the differentiability of $\hat{u}$.

Lemma 2.3 *There is some constant C_Ω depending only on the $\mathcal{C}^3$ regularity of $\partial \mathbf{C}$, such that*

$$d(z,z') \leq |z - z'| + C_\Omega|z - z'|^3 \qquad \forall z, z' \in \partial \mathbf{C} . \tag{2}$$

Proof of Lemma 2.3 : Throughout the proof of Lemma 2.3 we denote by $C_{\partial \mathbf{C}}$ any constant which depends only on the $\mathcal{C}^3$ regularity of $\mathbf{C}$.

Because of the compactness of $\mathbf{C}$, the quantity $\frac{d(z,z')-|z-z'|}{|z-z'|}$ is bounded when $|z - z'|$ is bounded from below, so we only need to show that (2) holds for $z, z' \in \partial \mathbf{C}$ sufficiently close, say in a ball B_η of radius $\eta > 0$, and such that $z \neq z'$. Let us set

$$p_z = (z' - z)/|z' - z| \text{ and } q_z = (D\delta(z) - \langle D\delta(z), p_z\rangle p_z)/|D\delta(z) - \langle D\delta(z), p_z\rangle p_z| .$$

We have:

$$\langle D\delta(z), p_z\rangle = \frac{\delta(z') - \delta(z) + o(|z - z'|)}{|z - z'|} = \varepsilon(|z - z'|),$$

$$\langle D\delta(z), q_z\rangle = \frac{1 - (\langle D\delta(z), p_z\rangle)^2}{|D\delta(z) - \langle D\delta(z), p_z\rangle p_z|} = \sqrt{1 - (\langle D\delta(z), p_z\rangle)^2}.$$

Note that, if η is sufficiently small using the continuity of $D\delta$ and the previous estimates, we have

$$\langle q_z, D\delta(\xi)\rangle \geq \frac{1}{2} \text{ and } |\langle p_z, D\delta(\xi)\rangle| \leq \frac{1}{8} \qquad \forall \xi \in B_\eta .$$

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Let $R = (8\|D^2\delta\|_{L^\infty(B_\eta)})^{-1}$ and $\alpha_z > 0$ such that $\sin(\alpha_z) = |z - z'|/(2R)$. We can also assume that η is so small that $\cos(\alpha_z) \geq 1/2$. Let

$$\sigma(s) = (z + z')/2 + R(\cos(\alpha_z s) - \cos(\alpha_z))q_z - R\sin(\alpha_z s)p_z \qquad s \in [-1, 1] .$$

Then $\sigma(1) = z$ and $\sigma(-1) = z'$. We claim that $\sigma(s) \in \Omega$ for any $s \in [-1, 1]$, i.e., that the map $f(s) = \delta(\sigma(s))$ is nonnegative on $[-1, 1]$. For this it is enough to show that f is concave on $[-1, 1]$ because $f(1) = f(-1) = 0$. We have

$$
\begin{aligned}
f''(s) &= \langle D^2\delta(\sigma(s))\sigma'(s), \sigma'(s)\rangle + \langle D\delta(\sigma(s)), \sigma''(s)\rangle \\
&\leq \alpha_z^2 R^2\|D^2\delta\|_\infty - \alpha_z^2 R(\cos(\alpha_z s)\langle D\delta(\sigma(s)), q_z\rangle + \sin(\alpha_z s)\langle D\delta(\sigma(s)), p_z\rangle) \\
&\leq \alpha_z^2 R^2(\|D^2\delta\|_\infty - 1/(8R)) \qquad \leq \qquad 0,
\end{aligned}
$$

where we have used that $-\cos(\alpha_z s) \leq -1/2$. This proves that $\sigma(s) \in \overline{\Omega}$ for any $s \in [-1, 1]$. Hence by expanding arcsin in the neighborhood of 0:

$$d(z, z') \leq \int_{-1}^{1} |\sigma'(s)|ds = 2R\alpha_z = 2R\arcsin(|z' - z|/2R) \leq |z' - z| + C_{\partial\mathbf{C}}|z' - z|^3 .$$

□

Lemma 2.4 *Let $\gamma : [0, 1] \to \overline{\Omega}$ be a constant speed geodesic curve such that:*

$$\gamma(0) = A \in \Omega, \ \gamma(1) = B \in \Omega .$$

Let $s \in (0, 1)$ and $x := \gamma(s)$. If $\xi : [0, 1] \to \overline{\Omega}$ is also a constant speed geodesic curve such that $\xi(0) = A$ and $\xi(s) = x$, then $\xi(t) = \gamma(t)$ for any $t \in [0, s]$.

Proof: Assume first that $x \in \partial\mathbf{C}$.

- Step 1: We first notice that $\xi'(s) = \gamma'(s)$. This is a consequence of the $C^{1,1}$ regularity (cf Lemma 2.2) of the geodesic curve ζ given by:

$$\zeta(t) = \xi(t) \text{ for } 0 \leq t < s, \ \zeta(t) = \gamma(t) \text{ for } s \leq t \leq 1.$$

- Step 2. Let $t_0, t_1 \in [0, s]$ be such that:

$$\gamma(t) \in \partial\mathbf{C} \ \forall t \geq t_0, \qquad \gamma(t) \notin \partial\mathbf{C} \ \forall t < t_0,$$

$$\xi(t) \in \partial\mathbf{C} \ \forall t \geq t_1, \qquad \xi(t) \notin \partial\mathbf{C} \ \forall t < t_1.$$

Note that γ and ξ are straight lines on $[0, t_0]$ and $[0, t_1]$ respectively. Set $p = \gamma'$ and $q = \xi'$. By Lemma 2.2, (γ, p) and (ξ, q) are both solutions of the following ODE on the time interval on $[t_0 \vee t_1, s]$:

$$
\begin{cases}
x' &= y, \\
y' &= -\langle D^2\delta(x(s))y(s), y(s)\rangle \, D\delta(x(s)), \\
x(s) &= x, \\
y(s) &= \zeta'(s).
\end{cases}
$$

So $\gamma = \xi$ on $[t_0 \vee t_1, s]$.

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- Step 3. Let us assume that $t_0 \leq t_1$ (this is without loss of generality since we can switch the roles of γ and ξ). Then, combining the fact that ξ is a straight line on $[0, t_1]$, that $\gamma(t_1) = \xi(t_1)$ and that γ is a geodesic, we get

$$|A - \gamma(t_1)| = d(A, \gamma(t_1)) = |A - \gamma(t_0)| + d(\gamma(t_0), \gamma(t_1)) \geq |A - \gamma(t_0)| + |\gamma(t_0) - \gamma(t_1)| \, ,$$

which proves that A, $\gamma(t_0)$ and $\gamma(t_1)$ are aligned. But then γ has to be a straight line between A and $\gamma(t_1)$. This proves that $\gamma = \xi$ on $[0, t_1]$, and therefore on $[0, s]$.

If $x \notin \partial \mathbf{C}$, then equality $\xi'(s) = \gamma'(s)$ remains true. If $\gamma([0, s])$ does not intersect $\partial \mathbf{C}$, the proof is obvious as $\gamma([0, s])$ and $\xi([0, s])$ are straight lines. Otherwise, applying the same arguments as in step 3, there exists $t_0 \in (0, s)$ such that γ and ξ are straight lines and coincide in $[t_0, s]$, with $\xi(t_0) = \gamma(t_0) \in \partial \mathbf{C}$. Then we can complete the proof as before by replacing (x, s) by $(\gamma(t_0), t_0)$. $\square$

Lemma 2.5 *(i) For any $\gamma \in \Gamma$, let $x = \gamma(0)$, $y = \gamma(1)$. Then*

$$\hat{u}(\gamma(s), s) = \hat{u}(\gamma(t), t) - c((\gamma(s), s), (\gamma(t), t)) \qquad \forall \, 0 \leq s \leq t \leq 1 \qquad (3)$$

or, equivalently,

$$\hat{u}(\gamma(s), s) = \hat{u}(\gamma(t), t) - \frac{1}{2} d^2(x, y)(t - s) \qquad \forall \, 0 \leq s \leq t \leq 1 \, . \qquad (4)$$

(ii) Equality $\hat{u}(x, s) = \check{u}(x, s)$ holds for any $(x, s) \in \mathcal{T}$.

(iii) The functions $\hat{u}$ and $\check{u}$ are differentiable at any point of $(x, s) \in \mathcal{T}$ with

$$D_{x,t} \hat{u}(x, s) = D_{x,t} \check{u}(x, s) = (\gamma'(s), -\frac{|\gamma'(s)|^2}{2}) \qquad \text{if } x \in \Omega$$

and

$$D_{x,t}^\tau \hat{u}(x, s) = D_{x,t}^\tau \check{u}(x, s) = (\gamma'(s), -\frac{|\gamma'(s)|^2}{2}) \qquad \text{if } x \in \partial \mathbf{C} \, ,$$

where $\gamma \in \Gamma$ is any geodesic curve such that $\gamma(s) = x$.

Proof : (i) Note first that, $\gamma : [0, 1] \to \overline{\Omega}$ being a constant speed geodesic, we have:

$$c((x, s), (y, t)) = (t - s) \frac{1}{2} d^2(x, y).$$

Now, using the definition of Γ, the triangle inequality for $c((.,.), (.,.))$ and the fact that $\hat{u}$ satisfies the constraint of (D_t), we get:

$$
\begin{aligned}
\hat{u}(\gamma(t), t) - \hat{u}(\gamma(s), s) \ &\leq \ c((\gamma(s), s), (\gamma(t), t)) \ = \ (t - s) \frac{d^2(x, y)}{2} \\
&= \ (t - 1) \frac{d^2(x, y)}{2} + \frac{d^2(x, y)}{2} - s \frac{d^2(x, y)}{2} \\
&\leq \ \hat{u}(\gamma(t), t) - \hat{u}(\gamma(1), 1) + \hat{u}(\gamma(1), 1) - \hat{u}(\gamma(0), 0) + \hat{u}(\gamma(0), 0) - \hat{u}(\gamma(s), s) \\
&= \ \hat{u}(\gamma(t), t) - \hat{u}(\gamma(s), s).
\end{aligned}
$$

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This proves (3).

(ii) Let $(x, s) \in \mathcal{T}$ and $\gamma \in \Gamma$ be such that $\gamma(s) = x$. As in (i) on can show that

$$\check{u}(\gamma(s), s) = \check{u}(\gamma(t), t) - \frac{1}{2}d^2(x, y)(t - s).$$

Then $\hat{u}(x, s) = \check{u}(x, s)$ easily follows from (3) and the above equality for $t = 1$ since $\hat{u}(\cdot, 1) = \check{u}(\cdot, 1) = u_1$.

