Congetture riguardanti alcuni problemi di evoluzione.

A Paper in Honor of John Nash

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Summary. I have greatly appreciated the invitation to write a paper in honor of John Nash, a scientist who has been a major source of new ideas for mathematicians. His work has been a clear example of how the most original mathematical ideas are often close to the fundamental problems of other disciplines. I believe that his example is very important for any student who is still motivated by the drive that the ancients called *Philosophy*, that is love for Wisdom.

The character of my paper is exploratory.
Some conjectures concerning "evolution problems" are presented. They are related to the "steepest descent", to the approximations of Newton's gravitation law, to hyperbolic non linear equations and to "descent movements" of manifolds.
The article identifies several questions, of which I do not know the answer, and points out a number of analogies between problems that are apparently far from each other. I believe that the study of these conjectures might provide an opportunity for scientists that are expert in different fields within pure and applied mathematics to get together and investigate on the connections existing among various mathematical concepts, such as linear vs. non-linear behaviour, stability vs. instability, or convergence of different approximation methods, as well as certain ideas well developed in physics, such as deterministic vs. non-deterministic behaviour, predictability vs. non-predictability, order and chaos, etc.

Problem 1. Steepest Descent Problems.

We begin with the analysis of some problems related to the "steepest descent method". We deal with functionals of the following kind

$$F(w,A)=\int_A g(w,\nabla w,\,\dots\,,\nabla^k w). \tag{1}$$

Here A is a bounded open subset of $\mathbf{R}^n$, w is a smooth real function, g is a smooth real function defined on the space of the real polynomials of n variables with degree not greater than k.

We say that a function $\psi\in C^\infty(\mathbf{R}^n)$ is the "Euler function" associated to F and to w, and we write

$$\psi=\mathcal{E}(F,w),$$

if for every bounded open set $A\subset\mathbf{R}^n$ and for every test function $\tau\in C_0^\infty(A)$ the following condition holds

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left(F(w+\lambda\tau,A)-F(w,A)\right)=\int_A \tau\,\psi. \tag{2}$$

Sometimes the function g may be neither convex nor enjoy any other condition close to convexity, see [4]. In these cases to study the differential equation

$$\frac{\partial u}{\partial t}=-\mathcal{E}(F,u(\cdot,t)) \tag{3}$$

with the initial condition

$$u(x,0)=\varphi(x) \tag{4}$$

in the cone $\mathbf{R}^n\times[0,+\infty)$ may be very difficult. Nevertheless a vanishing–viscosity type method seems to be suitable (see for instance [5]). It reads as follows: one associates F the functionals F_ε defined for every $\varepsilon\neq 0$, by the formula

$$F_\varepsilon(w,A)=F(w,A)+\varepsilon^2\int_A\left|\nabla^{k+1}w\right|^2$$

where as usual

$$|\nabla^s w|^2=\sum_{j_1=1}^n\dots\sum_{j_s=1}^n\Big(\frac{\partial^s u}{\partial x_{j_1}\dots\partial x_{j_s}}\Big)^2.$$

Hence the task is the approximation of the differential equation (3) with the equation

$$\frac{\partial u}{\partial t} = -\mathcal{E}(F_\varepsilon, u(\cdot, t)). \tag{5}$$

A simple but really not trivial example of this situation is the following: let $k = 1$, $n = 1$ (see [10,11]),

$$g\left(w, \frac{\partial w}{\partial x}\right) = \frac{1}{2}\log\left(1 + \left|\frac{\partial w}{\partial x}\right|^2\right).$$

In this case the equation (3) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{1 + \left|\frac{\partial u}{\partial x}\right|^2}\right) = \frac{\partial^2 u}{\partial x^2}\frac{1 - \left|\frac{\partial u}{\partial x}\right|^2}{\left(1 + \left|\frac{\partial u}{\partial x}\right|^2\right)^2}$$

The study of this equation may be very hard when the condition $\left|\frac{\partial \varphi}{\partial x}\right| > 1$ is fulfilled by the initial value φ in some subset of $\mathbf{R}$. Then the functional F_ε can be considered. This one leads to the "stabilized" equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\frac{1 - \left|\frac{\partial u}{\partial x}\right|^2}{\left(1 + \left|\frac{\partial u}{\partial x}\right|^2\right)^2} - \varepsilon^2\frac{\partial^4 u}{\partial x^4}, \tag{6}$$

and leads to state the first conjecture.

