Characterization of Sobolev and BV Spaces

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Abstract

The main results of this paper are new characterizations of $W^{1,p}(\Omega)$, $1 < p < \infty$, and $BV(\Omega)$ for $\Omega \subset \mathbb{R}^N$ an arbitrary open set. Using these results, we answer some open questions of Brezis [11] and Ponce [30].

1 Introduction

In the recent paper [6], Bourgain, Brezis, and Mironescu studied the limiting behavior of the semi-norm

$$|f|_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}} \, dydx \right)^{\frac{1}{p}},$$

of the fractional Sobolev spaces $W^{s,p}$, $0 < s < 1$, $1 < p < \infty$. This semi-norm was introduced by Gagliardo in [20], to characterize the space of traces of functions in $W^{1,p}$, $p > 1$. It is well known that $|f|_{W^{s,p}(\Omega)}$ does not converge to

$$|f|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla f|^p \, dx \right)^{\frac{1}{p}}$$

when $s \to 1^-$. Bourgain, Brezis, and Mironescu [6] recognized that this difficulty is a question of scaling. Indeed, they were able to show that when Ω is a smooth, bounded domain,

$$\lim_{s \to 1^-} (1 - s)|f|^p_{W^{s,p}(\Omega)} = K_{p,N} |f|^p_{W^{1,p}(\Omega)}, \tag{1.1}$$

for all $f \in L^p(\Omega)$, $1 < p < \infty$, where $|f|_{W^{1,p}} := \infty$ if $f \notin W^{1,p}(\Omega)$. Here, $K_{p,N} > 0$ only depends on p and N. This important result has been extended in several directions. Maz'ya and Shaposhnikova [26] proved that for $f \in \bigcup_{0<s<1} W_0^{s,p}(\mathbb{R}^N)$,

$$\lim_{s \to 0^+} s|f|^p_{W^{s,p}(\mathbb{R}^N)} = C_{p,N} ||f||^p_{L^p(\mathbb{R}^N)}. \tag{1.2}$$

Kolyada and Lerner [24] extended these results to general Besov spaces $B^s_{p,\theta}$ (see also [23]), while Milman [27] generalized (1.1) and (1.2) to the setting of interpolation spaces, by establishing continuity of the real and complex interpolation spaces at the endpoints.

Another important consequence of (1.1) is that the analysis led to a new characterization of the Sobolev spaces $W^{1,p}(\Omega)$, $1 < p < \infty$.

Consider the family of mollifiers

$$\rho_\epsilon \geq 0, \quad \int_{\mathbb{R}^N} \rho_\epsilon(x)\, dx = 1, \tag{1.3}$$

$$\lim_{\epsilon \to 0} \int_{|x|>\delta} \rho_\epsilon(x)\, dx = 0 \quad \text{for all } \delta > 0, \tag{1.4}$$

$$\rho_\epsilon \text{ is radial, that is, } \rho_\epsilon(x) = \hat{\rho}_\epsilon(|x|), \quad x \in \mathbb{R}^N. \tag{1.5}$$

In [6], Bourgain, Brezis, and Mironescu proved the following result.

Theorem 1.1 ([6], Theorem 2) *Suppose $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, $1 < p < \infty$, and ρ_ϵ satisfy* (1.3), (1.4), *and* (1.5). *Then for $f \in L^p(\Omega)$,*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(|x-y|)\, dydx = K_{p,N} |f|^p_{W^{1,p}(\Omega)}, \tag{1.6}$$

where $|f|_{W^{1,p}}(\Omega) := \infty$ if $f \notin W^{1,p}(\Omega)$.

Note that (1.1) follows from Theorem 1.1 by taking

$$\rho_\epsilon(x) := \chi_{[0,R]}(|x|) \frac{\epsilon c_\epsilon}{|x|^{N-p\epsilon}},$$

where $\epsilon = 1 - s$, $R > 0$ is chosen bigger than the diameter of Ω, and $c_\epsilon \to 1$ as $\epsilon \to 0$.

The case $p = 1$ is a little delicate, and if $f \in W^{1,1}(\Omega)$, then the equality (1.6) holds. However, assuming the left-hand-side of (1.6) is finite is not enough to conclude $f \in W^{1,1}(\Omega)$. The following theorem is the appropriate extension to $p = 1$.

Theorem 1.2 ([6], Theorem 3′) *Suppose $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and ρ_ϵ satisfy* (1.3), (1.4), *and* (1.5). *Then there exist constants $C_1, C_2 > 0$ such that for every $f \in L^1(\Omega)$,*

$$\begin{aligned} C_1 |Df|(\Omega) &\leq \liminf_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \hat{\rho}_\epsilon(|x-y|)\, dydx \\ &\leq \limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \hat{\rho}_\epsilon(|x-y|)\, dydx \leq C_2 |Df|(\Omega), \end{aligned}$$

where $|Df|(\Omega)$ is the total variation of the measure Df, the distributional derivative of f, and $|Df|(\Omega) = +\infty$ if $f \notin BV(\Omega)$.

In one dimension, Bourgain, Brezis, and Mironescu were able to obtain $C_1 = C_2 = 1$, so that the BV semi-norm is actually the limit as in the $W^{1,p}$ case. This limit characterization was completed for $N \geq 2$ independently by Ambrosio [2] and Dávila [17], who proved the following result (see also the recent work of Ambrosio, De Philippis, and Martinazzi [3] for some related results).

Theorem 1.3 ([17], Theorem 1) *Suppose* $\Omega \subset \mathbb{R}^N$ *be open, bounded domain with Lipschitz boundary and* ρ_ϵ *satisfy* (1.3), (1.4), *and* (1.5). *Then for* $f \in L^1(\Omega)$,

$$\lim_{\epsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\hat{\rho}_\epsilon(|x-y|)\,dydx = K_{1,N}|Df|(\Omega), \tag{1.7}$$

where $|Df|(\Omega) = +\infty$ *if* $f \notin BV(\Omega)$.

Note that for *smooth* domains, Theorems 1.1 and 1.3 give new characterizations of the spaces $W^{1,p}(\Omega)$, $1 < p < \infty$, and $BV(\Omega)$. However, these characterizations fail for arbitrary open, bounded sets, as Brezis [11, Remark 5] gives a construction of a bounded open set Ω and a function $f \in W^{1,\infty}(\Omega)$ such that

$$\lim_{\epsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\hat{\rho}_\epsilon(|x-y|)\,dydx = +\infty,$$

where ρ_ϵ satisfy (1.3), (1.4), and (1.5) (see a related construction in Theorem 1.14). Thus $f \in W^{1,p}(\Omega)$ for every p, and yet the iterated integral is infinite. In this construction, Ω is specifically chosen such that points close with respect to the Euclidean distance are far with respect to the geodesic distance d_Ω in Ω. This leads to the following questions of Brezis [11] and Ponce [30].

Open Question 1 (Brezis, [11])

For $\Omega \subset \mathbb{R}^N$ open and ρ_ϵ satisfying (1.3), (1.4), and (1.5), does $f \in L^p(\Omega)$ and

$$\limsup_{\epsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{d_\Omega(x,y)^p}\hat{\rho}_\epsilon(d_\Omega(x,y))\,dydx < +\infty \tag{1.8}$$

imply that $f \in W^{1,p}(\Omega)$?

Open Question 2 (Ponce, [30])

For $\Omega \subset \mathbb{R}^N$ open and ρ_ϵ satisfy (1.3), (1.4), and (1.5), does $f \in L^p(\Omega)$ and

$$\limsup_{\epsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\hat{\rho}_\epsilon(d_\Omega(x,y))\,dydx < +\infty \tag{1.9}$$

imply that $f \in W^{1,p}(\Omega)$?

The main purposes of this paper are to provide answers to these questions and to give a characterization of the spaces $W^{1,p}(\Omega)$, $1 < p < \infty$ and $BV(\Omega)$ for arbitrary domains.

Following the work of Ponce [30], we replace the hypothesis that ρ_ϵ are radial with a weaker condition. Precisely, we assume there exist $\{v_i\}_{i=1}^N \subset \mathbb{R}^N$ and a $\delta > 0$ such that for all $\sigma_i \in C_\delta(v_i)$ the set $\{\sigma_i\}_{i=1}^N$ is linearly independent, where

$$C_\delta(v) := \left\{ w \in \mathbb{R}^N\backslash\{0\} : \frac{v}{|v|}\cdot\frac{w}{|w|} > 1-\delta \right\},$$

and

$$\liminf_{\epsilon\to 0}\int_{C_\delta(v_i)} \rho_\epsilon(x)\,dx > 0 \text{ for all } i = 1,\dots,N. \tag{1.10}$$

Remark 1.4 *Given a linearly independent set* $\{v_i\}_{i=1}^N$*, by using the continuity of the determinant it is always possible to find a* $\delta > 0$ *small enough such that for all* $\sigma_i \in C_\delta(v_i)$ *the set* $\{\sigma_i\}_{i=1}^N$ *is linearly independent. However, we additionally require that condition* (1.10) *hold for these cones to ensure the coercivity of the limiting measure, so that it is, in a sense, equivalent to the Hausdorff surface measure and we can draw conclusions similar to the ones in the radial case.*

The first main result of the paper is the following characterization of $W^{1,p}(\Omega)$, $1 < p < \infty$, for *arbitrary* open sets Ω.