(iii) Let us prove the differentiability of $\hat{u}$ at a point $(x, s) \in \mathcal{T}$. Let $\gamma : [0, 1] \to \overline{\Omega}$ be a constant speed geodesic in Γ such that $\gamma(s) = x$ and let us set $A := \gamma(0)$ and $B := \gamma(1)$. Arguing as in [19] note that for any $(y, t) \in \overline{\Omega} \times]0, 1[$, it holds:

$$\hat{u}(y, t) - \hat{u}(A, 0) \leq \frac{d^2(y, A)}{2t}, \quad \hat{u}(B, 1) - \hat{u}(y, t) \leq \frac{d^2(y, B)}{2(1 - t)}.$$

Moreover equality holds replacing (y, t) by (x, s), consequently:

$$-\frac{d^2(y, B)}{2(1 - t)} + \frac{d^2(x, B)}{2(1 - s)} \leq \hat{u}(y, t) - \hat{u}(x, s) \leq \frac{d^2(y, A)}{2t} - \frac{d^2(x, A)}{2s}.$$

The differentiability of $\hat{u}$ in s easily follows:

$$\partial_t \hat{u}(x, s) = \partial_t \left(-\frac{d^2(x, B)}{2(1 - s)} \right) = \partial_t \left(\frac{d^2(x, A)}{2s} \right) = -\frac{|\gamma'(s)|^2}{2}.$$

If $x \in \Omega$, then we also get the differentiability of $\hat{u}$ with respect to x thanks to the semi-concavity of d. To prove this differentiability when $x \in \partial \mathbf{C}$, we need the following intermediate result. Let y, z be two points in $\partial \mathbf{C}$ and let γ_y be a constant speed geodesic linking A to y with $\gamma_y(s) = y$. We claim that

$$d(z, A) - d(y, A) - \langle \frac{\gamma_y'(s)}{|\gamma_y'(s)|}, z - y \rangle \leq C_{\partial \mathbf{C}} |z - y|^{4/3} . \tag{5}$$

Proof of (5) : Without loss of generality we assume that $|z - y| \leq 1$. Let us set $h = |z - y|$, $p = \frac{\gamma_y'(s)}{|\gamma_y'(s)|}$ and $\alpha = \langle p, z - y \rangle$. Note that $\alpha \leq h \leq h^{2/3}$. Then

$$d(z, A) - d(y, A) - \langle p, z - y \rangle$$
$$\leq d\left(z, \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|}) \right) + d\left(\gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|}), \gamma_y(0) \right) - d(y, A) - \alpha$$
$$= d\left(z, \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|}) \right) - h^{2/3} .$$

In order to estimate $d\left(z, \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|}) \right)$, we use Lemma 2.3 which states that:

$$d\left(z, \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|}) \right) \leq |z - \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|})| + C_\Omega |z - \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma_y'(s)|})|^3.$$

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Since, from Lemma 2.2, γ'_y is Lipschitz continuous with a Lipschitz constant bounded by $C_{\partial\Omega}|\gamma'_y(s)|$, we get, recalling the definitions of h and p,

$$
\begin{aligned}
|z - \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma'_y(s)|})| &= |z - y + y - \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma'_y(s)|})| \\
&\leq |z - y - (\alpha - h^{2/3})p| + C_{\partial\mathbf{C}}h^{4/3} \\
&\leq |z - y - \langle z - y, p\rangle p + h^{2/3}p| + C_{\partial\mathbf{C}}h^{4/3} \\
&\leq (h^2 + h^{4/3})^{1/2} + C_{\partial\mathbf{C}}h^{4/3} \leq h^{2/3} + C_{\partial\mathbf{C}}h^{4/3}
\end{aligned}
$$

where we have used the Taylor formula to get the first inequality. Finally:

$$
\begin{aligned}
d(z, A) - d(y, A) - \langle p, z - y\rangle &\leq d\left(z, \gamma_y(s + \frac{\alpha - h^{2/3}}{|\gamma'_y(s)|})\right) - h^{2/3} \\
&\leq C_{\partial\mathbf{C}}h^{4/3} .
\end{aligned}
$$

This completes the proof of claim (5).

Applying this claim to $x = \gamma(s)$ and to any $y \in \partial\mathbf{C}$ directly gives that

$$
d(y, A) - d(x, A) - \langle \frac{\gamma'(s)}{|\gamma'(s)|}, y - x\rangle \leq C_{\partial\mathbf{C}}h^{4/3} . \tag{6}
$$

For the reverse inequality, let us note that any constant speed geosedic γ_y such that $\gamma_y(0) = A$ and $\gamma_y(s) = y$ converges in the $\mathcal{C}^1$ norm to γ as $y \to x$ because of the $\mathcal{C}^{1,1}$ bound on all the geodesics of $\overline{\Omega}$ (Lemma 2.2) and Lemma 2.4. Hence, for any $\varepsilon > 0$ there is some $\eta > 0$ such that $y \in B(x, \eta) \cap \partial\mathbf{C}$ implies that $\left|\frac{\gamma'_y(s)}{|\gamma'_y(s)|} - \frac{\gamma'(s)}{|\gamma'(s)|}\right| \leq \varepsilon$. Applying claim (5) to y and x gives

$$
d(x, A) - d(y, A) - \langle \frac{\gamma'(s)}{|\gamma'(s)|}, x - y\rangle \leq C_{\partial\mathbf{C}}|x - y|^{4/3} + \varepsilon|y - x| . \tag{7}
$$

Combining (6) and (7) gives the differentiability of $d(A, \cdot)$ on $\partial\mathbf{C}$ with derivative given by $\frac{\gamma'(s)}{|\gamma'(s)|}$. $\square$

Note that, as a consequence of the previous lemma, if two geodesics in Γ meet in their interior at the same time, they cross tangentially.

Notations: Let us now introduce the main notations of this paper. Let E^+ be the set of point $x \in \Omega$ such that u_0 is differentiable at x and there exists $y \in \Omega$ such that

$$
u_1(y) - u_0(x) = \frac{1}{2}d^2(x, y) \qquad \text{and} \qquad d(x, y) > |x - y| .
$$

Note that the points x and y are connected by a geodesic which "bends around $\partial\mathbf{C}$" between x and y. Then we set

$$
Z = \{(x, y) \in \Omega \times \Omega \mid u_1(y) - u_0(x) = \frac{1}{2}d^2(x, y) \text{ and } u_0 \text{ is differentiable at } x\} ,
$$

$$Z_0 = \{(x, y) \in Z \mid x \in E^+\} \qquad \text{and} \qquad Z_1 = Z \backslash Z_0 .$$

Note that for $(x, y) \in Z_1$, there is a unique geodesic connecting x and y and this geodesic is a straight line. Let us also point out that $\mu(Z) = 1$ for any optimal transport plan μ because, on the one hand, of the primal-dual condition and, on another hand, u_0, being locally Lipschitz continuous, is differentiable a.e. and μ has a marginal f^+ which is absolutely continuous with respect to the Lebesgue measure.

Remark 2.6 *The set E^+ is clearly Borel measurable.*

Lemma 2.7 *Let $x \in E^+$, $y \in \Omega$ be such that $(x, y) \in Z_0$ and $\gamma \in \Gamma$ be such that $\gamma(0) = x$, $\gamma(1) = y$. Then $Du_0(x) \neq 0$ and $\gamma'(0) = Du_0(x)$. In particular $\gamma'(0)$ only depends on x.*

Proof : Let $x \in E^+$, $h > 0$ and $z \in \mathbb{R}^N$. The point x being outside of **C**, we may assume that h is small enough to get:

$$\gamma(h) = x + h\gamma'(0), \ \gamma'(h) = \gamma'(0), \quad d(x + hz, \gamma(h) + hz) = |h\gamma'(0)|.$$

Then, by (4) and the definition of $\hat{u}$, we have:

$$\hat{u}(\gamma(h), h) = u_0(x) + h\frac{|\gamma'(0)|^2}{2}, \ \hat{u}(\gamma(h) + hz, h) \leq u_0(x + hz) + \frac{|h\gamma'(0)|^2}{2h}.$$

Hence, using the differentiability of $\hat{u}$ given in Lemma 2.5, we get

$$
\begin{aligned}
\langle D_{x,t}\hat{u}(\gamma(h), h), (hz, 0)\rangle + o(h) &= \hat{u}(\gamma(h) + hz, h) - \hat{u}(\gamma(h), h) \\
&\leq u_0(x + hz) - u_0(x) = h\langle Du_0(x), z\rangle + o(h).
\end{aligned}
$$

Since $D\hat{u}(\gamma(h), h) = \gamma'(0)$, this leads to: $\langle Du_0(x), z\rangle \geq \langle \gamma'(0), z\rangle$. As this inequality also holds true for $-z$ instead of z, we have the desired equality. $\square$

As pointed out in the introduction, the interior of two geodesics might intersect. However the following Lemma states that, for fixed $(x, y) \in Z$, there is a unique geodesics linking x and y.