Conjecture 1. If $\varphi \in C_0^\infty(\mathbf{R})$ and for every $\varepsilon \neq 0$ u_ε is a solution of the equation (6) with the initial condition (4) and it is bounded in $\mathbf{R} \times [0, +\infty)$, then for almost all $(x, t) \in \mathbf{R} \times [0, +\infty)$ the limit of $u_\varepsilon(x, t)$ there exists as $\varepsilon \to 0$.

Remark 1. If Conjecture 1 is true, it could be strengthened by requiring the convergence *for all* (x, t), and then by requiring much stronger convergences leading to further regularity properties of the limit function. The convergence may be requested to have a certain degree of uniformity with respect to the initial value φ, and then the continuous dependence of the limit on the initial datum may be also requested.
On the other hand, if any counterexample to Conjecture 1 is found, the conjecture itself should be weakened. This can be done in several ways. E.g., the convergence of the solutions of equations (6) would already be satisfactory for initial data simply belonging to a dense set in some typical metric or topological space. For instance analytical functions spaces, classes of trigonometric polynomials or of functions built up by Gaussian, could be considered to this aim. Moreover these topological spaces could be endowed with some

natural measure μ, hence the convergence of the u_ε might simply be required for μ–almost all initial data.

Remark 2. A further weakening of Conjecture 1 could be obtained by relaxing the global condition of convergence for almost all the points of $\mathbf{R}\times[0,+\infty)$ to the less severe request of a local almost sure convergence. Namely: there exists $T>0$ such that u_ε converge for almost all the points of $\mathbf{R}\times[0,T)$. A first step towards the evidence of Conjecture 1 or the discovery of any counterexample can be the study of sequences of positive real numbers (ε_i) such that $\varepsilon_i\to 0$ and the sequence (u_{ε_i}) converges in some sense.

Remark 3. Several versions of Conjecture 1 may be considered: functions defined in $\mathbf{R}$ can be replaced by functions defined in some interval of $\mathbf{R}$ with the right boundary conditions, initial data with compact support and bounded solutions may be replaced respectively by periodic initial data and periodic solutions in the space variable x, bounded solutions can be replaced by solutions of (6) verifying some decay condition at infinity. More generally we can consider

$$F(w,A)=\frac{1}{2}\int_A \log(1+|\nabla w|^2)\ .$$

Similarly a new functional, in the case of only one space-variable, such as

$$F(w,A)=\int_A\left(1-\left|\frac{\partial w}{\partial x}\right|^2\right)^2\ ,$$

leads to consider the correspondent functional in $\mathbf{R}^n$

$$F(w,A)=\int_A\left(1-|\nabla w|^2\right)^2\ .$$

Interesting results could be obtained by studying other cases, for instance setting

$$F_\varepsilon(w,A)=\int_A \varepsilon^2 w^2+\left(1-|\nabla w|^2\right)^2+\varepsilon^2|\nabla^2 w|^2.$$

Eventually non-stationary and nonhomogeneous cases can be considered: the function g appearing in (1) should depend not only on w and on its derivatives but also on the space variables $(x_1,\dots,x_n)$ and possibly on a parameter t, that in (3), (5) is a time variable.

Problem 2. Approximate Gravitational Problem.

We now formulate a problem related to a reasonable approximation of Newton's law (see [3,7]); this problem will be called "approximate gravitational problem" and it can be formulated as follows:

Problem 2.1. Let μ be a finite positive σ-additive measure defined on the σ-algebra of Borel subsets of $\mathbf{R}^6$. The approximate gravitational problem associated to μ is a triplet of functions (s, v, u) satisfying the qualitative conditions **(A)** and the integral–differential equations **(B)** below:

(A)

$$s = s(\varepsilon, \xi, \eta, t) \in \left[C^1((\mathbf{R} \setminus \{0\}) \times \mathbf{R}^7)\right]^3,$$

$$v = v(\varepsilon, \xi, \eta, t) \in \left[C^1((\mathbf{R} \setminus \{0\}) \times \mathbf{R}^7)\right]^3,$$

$$u = u(\varepsilon, x, t) \in C^0((\mathbf{R} \setminus \{0\}) \times \mathbf{R}^4).$$

The scalar function u is differentiable with respect to the space variable $x = (x_1,\ x_2,\ x_3)$ and the spatial gradient $\nabla_x u$ belongs to $C^0((\mathbf{R} \setminus \{0\}) \times \mathbf{R}^4)^3$.