Theorem 1.5 *Let* $\Omega \subset \mathbb{R}^N$ *be open, let* ρ_ϵ *satisfy* (1.3), (1.4), *and* (1.10), *let* $1 < p < \infty$ *and* $1 \le q < \infty$, *with* $1 \le q \le \frac{N}{N-p}$ *if* $p < N$, *and let* $f \in L^1_{\mathrm{loc}}(\Omega)$. *Then* $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *and* $\nabla f \in L^p(\Omega; \mathbb{R}^N)$ *if and only if*

$$\lim_{\lambda \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx < +\infty. \tag{1.11}$$

Moreover, if ρ_ϵ *satisfy (*1.5*), then there exists*

$$\lim_{\lambda \to 0} \lim_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx = K_{p,q,N} \int_\Omega |\nabla f|^p \, dx, \tag{1.12}$$

where

$$K_{p,q,N} := \left(\int_{S^{N-1}} |e_1 \cdot \sigma|^{pq} \, d\mathcal{H}^{N-1}(\sigma) \right)^{\frac{1}{q}}.$$

Here, for $\lambda > 0$,

$$\Omega_\lambda := \left\{ x \in \Omega \ : \ |x| < \frac{1}{\lambda}, \ \ \mathrm{dist}(x, \partial\Omega) > \lambda \right\}.$$

Remark 1.6 *Without the hypothesis* (1.5), *we cannot in general expect convergence of the whole sequence. However, we can still prove that there exist a subsequence* $\{\epsilon_j\}$ *and a probability measure* $\mu \in M(S^{N-1})$ *such that*

$$\begin{aligned} &\lim_{\lambda \to 0} \lim_{j \to \infty} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &= \int_\Omega \left(\int_{S^{N-1}} (|\nabla f(x) \cdot \sigma|^p)^q \, d\mu(\sigma) \right)^{\frac{1}{q}} dx. \end{aligned}$$

Beyond the interest of the characterization, we will demonstrate that both (1.8) and (1.9) imply the condition (1.11) in the case $q = 1$, so that the proof of this theorem will imply the sufficiency of conditions (1.8) or (1.9). In this way we obtain the conjectures of [11] and [30], the substance of which is contained in the following two corollaries.

Corollary 1.7 *Let* $\Omega \subset \mathbb{R}^N$ *open,* $1 < p < \infty$, $\hat{\rho}_\epsilon$ *satisfy* (1.3), (1.4), *and* (1.5), $f \in L^p(\Omega)$ *and*

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dy dx < +\infty.$$

Then $f \in W^{1,p}(\Omega)$.

Corollary 1.8 *Let* $\Omega \subset \mathbb{R}^N$ *open,* $1 < p < \infty$*,* $\hat{\rho}_\epsilon$ *satisfy* (1.3)*,* (1.4)*, and* (1.5)*,* $f \in L^p(\Omega)$ *and*

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx < +\infty.$$

Then $f \in W^{1,p}(\Omega)$*.*

Analogous to the smooth boundary case, when $p = 1$ our result gives the following characterization of $BV(\Omega)$ for Ω an arbitrary open set.

Theorem 1.9 *Let* $\Omega \subset \mathbb{R}^N$ *be open, let* ρ_ϵ *satisfy* (1.3)*,* (1.4)*, and* (1.10)*, let* $1 \le q < \infty$ *with* $1 \le q \le \frac{N}{N-1}$ *if* $N > 1$*, and let* $f \in L^1_{\mathrm{loc}}(\Omega)$*. Then* $f \in BV_{\mathrm{loc}}(\Omega)$ *and* $Df \in M_b(\Omega; \mathbb{R}^N)$ *if and only if*

$$\lim_{\lambda \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx < +\infty. \tag{1.13}$$

Moreover, if ρ_ϵ *satisfy (*1.5*), then there exists*

$$\lim_{\lambda \to 0} \lim_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx = K_{1,q,N} |Df|(\Omega).$$

Remark 1.10 *Again, without the hypothesis* (1.5) *we are able to show that there exist a subsequence* $\{\epsilon_j\}$ *and a probability measure* $\mu \in M(S^{N-1})$ *such that*

$$\lim_{\lambda \to 0} \lim_{j \to \infty} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx$$
$$= \int_\Omega \left(\int_{S^{N-1}} \left(\left| \frac{dDf}{d|Df|}(x) \cdot \sigma \right| \right)^q d\mu(\sigma) \right)^{\frac{1}{q}} d|Df|(x),$$

where $\frac{dDf}{d|Df|}$ *is the Radon–Nikodym of* Df *with respect to* $|Df|$*.*

As before, we are able to apply Theorem 1.9 (with $q = 1$) to prove corresponding conjectures in BV.

Corollary 1.11 *Let* $\Omega \subset \mathbb{R}^N$ *open,* $\hat{\rho}_\epsilon$ *satisfy* (1.3)*,* (1.4)*, and* (1.5)*,* $f \in L^1(\Omega)$ *and*

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{d_\Omega(x,y)} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx < +\infty.$$

Then $f \in BV(\Omega)$*.*

Corollary 1.12 *Let* $\Omega \subset \mathbb{R}^N$ *open,* $\hat{\rho}_\epsilon$ *satisfy* (1.3)*,* (1.4)*, and* (1.5)*,* $f \in L^1(\Omega)$ *and*

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx < +\infty.$$

Then $f \in BV(\Omega)$*.*

These corollaries answer the question of the sufficiency of (1.8) and (1.9) for arbitrary domains, however, it is still of interest to consider the necessity. It turns out that neither condition is necessary, as we are able to give a counterexample that demonstrates the class of functions for which (1.8) or (1.9) is finite can be strictly contained in $W^{1,p}(\Omega)$ (or $BV(\Omega)$). As they have been proven equivalent for extension domains in $\mathbb{R}^N$, the key ingredient here is to examine issues of boundary regularity. Extension domains are precisely those for which the standard Sobolev embeddings can be expected, and so we examine a construction of Fraenkel [19] that shows for general domains the Sobolev embedding theorem fails. Extending his analysis to our problem, we are able to prove the following theorem.

Theorem 1.13 *There exists an open set* $\Omega \subset \mathbb{R}^2$*, an* $f \in W^{1,2}(\Omega)$*, and* $\hat{\rho}_\epsilon$ *satisfying* (1.3)*,* (1.4)*, and* (1.5) *such that*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx = +\infty.$$

We mention that it is not difficult to modify Theorem 1.13 to extend the proof to other values of p, including the case $p = 1$, such that the iterated integral is infinite. This is accomplished simply by changing the parameters in the construction, demonstrating that there is nothing special about the case $p = 2$.

Concerning the functional (1.9), we have a stronger result in that we are able to give a function which has much greater regularity for which the iterated integral is infinite, as the following theorem demonstrates.

Theorem 1.14 *There exists an open set* $\Omega \subset \mathbb{R}^2$*,* $\hat{\rho}_\epsilon$ *satisfying* (1.3)*,* (1.4)*, and* (1.5)*, and an* $f \in W^{1,\infty}(\Omega)$ *such that for every* $p > 1$*,*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx = +\infty.$$

In Theorem 1.14, we are able to construct an f with much more regularity precisely by exploiting the difference between $W^{1,\infty}$ and Lipschitz functions. It is not possible to do so in Theorem 1.13, as if $f \in W^{1,\infty}(\Omega)$, then f is necessarily Lipschitz with respect to the geodesic distance (see, for example, [9]).

As Theorems 1.5 and 1.9 have suggested, we will be concerned with the functional

$$J^{p,q}_{\epsilon,\lambda}(f) := \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx, \tag{1.14}$$

whose limit in epsilon and lambda characterizes $W^{1,p}(\Omega)$ for $p > 1$ and $BV(\Omega)$ for $p = 1$. The functional (1.14) is the same as the one introduced in [6], aside from two specific modifications. The first of which is the approach of Ω by subsets with compact closure and positive distance to the boundary ($\Omega_\lambda \subset\subset \Omega$), which along with a measure support truncation lemma is the key to allowing our proofs to go through for arbitrary open Ω. The second of these modifications is the addition of the variable q, which enables us to prove a localization result that has applications in image processing. This is because the non-local functionals we are concerned with are one class of examples of recently introduced non-local functionals in image processing by Gilboa and Osher [22], whose aim is to improve effectiveness in image denoising and reconstruction. Since for the purpose of the applications in imaging the domain can be assumed to be sufficiently regular (usually a rectangle), it is better to consider the functional

$$J_{\epsilon}^{p,q}(f) := \int_{\Omega} \left(\int_{\Omega} \left(\frac{|f(x) - f(y)|^p}{|x - y|^p} \right)^q \rho_{\epsilon}(x - y) \, dy \right)^{\frac{1}{q}} dx, \tag{1.15}$$

as it relates to the non-local imaging functionals. We give a brief summary of this relationship, and then proceed to the proofs of the above theorems.

Although the total variation model of Rudin, Osher, and Fatemi [32] has been highly successful in such problems, it has had notable difficulties in preserving fine structures, details, and textures (since blurring is common), as well as the highly undesirable staircase effect (where smooth affine regions are replaced by piecewise constant regions). This model is mathematically represented via the minimization problem

$$\min \left\{ |Df|(\Omega) + \lambda \int_{\Omega} |f(x) - f_0(x)|^2 \, dx : \, f \in BV(\Omega) \right\}, \tag{1.16}$$

where $f_0 \in L^2(\Omega)$ is given. Seeking to overcome the above difficulties, in some recent work, Gilboa and Osher [21], [22] (see also [7]) propose a systematic and coherent framework for non-local image and signal processing. They specifically address the problem of image reconstruction and segmentation for images with repetitive structures and fine textures, and introduce a non-local version of (1.16) to correct the blurring and staircasing problems mentioned. The idea is that any point in the image domain is (ideally) allowed to interact directly with any other point. The use of information beyond the local function value gives them some freedom in the reconstruction of an image. The gradient-based regularizing functional introduced by Gilboa and Osher in [22] takes the form

$$J(f) := \int_{\Omega} \phi \left(\int_{\Omega} |f(x) - f(y)|^2 w(x, y) \, dy \right) dx, \tag{1.17}$$

where $\Omega \subset \mathbb{R}^N$ is an open set (in imaging $N = 2$), $f : \Omega \to \mathbb{R}$, $\phi : [0, \infty) \to [0, \infty)$ is a function convex in $\sqrt{s}$ with $\phi(0) = 0$, and w is a positive and symmetric weight function that measures the interaction between different values of x, y. The prototype model for ϕ is the function $\phi(s) = \sqrt{s}$, which leads to the non-local functional

$$J_{NL-TV}(f) := \int_{\Omega} \sqrt{\int_{\Omega} |f(x) - f(y)|^2 \, w(x, y) \, dy} dx. \tag{1.18}$$

This corresponds to the functional (1.15) when $p = 1$, $q = 2$, and $w = w_{\varepsilon}(x, y) = \frac{\rho_{\varepsilon}(|x-y|)}{|x-y|^2}$. Thus, our result shows that the non-local functional (1.18) converges to a constant times the total variation, when the mass of $\{\rho_{\varepsilon}\}_{\varepsilon}$ concentrates at origin. This shows that the non-local minimization problem, in some sense, localizes to the Rudin, Osher, Fatemi model (see [5] for more relationships between non-local minimization problems and their corresponding local forms).