Lemma 2.8 *For any $(x, y) \in Z$ there is a unique geodesic $\gamma \in \Gamma$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proof : We only do the proof for $(x, y) \in Z_0$, the case $(x, y) \in Z_1$ being standard. Let γ_1 and γ_2 be two geodesics of Γ such that $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(1) = \gamma_2(1) = y$. Since $x \in \Omega$, γ_1 and γ_2 are straight line on some interval $[0, t_1]$ and $[0, t_2]$ respectively. Then, by Lemma 2.7, we have $\gamma_1'(0) = \gamma_2'(0) = Du_0(x)$ and this equality implies that $\gamma_1 = \gamma_2$ on $[0, t_1 \wedge t_2]$. The result easily follows using Lemma 2.4 as $\tilde{\gamma}_1(\cdot) = \gamma_1(1 - \cdot)$ and $\tilde{\gamma}_2(1 - \cdot)$ are two geodesic curves such that:

$$\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = y \text{ and } \tilde{\gamma}_1(t_1 \wedge t_2) = \tilde{\gamma}_2(t_1 \wedge t_2).$$

$\square$

Remark 2.9 *Note that the same arguments show that, if $x \in E^+$ and if ξ and γ are two geodesics of Γ such that $\gamma(0) = \xi(0) = x$, then $\xi = \gamma$ on $[0, t_0]$ with*

$$t_0 = \min\{t : \ \gamma(t) \in \partial \mathbf{C}\} = \min\{t : \ \xi(t) \in \partial \mathbf{C}\}.$$

Definition of $\pi^+(x)$ and $\pi^-(x,y)$: Let $(x,y) \in Z_0$ and $\gamma \in \Gamma$ be the unique geodesic such that $\gamma(0) = x$, $\gamma(1) = y$. Note that $\gamma([0,1])$ consists in two segments separated by a geodesic on $\partial \mathbf{C}$. From Lemma 2.7 we can set

$$t^+(x) = \min\{t \geq 0 \mid \gamma(t) \in \partial \mathbf{C}\} \qquad \text{and} \qquad t^-(x,y) = \max\{t \in [0,1] \mid \gamma(t) \in \partial \mathbf{C}\}$$

and

$$\pi^+(x) = (\gamma(t^+(x)), t^+(x)) \qquad \text{and} \qquad \pi^-(x,y) = (\gamma(t^-(x,y)), t^-(x,y)) \ .$$

Lemma 2.10 *The maps $\pi^+ = (\pi_x^+, \pi_t^+) : E^+ \to \partial \mathbf{C} \times (0,1)$ and $\pi^- = (\pi_x^-, \pi_t^-) : Z_0 \to \partial \mathbf{C} \times (0,1)$ are Borel measurable. Moreover*

$$u_0(x) = \hat{u}(\pi^+(x)) - \frac{1}{2}\frac{d^2(x, \pi_x^+(x))}{\pi_t^+(x)} = \check{u}(\pi^+(x)) - \frac{1}{2}\frac{d^2(x, \pi_x^+(x))}{\pi_t^+(x)} \qquad \forall x \in E^+$$

and

$$u_1(y) = \hat{u}(\pi^-(x,y)) + \frac{d^2(y, \pi_x^-(x,y))}{2(1 - \pi_t^-(x,y))} = \check{u}(\pi^-(x,y)) + \frac{d^2(y, \pi_x^-(x,y))}{2(1 - \pi_t^-(x,y))} \qquad \forall (x,y) \in Z_0 \ .$$

Proof : We only prove the measurability of π^+, the measurability of π^- following the same arguments and the rest of the Lemma being an easy consequence of (4).
Let us point out that Remark 2.9 gives a well defined map $G : x \in E^+ \mapsto \gamma \in \Gamma$ where $\gamma : t \in [0, t^+(x)] \to \overline{\Omega}$ is a right line of direction $Du_0(x)$ linking x to $\gamma(t^+(x)) \in \partial \mathbf{C}$. This map is continuous in the following sense: If $x_n \in E^+$ converges to x, then $G(x_n)$ can be extended in $[0,1]$ to a curve in $\gamma_n \in \Gamma$. By Lemma 2.2, $(\gamma_n)_n$ is an equilipschitz sequence, then by Ascoli (using $\gamma_n(0) \to x$), it converges uniformly to some γ. It is easily seen that this curve is in Γ, and satisfies $\gamma(0) = x$. Using Remark 2.9, we get $\gamma = G(x)$ in $[0, t^+(x)]$. Then, as $\delta(\gamma_n(t^+(x_n))) = 0$ we have $\gamma(\liminf_n t^+(x_n)) \in \partial \mathbf{C}$ and :

$$\liminf_n t^+(x_n) \geq \min\{t : \gamma(t) \in \partial \mathbf{C}\} = t^+(x).$$

So t^+ is lower semicontinuous and, therefore, measurable. It can be written as the limit of continuous functions $(t_k)_{k \in N}$ and by continuity of $G(x)$ for any $x \in E^+$:

$$\pi_x^+(\cdot) = G(\cdot)(\lim_k t_k(\cdot)) = \lim_k G(\cdot)(t_k(\cdot)).$$

The map $x \mapsto G(x)(t_k(x))$ being continuous thanks to the property of uniform continuity of G and the continuity of t_k, it is measurable and so is π_x^+. $\square$

We now collect some results on the derivative of $\hat{u}$ at points $\pi^+(x)$ and $\pi^-(x,y)$.

Lemma 2.11 *Let $(x, y) \in Z_0$. Then*

$$D^\tau \hat{u}(\pi^+(x)) = Du_0(x) = \frac{\pi_x^+(x) - x}{\pi_t^+(x)} ,$$

while

$$D^\tau \hat{u}(\pi^-(x, y)) = \frac{y - \pi_x^-(x, y)}{(1 - \pi_t^-(x, y))} .$$

Proof : Let $\gamma \in \Gamma$ be the geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$. Then γ is a straight line on $[0, t^+(x)]$, so that

$$D^\tau \hat{u}(\pi^+(x)) = \gamma'(t^+(x)) = \frac{\pi_x^+(x) - x}{\pi_t^+(x)} = \gamma'(0) = Du_0(x) .$$

The second part of the Lemma can be proved in the same way since γ is a straight line on $[t^-(x, y), 1]$ and $\hat{u}$ is differentiable at $\pi^-(x, y)$ with $D\hat{u}(\pi^-(x, y)) = \gamma'(t^-(x, y))$. $\square$

From now on we fix an optimal transport plan μ. Recall that $\mu(Z) = 1$.

Definition of μ_0 and μ_1 : We divide the measure μ into the sum of two measures:

$$\mu_0 = \mu \lfloor Z_0 \text{ and } \mu_1 = \mu \lfloor Z_1 .$$

Obviously we have

$$\mu = \mu_0 + \mu_1 .$$

Following Theorem 4.6 of [25], μ_0 is an optimal transport plan between $\hat{f}^+$ and $\hat{f}^-$, where

$$\hat{f}^+ = \pi_1 \sharp \mu_0 \text{ and } \hat{f}^- = \pi_2 \sharp \mu_0 ,$$

while (u_0, u_1) is an optimal potential pair between $\hat{f}^+$ and $\hat{f}^-$. Note also for later use that

$$\hat{f}^+(\Omega \backslash E^+) = 0 \tag{8}$$

because, by definition of E^+, Z_0 and Z_1, $\pi_1^{-1}(\Omega \backslash E^+) \cap Z = Z_1$.

Definition of $\hat{f}_\pi^\pm$:

$$\hat{f}_\pi^+ = \pi^+ \sharp \hat{f}^+ \qquad \text{and} \qquad \hat{f}_\pi^- = \pi^- \sharp \mu_0 .$$

By definition $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ are Borel measures on $\partial \mathbf{C} \times (0, 1)$.

Lemma 2.12 *$\hat{f}^+$ is a.c. with respect to the Lebesgue measure $\mathcal{L}^N$ and*

$$\hat{f}^+(E^+) = \mu(Z_0) .$$

Moreover

$$\hat{f}_\pi^+(\partial \mathbf{C} \times [0, 1]) = \hat{f}_\pi^-(\partial \mathbf{C} \times [0, 1]) = \mu(Z_0) .$$

Proof : Take N a Lebesgue negligible set, then:

$$\hat{f}^+(N) = \mu_0(N \times \Omega) \leq \mu(N \times \Omega) = \int_N df_0 = 0.$$

Moreover by definitions of E^+ and of μ_0:

$$\hat{f}^+(E^+) = \mu_0(E^+ \times \Omega) = \mu_0(Z_0) = \mu(Z_0).$$

Finally by definitions of π^+ and π^- we have $(\pi^+)^{-1}(\partial \Omega \times [0, 1]) = E^+$ and $(\pi^-)^{-1}(\partial \Omega \times [0, 1]) = Z_0$ which gives the last equality. $\square$

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3 The absolute continuity of $\hat{f}_\pi^+$

In this section we prove the absolute continuity of $\hat{f}_\pi^+$: this key property allows us to build a suitable optimal transport map between $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ in the next section.

3.1 Removing the flat part of the constraints

Let E_0^+ be the set of points $x \in E^+$ such that $\pi_x^+(x)$ is a point where $\partial \mathbf{C}$ is not strictly convex along the direction $x - \pi_x^+(x)$:

$$E_0^+ = \{x \in E^+ : \ \langle D^2\delta(\pi_x^+(x))(\pi_x^+(x) - x), (\pi_x^+(x) - x)\rangle = 0\}.$$

As for any $x \in E^+$, $D^2\delta(\pi_x^+(x))$ is symmetric and positive-semidefinite, one also has

$$E_0^+ = \{x \in E^+ : \ D^2\delta(\pi_x^+(x))(\pi_x^+(x) - x) = 0\}. \tag{9}$$

Lemma 3.1 *The set E_0^+ is Borel measurable and*

$$\mathcal{L}^N(E_0^+) = 0 \ .$$

Proof of Lemma 3.1 :   The map $x \mapsto \langle D^2\delta(\pi_x^+(x))(\pi_x^+(x) - x), (\pi_x^+(x) - x)\rangle$ and E^+ being measurable (see Remark 2.6), so is E_0^+.

   We denote by $T_{\partial \mathbf{C}}$ the tangent bundle of $\partial \mathbf{C}$. We consider the following application:

$$\begin{array}{rccc}
\Psi : & T_{\partial \mathbf{C}} & \to & \mathbb{R}^N \\
& (y, v) & \mapsto & y + v.
\end{array}$$

Let us show that E_0^+ is included in the set of critical values of Ψ which, using Sard Theorem (see for instance [3], Theorem 1.30) will give the thesis. Take $x \in E_0^+$ and set $y = \pi_x^+(x)$ and $v = x - \pi_x^+(x)$ so that $\Psi(y, v) = x$. It is easily seen that:

$$T_{(y,v)}(T_{\partial \mathbf{C}}) = \{(\dot{y}, \dot{v}) : \ \langle D\delta(y), \dot{y}\rangle = 0, \ \langle D^2\delta(y)\dot{y}, v\rangle + \langle D\delta(y), \dot{v}\rangle = 0\},$$

$$D\Psi(y, v)(\dot{y}, \dot{v}) = \dot{y} + \dot{v} \quad \forall (\dot{y}, \dot{v}) \in T_{(y,v)}(T_{\partial \mathbf{C}}).$$

Remembering (9), $T_{(y,v)}(T_{\partial \mathbf{C}})$ can be rewritten as:

$$T_{(y,v)}(T_{\partial \mathbf{C}}) = \{(\dot{y}, \dot{v}) : \ \langle D\delta(y), \dot{y}\rangle = 0, \ \langle D\delta(y), \dot{v}\rangle = 0\},$$

so that the image of $D\Psi(y, v)$ is included in $T_y\partial \mathbf{C}$ which is a proper subset of $\mathbb{R}^N$. From this we deduce that $x = \Psi(y, v)$ is a critical value of Ψ. $\square$

   A consequence of Lemma 3.1 is the following:

Lemma 3.2 *The set $\pi^+(E_0^+)$ is Borel measurable and $\hat{f}_\pi^+(\pi^+(E_0^+)) = 0$.*

Proof : To prove the Borel measurability of $\pi^+(E_0^+)$, it is enough to note that $g : (y, t) \mapsto y - t D^\tau \hat{u}(y, t)$ and $h : (y, t) \mapsto t^+(g(y, t)) - t$ are Borel measurable (cf Lemma 2.10) and that $\pi^+(E_0^+) = g^{-1}(E_0^+) \cap h^{-1}(0)$.