(B)

The vector functions s, v satisfy

$$\frac{\partial s}{\partial t} = v, \quad \frac{\partial v}{\partial t} = \alpha(\varepsilon, s(\varepsilon, \xi, \eta, t), t), \quad \alpha(\varepsilon, x, t) = \nabla_x u(\varepsilon, x, t) \tag{1}$$

with the initial conditions

$$s(\varepsilon, \xi, \eta, 0) = \xi, \qquad v(\varepsilon, \xi, \eta, 0) = \eta \tag{2}$$

whereas u satisfies the integral equation

$$u(\varepsilon, x, t) = \int_{\mathbf{R}^6} (|x - s(\varepsilon, \xi, \eta, t)|^2 + \varepsilon^2)^{-1/2} d\mu(\xi, \eta). \tag{3}$$

Remark 1. Intuitively, $s(\varepsilon, \xi, \eta, t)$ represents the position at time t of a particle located at $t = 0$ in the point ξ with speed η; $v(\varepsilon, \xi, \eta, t)$ represents the velocity at time t of the same particle and $\alpha(\varepsilon, x, t)$ represents the acceleration of a particle that at time t is located in x. Finally, u should give, when ε is small, a reasonable approximation of Newton's potential. The measure μ represents an initial distribution of masses (possibly endowed with some velocity) at time $t = 0$.

In order to understand more clearly the meaning of (1), (2), (3), it is convenient to consider the case of a measure μ which can be written as the sum of n Dirac measures, i.e., there exist (ξ_h, η_h) such that

$$\mu(B) = \sum_{h=1}^{n} m_h \delta(\xi_h, \eta_h)(B)$$

for any Borel set B, where

$$\begin{cases} \delta(\xi_h, \eta_h)(B) = 1 & \text{if } (\xi_h, \eta_h) \in B \\ \delta(\xi_h, \eta_h)(B) = 0 & \text{if } (\xi_h, \eta_h) \notin B. \end{cases}$$

In this case the potential u appearing in (3) becomes

$$u(\varepsilon, x, t) = \sum_{h=1}^{n} m_h (|x - s(\varepsilon, \xi_h, \eta_h, t)|^2 + \varepsilon^2)^{-1/2}.$$

Hence, the system of equations (1), (2), (3) appears to be a reasonable approximation of the classical n-body problem.

Remark 2. The case in which μ is not supported in a finite number of points is suggested by the universe, in which stars, planets, satellites, gas clouds and other forms of "diffuse mass" interact. In this case, (3) can be written in the form

$$(3^*) \qquad u(\varepsilon, x, t) = \int_{\mathbf{R}^6} (|x - \xi|^2 + \varepsilon^2)^{-1/2} d\mu_t^*(\xi, \eta),$$

where the measure μ_t^* is given for any $t \in \mathbf{R}$ by the formula

$$\mu_t^*(B) = \mu\left(\left\{(\xi, \eta) \in \mathbf{R}^6 : \ \big(\sigma(\varepsilon, \xi, \eta, t), v(\varepsilon, \xi, \eta, t)\big) \in B\right\}\right).$$

Intuitively, μ_t^* describes for any $t \in \mathbf{R}$ the distribution and the velocities at time t of the mass in the universe; the evolution of μ_t^* essentially describes what would be the evolution of the universe under an "approximate gravitational field" computable from the "elementary potential"

$$(4) \qquad V_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{-1/2} \ ,$$

which, for ε small, seems to be a reasonable approximation of the classical Newton's potential, proportional to

$$V(x) = \frac{1}{|x|}.$$

To my knowledge, in the literature there is not an exhaustive answer to the questions about the existence of limits, as $\varepsilon \to 0$, of the functions considered in the approximate gravitational problem. In particular, I am very interested in the proof or in a counterexample of the following conjecture.

