Linearized elasticity as Γ-limit of finite elasticity

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Abstract. Linearized elastic energies are derived from rescaled non-linear energies by means of Γ-convergence. For Dirichlet and mixed boundary value problems in a Lipschitz domain Ω, the convergence of minimizers takes place in the weak topology of $H^1(\Omega, \mathbf{R}^n)$ and in the strong topology of $W^{1,q}(\Omega, \mathbf{R}^n)$ for $1 \le q < 2$.

Key words: linearized elasticity, Γ-convergence.

1 Introduction

The stored energy of a hyperelastic material can be written in terms of the deformation gradient ∇v as

$$\int_\Omega W(x, \nabla v)\, dx\,, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is the reference configuration, and the energy density $W(x, F)$ is a function defined for $x \in \Omega$ and F in the space $\mathbf{M}^{n\times n}$ of $n\times n$ matrices. The stress tensor corresponding to the deformation gradient ∇v is then given by $T(x, \nabla v) = \partial_F W(x, \nabla v)$.

By frame indifference we can express $W(x, \nabla v)$ in terms of the right Cauchy-Green strain tensor $C(v) := \nabla v^T \nabla v$ or, equivalently, in terms of the Green-St.Venant tensor $\frac{1}{2}(C(v)-I)$, where I is the identity matrix. Thus we can write $W(x, \nabla v) = V(x, \frac{1}{2}(C(v) - I))$ for a suitable function $V(x, E)$ defined for $x \in \Omega$ and E in the space $\mathbf{M}^{n\times n}_{sym}$ of symmetric $n\times n$ matrices.

We prefer to express these quantities in terms of the displacement u, defined by $u(x) := v(x) - x$. As $\nabla v = I + \nabla u$ the Green-St.Venant tensor $\frac{1}{2}(C(v) - I)$ can be written as $E(u) := e(u) + \frac{1}{2}C(u)$, where $e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$ is the symmetric part of the displacement gradient.

We assume that the reference configuration is stress free, i.e., $T(x, I) = 0$, and thus $\partial_F W(x, I) = \partial_E V(x, 0) = 0$. As $W(x, \cdot)$ and $V(x, \cdot)$ are defined up to an additive constant, it is not restrictive to assume also that $W(x, 0) = V(x, 0) = 0$.

Since the displacement $u = 0$ is an equilibrium configuration when no external loads are applied, it is natural to expect small displacements for small external loads. It is then convenient to rescale the variables and to write the load as $\varepsilon\ell$ and the displacement as εu for a suitable (adimensional) small parameter $\varepsilon > 0$. Thus we have $v(x) = x + \varepsilon u(x)$, and the equilibrium configurations are stationary points of the total energy

$$\int_\Omega W(x, I+\varepsilon\nabla u)\,dx - \varepsilon^2 \int_\Omega \ell u\,dx\,. \tag{1.2}$$

As

$$W(x, I+\varepsilon\nabla u) = V(x, \varepsilon e(u) + \tfrac{1}{2}\varepsilon^2 C(u))\,, \tag{1.3}$$

if ∇u is bounded we have, by Taylor expansion,

$$W(x, I+\varepsilon\nabla u) = \varepsilon^2 \tfrac{1}{2}\partial_E^2 V(x,0)[e(u), e(u)] + o(\varepsilon^2)\,, \tag{1.4}$$

where $\partial_E^2 V(x,\cdot)$ denotes the second derivative of $V(x,\cdot)$ on $\mathbf{M}^{n\times n}_{sym}$, and $o(\varepsilon^2)$ is uniform with respect to x. The tensor $\mathsf{A}(x) := \partial_E^2 V(x,0)$ is called the elasticity tensor, and the linearized elastic energy is then defined as

$$\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\,dx\,.$$

The previous discussion shows that, if we rescale the total energy given by (1.2), we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\int_\Omega W(x, I+\varepsilon\nabla u)\,dx - \varepsilon^2\int_\Omega \ell u\,dx\Big) = \frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\,dx - \int_\Omega \ell u\,dx \tag{1.5}$$

for every Lipschitz function u. This equality is usually considered as the main justification of linearized elasticity.