We finally remark that there is a large body of work on related non-local functionals, including papers addressing compactness (see [6], [31]), applications to problems and further questions (see [10], [11]), extended looks at non-radial mollifiers (see [30], [31]), Γ-convergence of non-local functionals ([30]), and other characterizations of Sobolev spaces ([28], [29]). Our work is related to these papers, and all of them relate to the localization of non-local functionals. It is then natural that our techniques follow closely the work in [6], [11], [17], and [30], with the mentioned modifications specific to our aim and technical requirements for the proofs to work. Our organization will be as follows. We will begin with some notational preliminaries and preliminary results. This will lead us into theorems building to the main results, where the cases $p > 1$ and $p = 1$ will be done in parallel. This parallel structure will continue as we give proofs of the main results, and finally counterexamples.

2 Preliminaries

For a measurable set $E \subset \mathbb{R}^N$ and $r > 0$ we define a fattening of E and approach of E by compact subset as

$$E^r := \{x \in \mathbb{R}^N : \operatorname{dist}(x, E) < r\}, \tag{2.1}$$

$$E_r := \Big\{x \in E : |x| < \frac{1}{r}, \operatorname{dist}(x, \partial E) > r\Big\}, \tag{2.2}$$

so that $E_r \subset E \subset E^r$.

Fix $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \psi \, dx = 1$ and $\operatorname{supp} \psi \subset B(0,1)$. For $\delta > 0$ define

$$\psi_\delta(x) := \frac{1}{\delta^N} \psi\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^N.$$

Given an open set $\Omega \subset \mathbb{R}^N$ and a function $f \in L^1_{\text{loc}}(\Omega)$, for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \delta$ define the mollification of f by

$$f_\delta(x) := (f * \psi_\delta)(x) = \int_{\mathbb{R}^N} f(y) \psi_\delta(x - y) \, dy. \tag{2.3}$$

Since $\int_{\mathbb{R}^N} \psi_\delta \, dx = 1$, by Jensen's inequality for $\omega : [0, \infty) \to [0, \infty)$ convex, we have

$$\begin{aligned} \omega(|f_\delta(x)|) &= \omega\left(\left|\int_{\mathbb{R}^N} f(x-y)\psi_\delta(y) \, dy\right|\right) \\ &\leq \int_{\mathbb{R}^N} \omega(|f(x-y)|)\psi_\delta(y) \, dy = (\omega \circ |f|)_\delta(x), \end{aligned}$$

for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \delta$.

As stated in the introduction, we assume $\{\rho_\epsilon\} \subset L^1(\mathbb{R}^N)$ satisfy (1.3), (1.4), and (1.10), so that if $E \subset \mathbb{R}^N$ is bounded and measurable, then

$$\lim_{\epsilon \to 0} \int_E |x| \rho_\epsilon(x) \, dx = 0, \tag{2.4}$$

since fixing $\delta > 0$, by (1.3) we have

$$\begin{aligned} \limsup_{\epsilon \to 0} \int_E |x|\rho_\epsilon(x) \, dx &\leq \limsup_{\epsilon \to 0} \left(\int_{\{|x| > \delta\} \cap E} |x|\rho_\epsilon(x) \, dx + \int_{|x| \leq \delta} |x|\rho_\epsilon(x) \, dx \right) \\ &\leq C \lim_{\epsilon \to 0} \int_{|x| > \delta} \rho_\epsilon(x) \, dx + \delta \end{aligned}$$

and (2.4) follows by sending $\delta \to 0$, along with the equality (1.4).

We are interested in utilizing the coercivity condition (1.10) to understand the behavior of a family $\{\rho_\epsilon\}$ which is not necessarily radial. This condition (1.10) implies the following lemma establishing some coercivity with respect to the uniform measure (see [30] for the introduction of this condition).

Lemma 2.1 *Let ρ_ϵ satisfy* (1.3)*,* (1.4)*, and* (1.10)*, and let $\{\mu_\epsilon\} \subset M(S^{N-1})$ be the measures defined by*

$$\mu_\epsilon(F) := \int_F \int_0^\infty \rho_\epsilon(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) \tag{2.5}$$

for $F \subset S^{N-1}$ Borel. Then there exist a subsequence $\{\epsilon_j\}$, with $\epsilon_j \to 0^+$, and μ in $M(S^{N-1})$ such that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Moreover, for every $p > 0$ there exists $\alpha > 0$ such that for every $v \in \mathbb{R}^N$, we have

$$\int_{S^{N-1}} |v \cdot \sigma|^p \, d\mu(\sigma) \geq \alpha |v|^p. \tag{2.6}$$

Proof. Using polar coordinates and (1.3), we have that

$$\mu_\epsilon(S^{N-1}) = \int_{S^{N-1}} \int_0^\infty \rho_\epsilon(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) = 1.$$

Thus, $||\mu_\epsilon||_{M(S^{N-1})} = 1$ and so up to a subsequence, $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$ with $||\mu||_{M(S^{N-1})} = 1$ (since $1 \in C(S^{N-1})$). Let $\{v_i\}_{i=1}^N$ be the linearly independent set of vectors given in (1.10). We claim there exists an $\epsilon_0 > 0$ with the property that for all $v \in \mathbb{R}^N$ there exists an i such that

$$|v \cdot \sigma| \geq \epsilon_0 |v| \tag{2.7}$$

for all $\sigma \in C_\delta(v_i) \cap S^{N-1}$. By rescaling we restrict ourselves to the case $v \in S^{N-1}$, and we proceed by contradiction. If not, then there exist a sequence $\{\epsilon_n\}$ tending to zero, $w_n \in S^{N-1}$, and $\sigma_{i,n} \in C_\delta(v_i)$, $i = 1, \ldots, N$, so that up to a subsequence, which we will not relabel, $w_n \to w_0 \in S^{N-1}$ and $\sigma_{i,n} \to \sigma_{i,0} \in C_\delta(v_i)$, with

$$|w_0 \cdot \sigma_{i,0}| = 0$$

for all $i = 1, \ldots, N$. However, since the $\{\sigma_{i,0}\}_{i=1}^N$ form a linearly independent set (see Remark 1.4), we have a contradiction. Thus, (2.7) holds. Define

$$c := \min_i \liminf_{j \to \infty} \int_{C_\delta(v_i)} \rho_{\epsilon_j}(x)\, dx.$$

By (1.10), we have that $c > 0$. Given $v \in \mathbb{R}^N$, let i be such that (2.7) holds; then by (2.5) and Tonelli's theorem we compute

$$\begin{aligned}
\int_{S^{N-1}} |v \cdot \sigma|^p \, d\mu_{\epsilon_j}(\sigma) &\geq \int_{C_\delta(v_i) \cap S^{N-1}} |v \cdot \sigma|^p \, d\mu_{\epsilon_j}(\sigma) \\
&\geq (\epsilon_0 |v|)^p \int_{C_\delta(v_i) \cap S^{N-1}} \int_0^\infty \rho_{\epsilon_j}(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) \\
&= (\epsilon_0 |v|)^p \int_{C_\delta(v_i)} \rho_{\epsilon_j}(x)\, dx.
\end{aligned}$$

Letting $j \to \infty$, using the fact that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, and the definition of c, we have

$$\int_{S^{N-1}} |v \cdot \sigma|^p \, d\mu(\sigma) \geq c\epsilon_0^p |v|^p.$$

Define $\alpha := c\epsilon_0^p$, and the result is demonstrated. ■

Remark 2.2 *If ρ_ϵ satisfy* (1.5), *then $\mu_\epsilon = \mu = \mathcal{H}^{N-1}$ and there is no need to pass to a subsequence, since we may rewrite equation* (2.5) *as*

$$\begin{aligned}\mu_\epsilon(F) &= \int_F \int_0^\infty \rho_\epsilon(t\sigma) t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \\ &= \int_F \int_0^\infty \hat{\rho}_\epsilon(t|\sigma|) t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \\ &= \int_F d\mathcal{H}^{N-1}(\sigma),\end{aligned}$$

where we have used (1.3).