Conjecture 1. If (s, v, u) is a solution of the approximate gravitational problem associated to μ, then for almost every point $(x, t) \in \mathbf{R}^4$ there exists

$$\lim_{\varepsilon \to 0} u(\varepsilon, x, t) \in \mathbf{R};$$

moreover, for almost every $(\xi, \eta, t) \in \mathbf{R}^7$, there exist the limits

$$\lim_{\varepsilon \to 0} s(\varepsilon, \xi, \eta, t),$$

$$\lim_{\varepsilon \to 0} v(\varepsilon, \xi, \eta, t).$$

Remark 3. It is very likely that Conjecture 1 holds for large classes of measures μ but not for all measures μ considered in Problem 2.1. Up to now, I didn't find in the literature any counterexample; I think that a counterexample may be looked for in the class of measures μ satisfying the conditions

$$\mu = \sum_{h=1}^{+\infty} m_h \delta(\xi_h, \eta_h), \qquad \sum_{h=1}^{+\infty} m_h < +\infty, \qquad \sum_{h=1}^{+\infty} |\eta_h| = +\infty.$$

More difficult but also more interesting is the case

$$\mu = \sum_{h=1}^{+\infty} m_h \delta(\xi_h, \eta_h), \qquad \sum_{h=1}^{+\infty} m_h < +\infty, \qquad \sup_h \left(|\xi_h| + |\eta_h|\right) < +\infty$$

corresponding to a situation in which initially all the masses are confined in a bounded region in space and the set of their initial velocities is bounded as well.

We can also state a weaker conjecture.

Conjecture 2. Let $\mathcal{M}(\mathbf{R}^6)$ be the set of positive σ-additive measures μ defined on Borel subsets of $\mathbf{R}^6$ and satisfying the condition $\mu(\mathbf{R}^6) < +\infty$. Endowing $\mathcal{M}(\mathbf{R}^6)$ with the distance

$$\delta(\mu, \nu) = \sup_B \left(\mu(B) - \nu(B)\right) + \sup_B \left(\nu(B) - \mu(B)\right),$$

the set of measures μ satisfying Conjecture 1 is dense.

Moreover, it would be interesting to evaluate the stability with respect to μ of the limits as $\varepsilon \to 0$ of the solutions of the approximate gravitational problems, in the cases where the limit exist. Very likely there is not always an high stability of the limits, hence it might be convenient to introduce a weaker condition, called α–stability, with the following definition.

Definition 1. Given a measure μ, the functions (s, v, u) solving the approximate gravitational problem associated to μ and a real number $\alpha \geq 0$, we say that (s, v, u) is an $\alpha-$stable solution of the approximate gravitational problem associated with μ if the limits

$$\lim_{\varepsilon \to 0} u(\varepsilon, x, t),$$

$$\lim_{\varepsilon \to 0} s(\varepsilon, \xi, \eta, t),$$

$$\lim_{\varepsilon \to 0} v(\varepsilon, \xi, \eta, t),$$

exist for almost every $(x, t) \in \mathbf{R}^4$ and almost every $(\xi, \eta, t) \in \mathbf{R}^7$. Moreover, we ask that for any infinitesimal sequence (ε_i), and any sequence of measures (μ_i) satisfying the condition

$$\lim_{i \to +\infty} {\varepsilon_i}^{-\alpha} \delta(\mu_i, \mu) = 0,$$

denoting by (s_i, v_i, u_i) the solutions of the approximate gravitational problem corresponding to μ_i, we have

$$\lim_{i \to +\infty} u_i(\varepsilon_i, x, t) = \lim_{\varepsilon \to 0} u(\varepsilon, x, t),$$

for almost every $(x, t) \in \mathbf{R}^4$, and, for almost every $(\xi, \eta, t) \in \mathbf{R}^7$ we ask that

$$\lim_{i \to +\infty} s_i(\varepsilon_i, \xi, \eta, t) = \lim_{\varepsilon \to 0} s(\varepsilon, \xi, \eta, t),$$

$$\lim_{i \to +\infty} v_i(\varepsilon_i, \xi, \eta, t) = \lim_{\varepsilon \to 0} v(\varepsilon, \xi, \eta, t).$$

We can now formulate a meaningful reinforcement of Conjecture 2 with the following

Conjecture 3. There exist positive numbers α such that the set of all measures μ corresponding to $\alpha-$stable solutions of the approximate gravitational problem is dense in $\mathcal{M}(\mathbf{R}^6)$.