Note that this argument does not prove that the minimizers u_ε of (1.2), satisfying suitable boundary conditions, actually converge to the minimizer of the corresponding limit problem

$$\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\,dx - \int_\Omega \ell u\,dx\,.$$

Indeed we shall see that this is not always true (see Example 3.5).

In this paper, given a load $\ell \in L^2(\Omega, \mathbf{R}^n)$, a boundary value $g \in W^{1,\infty}(\Omega, \mathbf{R}^n)$, and a closed subset $\partial\Omega_D$ of $\partial\Omega$ with $\mathcal{H}^{n-1}(\partial\Omega_D) > 0$, we consider the minimum problems

$$\min_{u\in H^1_{g,\partial\Omega_D}} \Big\{\int_\Omega W(x, I+\varepsilon\nabla u)\,dx - \varepsilon^2\int_\Omega \ell u\,dx\Big\}\,, \tag{1.6}$$

where $H^1_{g,\partial\Omega_D}$ denotes the closure in $H^1(\Omega, \mathbf{R}^n)$ of the space of functions $u \in W^{1,\infty}(\Omega, \mathbf{R}^n)$ such that $u = g$ on $\partial\Omega_D$. Suppose that, for every $\varepsilon > 0$, there exists a solution u_ε of (1.6) which satisfies the orientation preserving condition $\det(I + \varepsilon\nabla u_\varepsilon) > 0$. Under some natural hypotheses on the function V, we prove that u_ε converges weakly in $H^1(\Omega, \mathbf{R}^n)$ to the (unique) minimizer u_0 of the problem

$$\min_{u\in H^1_{g,\partial\Omega_D}} \Big\{\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\,dx - \int_\Omega \ell u\,dx\Big\}\,.$$

Moreover we prove the convergence of the rescaled energies, i.e.,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big\{\int_\Omega W(x, I+\varepsilon\nabla u_\varepsilon)\,dx - \varepsilon^2\int_\Omega \ell u_\varepsilon\,dx\Big\} = \frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u_0), e(u_0)]\,dx - \int_\Omega \ell u_0\,dx\,. \tag{1.7}$$

More generally, the same results hold if $\det(I+\varepsilon\nabla u_\varepsilon)>0$ and

$$\int_\Omega W(x, I+\varepsilon\nabla u_\varepsilon)\,dx - \varepsilon^2\int_\Omega \ell u_\varepsilon\,dx = \mathcal{J}_\varepsilon + o(\varepsilon^2)\,,$$

where $\mathcal{J}_\varepsilon$ is the (possibly not attained) infimum of problem (1.6). This provides a full variational justification of linearized elasticity.

These results are proved under the following additional hypotheses on V:

(a) $\inf\limits_{|E|\geq\rho}\ \inf\limits_{x\in\Omega} V(x,E)>0$ for every $\rho>0$;

(b) there exist $\alpha>0$ and $\rho>0$ such that $\inf\limits_{x\in\Omega} V(x,E)\geq\alpha|E|^2$ for every $|E|\leq\rho$;

(c) $\liminf\limits_{|E|\to+\infty}\dfrac{1}{|E|}\inf\limits_{x\in\Omega} V(x,E)>0$.

These conditions say that 0 is the unique minimizer of $V(x,\cdot)$ (with a uniform estimate with respect to x) and that $V(x,\cdot)$ grows more than quadratically near the origin and more than linearly at infinity.

If (c) is replaced by the slightly stronger condition

(c') $\liminf\limits_{|E|\to+\infty}\dfrac{1}{|E|^p}\inf\limits_{x\in\Omega} V(x,E)>0$

for some exponent $p>1$, then we prove also that u_ε converges to u_0 strongly in $W^{1,q}(\Omega,\mathbf{R}^n)$ for every $q<2$.

The proof is obtained in two steps. First we show that the sequence u_ε is compact in the weak topology of $H^1(\Omega,\mathbf{R}^n)$, using a recent lemma proved by Friesecke, James and Müller [4]. Then we prove that the functionals

$$\mathcal{F}_\varepsilon(u)=\frac{1}{\varepsilon^2}\int_\Omega V(x,\varepsilon e(u)+\tfrac{1}{2}\varepsilon^2 C(u))\,dx$$

Γ-converge to the functional

$$\mathcal{F}(u)=\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u),e(u)]\,dx\,.$$

These two facts lead to the weak convergence of the solutions in $H^1(\Omega,\mathbf{R}^n)$ and to the convergence of the rescaled energies expressed by (1.7). The strong convergence in $W^{1,q}(\Omega,\mathbf{R}^n)$ for $q<2$ is obtained from (1.7).