We now prove $\hat{f}_\pi^+(\pi^+(E_0^+)) = 0$. We first claim that π^+ is injective. Indeed, let $x, x' \in E^+$ such that $\pi^+(x) = \pi^+(x')$. Let $(y, t) = \pi^+(x)$. Since $\hat{u}$ is differentiable at (y, t) with $D\hat{u}(y, t) = D\hat{u}(x, 0) = D\hat{u}(x', 0)$, we have

$$x = y - t D\hat{u}(x, 0) = x' \ \ .$$

With this in mind we get

$$\hat{f}_\pi^+(\pi^+(E_0^+)) = \hat{f}^+((\pi^+)^{-1}(\pi^+(E_0^+))) = \hat{f}^+(E_0^+) = 0$$

because, from Lemma 3.1, $\mathcal{L}^N(E_0^+) = 0$ and $\hat{f}^+$ is absolutely continuous. $\square$

3.2 Countable Lipschitz continuity of $D\hat{u}$

We aim at understanding some properties of transport rays which remain on $\partial \mathbf{C}$ for a while. All what follows has very much to do with the classical property that "$D\hat{u}$ is Lipschitz continuous inside the rays" (see section 3 of [19]). It turns out that this Lipschitz continuity fails for state-constraints problems. Namely, for fixed $\varepsilon > 0$, the map $D\hat{u}$, restricted to the points which are at a distance at least ε from the end points of the rays, need not be Lipschitz continuous in a neighborhood of the points of $\partial \mathbf{C}$. To overcome this difficulty, we are going to show that $D^\tau \hat{u}$ restricted to some subset of $\partial \mathbf{C}$ is Lipschitz continuous.

To define this set, let us denote, for any $\varepsilon > 0$, by Γ_ε the set of $\gamma \in \Gamma$ such that $x := \gamma(0) \in E^+ \backslash E_0^+$ and such that

$$|\gamma'(0)| \geq \varepsilon, \ t^+(x) \geq \varepsilon \text{ and } \gamma(s) \in \partial \mathbf{C} \text{ for } s \in [t^+(x), t^+(x) + \varepsilon]$$

and

$$\langle D^2 \delta(\gamma(t^+(x)))(\gamma(t^+(x)) - x), (\gamma(t^+(x)) - x) \rangle \geq \varepsilon \ .$$

Let us set

$$\mathcal{T}_\varepsilon^0 = \{(\gamma(t), t) \in \partial \mathbf{C} \times (0, 1) \ ; \ \gamma \in \Gamma_\varepsilon, \ t = t^+(\gamma(0))\}$$

and

$$\mathcal{T}_\varepsilon = \{(\gamma(t), t) \in \partial \mathbf{C} \times (0, 1) \ ; \ \gamma \in \Gamma_\varepsilon, \ t \in [t^+(\gamma(0)), t^+(\gamma(0)) + \frac{\varepsilon}{2}]\} \ .$$

Lemma 3.3 $\mathcal{T}_\varepsilon^0$ and $\mathcal{T}_\varepsilon$ are Borel measurable.

Proof: Without loss of generality we can assume that E^+ is $\sigma-$compact: there is a sequence K_n of compact subsets of $\bar{\Omega}$ such that $E^+ = \bigcup_n K_n$ (see [16] Theorem 4). Recall that the restriction of t^+ to each K_n is lower semi-continuous (see the proof of Lemma 2.10). Let Γ_ε^n be the set of $\gamma \in \Gamma_\varepsilon$ such that $\gamma(0) \in K_n$. Then clearly $\bigcup_n \Gamma_\varepsilon^n = \Gamma_\varepsilon$.

We are going to show that Γ_ε^n is compact and that t^+ is continuous on the set $\{\gamma(0), \gamma \in \Gamma_\varepsilon^n\}$. Let $\gamma_p \in \Gamma_n$. Without loss of generality we can assume that γ_p converges to some $\gamma \in \Gamma$ because Γ is compact. Let $x = \gamma(0)$, $x_p = \gamma_p(0)$, $t^+ = t^+(x)$ and $t_p^+ = t^+(x_p)$.

Still without loss of generality we can suppose that t_p^+ has a limit, denoted $\bar{t}^+$. Note that $t^+ \leq \bar{t}^+$.

Let us now prove that $t^+ = \bar{t}^+$. For this we argue by contradiction, assuming that $t^+ < \bar{t}^+$. Then, since the γ_p are straight lines on $[0, t_p^+]$, γ is a straight line on $[t^+, \bar{t}^+]$ and this straight line is contained in $\partial\mathbf{C}$. On another hand, we have

$$\langle D^2\delta(\gamma_p(t_p^+))(\gamma_p(t_p^+) - x_p), (\gamma(t_p^+) - x_p)\rangle \geq \varepsilon \,,$$

so that, letting $p \to +\infty$, we get

$$\langle D^2\delta(\gamma(\bar{t}^+))(\gamma(\bar{t}^+) - x), (\gamma(\bar{t}^+) - x)\rangle \geq \varepsilon \,.$$

This contradicts the fact that the restriction of γ to $[t^+, \bar{t}^+]$ is straight line contained in $\partial\mathbf{C}$. Therefore we have proved that t^+ is continuous on the set $\{\gamma(0), \ \gamma \in \Gamma_\varepsilon^n\}$, which easily implies that Γ_ε^n is compact.

If we set

$$\mathcal{T}_\varepsilon^{0,n} = \{(\gamma(t), t) \in \partial\mathbf{C} \times (0,1) \ ; \ \gamma \in \Gamma_\varepsilon^n, \ t = t^+(\gamma(0))\}$$

and

$$\mathcal{T}_\varepsilon^n = \{(\gamma(t), t) \in \partial\mathbf{C} \times (0,1) \ ; \ \gamma \in \Gamma_\varepsilon^n, \ t \in [t^+(\gamma(0)), t^+(\gamma(0)) + \frac{\varepsilon}{2}]\}$$

then it is clear that $\mathcal{T}_\varepsilon^{0,n}$ and $\mathcal{T}_\varepsilon^n$ are compact and that $\mathcal{T}_\varepsilon^0 = \bigcup_n \mathcal{T}_\varepsilon^{0,n}$ while $\mathcal{T}_\varepsilon = \bigcup_n \mathcal{T}_\varepsilon^n$.
$\square$

Let us also point out for later use that, because of Lemma 3.2,

$$\lim_{\varepsilon \to 0^+} \hat{f}_\pi^+((\partial\mathbf{C} \times (0,1))\backslash\mathcal{T}_\varepsilon^0) = 0 \,. \tag{10}$$

We intend to show that $D^\tau\hat{u} = D^\tau\check{u}$ is Lipschitz continuous on $\mathcal{T}_\varepsilon$. To do so we are going to check that $\hat{u}$ is semi-concave on $\mathcal{T}_\varepsilon$ while $\check{u}$ is semi-convex on this set. Our first step consists in proving that one can "touch" the map $\hat{u}$ from above by a smooth map at any point $(\bar{y}, \bar{t}) \in \mathcal{T}_\varepsilon^0$.

Lemma 3.4 *There are $\eta_\varepsilon > 0$ and $C_\varepsilon > 0$, depending only on ε and on the obstacle $\mathbf{C}$ and, for any $(\bar{y}, \bar{t}) \in \mathcal{T}_\varepsilon^0$, there is a smooth map $\bar{\phi} : \partial\mathbf{C} \to \mathbb{R}$ such that $\hat{u}(\bar{y}, \bar{t}) = \bar{\phi}(\bar{y}, \bar{t})$, $\mathrm{Lip}\left(D_{x,t}^\tau\bar{\phi}\right) \leq C_\varepsilon$ and*

$$\hat{u}(y, t) \leq \bar{\phi}(y, t) \qquad \forall(y, t) \in (\partial\mathbf{C} \times (0,1)) \cap B((\bar{y}, \bar{t}), \eta_\varepsilon) \,. \tag{11}$$

Remark 3.5 This result is somewhat unexpected, because it says that $\hat{u}$ enjoys some "tangential semiconcavity" property. Recall however that $\hat{u}$ is not semiconcave in a neighborhood on $\partial\mathbf{C}$, as shows the counterexample given in [14].

Proof of Lemma 3.4 : Let $\gamma \in \Gamma_\varepsilon$ be such that $\bar{t} = t^+(\gamma(0))$ and $\bar{y} = \gamma(\bar{t})$. Let us set $\bar{x} = \gamma(0)$.

Let us choose $0 < \eta < \varepsilon$ such that δ is of class $\mathcal{C}^3$ in $B(\bar{y}, \eta)$ and such that

$$\langle D^2\delta(\xi)(z - \bar{x}), (z - \bar{x})\rangle \geq \frac{1}{2}\varepsilon \qquad \forall\xi, z \in B(\bar{y}, \eta) \,. \tag{12}$$

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We note that

$$\hat{u}(y,t) \leq u_0(\bar{x}) + \frac{d^2(\bar{x},y)}{2t} \qquad \forall (y,t) \in \partial\mathbf{C} \times (0,1) , \tag{13}$$

with an equality at $(y,t) = (\bar{y},\bar{t})$. We claim that

$$d(\bar{x},y) \leq |\bar{x} - y| + C_\varepsilon \langle D\delta(y), y - \bar{x}\rangle_+^3 |\bar{x} - y|^3 \qquad \forall y \in \partial\mathbf{C} . \tag{14}$$

Let us fix $y \in \partial\mathbf{C} \cap B(\bar{y},\eta)$. We consider the map $\varphi(s) = \delta((1-s)\bar{x} + sy)$. It satisfies $\varphi(1) = 0$. Let us first assume that $\varphi'(1) = \langle D\delta(y), y - \bar{x}\rangle \leq 0$. Then the segment $]\bar{x}, y[$ is contained in Ω, so that equality $|y - \bar{x}| = d(\bar{x},y)$ holds. In this case (14) is obvious.