Remark 4. If Conjecture 3 were true, it would be interesting to compute the infimum of the numbers α for which the density property is true; this infimum is very likely strictly positive. Taking into account that $\alpha-$stability becomes weaker as α increases, the infimum would give an indication of the global degree of stability of the approximate gravitational problem.

3. Some Variants of the Approximate Gravitational Problem.

We can consider many variants of the gravitational problem shown in §2. We could imagine that the force acting between 2 bodies is a sum of two parts, one attractive and the other repulsive, for instance replacing the equation (3) of §1 with

$$u(\varepsilon,x,t)=\int_{\mathbf{R}^6}(|x-s|^2+\varepsilon^2)^{-1/2}-c^2(|x-s|^2+\varepsilon^2)^{-1}d\mu(\xi,\eta)$$

where c is a positive constant.

After the limits for $\varepsilon\to 0$, the convergence of such limits when $c\to 0$ should also be considered; this means, in some sense, that the repulsive forces act only when the distance is very small. Finally, we could imagine even more complex expressions, corresponding to forces depending not only on the position but also on the velocity of the particles, for instance studying integrals like

$$\alpha(\varepsilon,x,y,t)=\int_{\mathbf{R}^6}\phi(\varepsilon,x,y,s(\varepsilon,\xi,\eta,t),v(\varepsilon,\xi,\eta,t))d\mu(\xi,\eta)$$

hence differential equations,

$$\frac{\partial v}{\partial t}=\alpha(\varepsilon,s,v,t).$$

Clearly, in modelling very complex physical theories, many further complications may arise and these are beyond the aim of this note, which is only directed to underline some kinds of mathematical problems that subsequently can also produce interesting physical models.

Coming back to the approximate gravitational problem, particularly to Remark 2 of §2, it can be studied the case in which the mass μ^* is continuously distributed, that is, there exists a function $\varphi=\varphi(\varepsilon,x,y,t)\in C^\infty(\mathbf{R}\setminus\{0\}\times\mathbf{R}^7)$ such that

$$\mu_t^*(B)=\int_B\varphi(\varepsilon,x,y,t)dxdy; \tag{1}$$

then the function $u(\varepsilon,x,t)$ will be given by the formula

$$u(\varepsilon,x,t)=\int_{\mathbf{R}^6}\left(|x-\xi|^2+\varepsilon^2\right)^{-1/2}\ \varphi(\varepsilon,\xi,\eta,t)\,d\xi\,d\eta,$$

moreover φ will satisfies the first order linear differential equation

$$\frac{\partial\varphi}{\partial t}=-\sum_{h=1}^{3}\left(\frac{\partial\varphi}{\partial x_h}y_h+\frac{\partial\varphi}{\partial y_h}\frac{\partial u}{\partial x_h}\right)$$

whose characteristics are determined by equations (1), (2) of Problem 2.1. Hence, we can state the problem of existence of the limits as $\varepsilon\to 0$ of the function $\varphi(\varepsilon,x,y,t)$ and we can study the possible relations with the limits considered in the Conjectures 1, 2, 3 of §2. Probably in this analysis, interesting phenomena may happen, concerning the solutions (generalized or not) of the first order partial differential equations.

4. Limits of Variational Problems Related to Hyperbolic Equations.

In the research of suitable approximations of difficult and unstable problems with easier and more stable ones, we can include the idea of finding solutions of evolution problems as a limit of solutions of minimizing problems. An interesting example where this approach was successfully applied is in the paper [8]. A possible variant of Ilmanen's result is given by the following conjecture.