Since boundary regularity is not assumed, we must avoid calculations which might be near the boundary. Thus, the following measure truncation lemma is an essential tool in our proof of Theorems 1.5 and 1.9. We demonstrate that restricting the support (truncation in the domain) of ρ_ϵ gives the same measure in the weak-star limit. More precisely, consider, for every fixed $\eta > 0$, ρ_ϵ^η defined by

$$\rho_\epsilon^\eta := \rho_\epsilon \chi_{B(0,\eta)}. \tag{2.8}$$

This gives rise to a measure μ_ϵ^η defined by

$$\mu_\epsilon^\eta(F) := \int_F \int_0^\infty \rho_\epsilon^\eta(t\sigma) t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \tag{2.9}$$

for $F \subset S^{N-1}$ Borel, so that again applying the Radon–Nikodym theorem, for $\mathcal{H}^{N-1}$ a.e. $\sigma \in S^{N-1}$,

$$\frac{d\mu_\epsilon^\eta}{d\mathcal{H}^{N-1}}(\sigma) = \int_0^\infty \rho_\epsilon^\eta(t\sigma) t^{N-1} \, dt = \int_0^\eta \rho_\epsilon(t\sigma) t^{N-1} \, dt.$$

Lemma 2.3 *Let ρ_ϵ satisfy* (1.3) *and* (1.4), *and let $\{\mu_\epsilon\} \subset M(S^{N-1})$ be the corresponding measures defined in* (2.5). *Let $\epsilon_j \to 0^+$ and assume that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Then for every $\eta > 0$, $\mu_{\epsilon_j}^\eta \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, where $\mu_{\epsilon_j}^\eta$ are the measures defined in* (2.9).

Proof. We begin by proving that $\mu_{\epsilon_j} - \mu_{\epsilon_j}^\eta \to 0$ in $M(S^{N-1})$. For $f \in C(S^{N-1})$, with $\max_{S^{N-1}} |f| = 1$, using spherical coordinates we have

$$\begin{aligned}\left| \int_{S^{N-1}} f \, d\mu_{\epsilon_j}^\eta - \int_{S^{N-1}} f \, d\mu_{\epsilon_j} \right| &= \left| \int_{S^{N-1}} \int_\eta^\infty f(\sigma) \rho_{\epsilon_j}(t\sigma) t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \right| \\ &\leq \max_{S^{N-1}} |f| \int_{|x|>\eta} \rho_{\epsilon_j}(x) \, dx = \int_{|x|>\eta} \rho_{\epsilon_j}(x) \, dx.\end{aligned}$$

Taking the supremum over all such f, we get

$$||\mu^\eta_{\epsilon_j} - \mu_{\epsilon_j}||_{M(S^{N-1})} \le \int_{|x|>\eta} \rho_{\epsilon_j}(x)\, dx \to 0$$

as $j \to \infty$ by (1.4). Thus, $\mu_{\epsilon_j} - \mu^\eta_{\epsilon_j} \to 0$ in $M(S^{N-1})$. Since $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, it follows that $\mu^\eta_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. ■

Note also that by the definition of ρ^η_ϵ, the fact that $\rho^\eta_\epsilon \le \rho_\epsilon$, (1.3), (1.4), and (2.4), we have that the following properties of ρ^η_ϵ hold

$$\rho^\eta_\epsilon \ge 0, \quad \int_{\mathbb{R}^N} \rho^\eta_\epsilon(x)\, dx \le 1, \tag{2.10}$$

$$\lim_{\epsilon\to 0^+} \int_{|x|>\delta} \rho^\eta_\epsilon(x)\, dx = 0 \quad \text{for all } \delta > 0, \tag{2.11}$$

$$\lim_{\epsilon\to 0^+} \int_E |x|\rho^\eta_\epsilon(x)\, dx = 0 \tag{2.12}$$

for every $E \subset \mathbb{R}^N$ bounded and measurable.

For definitions and properties of the Sobolev space $W^{1,p}(\Omega)$, the space of functions of bounded variations $BV(\Omega)$, the fractional Sobolev space $W^{s,p}(\Omega)$, and the Besov space $B^s_{p,\theta}(\Omega)$ we refer to [1, 4, 18, 25]

3 Preliminary Results

In this section we prove some preliminary results, which will be used in the sequel. Proofs of variants of these results can be found in [6], [11], and [30]. We adapt these proofs to our setting allowing for truncated mollifiers and for an additional q in the integrand, and present the proofs for the convenience of the reader. We use the notation (2.1) and (2.2).

Lemma 3.1 *Let $A \subset \mathbb{R}^N$ be open and bounded and let $f \in C^2(\overline{A^\eta})$ for some $\eta > 0$. Then*

$$|f(x) - f(y) - \nabla f(x) \cdot x - y| \le C^f |x-y|^2$$

for all $x \in A$ and $y \in A^\eta$, where C^f depends upon $||f||_{C^2(\overline{A^\eta})}$.

Proof. Fix $x \in A$ and $y \in A^\eta$. If $|x-y| < \eta$, then the segment of endpoints x and y is contained in $\overline{A^\eta}$, and so we may apply Taylor's formula to obtain

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \le C(N)||\nabla^2 f||_{L^\infty(\overline{A^\eta})}|x-y|^2.$$

On the other hand, if $|x-y| > \eta$, we may estimate

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \le \left(\frac{2}{\eta^2}||f||_{L^\infty(\overline{A^\eta})} + \frac{1}{\eta}||\nabla f||_{L^\infty(\overline{A^\eta})}\right)|x-y|^2,$$

and defining

$$C^f := C(N)||\nabla^2 f||_{L^\infty(\overline{A^\eta})} + \frac{2}{\eta^2}||f||_{L^\infty(\overline{A^\eta})} + \frac{1}{\eta}||\nabla f||_{L^\infty(\overline{A^\eta})},$$

the result is demonstrated. ■

Lemma 3.2 *Let Ω and ρ_ϵ be as in Theorem* 1.5*, let $A \subset \Omega$ be open and bounded with* $\operatorname{dist}(A, \partial\Omega) > 0$*, let $r \geq 1$ and let $f \in C^2(\overline{A^\eta})$, where* $0 < \eta < \operatorname{dist}(A, \partial\Omega)$*. Let $\epsilon_j \to 0^+$ and assume that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Then for every $x \in A$, we have*

$$\lim_{j\to\infty} \int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r \rho^\eta_{\epsilon_j}(x-y)\, dy = \int_{S^{N-1}} |\nabla f(x) \cdot \sigma|^r \, d\mu(\sigma),$$

where $\rho^\eta_{\epsilon_j}$ is the family of truncated mollifiers introduced in (2.8)*.*

Proof. First, we demonstrate that in the limit, the difference quotient averages over A^η behave like the derivative averages over A^η. We then use Tonelli's theorem and the weak-star convergence of the measures to prove the result.

Step 1: We prove that for $x \in A$,

$$\begin{aligned} &\limsup_{j\to\infty} \int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r \rho^\eta_{\epsilon_j}(x-y)\, dy \\ &\quad = \limsup_{j\to\infty} \int_{A^\eta} \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy. \end{aligned}$$

Set $M_f := ||\nabla f||_{L^\infty(\overline{A^\eta})}$. By the mean value theorem, for all $s, t \in [0, M_f]$,

$$|s^r - t^r| \leq r M_f^{r-1} |s - t|.$$

Thus, for $x \in A$ and $y \in A^\eta$ we can estimate the difference

$$\begin{aligned} &\left| \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r - \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \right| \\ &\leq r M_f^{r-1} \frac{|f(x) - f(y) - \nabla f(x) \cdot x - y|}{|x-y|} \leq r M_f^{r-1} C^f |x-y|, \end{aligned}$$

where C^f is the constant given in Lemma 3.1. Therefore,

$$\begin{aligned} &\int_{A^\eta} \left| \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r - \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \right| \rho^\eta_{\epsilon_j}(x-y)\, dy \\ &\quad \leq C^f_r \int_{A^\eta} |x-y| \rho^\eta_{\epsilon_j}(x-y)\, dy, \end{aligned}$$

where $C^f_r := r M_f^{r-1} C^f$. Making the change of variables $h = x - y$ and using monotonicity of the integral, we obtain that the right-hand side of the previous inequality is less than or equal to

$$C^f_r \int_{|h|<\eta} |h| \rho^\eta_{\epsilon_j}(h)\, dh.$$

Using (2.12) and sending $j \to \infty$, we conclude

$$\begin{aligned} &\limsup_{j\to\infty} \int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r \rho^\eta_{\epsilon_j}(x-y)\, dy \\ &\quad = \limsup_{j\to\infty} \int_{A^\eta} \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy. \end{aligned}$$

Step 2: We will show that for each $x \in A$,

$$\lim_{j\to\infty} \int_{A^\eta} \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy = \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r\, d\mu(\sigma). \tag{3.1}$$

Since $\rho^\eta = 0$ if $|x-y| > \eta$, we may use polar coordinates to write

$$\int_{A^\eta} \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy = \int_{B(x,\eta)} \left| \nabla f(x) \cdot \frac{x-y}{|x-y|} \right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy$$
$$= \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \int_0^\eta \rho^\eta_{\epsilon_j}(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma),$$

and using the definition of $\mu^\eta_{\epsilon_j}$ (see (2.9)), we have

$$\int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \int_0^\eta \rho^\eta_{\epsilon_j}(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) = \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r\, d\mu^\eta_{\epsilon_j}(\sigma).$$

Now, since the function $\sigma \mapsto |\nabla f(x)\cdot\sigma|^r$ is continuous, we may let $j\to\infty$ and Lemma 2.3 to obtain (3.1). ■

Lemma 3.3 *Let Ω and ρ_ϵ be as in Theorem 1.5, let $A \subset \Omega$ be open and bounded, with $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, let $1 \le p, q < \infty$, and let $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then for all $0 < \eta < \frac{\gamma}{3}$ we have*

$$\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho^\eta_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx$$
$$\le \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f(y) \cdot \frac{h}{|h|} \right|^{pq} \rho^\eta_\epsilon(h)\, dh \right)^{\frac{1}{q}} dy.$$