2 The main results

Let the reference configuration be an open, bounded, connected domain $\Omega\subset\mathbf{R}^n$, for $n\geq 2$, having Lipschitz boundary. Let $|F|^2=\sum_{i,j}|F_{ij}|^2$ be the norm in the space $\mathbf{M}^{n\times n}$ and let $SO(n)$ be the subset of rotations (orthogonal matrices with positive determinant).

We will assume that the material is hyperelastic, i.e., there exists a stored energy density $W:\Omega\times\mathbf{M}^{n\times n}\to[0,+\infty]$ such that for a.e. $x\in\Omega$ we have

$$W(x,F)=+\infty\quad\text{if }\det F\leq 0 \tag{2.1}$$

(orientation preserving condition), and such that for a.e. $x \in \Omega$

$$W(x, F) < +\infty \tag{2.2}$$

for F in a neighborhood U of the identity I independent of x (so that small deformation of the reference configuration have finite energy). By frame indifference the stored energy density can be written as

$$W(x, F) = V(x, \tfrac{1}{2}(F^T F - I)) . \tag{2.3}$$

where F^T denotes the transpose of the matrix F. We suppose that $V : \Omega \times \mathbf{M}^{n\times n}_{sym} \to \mathbf{R}$ is $\mathcal{L}^n \times \mathcal{B}^n$-measurable (where $\mathcal{L}^n$ and $\mathcal{B}^n$ are the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbf{R}^n$) and that, for some $\delta > 0$, the function $B \to V(x, B)$ is of class C^2 for $|B| < \delta$ and for a.e. $x \in \Omega$. Moreover we will assume that the reference configuration has zero energy and is stress free, which means that for a.e. $x \in \Omega$

$$V(x, 0) = 0 \qquad \partial_E V(x, 0) = 0 . \tag{2.4}$$

Finally we require the coercivity assumptions (a), (b), (c) and for a.e. $x \in \Omega$ the upper bound

$$|\partial^2_E V(x, E)[T, T]| \leq 2\gamma |T|^2 \quad \text{for } |E| < \delta \text{ and } T \in \mathbf{M}^{n\times n}_{sym} \ , \tag{2.5}$$

for some constant $\gamma > 0$ independent of x.

From (2.4) it is easy to deduce by Taylor expansion that for a.e. $x \in \Omega$

$$V(x, E) = \frac{1}{2}\partial^2_E V(x, tE)[E, E] \tag{2.6}$$

for some $t \in (0, 1)$ depending on x, hence

$$|V(x, E)| \leq \gamma |E|^2 \quad \forall E \in \mathbf{M}^{n\times n}_{sym} \text{ with } |E| < \delta \ . \tag{2.7}$$

Let $\mathsf{A}(x) := \partial^2_E V(x, 0)$. From (2.6) and (b) it follows that for a.e. $x \in \Omega$

$$\mathsf{A}(x)[E, E] = \partial^2_E V(x, 0)\,[E, E] \geq 2\alpha |E|^2 \quad \forall E \in \mathbf{M}^{n\times n}_{sym} . \tag{2.8}$$

Finally for every $x \in \Omega$ and $F \in \mathbf{M}^{n\times n}$ let $F_{sym} = (F + F^T)/2$ and

$$W_\varepsilon(x, F) := \frac{1}{\varepsilon^2} W(x, I + \varepsilon F) = \frac{1}{\varepsilon^2} V(x, \varepsilon\, F_{sym} + \tfrac{1}{2}\varepsilon^2 F^T F) . \tag{2.9}$$

It is easy to see that for a.e. $x \in \Omega$

$$\begin{aligned} \lim_{\varepsilon\to 0} W_\varepsilon(x, F) &= \frac{1}{2}\partial^2_F W(x, I)[F, F] \\ &= \frac{1}{2}\partial^2_E V(x, 0)[F_{sym}, F_{sym}] = \frac{1}{2}\mathsf{A}(x)[F_{sym}, F_{sym}] . \end{aligned} \tag{2.10}$$