Conjecture 1. Let be given $\varphi, \psi \in C_0^\infty(\mathbf{R}^n)$, $k > 1$ integer. For every $\lambda > 0$ let $w_\lambda = w_\lambda(x_1, \dots, x_n, t)$ be a minimizer of the functional

$$F_\lambda(u) = \int_{\mathbf{R}^n \times [0,+\infty[} e^{-\lambda t} \left[\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + \lambda^2 |\nabla_x u|^2 + \lambda^2 u^{2k} \right] dx\, dt \tag{1}$$

in the class of all functions u satisfying the initial conditions

$$u(x,0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x).$$

Then, there exists $\lim_{\lambda \to +\infty} w_\lambda(x,t) = w_0(x,t)$, and satisfies the equation

$$\frac{\partial^2 w_0}{\partial t^2} = \Delta_x w_0 - k w_0^{2k-1}. \tag{2}$$

Remark 1. If this conjecture is false, then we can imagine various modifications, for instance, asking whether it holds for a dense set of initial data, replacing C^∞ with the analytic functions rapidly decreasing, considering only periodic functions, in particular trigonometric polynomials, or even considering functionals F_λ different from the one defined in (1) which share minimizers converging to solutions of equation (2). Otherwise, if the conjecture can be proved, then we can think to a large number of generalizations, starting from a functional F of the calculus of variations and associating to it the functional

$$F_\lambda(u) = \int_{\mathbf{R}^n \times [0,+\infty[} e^{-\lambda t} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt + \lambda^2 \int_0^{+\infty} e^{-\lambda t} F\big(u(\cdot, t)\big)\, dt.$$

5. Motion of Manifolds According to Euler Functions.

In order to define the motion of embedded manifolds of any dimension and codimension, we establish some notations.
For every open set $A \subset \mathbf{R}^n$ we denote as usual by $C^\infty(A)$ the space of real smooth functions on A. If E is not open, we say that $f : E \to \mathbf{R}$ belongs to $C^\infty(E)$ if f admits a smooth extension to some open set A which includes E. Given open sets A, Ω, we write $A \subset\subset \Omega$ when A is relatively compact in Ω. Finally, for every $E \subset \mathbf{R}^n$ we define the function $\eta_E : \mathbf{R}^n \to [0, \infty)$ as follows:

$$\eta_E(x) = \frac{1}{2}[\mathrm{dist}(x, E)]^2 = \inf\left\{\frac{|x-\xi|^2}{2};\ \xi \in E\right\}. \tag{1}$$

Now we can give the following definition of h-dimensional manifolds of class C^∞ in an open set Ω.

Definition 1. Let Ω be an open subset of $\mathbf{R}^n$, and let h be an integer with $0 \le h \le n$. We say that a set E is an h-dimensional manifold of class C^∞ in Ω, and we write $E \in V_h C^\infty(\Omega)$, when the following three conditions hold:

(a) $$\Omega \cap E = \{x \in \Omega : \eta_E(x) = 0\};$$

(b) there exists an open set $A \supset \Omega \cap E$ such that $\eta_E \in C^\infty(A)$;

(c) for every $x \in E \cap \Omega$ the Hessian matrix $\nabla^2 \eta_E(x)$ has rank $n - h$.

When g is a smooth function defined on the class of all real polynomials of n-variables with degree $\le k$, and $E \in V_h C^\infty(\Omega)$, then for every open set $A \subset\subset \Omega$ we define the functional

$$F(E, A) = \int_{A \cap E} g\big(\eta_E(x), \nabla \eta_E(x), \ldots, \nabla^k \eta_E(x)\big)\, d\mathcal{H}^h(x), \tag{2}$$

where $\mathcal{H}^h$ is the h-dimensional Hausdorff measure. Since both η_E and $\nabla \eta_E$ vanish on E, this functional is non-trivial when $k \ge 2$.

We give now the definition of Euler function associated with E and F.