Proof. Making the change of variables $y = x+h$, and using the fact that $\rho^\eta_\epsilon = 0$ outside $B(0,\eta)$, we have

$$\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho^\eta_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx$$
$$= \int_{A^\eta} \left(\int_{B(0,\eta)} \left(\frac{|f(x+h)-f(x)|^p}{|h|^p} \right)^q \rho^\eta_\epsilon(h)\, dh \right)^{\frac{1}{q}} dx.$$

For $0 < \delta < \eta < \frac{\gamma}{3}$, we have that f_δ (see (2.3)) is well defined in A^η, and so we may apply the fundamental theorem of calculus to f_δ to write

$$\int_{A^\eta} \left(\int_{B(0,\eta)} \left(\frac{|f_\delta(x+h)-f_\delta(x)|^p}{|h|^p} \right)^q \rho^\eta_\epsilon(h)\, dh \right)^{\frac{1}{q}} dy$$
$$= \int_{A^\eta} \left(\int_{B(0,\eta)} \left(\int_0^1 \left| \nabla f_\delta(x+th) \cdot \frac{h}{|h|} \right| dt \right)^{pq} \rho^\eta_\epsilon(h)\, dh \right)^{\frac{1}{q}} dx =: I.$$

Then Jensen's inequality and Minkowski's inequality for integrals imply

$$I \le \int_{A^\eta} \int_0^1 \left(\int_{B(0,\eta)} \left(\left| \nabla f_\delta(x+th) \cdot \frac{h}{|h|} \right| \right)^{pq} \rho^\eta_\epsilon(h)\, dh \right)^{\frac{1}{q}} dt dx,$$

while Tonelli's theorem and the change of variables $y = x + th$ yield

$$I \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(y) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dy, \tag{3.2}$$

where we have used the fact that $|h| < \eta$ and that the integrand is non-negative. We thus conclude that

$$\begin{aligned} &\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(y) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dy, \end{aligned}$$

and letting $\delta \to 0$, by Fatou's lemma and Lebesgue dominated convergence theorem, we obtain the result. ■

Remark 3.4 *The hypothesis $0 < \eta < \frac{\gamma}{3}$ is a technical assumption to ensure f_δ is well defined in the region being considered. In the case where Ω is an extension domain for $W^{1,p}$ (and hence we can extend f to all of $\mathbb{R}^N$) we have no need for this assumption. We also note that it is here that we have implicitly used the truncation of ρ_ϵ, ρ_ϵ^η, to ensure that the change of variables does not leave the domain of definition of the function f. This can also be bypassed in the case Ω is an extension domain.*

Next we extend the previous lemma to the BV case. We remark that the calculations in the next proof are identical to those of the previous one until the final limiting step, where Df is only a measure and not a function, and so we use the Reshetnyak continuity theorem instead of Lebesgue dominated convergence theorem to pass the limit.

Lemma 3.5 *Let Ω and ρ_ϵ be as in Theorem 1.5 with $p = 1$, let $A \subset \Omega$ be open and bounded, with $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, let $1 \leq q < \infty$, and let $f \in BV_{\mathrm{loc}}(\Omega)$. Then for all $0 < \eta < \frac{\gamma}{3}$ such that $|Df|(\partial A^{2\eta}) = 0$, we have*

$$\begin{aligned} &\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \frac{dDf}{d|Df|}(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} d|Df|(x), \end{aligned}$$

where $\frac{dDf}{d|Df|}$ is the Radon-Nikodym derivative of Df with respect to $|Df|$.

Proof. We proceed as in the previous proof with $p = 1$ up to (3.2). Thus, we have

$$I \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dx. \tag{3.3}$$

Consider the Radon measures $\nu_\delta \in M_b(A^\eta; \mathbb{R}^N)$ defined by

$$\nu_\delta(F) := \int_F \nabla f_\delta(x)\, dx$$

for $F \subset A^\eta$ Borel, and let $\Psi_{\epsilon,\eta} : \mathbb{R}^N \to [0, \infty)$ defined by

$$\Psi_{\epsilon,\eta}(v) := \left(\int_{B(0,\eta)} \left| v \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}}, \quad v \in \mathbb{R}^N. \tag{3.4}$$

Then by 1-homogeneity inequality (3.3) can be rewritten as

$$I \leq \int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{d\nu_\delta}{d|\nu_\delta|}(x) \right) d|\nu_\delta|(x).$$

Now, since $\nu_\delta \overset{*}{\rightharpoonup} Df$ in $M_b(\Omega_\lambda^{2\eta}; \mathbb{R}^N)$, $|\nu_\delta|(A^{2\eta}) \to |Df|(A^{2\eta})$ (as a result of the assumption $|Df|(\partial A^{2\eta}) = 0$), it follows by Reshetnyak's continuity theorem (see [4] and [33]) that

$$\int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{d\nu_\delta}{d|\nu_\delta|}(x) \right) d|\nu_\delta|(x) \to \int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{dDf}{d|Df|}(x) \right) d|Df|(x).$$

Combining this convergence with Fatou's lemma as in the proof of Lemma 3.3, we have that

$$\begin{aligned} &\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x - y|} \right)^q \rho_\epsilon^\eta(x - y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \frac{dDf}{d|Df|}(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} d|Df|(x), \end{aligned}$$

which concludes the proof. ■

The next two Lemmas are due to an observation of Stein (see [11] or [30]), adapted to our setting.

Lemma 3.6 *Let Ω and ρ_ϵ be as in Theorem 1.5, let $A \subset \Omega$ be open and bounded, with $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, $1 \leq p, q < \infty$, and $f \in L^1_{\text{loc}}(\Omega)$. Then for all $0 < \delta < \eta < \frac{\gamma}{3}$ we have*

$$\begin{aligned} &\int_A \left(\int_A \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x - y|^p} \right)^q \rho_\epsilon^\eta(x - y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \leq \int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x) - f(y)|^p}{|x - y|^p} \right)^q \rho_\epsilon^\eta(x - y)\, dy \right)^{\frac{1}{q}} dx, \end{aligned}$$

where f_δ is the mollification of f (see (2.3)).

Proof. We begin by writing

$$\int_A \left(\int_A \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon^\eta(x-y)\, dy\right)^{\frac{1}{q}} dx$$

$$= \int_A \left(\int_A \left(\frac{|\int_{B(0,\delta)}[f(x-z) - f(y-z)]\psi_\delta(z)\, dz|}{|x-y|}\right)^{pq} \rho_\epsilon^\eta(x-y)\, dy\right)^{\frac{1}{q}} dx$$

$$\leq \int_A \left(\int_A \left(\int_{B(0,\delta)} \frac{|f(x-z) - f(y-z)|}{|x-y|}\psi_\delta(z)\, dz\right)^{pq} \rho_\epsilon^\eta(x-y)\, dy\right)^{\frac{1}{q}} dx =: I,$$

Applying Jensen's inequality and Minkowski's inequality for integrals as in Lemma 3.3, followed by Tonelli's theorem, we have

$$I \leq \int_{B(0,\delta)} \int_A \left(\int_A \left(\frac{|f(x-z) - f(y-z)|}{|x-y|}\right)^{pq} \rho_\epsilon^\eta(x-y)\, dy\right)^{\frac{1}{q}} \psi_\delta(z)\, dxdz.$$

Then making the change of variables $w = x+z$, $v = y+z$, for $z \in B(0,\delta)$, along with non-negativity of the integrand, we have

$$I \leq \int_{B(0,\delta)} \int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(w) - f(v)|^p}{|w-v|^p}\right)^q \rho_\epsilon^\eta(w-v)\, dv\right)^{\frac{1}{q}} \psi_\delta(z)\, dwdz.$$

Finally, integrating in z and using $\int_{B(0,\delta)} \psi_\delta(z)\, dz = 1$, we obtain the result. ■

4 Proof of the Main Theorems

In this section, we prove several results of independent interest that lead up to a characterization of $W^{1,p}(\Omega)$ and $BV(\Omega)$ for Ω an arbitrary open set. We begin by proving the sufficiency of conditions (1.11) and (1.13) in Theorems 1.5 and 1.9.

Theorem 4.1 *Let Ω and ρ_ϵ be as in Theorem* 1.5*, let $1 < p < \infty$, $1 \leq q < \infty$, and let $f \in L^1_{\text{loc}}(\Omega)$. Assume*

$$\lim_{\lambda \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon(x-y)\, dy\right)^{\frac{1}{q}} dx < +\infty.$$

Then $f \in W^{1,p}_{\text{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$. Moreover, there exist $\epsilon_j \to 0^+$ and a probability measure $\mu \in M(S^{N-1})$ such that for all $0 < \eta < \frac{\lambda}{3}$,

$$\lim_{\lambda \to 0} \liminf_{j \to \infty} \int_{\Omega^\eta_\lambda} \left(\int_{\Omega^{2\eta}_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\, dy\right)^{\frac{1}{q}} dx$$

$$\geq \int_\Omega \left(\int_{S^{N-1}} (|\nabla f(x) \cdot \sigma|^p)^q \, d\mu(\sigma)\right)^{\frac{1}{q}} dx.$$

Proof. Define

$$C := \lim_{\lambda \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon(x-y)\, dy\right)^{\frac{1}{q}} dx < \infty.$$

By the monotonicity of the integrals over Ω_λ we have that for any $\eta < \frac{\lambda}{3}$,

$$\limsup_{\epsilon\to 0} \int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \le C,$$

where $\Omega_\lambda^\eta := (\Omega_\lambda)^\eta$. But since $\rho_\epsilon^\eta \le \rho_\epsilon$, we have that

$$\limsup_{\epsilon\to 0} \int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \le C. \tag{4.1}$$

Fix $0 < \eta < \frac{\lambda}{3}$, and for any $0 < \delta < \eta$ apply Lemma 3.6 to obtain

$$\begin{aligned} &\int_{\Omega_\lambda} \left(\int_{\Omega_\lambda^\eta} \left(\frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \le \int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx. \end{aligned}$$