We now assume that $\langle D\delta(y), y - \bar{x}\rangle > 0$. Then, by convexity of $\mathbf{C}$ there is a unique $\theta \in (0,1)$ such that the point $z = (1-\theta)\bar{x} + \theta y$ belongs to $\partial\mathbf{C}$. The sign of φ varies as follows:

$$\varphi(s) > 0 \; \forall s \in [0,\theta), \quad \varphi(s) < 0 \; \forall s \in (\theta,1), \quad \varphi(\theta) = \varphi(1) = 0. \tag{15}$$

Note that, because of (12), we have $\varphi''(t) \geq \varepsilon/2$ on $[1 - \eta/|\bar{y} - \bar{x}|, 1]$. On this interval we have

$$\varphi(s) \geq \varphi(1) - \varphi'(1)(1-s) + 1/(4\varepsilon)(1-s)^2 \geq -\langle D\delta(y), y - \bar{x}\rangle(1-s) + (1-s)^2\varepsilon/4 . \tag{16}$$

Taking $\eta_\varepsilon < \eta$ sufficiently small and recalling $\langle D\delta(\bar{y}), \bar{y} - \bar{x}\rangle = 0$, we may assume $\eta/|\bar{y} - \bar{x}| > 4\langle D\delta(y), y - \bar{x}\rangle/\varepsilon$ for any $y \in B(\bar{y}, \eta_\varepsilon)$. Then (16) implies that $\varphi(s) > 0$ as soon as $1 - \eta/|\bar{y} - \bar{x}| \leq s < 1 - 4\langle D\delta(y), y - \bar{x}\rangle/\varepsilon$. This information together with (15) gives

$$1 - \theta \leq 4\langle D\delta(y), y - \bar{x}\rangle/\varepsilon . \tag{17}$$

Then, using Lemma 2.3, we get

$$\begin{aligned} d(\bar{x},y) \leq |z - \bar{x}| + d(z,y) &\leq & |z - \bar{x}| + |z - y| + C_\Omega|z - y|^3 \\ &\leq & |y - \bar{x}| + C_\Omega(1-\theta)^3|y - \bar{x}|^3 \\ &\leq & |y - \bar{x}| + C_\varepsilon\langle D\delta(y), y - \bar{x}\rangle^3|y - \bar{x}|^3 \end{aligned}$$

This proves (14) in the case $\langle D\delta(y), y - \bar{x}\rangle > 0$.

Combining (13) with (14), we get

$$\hat{u}(y,t) \leq u_0(\bar{x}) + \frac{(|\bar{x} - y| + C_\varepsilon\langle D\delta(y), y - \bar{x}\rangle_+^3 |\bar{x} - y|^3)^2}{2t}$$

for any $(y,t) \in \partial\mathbf{C} \times (0,1) \cap B((\bar{y},\bar{t}), \eta_\varepsilon)$, with an equality at $(y,t) = (\bar{y},\bar{t})$. We note that the right-hand side of the above inequality is smooth in a neighborhood of $(\bar{y},\bar{t})$ of size $\varepsilon/2$, and this proves the result. $\square$

Next we extend the previous Lemma to points in $\mathcal{T}_\varepsilon$.

Lemma 3.6 *There are $\eta_\varepsilon > 0$ and $C_\varepsilon > 0$, depending only on ε and on the obstacle $\mathbf{C}$ and, for any $(y,t) \in \mathcal{T}_\varepsilon$, there is a smooth map $\phi : \partial\mathbf{C} \to \mathbb{R}$ such that $\hat{u}(y,t) = \phi(y,t)$, $\mathrm{Lip}\left(D_{x,t}^\tau \phi\right) \leq C_\varepsilon$ and*

$$\hat{u}(z,s) \leq \phi(z,s) \qquad \forall (z,s) \in (\partial\mathbf{C} \times (0,1)) \cap B((y,t), \eta_\varepsilon) .$$

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Proof : Let $(y, t) \in \mathcal{T}_\varepsilon$. From the definition of $\mathcal{T}_\varepsilon$, there is some $\gamma \in \Gamma_\varepsilon$ such that $\gamma(t) = y$. Let $\bar{x} = \gamma(0)$, $\bar{t} = t^+(\bar{x})$ and $\bar{y} = \gamma(\bar{t})$. From Lemma 3.4 there is some smooth map $\bar\phi$ such that $\hat{u}(\bar{y}, \bar{t}) = \bar\phi(\bar{y}, \bar{t})$, $\mathrm{Lip}\left(D_{x,t}^\tau \bar\phi\right) \le \bar{C}_\varepsilon$ and

$$\hat{u}(y, t) \le \bar\phi(y, t) \qquad \forall (y, t) \in (\partial\mathbf{C} \times (0, 1)) \cap B((\bar{y}, \bar{t}), \bar\eta_\varepsilon) \ ,$$

where $\bar\eta_\varepsilon, \bar{C}_\varepsilon$ only depend on ε and $\partial\mathbf{C}$. Let $\xi(z, \sigma)$ be the map which associates to any point $z \in \partial\mathbf{C}$ in a neighborhood of y the solution at time τ of the geodesic flow on $\partial\mathbf{C}$ starting from z with an initial velocity given by the projection over $T_z\partial\mathbf{C}$ of $-\gamma'(t)$. For $\sigma = t - \bar{t}$, we simply abbreviate $\xi(z, t - \bar{t})$ by $\xi(z)$: $\xi(z) = \xi(z, t - \bar{t})$. We note that

$$\hat{u}(z, s) \le \hat{u}(\xi(z), s - t + \bar{t}) + \frac{d^2(\xi(z), z)}{2(t - \bar{t})} \ .$$

If we choose η_ε sufficiently small, we have that $(\xi(z), s - t + \bar{t}) \in B((\bar{y}, \bar{t}), \bar\eta_\varepsilon)$ for any $(z, s) \in B((y, t), \eta_\varepsilon)$ because $\xi(y) = \bar{y}$. So

$$\hat{u}(z, s) \le \bar\phi(\xi(z), s - t + \bar{t}) + \frac{d^2(\xi(z), z)}{2(t - \bar{t})} \qquad \forall (z, s) \in \partial\mathbf{C} \cap B((y, t), \eta_\varepsilon) \ .$$

It remains to show that the right-hand side $\phi = \phi(z, s)$ is smooth with $\mathrm{Lip}\left(D_{x,t}^\tau \phi\right) \le C_\varepsilon$ in $\partial\mathbf{C} \cap B((y, t), \eta_\varepsilon)$. This is clear for the term $(z, s) \to \phi(\xi(z), s - t + \bar{t})$. Let us now consider the term $(z, s) \to \frac{d^2(\xi(z), z)}{2(t - \bar{t})}$. Note that the projection over $T_z\partial\mathbf{C}$ of $-\gamma'(t)$ is given by $-\gamma'(t) + \langle D\delta(z), \gamma'(t)\rangle D\delta(z)$. Then as $\tau \mapsto \xi(z, \tau)$ is a constant speed geodesic between z and $\xi(z)$ we have:

$$d(\xi(z), z) = (t - \bar{t}) | -\gamma'(t) + \langle D\delta(z), \gamma'(t)\rangle D\delta(z)| \ ,$$

so that

$$\frac{d^2(\xi(z), z)}{2(t - \bar{t})} = (t - \bar{t}) \frac{|\gamma'(t)|^2 - |\langle D\delta(z), \gamma'(t)\rangle|^2}{2} \ .$$

The right-hand side is of class $\mathcal{C}^2$, with a $\mathcal{C}^2$ norm depending only on $\partial\mathbf{C}$, thanks to the $\mathcal{C}^3$ regularity of δ. $\square$

Next we show that the map $\check{u}$ can be touched from below by smooth maps at point of $\mathcal{T}_\varepsilon$.

Lemma 3.7 *There are $\eta_\varepsilon, C_\varepsilon > 0$ such that, for any $(y, t) \in \mathcal{T}_\varepsilon$, there is some smooth map ψ such that $\check{u}(y, t) = \psi(y, t)$, $\mathrm{Lip}\left(D_{x,t}^\tau \psi\right) \le C_\varepsilon$ and*

$$\check{u}(z, s) \ge \psi(z, s) \qquad \forall (z, s) \in (\partial\mathbf{C} \times (0, 1)) \cap B((y, t), \eta_\varepsilon) \ .$$

Proof : Let $(y, t) \in \mathcal{T}_\varepsilon$, $\gamma \in \Gamma_\varepsilon$ such that $\gamma(t) = y$ and let us set $\bar{t}_\varepsilon = \tau^+(\gamma(0)) + \varepsilon$ and $\bar{y}_\varepsilon = \gamma(\bar{t}_\varepsilon)$. Then

$$\check{u}(z, s) \ge \check{u}(\bar{y}_\varepsilon, \bar{t}_\varepsilon) - \frac{d^2(z, \bar{y}_\varepsilon)}{2(\bar{t}_\varepsilon - s)} \qquad \forall (z, s) \in B((y, t), \eta_\varepsilon)$$

provided $\eta_\varepsilon \in (0, \varepsilon/4)$ is sufficiently small. Then the map $(z, s) \to \frac{d^2(z, \bar{y}_\varepsilon)}{2(\bar{t}_\varepsilon - s)}$ is smooth with a $\mathcal{C}^2$ derivative bounded by some C_ε, which proves our claim. $\square$

Lemma 3.8 *The map $D_{x,t}^\tau \hat{u}$ is locally Lipschitz continuous on $\mathcal{T}_\varepsilon$.*

Remark 3.9 The difficult part of the result (the Lipschitz continuity of $D_{x,t}^\tau u$ on $\mathcal{T}_\varepsilon^0$) plays a key role for proving the absolute continuity of $\hat{f}_\pi^+$. Its more standard part (the propagation of this Lipschitz continuity to $\mathcal{T}_\varepsilon$) is used in the construction of the optimal transport plan between $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ (section 4).