We consider the functional $\mathcal{F}_\varepsilon : H^1(\Omega, \mathbf{R}^n) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \int_\Omega W_\varepsilon(x, \nabla u)\, dx , \tag{2.11}$$

and the functional $\mathcal{F} : H^1(\Omega, \mathbf{R}^n) \to [0, +\infty)$ given by

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega \mathsf{A}(x)\,[e(u), e(u)]\, dx . \tag{2.12}$$

Let $\partial\Omega_D$ a closed subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\partial\Omega_D) > 0$ and let $g \in W^{1,\infty}(\Omega, \mathbf{R}^n)$. Let $H^1_{g,\partial\Omega_D}$ be the closure in $H^1(\Omega, \mathbf{R}^n)$ of the space of functions $u \in W^{1,\infty}(\Omega, \mathbf{R}^n)$ such that $u = g$ on $\partial\Omega_D$. By strong (resp. weak) topology in $H^1_{g,\partial\Omega_D}$ we mean the restriction of the strong (resp. weak) topology of $H^1(\Omega, \mathbf{R}^n)$. Let $\mathcal{L} : H^1(\Omega, \mathbf{R}^n) \to \mathbf{R}$ be a continuous linear operator, representing the work of the (rescaled) loads. We define the functionals $\mathcal{G}_\varepsilon, \mathcal{G} : H^1_{g,\Omega_D} \to [0, +\infty]$ as $\mathcal{G}_\varepsilon(u) = \mathcal{F}_\varepsilon(u) - \mathcal{L}(u)$ and $\mathcal{G}(u) = \mathcal{F}(u) - \mathcal{L}(u)$.

The main convergence results, proved in Section 5, are the following.

Theorem 2.1 *Assume that $V : \Omega \times \mathbf{M}^{n\times n}_{sym} \to [0, +\infty]$ satisfies conditions (a), (b), (c), (2.1), (2.4), and (2.5). If u_ε satisfies*

$$\mathcal{G}(u_\varepsilon) = \inf_{u \in H^1_{g,\partial\Omega_D}} \mathcal{G}_\varepsilon(u) + o(1) \tag{2.13}$$

then u_ε converges weakly to the (unique) solution u_0 of

$$\min_{u \in H^1_{g,\partial\Omega_D}} \mathcal{G}(u) \, .$$

Theorem 2.2 *Under the hypotheses of the previous theorem, if condition (c') is satisfied then u_ε converges to u_0 strongly in $W^{1,q}(\Omega, \mathbf{R}^n)$ for $1 \le q < 2$.*

The proof follows basically from the following results, contained in Section 3 and 4 respectively.

Proposition 2.3 *If $\varepsilon_j \to 0$ and $u_{\varepsilon_j} \in H^1_{g,\partial\Omega_D}$ is a sequence such that*

$$\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \le C < +\infty \, ,$$

then u_{ε_j} is equibounded in $H^1(\Omega, \mathbf{R}^n)$.

Proposition 2.4 *Let $\varepsilon_j \to 0$. The functionals $\mathcal{G}_{\varepsilon_j}$ Γ-converge to $\mathcal{G}$ in the weak topology of $H^1_{g,\partial\Omega_D}$.*

3 Compactness

From conditions (a), (b) and (c) it follows easily that there exists a non-decreasing, continuous function $\phi(t)$, of the form

$$\phi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \le t \le c \\ \alpha c^2 & \text{for } c \le t \le d \\ (\alpha c^2 d^{-1}) t & \text{for } d \le t \, , \end{cases}$$

such that $\phi(|E|) \le V(x, E)$ for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$. For a positive β let $\psi(t)$ be the function defined as

$$\psi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \le t \le \beta \, , \\ (2\alpha\beta)t - (\alpha\beta^2) & \text{for } t \ge \beta \, . \end{cases} \tag{3.1}$$

It is easy to check that $\psi(t)$ is increasing, C^1, and convex. Moreover, since

$$\lim_{\beta \to 0} 2\alpha\beta = 0 \, ,$$

for β sufficiently small we have $\psi(t) \leq \phi(t)$ and then

$$V(x, E) \geq \psi(|E|) \tag{3.2}$$

for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$.