Let μ_ϵ be the measures defined in (2.5). By Lemma 2.1 there exist a subsequence $\{\epsilon_j\}$, with $\epsilon_j \to 0^+$, and a probability measure $\mu \in M(S^{N-1})$ such that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Since $f_\delta \in C^2\left(\overline{\Omega_\lambda^{2\eta}}\right)$ with $\Omega_\lambda^{2\eta}$ open and bounded, by Lemma 3.2 for every $x \in \Omega_\lambda$,

$$\lim_{j\to\infty} \int_{\Omega_\lambda^\eta} \left(\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|} \right)^{pq} \rho_{\epsilon_j}^\eta(x-y)\, dy = \int_{S^{N-1}} |\nabla f_\delta(x)\cdot\sigma|^{pq}\, d\mu(\sigma).$$

Thus, applying Fatou's lemma and the fact that $t^{\frac{1}{q}}$ is continuous, we have that

$$\begin{aligned} &\int_{\Omega_\lambda} \left(\int_{S^{N-1}} |\nabla f_\delta(x)\cdot\sigma|^{pq}\ d\mu(\sigma) \right)^{\frac{1}{q}} dx \qquad (4.2) \\ &\quad \le \liminf_{j\to\infty} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda^\eta} \left(\frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad \le \liminf_{j\to\infty} \int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \le C \end{aligned}$$

so that

$$\int_{\Omega_\lambda} \left(\int_{S^{N-1}} |\nabla f_\delta(x)\cdot\sigma|^{pq}\ d\mu(\sigma) \right)^{\frac{1}{q}} dx \le C.$$

However, Lemma 2.1 implies

$$\int_{\Omega_\lambda} |\nabla f_\delta(x)|^p\, dx \le \frac{C}{\alpha} \tag{4.3}$$

for some constant $\alpha > 0$ (independent of λ). Since as $\delta \to 0$, $f_\delta \to f$ in $L^1_{\mathrm{loc}}(\Omega)$, these bounds on ∇f_δ imply $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega_\lambda;\mathbb{R}^N)$. Finally, letting $\lambda \to 0$ we obtain $\nabla f \in L^p(\Omega;\mathbb{R}^N)$.

To prove the last part of the statement, let $\delta \to 0$ in (4.2) (utilizing $\rho^{\eta}_{\epsilon_j} \le \rho_{\epsilon_j}$) and use Fatou's lemma to obtain

$$\int_{\Omega_\lambda} \left(\int_{S^{N-1}} |\nabla f(x) \cdot \sigma|^{pq} \; d\mu(\sigma) \right)^{\frac{1}{q}} dx$$
$$\le \liminf_{j \to \infty} \int_{\Omega^{\eta}_{\lambda}} \left(\int_{\Omega^{2\eta}_{\lambda}} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}(x-y) \, dy \right)^{\frac{1}{q}} dx. \qquad (4.4)$$

It now suffices to let $\lambda \to 0$ and use Lebesgue monotone convergence theorem. ■

The analogous result for $p = 1$ is the following theorem.

Theorem 4.2 *Let Ω and ρ_ϵ be as in Theorem 1.5, let $1 \le q < \infty$, and let $f \in L^1_{\text{loc}}(\Omega)$. Assume*

$$\lim_{\lambda \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_\epsilon(x-y) \, dy \right)^{\frac{1}{q}} dx < +\infty.$$

Then $f \in BV_{\text{loc}}(\Omega)$ and $Df \in M_b(\Omega; \mathbb{R}^N)$. Moreover, there exist $\epsilon_j \to 0^+$ and a probability measure $\mu \in M(S^{N-1})$ such that for all $0 < \eta < \frac{\lambda}{3}$,

$$\lim_{\lambda \to 0} \lim_{j \to \infty} \int_{\Omega^{\eta}_{\lambda}} \left(\int_{\Omega^{2\eta}_{\lambda}} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_{\epsilon_j}(x-y) \, dy \right)^{\frac{1}{q}} dx$$
$$\ge \int_{\Omega} \left(\int_{S^{N-1}} \left| \frac{dDf}{d|Df|}(x) \cdot \sigma \right|^q d\mu(\sigma) \right)^{\frac{1}{q}} d|Df|(x),$$

where $\frac{dDf}{d|Df|}$ is the Radon–Nikodym of Df with respect to $|Df|$.

Proof. We proceed as in the previous theorem up to (4.3), which now becomes

$$\int_{\Omega_\lambda} |\nabla f_\delta(x)| \, dx \le \frac{C}{\alpha},$$

and again, since as $\delta \to 0$, $f_\delta \to f$ in $L^1_{\text{loc}}(\Omega)$, these bounds on ∇f_δ imply $f \in BV_{\text{loc}}(\Omega)$ and $Df \in M_b(\Omega_\lambda; \mathbb{R}^N)$. Finally, letting $\lambda \to 0$ we obtain $f \in BV_{\text{loc}}(\Omega)$ with $Df \in M_b(\Omega; \mathbb{R}^N)$.

To prove the last part of the statement, observe that the function $\Psi : \mathbb{R}^N \to [0, \infty)$, defined by

$$\Psi(v) := \left(\int_{S^{N-1}} |v \cdot \sigma|^q \; d\mu(\sigma) \right)^{\frac{1}{q}}, \quad v \in \mathbb{R}^N, \qquad (4.5)$$

is convex and positively homogeneous of degree one, and again consider the Radon measures $\nu_\delta \in M_b(\Omega_\lambda; \mathbb{R}^N)$ defined by

$$\nu_\delta(F) := \int_F \nabla f_\delta(x) \, dx$$

for $F \subset \Omega_\lambda$ Borel. We rewrite (4.2) (again utilizing $\rho^\eta_{\epsilon_j} \le \rho_{\epsilon_j}$) as

$$\int_{\Omega_\lambda} \Psi\left(\frac{d\nu_\delta}{d|\nu_\delta|}(x)\right) d|\nu_\delta|(x) \\ \le \liminf_{j\to\infty} \int_{\Omega^\eta_\lambda} \left(\int_{\Omega^{2\eta}_\lambda} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx.$$

Since $\nu_\delta \overset{*}{\rightharpoonup} Df$ in $M_b(\Omega_\lambda;\mathbb{R}^N)$, it follows by Reshetnyak's lower semicontinuity theorem (see [4] and [33]) that

$$\int_{\Omega_\lambda} \Psi\left(\frac{dDf}{d|Df|}(x)\right) d|Df|(x) \\ \le \liminf_{j\to\infty} \int_{\Omega^\eta_\lambda} \left(\int_{\Omega^{2\eta}_\lambda} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx. \tag{4.6}$$

It now suffices to let $\lambda \to 0$ and use Lebesgue monotone convergence theorem. ■

Using Theorems 4.1 and 4.2 we can now prove Corollaries 1.7, 1.8, 1.11, and 1.12.

Proof of Corollary 1.7. Let $f \in L^p(\Omega)$ satisfy (1.8). Then for every $\eta < \frac{\lambda}{3}$ and every $\Omega_\lambda \subset \Omega$ we have

$$\limsup_{\epsilon\to 0} \int_{\Omega^\eta_\lambda} \int_{\Omega^{2\eta}_\lambda} \frac{|f(x)-f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}^\eta_\epsilon(d_\Omega(x,y))\, dydx < +\infty.$$

Now, since $\hat{\rho}^\eta_\epsilon = 0$ if $d_\Omega(x,y) > \eta$, for each x we can restrict ourselves to integration over y such that $d_\Omega(x,y) \le \eta$. Then since $|x-y| \le d_\Omega(x,y) \le \eta$, this implies that

$$\limsup_{\epsilon\to 0} \int_{\Omega^\eta_\lambda} \int_{\Omega^{2\eta}_\lambda} \frac{|f(x)-f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}^\eta_\epsilon(d_\Omega(x,y))\, dydx < +\infty$$

However, since $\eta < \frac{\lambda}{3}$, for $x \in \Omega^\eta_\lambda$ and $y \in \Omega^{2\eta}_\lambda$, we have that the segment containing x and y is contained in Ω, so that $d_\Omega(x,y) = |x-y|$, and thus

$$\int_{\Omega^\eta_\lambda} \int_{\Omega^{2\eta}_\lambda} \frac{|f(x)-f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}^\eta_\epsilon(d_\Omega(x,y))\, dydx \\ = \int_{\Omega^\eta_\lambda} \int_{\Omega^{2\eta}_\lambda} \frac{|f(x)-f(y)|^p}{|x-y|^p} \hat{\rho}^\eta_\epsilon(|x-y|)\, dydx,$$

so that

$$\limsup_{\epsilon\to 0} \int_{\Omega^\eta_\lambda} \int_{\Omega^{2\eta}_\lambda} \frac{|f(x)-f(y)|^p}{|x-y|^p} \hat{\rho}^\eta_\epsilon(|x-y|)\, dydx < +\infty.$$

We can now proceed as in the proof of Theorem 4.1 starting from equation (4.1) to conclude that $\nabla f \in L^p(\Omega;\mathbb{R}^N)$, and therefore, $f \in W^{1,p}(\Omega)$. ■

Remark 4.3 *Lemma 2.3 is essential in the proof of Corollary 1.7, as it ensures that truncation of the mollifiers does not destroy coercivity of the limiting* *measure, which was necessary for our comparison of the geodesic and Euclidean distances (our analysis hinged on the equality $d_\Omega(x,y) = |x-y|$ for certain x and y). This analysis implies that the same argument applies to Corollaries 1.8, 1.11, and 1.12, where in the BV case we invoke the argument of Theorem 4.2 instead of Theorem 4.1.*

Next we prove the necessity of conditions (1.11) and (1.13) in Theorems 1.5 and 1.9.