Proof : Let us fix some point $(\bar{y}, \bar{t}) \in \partial\mathbf{C} \times (0,1)$. For any point $(y,t) \in \mathcal{T}_\varepsilon \cap B((\bar{y}, \bar{t}), \eta_\varepsilon/2)$, there are $\phi_{y,t}$ and $\psi_{y,t}$ such that

$$\check{u}(z,s) \geq \psi_{y,t}(z,s), \ \hat{u}(z,s) \leq \phi_{y,t}(z,s), \ \text{Lip}\,(D_{x,t}\phi_{y,t}), \text{Lip}\,(D_{x,t}\psi_{y,t}) \leq C_\varepsilon$$

for all $(z,s) \in (\partial\mathbf{C} \times (0,1)) \cap B((y,t), \eta_\varepsilon/2)$. Since $\check{u} \leq \hat{u}$ we get

$$\Phi(z,s) := \inf_{y,t} \phi_{y,t}(z,s) \geq \hat{u}(z,s) \geq \check{u}(z,s) \geq \sup_{y,t} \psi_{y,t}(z,s) =: \Psi(z,s)$$

(where the supremum and the infimum are taken on $\mathcal{T}_\varepsilon \cap B((\bar{y}, \bar{t}), \eta_\varepsilon/2)$) with an equality on $\mathcal{T}_\varepsilon \cap B((\bar{y}, \bar{t}), \eta_\varepsilon/2)$ (thanks to Lemma 2.5, ii)). Since Ψ is semi-convex and Φ is semi-concave, Ilmanen Lemma [20] states that there is a $\mathcal{C}^{1,1}$ map Ξ such that

$$\Psi(z,s) \leq \Xi(z,s) \leq \Phi(z,s) \qquad \forall (z,s) \in (\partial\mathbf{C} \times (0,1)) \cap B((\bar{y}, \bar{t}), \eta_\varepsilon/2) \ .$$

Hence $D_{x,t}^\tau \hat{u}$ coincides with $D_{x,t}^\tau \Xi$ on $\mathcal{T}_\varepsilon \cap B((\bar{y}, \bar{t}), \eta_\varepsilon/2)$, which proves that $D_{x,t}^\tau \hat{u}$ is Lipschitz continuous on this set. $\square$

Corollary 3.10 *There is a Borel measurable set $\mathcal{T}^0$ on which $\hat{f}_\pi^+$ is concentrated and on which $D_{x,t}^\tau \hat{u}$ has the countable Lipschitz property: there are compacts sets (K_n) such that $\mathcal{T}^0 = \bigcup_n K_n$ and $D_{x,t}^\tau \hat{u}$ is Lipschitz continuous on each K_n.*

Proof : Indeed, we know from Lemma 3.8 that $D_{x,t}^\tau \hat{u}$ is locally Lipschitz continuous on $\mathcal{T}_\varepsilon$ and we also know that

$$\lim_{\varepsilon \to 0^+} \hat{f}_\pi^+((\partial\mathbf{C} \times (0,1)) \backslash \mathcal{T}_\varepsilon^0) = 0 \ .$$

So it is just enough to write the increasing union $\bigcup_n \mathcal{T}_{1/n}$ as an enumerable union of compact sets (K_n), each K_n being contained in some $\mathcal{T}_{1/n}$. $\square$

3.3 Absolute continuity of $\hat{f}_\pi^+$

Let us recall that $\hat{u}$ is differentiable on the Borel set $\pi^+(E^+)$. We define the map $\theta^+ : \pi^+(E^+) \to \Omega$ by

$$\theta^+(y,t) = y - tD^\tau \hat{u}(y,t) \ .$$

Lemma 3.11 *We have*

$$\theta^+ \circ \pi^+(x) = x \qquad \forall x \in E^+$$

and θ^+ has the countable Lipschitz property on the set $\mathcal{T}^0$ defined in Corollary 3.10.

Proof :   Let $x \in E^+$ and $(y,t) = \pi^+(x)$. Then we know from Lemma 2.11 that $D^\tau \hat{u}(y,t) = Du_0(x)$. Since moreover $t = t^+(x)$, $y = x + t^+(x)Du_0(x)$, we easily gets $x = y - tD^\tau \hat{u}(y,t) = \theta^+(y,t)$.

The second assertion is a straightforward application of Corollary 3.10. $\square$

As a consequence, we have:

Corollary 3.12 $\hat{f}_\pi^+$ *is absolutely continuous with respect to* $\mathcal{H}^N \lfloor (\partial \mathbf{C} \times (0,1))$.

Proof :   Let (K_n) be an increasing sequence of compact subsets of $\mathcal{T}^0$ such that θ^+ is Lipschitz continuous on K_n and $\lim_n \hat{f}_\pi^+((\partial \mathbf{C} \times (0,1)) \backslash K_n) = 0$. Let $E \subset \partial \mathbf{C} \times (0,1)$ be of zero measure for $\mathcal{H}^N$. Let us set $E_n = E \cap K_n$. Then

$$\hat{f}_\pi^+(E_n) = \hat{f}^+((\pi^+)^{-1}(E_n)) = \hat{f}^+(\theta^+(E_n))$$

because $\theta^+ \circ \pi^+(x) = x$ for any $x \in E^+$. But

$$\mathcal{L}^N(\theta^+(E_n)) \leq \mathrm{Lip}(\theta^+_{|K_n})\mathcal{H}^N(E_n) = 0 \ ,$$

where $\mathrm{Lip}(\theta^+_{|K_n})$ is the Lipschitz constant of the restriction of θ^+ to K_n. So $\hat{f}^+(\theta^+(E_n)) = 0$ because $\hat{f}^+$ is absolutely continuous. Therefore $\hat{f}_\pi^+(E) = \sup_n \hat{f}_\pi^+(E_n) = 0$. $\square$

4   A transport map between $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$

Lemma 4.1 *There is a Borel measurable map* $\tilde{T} = (\tilde{T}_x, \tilde{T}_t) : \partial \mathbf{C} \times (0,1) \to \partial \mathbf{C} \times (0,1)$ *such that*

$$\tilde{T} \sharp \hat{f}_\pi^+ = \hat{f}_\pi^-$$

and such that

$$\hat{u}(\tilde{T}(x,s)) - \hat{u}(x,s) = c((x,s),\tilde{T}(x,s)) \qquad \text{for } \hat{f}_\pi^+ - \text{a.e. } (x,s) \in \partial \mathbf{C} \times (0,1) \ .$$

The idea is to build $\tilde{T}$ as an optimal transport between $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$. This construction is now standard (see for instance [19], [4], or [26] and the reference therein for more details). The only (small) difference here is that our cost c is unusual.

Proof :   In the proof, balls are always *geodesic* balls, but are still denoted $B(\xi, r)$.
Step 1: Let $\tilde{Z}$ and $\tilde{c}((\cdot, \cdot), (\cdot, \cdot))$ be

$$\tilde{Z} = \left\{ ((x,s),(y,t)) \in (\partial \mathbf{C} \times (0,1))^2 \text{ with } \hat{u}(y,t) - \hat{u}(x,s) = c((x,s),(y,t)) \right\} \ ,$$

$$\tilde{c}((x,s),(y,t)) = \begin{cases} [c((x,s),(y,t))]^2 & \text{if } ((x,s),(y,t)) \in \tilde{Z}, \\ +\infty & \text{otherwise.} \end{cases}$$

Following [4] we now consider the transport problem

$$\min_{\pi \in \Pi(\hat{f}_\pi^+, \hat{f}_\pi^-)} \int_{(\partial \mathbf{C} \times (0,1))^2} \tilde{c}((x,s),(y,t))d\pi((x,s),(y,t)). \tag{18}$$

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We first claim that the integral is finite for the transfer plan $\pi = (\pi^+, \pi^-)\sharp\mu_0$ where (π^+, π^-) stands for the map $(x, y) \mapsto (\pi^+(x), \pi^-(x, y))$. Indeed, by definition of $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ we have $\pi \in \Pi(\hat{f}_\pi^+, \hat{f}_\pi^-)$. Moreover for μ_0 a.e. $(x, y) \in \Omega^2$ we have

$$\hat{u}(y, 1) - \hat{u}(x, 0) = c((x, 0), (y, 1)) .$$

So, from Lemma 2.10, we get

$$\hat{u}(\pi^-(x, y)) - \hat{u}(\pi^+(x)) = c(\pi^+(x), \pi^-(x, y)) \qquad \text{for } \mu_0-\text{a.e. } (x, y) \in \Omega^2 .$$

This in turn implies that

$$\hat{u}(y, t) - \hat{u}(x, s) = c((x, s), (y, t)) \qquad \text{for } \pi-\text{almost all } ((x, s), (y, t)) \in (\partial\mathbf{C} \times (0, 1))^2 .$$

So π provides a finite value in the new transport problem. Then it is known that problem (18) has a solution which we denote by $\tilde{\pi}$ (see for instance [1] Theorem 2.1).

It remains to show that $\tilde{\pi}$ is concentrated on a graph. For this we use a strategy of proof initiated by Champion & De Pascale [11].

Step 2: Recall first that $\tilde{\pi}$ is concentrated on a monotone set for $\tilde{c}$ (see for instance [1], Theorem 2.2.). Namely there is a Borel set $\mathcal{Y} \subset (\partial\mathbf{C} \times (0, 1))^2$ of full measure for $\tilde{\pi}$ such that, for any $((x^j, s^j), (y^j, t^j)) \in \mathcal{Y}$ (where $j = 1, 2$), we have

$$\tilde{c}((x^1, s^1), (y^1, t^1)) + \tilde{c}((x^2, s^2), (y^2, t^2)) \leq \tilde{c}((x^1, s^1), (y^2, t^2)) + \tilde{c}((x^2, s^2), (y^1, t^1)) . \quad (19)$$

In order to give sense to this property and be able to use it in step 4, we show some properties of $\mathcal{Y}$.

Property 1: Let $((\bar{x}, \bar{s}), (\bar{y}, \bar{t})) \in \mathcal{Y}$. Then the constant speed geodesic $\gamma : [\bar{s}, \bar{t}] \to \partial\mathbf{C}$ such that $\gamma(\bar{s}) = \bar{x}$, $\gamma(\bar{t}) = \bar{y}$ satisfies:

$$\forall \bar{s} \leq s \leq t \leq \bar{t}, \quad ((\gamma(s), s), (\gamma(t), t)) \in \tilde{Z} . \quad (20)$$

Moreover,

$$\hat{u} \text{ is differentiable at } (\gamma(s), s) \text{ for any } s \in [\bar{s}, \bar{t}], \text{ with } D^\tau \hat{u}(\gamma(s), s) = \gamma'(s). \quad (21)$$

Proof of Property 1: Arguing as in the proof of Lemma 2.5 we get (20) and (21) inside $]\bar{s}, \bar{t}[$. Moreover, as $(\bar{x}, \bar{s})$ belongs to $\mathcal{T}$, by Lemma 2.5, $\hat{u}$ is differentiable at $(\bar{x}, \bar{s})$ and we also have (21) for $s = \bar{s}$. Indeed, notice that thanks to (20) we have:

$$\hat{u}(\gamma(\bar{s} + h), \bar{s} + h) - \hat{u}(\gamma(\bar{s}), \bar{s}) = h\frac{|\gamma'(s)|^2}{2},$$

then using Proposition 2.1 we get the claim. The same holds at $(\bar{y}, \bar{t})$.