Lemma 3.1 *Let $\varepsilon > 0$ and $u_\varepsilon \in H^1(\Omega, \mathbf{R}^n)$. Denote the rescaled deformation $x + \varepsilon u_\varepsilon(x)$ by $v_\varepsilon(x)$. Then there exists a function $R_\varepsilon : \Omega \to SO(n)$ such that*

$$\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|^2 \, dx \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,, \tag{3.3}$$

where C depends only on the function ψ (in particular it does not depend on ε or v_ε).

Proof. We may assume $\mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$, so that $\det \nabla v_\varepsilon > 0$ a.e. in Ω by (2.1). Considering that

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \int_\Omega W_\varepsilon(x, \nabla u_\varepsilon) \, dx = \frac{1}{\varepsilon^2} \int_\Omega V(x, \tfrac{1}{2}(\nabla v_\varepsilon^T \nabla v_\varepsilon - I)) \, dx \tag{3.4}$$

and using (3.2) we get

$$\int_\Omega \psi(\tfrac{1}{2}|\nabla v_\varepsilon^T \nabla v_\varepsilon - I|) \, dx \leq \int_\Omega V(x, \tfrac{1}{2}(\nabla v_\varepsilon^T \nabla v_\varepsilon - I)) \leq \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,. \tag{3.5}$$

As $\det \nabla v_\varepsilon > 0$ a.e. in Ω by polar decomposition (see for instance [2]) for a.e. $x \in \Omega$ there exists a rotation R_ε and a symmetric positive definite matrix U_ε such that $\nabla v_\varepsilon = R_\varepsilon U_\varepsilon$. In particular $\nabla v_\varepsilon^T \nabla v_\varepsilon = U_\varepsilon^2$, hence

$$|\nabla v_\varepsilon^T \nabla v_\varepsilon - I| = |U_\varepsilon^2 - I|\,. \tag{3.6}$$

Since U_ε is symmetric and positive definite, using an orthonormal basis in which U_ε is diagonal, we can prove that

$$|U_\varepsilon - I| \leq |U_\varepsilon^2 - I|\,.$$

Thus, by the definition of ψ, it follows that for $\frac{1}{2}|U_\varepsilon^2 - I| \leq \beta$

$$\tfrac{\alpha}{4}|U_\varepsilon - I|^2 = \psi(\tfrac{1}{2}|U_\varepsilon^2 - I|)\,.$$

Moreover for a suitable constant c_1, depending on β,

$$c_1|U_\varepsilon - I|^2 \leq |U_\varepsilon^2 - I| \qquad \text{for } \tfrac{1}{2}|U_\varepsilon^2 - I| \geq \beta\,.$$

Indeed, using again the diagonal form, we can write

$$\begin{aligned}\sum_{i=1}^n (\lambda_i - 1)^2 &\leq \sum_{i=1}^n \lambda_i^2 + n = \\ &= \sum_{i=1}^n (\lambda_i^2 - 1) + \frac{2n}{\beta}\beta \leq \left(1 + \frac{n}{\beta}\right) \sum_{i=1}^n |\lambda_i^2 - 1|\,.\end{aligned}$$

Moreover there is a constant c_2 such that $2c_2 t \leq \psi(t)$ for $t \geq \beta$, hence for $\frac{1}{2}|U_\varepsilon^2 - I| \geq \beta$

$$c_1 c_2 |U_\varepsilon - I|^2 \leq c_2|U_\varepsilon^2 - I| \leq \psi(\tfrac{1}{2}|U_\varepsilon^2 - I|)\,.$$

By this inequality and by (3.5) and (3.6) there exists a constant c_3, depending only on ψ, such that

$$\int_\Omega |U_\varepsilon - I|^2 \, dx \leq c_3 \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) \, .$$

Finally considering that for a.e. $x \in \Omega$ we have $\nabla v_\varepsilon = R_\varepsilon U_\varepsilon$ we can write

$$\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|^2 \, dx = \int_\Omega |U_\varepsilon - I|^2 \, dx \leq c_3 \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) \, .$$

which is the required estimate. ■

The following Lemma (for which we refer to [4]) will be crucial in our proof.