Theorem 4.4 *Let Ω and ρ_ϵ be as in Theorem 1.5, let $1 < p < \infty$ and $1 \leq q < \infty$, with $1 \leq q \leq \frac{N}{N-p}$ if $p < N$, let $0 < \eta < \frac{\lambda}{3}$, and let $f \in W^{1,p}_{\text{loc}}(\Omega)$ with $\nabla f \in L^p(\Omega;\mathbb{R}^N)$. Then*

$$\int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx$$
$$\leq \int_\Omega |\nabla f(x)|^p\, dx + C_{p,q,\eta,\lambda} ||f||^p_{L^{pq}(\Omega_\lambda)} \left(\int_{|x|>\frac{\eta}{2}} \rho_\epsilon(x)\, dx \right)^{\frac{1}{q}}.$$

Proof. Fix $\lambda > 0$, and let $0 < \eta < \frac{\lambda}{3}$. Consider

$$\begin{aligned}
&\int_{\Omega_\lambda^\eta} \left(\int_{\Omega_\lambda^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&= \int_{\Omega_\lambda^\eta} \left(\int_{|x-y|<\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\quad + \int_{\Omega_\lambda^\eta} \left(\int_{|x-y|>\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&=: I \;+\; II.
\end{aligned}$$

Considering II, we have

$$II \leq \frac{2^{p-1}}{\eta^p} \int_{\Omega_\lambda^\eta} \left(\int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
II &\leq \frac{2^{p-1}}{\eta^p} |\Omega_\lambda^\eta|^{1-\frac{1}{q}} \left(\int_{\Omega_\lambda^\eta} \int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dydx \right)^{\frac{1}{q}} \\
&\leq \frac{2^{p-1}}{\eta^p} |\Omega_\lambda^\eta|^{1-\frac{1}{q}} \left(\int_{\Omega_\lambda}^{\eta} \int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dydx \right)^{\frac{1}{q}}.
\end{aligned}$$

Separating terms and applying Tonelli's theorem, we have that

$$\int_{\Omega_\lambda^\eta}\int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq}\rho_\epsilon(x-y)\,dydx$$
$$= \int_{\Omega_\lambda^\eta} |f(x)|^{pq}\int_{|x-y|>\eta}\rho_\epsilon(x-y)\,dydx$$
$$+ \int_{\Omega_\lambda^\eta} |f(y)|^{pq}\int_{|x-y|>\eta}\rho_\epsilon(x-y)\,dxdy$$

so that we may bound

$$II \leq \frac{2^{p-1}}{\eta^p}|\Omega_\lambda^\eta|^{1-\frac{1}{q}}||f||^p_{L^{pq}(\Omega_\lambda^\eta)}\left(2\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.$$

Thus, if we define $C_{p,q,\eta,\lambda} := |\Omega_\lambda^\eta|^{1-\frac{1}{q}}\frac{2^p}{\eta^p}$, we obtain the necessary estimate for II. As for I, we may apply Lemma 3.3 to conclude that

$$\begin{aligned} I &\leq \int_{\Omega_\lambda^\eta}\left(\int_{|x-y|<\eta}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_\epsilon^\eta(x-y)\,dy\right)^{\frac{1}{q}}dx \\ &\leq \int_{\Omega_\lambda^{2\eta}}\left(\int_{B(0,\eta)}\left|\nabla f(x)\cdot\frac{h}{|h|}\right|^{pq}\rho_\epsilon^\eta(h)\,dh\right)^{\frac{1}{q}}dx \qquad (4.7) \\ &\leq \int_\Omega |\nabla f(x)|^p\,dx \end{aligned}$$

and the result is demonstrated. ■

Remark 4.5 *We will later use the fact that if the domain of integration in the inner integral is increased, it does not change the estimate (4.7).*

Theorem 4.6 *Let Ω and ρ_ϵ be as in Theorem 1.5, let $1 \leq q < \infty$ with $1 \leq q \leq \frac{N}{N-1}$ if $N > 1$, and let $f \in BV_{\text{loc}}(\Omega)$ with $Df \in M_b(\Omega;\mathbb{R}^N)$. Then for all $0 < \eta < \frac{\lambda}{3}$ such that $|Df|(\partial\Omega_\lambda^\eta) = 0$ we have*

$$\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^\eta}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx$$
$$\leq |Df|(\Omega) + C_{1,q,\eta,\lambda}\left(||f||_{L^q(\Omega_\lambda)}\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.$$

Proof. We proceed as in Theorem 4.4, and we must again obtain bounds on I and II. We can use the bounds on I as Theorem 4.4, while for II we utilize Lemma 3.5 to conclude that

$$\begin{aligned} II &= \int_{\Omega_\lambda^\eta}\left(\int_{|x-y|<\eta}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon^\eta(x-y)\,dy\right)^{\frac{1}{q}}dx \\ &\leq \int_{\Omega_\lambda^{2\eta}}\left(\int_{B(0,\eta)}\left|\frac{dDf}{d|Df|}(x)\cdot\frac{h}{|h|}\right|^q\rho_\epsilon^\eta(h)\,dh\right)^{\frac{1}{q}}d|Df|(x) \leq |Df|(\Omega), \qquad (4.8) \end{aligned}$$

and the result is demonstrated. ■

We are now able to prove Theorems 1.5 and 1.9.

Proof of Theorem 1.5. Let $f \in L^1_{\rm loc}(\Omega)$ be such that

$$\lim_{\lambda\to 0}\limsup_{\epsilon\to 0}\int_{\Omega_\lambda}\left(\int_{\Omega_\lambda}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx<+\infty.$$

Then applying Theorem 4.1 we have that $f \in W^{1,p}_{\rm loc}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$, with the inequality (4.4),

$$\int_{\Omega_\lambda}\left(\int_{S^{N-1}}(|\nabla f(x)\cdot\sigma|^p)^q\;d\mu(\sigma)\right)^{\frac{1}{q}}dx$$
$$\leq\liminf_{j\to\infty}\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^{2\eta}_\lambda}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx.$$

Conversely, let $f \in W^{1,p}_{\rm loc}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$. Then applying Theorem 4.4 we have that

$$\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^\eta_\lambda}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx$$
$$\leq\int_\Omega|\nabla f(x)|^p\,dx+C_{p,q,\eta,\lambda}||f||^p_{L^{pq}(\Omega^\eta_\lambda)}\left(\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.$$

By the standard embedding theorems, and utilizing the convergence

$$\int_{|h|>\eta}\rho_\epsilon(h)\,dh\to 0,$$

by (1.4), we have that the second term vanishes as $\epsilon\to 0$. Combining this with the estimate from (4.7) in Theorem 4.4, we have

$$\liminf_{j\to\infty}\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^{2\eta}_\lambda}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx$$
$$\leq\liminf_{j\to\infty}\int_{\Omega^{2\eta}_\lambda}\left(\int_{B(0,\eta)}\left|\nabla f(x)\cdot\frac{h}{|h|}\right|^{pq}\rho^\eta_{\epsilon_j}(h)\,dh\right)^{\frac{1}{q}}dx$$
$$=\liminf_{j\to\infty}\int_{\Omega^{2\eta}_\lambda}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\;d\mu^\eta_{\epsilon_j}(\sigma)\right)^{\frac{1}{q}}dx$$
$$=\int_{\Omega^{2\eta}_\lambda}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\;d\mu(\sigma)\right)^{\frac{1}{q}}dx.$$

Combining these two estimates we have

$$\begin{aligned}
&\int_{\Omega_\lambda}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} dx\\
&\quad\le \liminf_{j\to\infty}\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx\\
&\quad\le \int_{\Omega_\lambda^{2\eta}}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} dx,
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\Omega_\lambda}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} dx\\
&\quad\le \liminf_{j\to\infty}\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx\\
&\quad\le \int_{\Omega}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} dx,
\end{aligned}$$

and finally sending $\lambda\to 0$ the result is demonstrated.

When ρ_ϵ satisfy (1.5), by Remark 2.2 we have that $\mu=\mathcal{H}^{N-1}$. In this case, utilizing rotational invariance of $\mathcal{H}^{N-1}$ on the sphere S^{N-1} we have

$$\begin{aligned}
\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} &= |\nabla f(x)|^p\left(\int_{S^{N-1}}|e_1\cdot\sigma|^{pq}\ d\mathcal{H}^{N-1}(\sigma)\right)^{\frac{1}{q}}\\
&= K_{p,q,N}|\nabla f(x)|^p.
\end{aligned}$$

Since Remark 2.2 asserts the convergence of the full sequence in the radial case, we are hence able to conclude the limit (1.12) exists. ■

Proof of Theorem 1.9. Let $f\in L^1_{\mathrm{loc}}(\Omega)$ be such that

$$\lim_{\lambda\to0}\limsup_{\epsilon\to0}\int_{\Omega_\lambda}\left(\int_{\Omega_\lambda}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}} dx<+\infty.$$

Then applying Theorem 4.2 we have that $f\in BV_{\mathrm{loc}}(\Omega)$ and $Df\in M_b(\Omega;\mathbb{R}^N)$, with the inequality (4.6)

$$\begin{aligned}
&\int_{\Omega_\lambda}\left(\int_{S^{N-1}}\left|\frac{dDf}{d\,|Df|}(x)\cdot\sigma\right|^q\ d\mu(\sigma)\right)^{\frac{1}{q}} d\,|Df|\,(x)\\
&\quad\le \liminf_{j\to\infty}\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^{2\eta}}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx.
\end{aligned}$$

Conversely, let $f \in BV_{\rm loc}(\Omega)$ and $Df \in M_b(\Omega;\mathbb{R}^N)$. Then applying Theorem 4.6 we have that

$$\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^\eta}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx$$

$$\leq |Df|(\Omega)+C_{p,q,\eta,\lambda}\left(||f||_{L^q(\Omega_\lambda^\eta)}\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.$$

Taking the limit as $\epsilon\to 0$, the Sobolev-Gagliardo-Niremberg embedding theorem implies that the second right-hand-side term vanishes, so that letting $\lambda\to 0$, we see that the left-hand-side is finite. To prove the final part of the statement, we reason as in the proof of Theorem 1.5. Given λ, choose η such that $|Df|(\partial\Omega_\lambda^{2\eta})=0$. Combining inequality (4.6) with (4.8) (and using non-negativity of the integrand as in the last inequalities in Theorem 1.5), we have

$$\int_{\Omega_\lambda}\left(\int_{S^{N-1}}\left|\frac{dDf}{d|Df|}(x)\cdot\sigma\right|^q d\mu(\sigma)\right)^{\frac{1}{q}}d|Df|$$

$$\leq \liminf_{j\to\infty}\int_{\Omega_\lambda^\eta}\left(\int_{\Omega_\lambda^{2\eta}}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx$$

$$\leq \int_{\Omega}\left(\int_{S^{N-1}}\left|\frac{dDf}{d|Df|}(x)\cdot\sigma\right|^q d\mu(\sigma)\right)^{\frac{1}{q}}d|Df|(x).$$

Finally, sending $\lambda\to 0$ the result is demonstrated.