Definition 2. Let Ω be an open subset of $\mathbf{R}^n$, $E \in V_h C^\infty(\Omega)$, $\psi \in (C^\infty(E))^n$, and let F be given as in (2). We say that the vector function ψ is the *Euler function* of F on E, and we write $\psi \in \mathcal{E}(F, E)$, if for every open set $A \subset\subset \Omega$ and every $\tau \in (C_0^\infty(A))^n$ there holds

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\big(F(E_\lambda, A) - F(E, A)\big) = \int_{E \cap A} \sum_{j=1}^n \tau_j(x) \psi_j(x)\, d\mathcal{H}^h(x),$$

where we have set $E_\lambda = \{x + \lambda \tau(x);\ x \in A \cap E\}$.

Now we can give the definition of gradient flow according to the Euler function.

Definition 3. Let be given $E \subset \mathbf{R}^n$, $T \in]0, +\infty]$, $u = u(x_1, \ldots, x_n, t) \in C^\infty(E \times [0, T[)$, and let F be defined as in (2). We say that u is a F-*gradient flow* starting from E, and we write $u \in \mathcal{MD}(F, E, T)$, if the following conditions are fulfilled:

(A) $$u(x,0)=x \text{ for every } x\in E;$$
(B) for every $(x,t)\in E\times[0,T[$ there exists an open set $A(x,t)\subset\mathbf{R}^n$ such that $x\in A(x,t)$, and if we set $L(x,t)=\{u(x,t);\ x\in E\cap A(x,t)\}$,

$$\frac{\partial u}{\partial t}(x,t)=\mathcal{E}\big(F,L(x,t)\big)(u(x,t)) \qquad \forall\,(x,t)\in E\times[0,T[\,.$$

Remark 1. When g is a positive constant, $F(E,A)$ is proportional to $\mathcal{H}^h(E\cap A)$, and then the previous definition gives the usual motion by mean curvature (see for instance [1]). In general we do not expect that the gradient flow exists for every F and every initial data E, but we can still try to approximate the functional F as we did in §1. More precisely, if F is defined by (2), we can set for every $\varepsilon>0$

$$F_\varepsilon(E,A)=F(E,A)+\varepsilon^2\int_{E\cap A}|\nabla^{k+1}\eta_E|^2\,d\mathcal{H}^h\ . \tag{3}$$

Then we have the following

Conjecture 1. If E is a compact set in $V_hC^\infty(\mathbf{R}^n)$, and F is given as in (2), for every $\varepsilon>0$ there exist $T_\varepsilon>0$ and $u_\varepsilon\in\big(C^\infty(E\times[0,T_\varepsilon[)\big)^n$ such that $u_\varepsilon\in\mathcal{MD}(F_\varepsilon,E,T_\varepsilon)$.

When g is a positive constant, we think that T_ε does not depends on ε, as stated in the following

Conjecture 2. In the hypothesis of Conjecture 1, if we assume moreover that $k>h$ and g is a positive constant, then $T_\varepsilon\equiv+\infty$ and then $u_\varepsilon\in\big(C^\infty(E\times[0,+\infty[)\big)^n$.

Remark 2. Conjecture 2 basically gives a standard procedure to approximate the curvature flow. Regarding the limiting behaviour (as $\varepsilon\to0$) of these approximating flows, the problems should be similar to those described in the previous sections. To begin, we suggest the following result, which is more or less a uniqueness result for the curvature flow defined by the "viscosity method".

Conjecture 3. In the hypothesis of Conjecture 2, for $\mathcal{H}^{h+1}$-almost every $(x,t)\in E\times[0,+\infty)$ there exists the limit $\lim_{\varepsilon\to0}u_\varepsilon(x,t)$.

Remark 3. Conjecture 3 can be weakened in many different ways, e.g. by assuming that E belongs to $V_h^\omega(\mathbf{R}^n)$ (and not only $V_h^\infty(\mathbf{R}^n)$), that is, assuming that the function η_E in Definition 1 is analytic on the open set A. Also we could try to verify Conjecture 3 for all E in a some dense subset of the class of all compact elements of $V_hC^\omega(\mathbf{R}^n)$, endowed with a suitable topology.
If Conjecture 3 is true, examples of non-constant g should be examined. From a physical viewpoint, it would be interesting to consider equations of the form

$$\frac{\partial^2 u}{\partial t^2}=\mathcal{E}\big(F,L(x,t)\big)(u(x,t)).$$

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