Property 2: Let $\tau > 0$ and $((x^1, s^1), (y^1, t^1)) \in \tilde{Y}$. Let $\gamma : [s^1, t^1] \to \partial\mathbf{C}$ be the geodesic curve defined by Property 1 and $((\gamma(s^1 + \tau), s^1 + \tau), (y^2, t^2)) \in \mathcal{Y}$ for some $\tau > 0$ with

$s^1 + \tau \le t^1$.

Then γ can be extended to $[s^1, \max\{t^1, t^2\}]$ in such way that $\gamma(t^2) = y^2$ and the new γ satisfies the properties (20) and (21) on $[s^1, \max\{t^1, t^2\}]$.

Proof of Property 2: Let $\tilde{\gamma}$ a constant speed geodesic curve such that $\tilde{\gamma}(s^1 + \tau) = \gamma(s^1 + \tau)$ and $\tilde{\gamma}(t^2) = y_2$. Then applying Property 1 to γ and $\tilde{\gamma}$, we get $\gamma'(s^1 + \tau) = \tilde{\gamma}'(s^1 + \tau)$. Therefore (γ, γ') and $(\tilde{\gamma}, \tilde{\gamma}')$ both satisfy the ODE given in Lemma 2.2 (up to a re-parametrization) on $[s^1 + \tau, \min\{t^1, t^2\}]$ with the same initial condition at $s^1 + \tau$. So $\gamma = \tilde{\gamma}$ on $[s^1 + \tau, \min\{t^1, t^2\}]$. Property 2 follows by gluing together γ and $\tilde{\gamma}$ and by using again Property 1.

As a consequence of Properties 1 and 2 and (19), we have:

Property 3: Let $\tau > 0$, $s^2 = s^1 + \tau$ and t^1, t^2 such that $s^1 \le t^2$, $s^2 \le t^1$ and the following holds:

i) $((x^j, s^j), (y^j, t^j)) \in \mathcal{Y}$, $j = 1, 2$

ii) there exists a constant speed geodesic γ such that

$$D^\tau u(x^1, s^1) = \gamma'(s^1), \ \gamma(s^1) = x^1, \ \gamma(s^2) = x^2.$$

Then we have:

$$(s^2 - s^1)(t^2 - t^1) \ge 0 \ . \tag{22}$$

Step 3: Let us denote by $\mathcal{Y}^{-1}(y, t) = \{(x, s) \in \mathcal{Y} \ , \ ((x, s), (y, t)) \in \mathcal{Y}\}$. By Lemma 4.3 of [11], $\tilde{\pi}$ is concentrated on a $\sigma-$compact subset $R(\mathcal{Y})$ of $\mathcal{Y}$ such that for any $((x, s), (y, t)) \in R(\mathcal{Y})$, (x, s) is a Lebesgue point of $\mathcal{Y}^{-1}(B((y, t), r)$:

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^N(\mathcal{Y}^{-1}(B((y, t), r)) \cap B((\bar{x}, \bar{t}), \rho))}{\mathcal{H}^N(B((\bar{x}, \bar{t}), \rho))} = 1 \ .$$

Recall that $\hat{f}_\pi^+$ is concentrated on the set $\mathcal{T}^0 = \bigcup_n K_n$, where, for each n, K_n is compact and contained in some $\mathcal{T}_\varepsilon$.

Step 4: Let us fix some n and let $(\bar{x}, \bar{s})$ be a Lebesgue point of $K_n \cap \mathcal{T}^0$. Let us also assume that there are $(\bar{y}_1, \bar{t}_1) \ne (\bar{y}_2, \bar{t}_2)$ with $((\bar{x}, \bar{s}), (\bar{y}_i, \bar{t}_i)) \in R(\mathcal{Y})$. We are going to show that this assumption leads to a contradiction.

First note that $\bar{t}_1 \ne \bar{t}_2$, say $\bar{t}_1 < \bar{t}_2$. Let us choose $r < \frac{d((\bar{y}_1, \bar{t}_1), (\bar{y}_2, \bar{t}_2))}{2}$ and let us set, for any $\rho > 0$,

$$\Theta_\rho = \left\{(x, s) \in \mathcal{T}_\varepsilon^0 \cap K_n \cap B((\bar{x}, \bar{s}), \rho) \ , \ \exists(y_i, t_i) \in B((\bar{y}_i, \bar{t}_i), r) \text{ with } ((x, s), (y_i, t_i)) \in \mathcal{Y}, \ i = 1, 2\right\} \ .$$

Then, since $(\bar{x}, \bar{s})$ is a Lebesgue point of K_n, of $\mathcal{Y}^{-1}(B((\bar{y}_1, \bar{t}_1), r))$ and of $\mathcal{Y}^{-1}(B((\bar{y}_2, \bar{t}_2), r))$, we have

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^N(\Theta_\rho)}{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho))} = 1 \ . \tag{23}$$

For $\tau \in (0, \varepsilon/4)$, let $\Phi_\tau : \mathcal{T}_\varepsilon^0 \to \mathcal{T}_\varepsilon$ be the map which associates to any $(x, s) \in \mathcal{T}_\varepsilon^0$ the pair $(\gamma(\tau), s + \tau)$ where γ is the geodesic on $\partial \mathbf{C}$ starting from x with direction $D^\tau \hat{u}(x, s)$. We note that Φ_τ is one-to-one and that its inverse is Lipschitz continuous with a constant independent of τ. Indeed $\Phi_\tau^{-1}(y, t)$ is the pair $(\tilde{\gamma}(\tau), t - \tau)$ where $\tilde{\gamma}$ is the geodesic on $\partial \mathbf{C}$ starting from y with direction $-D^\tau \hat{u}(y, t)$; $D^\tau \hat{u}$ being Lipschitz continuous on $\mathcal{T}_\varepsilon$, this map is Lipschitz continuous. So

$$\mathcal{H}^N(\Phi_\tau(\Theta_\rho)) \geq \operatorname{Lip}(\Phi_\tau^{-1}) \mathcal{H}^N(\Theta_\rho) . \tag{24}$$

Since $D^\tau \hat{u}$ is globally bounded, there is some κ such that $\Phi_\tau(\Theta_\rho) \subset B((\bar{x}, \bar{s}), \rho + \kappa \tau)$. Let us choose $\tau = \rho^2$. Then as $\frac{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho))}{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho + \kappa \rho^2))} \to 1$ when $\rho \to 0$, combining (23) with (24) and recalling $\operatorname{Lip}(\Phi_\tau^{-1})$ is independent of τ, we get:

$$\lim_{\rho \to 0} \frac{\mathcal{H}^N(\Phi_{\rho^2}(\Theta_\rho))}{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho + \kappa \rho^2))} \geq \operatorname{Lip}(\Phi_\tau^{-1}). \tag{25}$$

On the other hand using (23) with $\rho + \kappa \rho^2$ instead of ρ:

$$\lim_{\rho \to 0} \frac{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho + \kappa \rho^2) \backslash \Theta_{\rho + \kappa \rho^2})}{\mathcal{H}^N(B((\bar{x}, \bar{s}), \rho + \kappa \rho^2))} = 0.$$

Since $\Phi_{\rho^2}(\Theta_\rho) \subset B((\bar{x}, \bar{s}), \rho + \kappa \rho^2)$, this last equality combined with (25) shows that there is some $\rho > 0$ such that $\Phi_{\rho^2}(\Theta_\rho) \cap \Theta_{\rho + \kappa \rho^2} \neq \emptyset$.

Let $(x^2, s^2) \in \Phi_{\rho^2}(\Theta_\rho) \cap \Theta_{\rho + \kappa \rho^2}$ and $(x^1, s^1) \in \Theta_\rho$ be such that $(x^2, s^2) = \Phi_{\rho^2}(x^1, s^1)$. By definition of Θ_ρ and of $\Theta_{\rho + \kappa \rho^2}$, there are some $(y_i^j, t_i^j) \in B((\bar{y}_i, \bar{t}_i), r)$ with $((x^j, s^j), (y_i^j, t_i^j)) \in \mathcal{Y}$ for $i = 1, 2$ and $j = 1, 2$. Then Property 3 applies to $((x^1, s^1), (y_2^1, t_2^1))$ and $((x^2, s^2), (y_1^2, t_1^2))$. So:

$$(s^2 - s^1)(t_1^2 - t_2^1) \geq 0.$$

Since $s^1 < s^2$ and $t_1^2 < t_2^1$, we get a contradiction.

Accordingly we have proved that, for $\hat{f}_\pi^+ -$almost all points $(\bar{x}, \bar{s})$ there is at most one point $(\bar{y}, \bar{t})$ such that $((\bar{x}, \bar{s}), (\bar{y}, \bar{t})) \in R(\mathcal{Y})$, which just means that $\tilde{\pi}$ is concentrated on the graph $R(\mathcal{Y})$ of a Borel measurable map $\tilde{T}$. $\square$

5 Construction of an optimal transport

Let us define the Borel measurable map $\theta^- : \partial \mathbf{C} \times (0, 1) \to \Omega$ by

$$\theta^-(y, t) = y + (1 - t) D^\tau \hat{u}(y, t) \qquad \text{if } D^\tau \hat{u}(y, t) \text{ exists}$$

and $\theta^-(y, t) = x_0$ for some fixed $x_0 \in \Omega$ otherwise.