Under the assumption ρ_ϵ satisfy (1.5), we reason as in the previous proof to conclude the corresponding convergence of the functional to $K_{1,q,N}|Df|(\Omega)$. ■

To conclude, we provide proofs to Theorems 1.13 and 1.14.

Proof of Theorem 1.13. This is based on a counterexample of Fraenkel [19], constructed from work by Courant and Hilbert. We find a set Ω, and $f\in W^{1,2}(\Omega)$ such that

$$\liminf_{\epsilon\to 0}\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^2}{d_\Omega(x,y)^2}\hat\rho_\epsilon(d_\Omega(x,y))\,dydx=+\infty,$$

where $\hat\rho_\epsilon$ are radial mollifiers satisfying (1.3) and (1.4). Initially posed as an example of when the embedding of $W^{1,2}$ into L^q fails for $q>2$, due to a lack of regularity of the boundary, the construction from Section 2.2, Example (i) of [19] is as follows. Let $N=2$, $p=2$, and construct Ω as follows. Let

$$h_j:=j^{-\frac{3}{2}},\quad \delta_j:=j^{-\frac{5}{2}},\quad c_n:=\sum_{i=1}^n h_i, \tag{4.9}$$

and use these sequences to define *rooms* R_j and *passages* P_{j+1},

$$R_j:=(c_j-h_j,c_j)\times\left(-\frac{1}{2}h_j,\frac{1}{2}h_j\right),$$

$$P_{j+1}:=[c_j,c_j+h_{j+1}]\times\left(-\frac{1}{2}\delta_{j+1},\frac{1}{2}\delta_{j+1}\right),$$

$$\Omega:=\bigcup_{i\text{ odd}}R_i\cup P_{i+1},$$

so that Ω is open. Given Ω, we define for j odd

$$f(x) := \begin{cases} K_j := \frac{j}{\log 2j} & x \in R_j \\ K_j + (K_{j+2} - K_j)\frac{x-c_j}{h_j+1} & x \in P_{j+1}. \end{cases}$$

As mentioned, in [19] Fraenkel demonstrates that $f \in W^{1,2}(\Omega)$, but $f \notin L^q(\Omega)$ for $q > 2$, so that Ω is not an extension domain. We continue this example, letting $\hat{\rho}(x) = \frac{1}{\alpha_2}\chi_{[0,1)}(|x|)$, and $\hat{\rho}_\epsilon(x) = \frac{1}{\epsilon^2}\hat{\rho}(\frac{x}{\epsilon})$. Consider

$$\begin{aligned} I &:= \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx \\ &\geq \sum_{4h_i<\epsilon} \int_{R_i} \int_{R_{i+2}} \frac{|f(x) - f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx, \end{aligned}$$

where we have thrown away the integral for all but neighboring rooms, and have began summing for $4h_i < \epsilon$. Since $d_\Omega(x,y) < 4h_i < \epsilon$ for $x \in R_i$, $y \in R_{i+2}$, we have $\hat{\rho}_\epsilon(d_\Omega(x,y)) = \frac{C}{16\epsilon^2}$, and $\frac{1}{d_\Omega(x,y)^2} \geq \frac{1}{(4h_i)^2}$, so that

$$I \geq \frac{C}{16\epsilon^2} \sum_{4h_i<\epsilon} \int_{R_i} \int_{R_{i+2}} \frac{|\frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)}|^2}{(4h_i)^2}\, dydx.$$

Now, $|R_i| = h_i^2$, while $|R_{i+2}| = h_{i+2}^2$, so that

$$I \geq \frac{C}{\epsilon^2} \sum_{4h_i<\epsilon} h_{i+2}^2 \left| \frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)} \right|^2.$$

By (4.9), and solving the equation $4h_i < \epsilon$ for i in terms of ϵ, we have

$$I \geq \frac{C}{\epsilon^2} \sum_{i>(\frac{4}{\epsilon})^{\frac{2}{3}}} \frac{1}{(i+2)^3} \left| \frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)} \right|^2.$$

However, considering the square term, we find a common denominator and expand to see that

$$\begin{aligned} \left| \frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)} \right|^2 &= \left| \frac{i\left(\log(2i) + \log(1+\frac{2}{i})\right) - (i+2)\log(2i)}{\log(2i)\log(2i+4)} \right|^2 \\ &= \left| \frac{\log(1+\frac{2}{i})^i}{\log(2i)\log(2i+4)} - \frac{2}{\log(2i+4)} \right|^2 \\ &\geq \frac{-4\log(1+\frac{2}{i})^i}{\log(2i)(\log(2i+4))^2} + \frac{4}{(\log(2i+4))^2} \\ &= \frac{2\log(2i) - 4\log(1+\frac{2}{i})^i}{\log(2i)(\log(2i+4))^2} + \frac{2}{(\log(2i+4))^2} \\ &\geq \frac{2}{(\log(2i+4))^2}, \end{aligned}$$

whenever i is large enough. Using this lower bound with the above inequality for I, we have

$$I \geq \frac{C}{16\epsilon^2} \sum_{i>(\frac{4}{\epsilon})^{\frac{2}{3}}} \frac{2}{(i+2)^3(\log(2i+4))^2},$$

for ϵ small. Now, the function $i \mapsto \frac{2}{(i+2)^3(\log(2i+4))^2}$ is decreasing, and so we may use the integral test to determine the convergence of the series. Thus,

$$\begin{aligned} I &\geq \frac{C}{16\epsilon^2} \int_{(\frac{4}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{(x+2)^3(\log(2x+4))^2} \, dx \\ &\geq \frac{C}{16\epsilon^2} \int_{(\frac{8}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{x^3(\log(x))^2} \, dx \end{aligned}$$

We utilize L'Hôpital's rule to calculate the limit of the right hand side

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{C}{16\epsilon^2} \int_{(\frac{8}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{x^3(\log(x))^2} \, dx &= \lim_{\epsilon \to 0} \frac{C}{32\epsilon} \frac{2}{4^3\epsilon^{-2}(\log(4\epsilon^{\frac{-2}{3}}))^2} \frac{8}{3}\epsilon^{\frac{-5}{3}} \\ &= \lim_{\epsilon \to 0} \tilde{C} \frac{1}{\epsilon^{\frac{2}{3}}(\log(4\epsilon^{\frac{-2}{3}}))^2}, \end{aligned}$$

and since $x \log^2(x) \to 0$ as $x \to 0$, we conclude that the limit of the right hand side is $+\infty$, so that

$$\liminf_{\epsilon \to 0} I = +\infty,$$

and the result is demonstrated. ■

Proof of Theorem 1.14. If we consider in $N = 2$ the unit disc without the positive x-axis, and the function

$$f(x, y) = \begin{cases} \operatorname{sign} y & \frac{1}{3} < x < 1, \\ 0 & x < 0, \end{cases}$$

and connected by affine functions between. Then for ρ_ϵ of the form

$$\hat{\rho}_\epsilon(|x|) = \tilde{\rho_\epsilon}(|x|) + \chi_{[1-\epsilon,1]}(|x|),$$

where $\int \tilde{\rho}_\epsilon = 1 - C(\epsilon)$ where $C(\epsilon) \to 0$ as $\epsilon \to 0$, and $\int \chi_{[1-\epsilon,1]} = C(\epsilon)$. Then $\int \hat{\rho}_\epsilon = 1$ and $\hat{\rho}_\epsilon = 1$ in a small tubular neighborhood of points of distance approximately 1. Let $Q_\epsilon(x, y)$ be a cube centered at (x, y) with side length ϵ, and $Q_\epsilon^+, Q_\epsilon^-$ be the upper and lower halves of such a cube. Then we can calculate

$$\begin{aligned} &\liminf_{\epsilon \to 0} \int_{Q_\epsilon^+(\frac{1}{2},0)} \int_{Q_\epsilon^-(\frac{1}{2},0)} \frac{2}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y)) \, dy dx \\ &\leq \limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y)) \, dy dx \end{aligned}$$

we can see that for every $\epsilon > 0$ we can find a small region inside the first cube where the inner integral is infinite (since $\hat{\rho}_\epsilon$ is 1 in a small region and $\frac{1}{|h|^p}$ is not integrable in two dimensions for any $p > 1$). This family ρ_ϵ is also an example of why we require the specific bounds on q as given in the main results of the paper. If ρ_ϵ are radial scalings of one function, or more generally, we can get control on the support as a function of ϵ, then we can relax the hypothesis on q to be $1 \leq q < \infty$ for any value of p. ■

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