Lemma 5.1 *We have*

$$\theta^- \circ \pi^-(x, y) = y \qquad \forall (x, y) \in Z_0$$

and

$$u_1(\theta^-(z, t)) = \hat{u}(z, t) + \frac{d^2(z, \theta^-(z, t))}{2(1 - t)} \qquad \forall (z, t) \in \pi^-(Z_0) . \tag{26}$$

Proof: Let $(z, t) \in \pi^-(Z_0)$. Let $(x, y) \in Z_0$ be such that $\pi^-(x, y) = (z, t)$ and let $\gamma \in \Gamma$ such that $\gamma(0) = x$, $\gamma(1) = y$. From Lemma 2.5 we know that $\hat{u}$ is differentiable at (z, t) and, from Lemma 2.11 that

$$D^\tau \hat{u}(z, t) = \frac{y - z}{(1 - t)} \ .$$

Hence

$$y = z + (1 - t)D^\tau \hat{u}(z, t) = \theta^-(z, t) \ .$$

From Lemma 2.5, we have, keeping the above notations,

$$u_1(y) = \hat{u}(z, t) + \frac{d^2(z, y)}{2(1 - t)}$$

which gives (26) since $\theta^-(z, t) = y$. $\square$

We now define

$$T(x) = \begin{cases} \theta^- \circ \tilde{T} \circ \pi^+(x) & \text{if } x \in E^+, \\ x + D\hat{u}(x) & \text{if } x \notin E^+ \text{ and } D\hat{u}(x) \text{ exists}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.2 *T is an optimal transport map for $(MK2)$.*

Proof : We first claim that, for $\hat{f}^+$−almost all $x \in E^+$, we have

$$u_1(T(x)) = \hat{u}(\tilde{T} \circ \pi^+(x)) - \frac{d^2(\tilde{T}_x \circ \pi^+(x), T(x))}{2(1 - \tilde{T}_t \circ \pi^+(x))} \ .$$

Proof of the claim : Let us denote by F the Borel set of points $x \in E^+$ such that

$$u_1(T(x)) \neq \hat{u}(\tilde{T} \circ \pi^+(x)) - \frac{d^2(\tilde{T}_x \circ \pi^+(x), T(x))}{2(1 - \tilde{T}_t \circ \pi^+(x))}.$$

and by F' the Borel set of points $(y, s) \in \mathcal{T} \cap (\partial \mathbf{C} \times (0, 1))$ such that

$$u_1(\theta^-(y, s)) \neq \hat{u}(y, s) - \frac{d^2(y, \theta^-(y, s))}{2(1 - s)} \ .$$

Then

$$\hat{f}^+(F) = \hat{f}_\pi^-(F') = \pi^- \sharp \mu_0(F') = \mu(Z_0 \cap (\pi^-)^{-1}(F')) = 0$$

because for any $(x, y) \in Z_0$, $\pi^-(x, y) \in \mathcal{T}$ by (26) in Lemma 5.1. This completes the proof of the claim.

From the claim we have for $\hat{f}^+-$almost every $x \in E^+$,

$$
\begin{aligned}
u_0(x) &= \hat{u}(x,0) \\
&= \hat{u}(\pi^+(x)) - \frac{d^2(x,\pi_x^+(x))}{2\pi_t^+(x)} \\
&= \hat{u}(\tilde{T} \circ \pi^+(x)) - \frac{d^2(\pi_x^+(x), \tilde{T}_x \circ \pi^+(x))}{2(\tilde{T}_t \circ \pi^+(x) - \pi_t^+(x))} - \frac{d^2(x,\pi_x^+(x))}{2\pi_t^+(x)} \\
&\leq \hat{u}(\theta^- \circ \tilde{T} \circ \pi^+(x)) - \frac{d^2(\tilde{T}_x \circ \pi^+(x), \pi^- \circ \tilde{T} \circ \pi^+(x))}{2(1 - \tilde{T}_t \circ \pi^+(x))} - \frac{d^2(x, \tilde{T}_x \circ \pi^+(x))}{2\tilde{T}_t \circ \pi^+(x)} \\
&\leq u_1(T(x)) - \frac{d^2(x,T(x))}{2}.
\end{aligned}
$$

Since the reverse inequality always holds, we get

$$
u_0(x) = u_1(T(x)) - \frac{1}{2}d^2(x,T(x)) \qquad \text{for } \hat{f}^+-\text{almost every } x \in \Omega.
$$

Moreover

$$
T\sharp\hat{f}^+ = (\theta^- \circ \tilde{T} \circ \pi^+)\sharp\hat{f}^+ = (\theta^- \circ \tilde{T})\sharp\hat{f}_\pi^+ = \theta^-\sharp\hat{f}_\pi^- = (\theta^- \circ \pi^-)\sharp\mu_0 = \pi_2\sharp\mu_0 = \hat{f}^-
$$

since from Lemma 5.1 we have $\theta^- \circ \pi^- = \pi_2$.

Next we set $\tilde{f}^+ = \pi_1\sharp\mu_1$ and $\tilde{f}^- = \pi_2\sharp\mu_1$. Then μ_1 is an optimal transport plan between $\tilde{f}^+$ and $\tilde{f}^-$. From standard arguments in the quadratic case (see [8]), we have

$$
u_0(x) = u_1(T(x)) - \frac{d^2(x,T(x))}{2} \qquad \text{for } \tilde{f}^+-\text{almost all } x \in \Omega
$$

and

$$
y = T(x) \qquad \text{for } \mu_1-\text{almost all } (x,y) \in \Omega \times \Omega.
$$

In particular $\mu_1 = (id, T)\sharp\tilde{f}^+$ and $\tilde{f}^- = T\sharp\tilde{f}^+$.

We finally have

$$
u_0(x) = u_1(T(x)) - \frac{d^2(x,T(x))}{2} \qquad \text{for } f^+-\text{almost all } x \in \Omega
$$

and, from (8),

$$
T\sharp f^+ = T_{|\Omega \setminus E^+}\sharp f^+ + T_{|E^+}\sharp f^+ = \tilde{f}^- + \hat{f}^- = f^- .
$$

Therefore T is optimal. $\square$

We finally address the uniqueness issue for the optimal transport map. For this we work under the additional assumption that f^- is also absolutely continuous with respect to the Lebesgue measure. We discuss this condition below. In this case, the measures $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ are intrinsic.

Lemma 5.3 *If f^- is absolutely continuous with respect to the Lebesgue measure, then the maps π^+ and θ^+ do not depend on the specific choice of the optimal pair of potential (u_0, u_1) nor on the optimal transport plan μ. This is also the case for the maps π^- and θ^- which moreover only depend on y.*

As a consequence, the measures $\hat{f}_\pi^+$ and $\hat{f}_\pi^-$ do not depend on the specific choice of the optimal pair of potential (u_0, u_1) nor on the optimal transport plan μ.

Proof: Let (u_0, u_1), (u'_0, u'_1) be optimal potentials and μ, μ' be optimal transport plans between f^+ and f^-. Let $(\bar{x}, \bar{y})$ be such that u_0 and u'_0 are differentiable at $\bar{x}$, u_1 and u'_1 are differentiable at $\bar{y}$ and equality

$$u_1(\bar{y}) - u_0(\bar{x}) = u'_1(\bar{y}) - u'_0(\bar{x}) = \frac{1}{2}d^2(\bar{x}, \bar{y})$$

holds. We know that the set of such points is a full measure for μ and μ'. Since

$$u_1(\bar{y}) \leq u_0(x) + \frac{1}{2}d^2(x, \bar{y}) \qquad \forall x \in \Omega$$

with an equality at $\bar{x}$, the semi-concave map $d(\cdot, \bar{y})$ is differentiable at $\bar{x}$, so that

$$Du_0(\bar{x}) = Du'_0(\bar{x}) = -d(\bar{x}, \bar{y})D_x d(\bar{x}, \bar{y}) \ .$$

In particular, the set

$$E^+ = \{\bar{x} \in \Omega \ , \ Du_0(x) \text{ exists and } [\bar{x}, \bar{x} + Du_0(\bar{x})] \cap \mathbf{C} \neq \emptyset\}$$

is defined independently of the choice of (u_0, u_1) and of μ, up to a set of measure 0. For any $x \in E^+$ we set (as before)

$$t^+(\bar{x}) = \min\{t \in [0, 1] \ , \ \bar{x} + tDu_0(\bar{x}) \in \mathbf{C}\}$$

and

$$\pi^+(\bar{x}) = (\bar{x} + t^+(\bar{x})Du_0(\bar{x}), t^+(\bar{x})) \ .$$

Then π^+ is intrinsically defined. So is θ^+, which is just the inverse of π^+.

Arguing in a symmetric way allows to define $\pi^- = \pi^-(y)$ on some set E^- and $\theta^- = \theta^-(y)$ in an intrinsic way. Then we can set

$$\hat{f}^+_\pi = \pi^+\sharp(f^+\mathbf{1}_{E^+}) \qquad \text{and} \qquad \hat{f}^-_\pi = \pi^-\sharp(f^-\mathbf{1}_{E^-}) \ .$$

$\square$

For simplicity, we now assume without loss of generality that $\hat{f}^+_\pi$ and $\hat{f}^-_\pi$ are probability measures (i.e., all the optimaly transported mass passes through $\partial\mathbf{C}$). Then there is a simple one-to-one correspondence between optimal transport plans between f^+ and f^- and optimal transport plans between $\hat{f}^+_\pi$ and $\hat{f}^-_\pi$.

Proposition 5.4 *Let us assume that f^- is absolutely continuous with respect to the Lebesgue measure. Then, for any optimal transport plan ν between $\hat{f}^+_\pi$ and $\hat{f}^-_\pi$, the measure*

$$\mu := (\pi^+, \pi^-)\sharp\nu$$

is an optimal transport plan between f^+ and f^-.

Conversely any optimal transport plan μ between f^+ and f^- defines an optimal transport plan

$$\nu := (\theta^+, \theta^-)\sharp\mu$$

between $\hat{f}^+_\pi$ and $\hat{f}^-_\pi$.

In other words, the lack of uniqueness in the Monge problem just comes from the lack of uniqueness in the transportation of $\hat{f}_\pi^+$ to $\hat{f}_\pi^-$ in $\partial\mathbf{C}$.

Proof: The Proposition is a straightforward consequence of the primal-dual necessary and sufficient optimality condition. $\square$

We conclude this discussion by noting that the above Proposition strongly relies on the fact that f^- is absolutely continuous. Indeed, otherwise, the measure $\hat{f}_\pi^-$ generally depends on the choice of the optimal transport plan μ, as shows the following example:

Example 5.5 *In dimension 2, we assume that the obstacle is a disc K centered at 0, $f^+ = dx \llcorner (C_1 \cup C_2)$ and $f^- = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$ as in the figure. The sets C_1 and C_2 are discs of area $1/2$ and A, B are two points on the $Oy-axis$. The figure is symmetric with respect to the $Oy-axis$. We assume that any geodesic connecting a point of $C_1 \cup C_2$ to A or B passes through the obstacle. Then it can easily be checked that the projected measure $\hat{f}_\pi^-$*
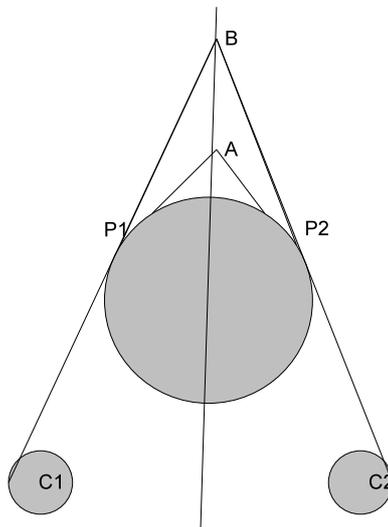
is not intrinsically defined but depends on the choice of the optimal transport plan.

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