Lemma 3.2 *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with Lipschitz boundary. There exists a constant C such that for every $v \in H^1(\Omega, \mathbf{R}^n)$ there exists a constant rotation $R \in SO(n)$ such that*

$$\int_\Omega |\nabla v(x) - R|^2 \, dx \leq C \int_\Omega \mathrm{dist}(\nabla v(x), SO(n))^2 \, dx \, , \tag{3.7}$$

where $\mathrm{dist}(F, SO(n))$ denotes the distance from the matrix F to the set $SO(n)$.

Moreover we will need the following result.

Lemma 3.3 *Let $S \subset \mathbf{R}^n$ be a bounded $\mathcal{H}^n$-measurable set with $0 < \mathcal{H}^m(S) < +\infty$, for some $m > 0$. Then*

$$|F|_S := \Big(\min_{\zeta \in \mathbf{R}^n} \int_S |Fx - \zeta|^2 d\mathcal{H}^m(x) \Big)^{\frac{1}{2}}$$

is a seminorm on $\mathbf{M}^{n \times n}$.

Let S_0 be the set of points $x \in S$ such that $\mathcal{H}^m(S \cap B_\rho(x)) > 0$, and let $\mathrm{aff}(S_0)$ be the smallest affine space containing S_0. Let $\mathbf{K} \subset \mathbf{M}^{n \times n}$ be a closed cone such that for every $F \in \mathbf{K}$ with $F \neq 0$

$$\dim(\ker(F)) < \dim(\mathrm{aff}(S_0)) \, . \tag{3.8}$$

Then there exists a constant $C > 0$ such that

$$C|F| \leq |F|_S \tag{3.9}$$

for every $F \in \mathbf{K}$.

Proof. It is not difficult to check that $|F|_S$ is a seminorm and the minimum is attained for $\zeta = \fint_S Fx \, d\mathcal{H}^m$. We will prove (3.9) by contradiction. Suppose that for every integer k it is possible to find a matrix $F_k \in \mathbf{K}$ with $|F_k| = 1$ such that

$$\frac{1}{k} = \frac{1}{k}|F_k|^2 > \int_S |F_k x - \zeta_k|^2 d\mathcal{H}^m \geq 0 \, , \tag{3.10}$$

with $\zeta_k := \fint_S F_k x \, d\mathcal{H}^m$. It is not restrictive to assume that F_k converges to $F \in \mathbf{K}$, with $|F| = 1$. Then by (3.10) and by continuity it follows that

$$\int_S |Fx - \zeta|^2 d\mathcal{H}^m = 0 \, .$$

for $\zeta = \fint_S Fx \, d\mathcal{H}^m$. Then $Fx = \zeta$ for $\mathcal{H}^m$-a.e. $x \in S$ and hence for every $x \in S_0$. By continuity and linearity $Fx = \zeta$ for every $x \in \mathrm{aff}(S_0)$. Then $\dim(\ker(F)) \geq \dim(\mathrm{aff}(S_0))$ and thus, by (3.8), $F = 0$. This is clearly impossible because $|F| = 1$. ■

Now we are ready to prove the following compactness result.

Proposition 3.4 *Let u_ε be a sequence in $H^1_{g,\partial\Omega_D}$. Then*

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx \le C\mathcal{F}_\varepsilon(u_\varepsilon) + C\int_{\partial\Omega_D} |g|^2\,d\mathcal{H}^{n-1} \tag{3.11}$$

where C depends only on ψ, Ω, and $\partial\Omega_D$.

Proof. By Lemma 3.1 we have

$$\int_\Omega \mathrm{dist}(\nabla v_\varepsilon(x), SO(n))^2\,dx \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon)$$

and by Lemma 3.2 there exists a constant rotation R_ε such that

$$\int_\Omega |\nabla v_\varepsilon(x) - R_\varepsilon|^2\,dx \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon)\,. \tag{3.12}$$

If $\zeta_\varepsilon = \fint_\Omega (v_\varepsilon(x) - R_\varepsilon x)\,dx$, then by the Poincaré inequality

$$\|v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C\int_\Omega |\nabla v_\varepsilon(x) - R_\varepsilon|^2\,dx \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon)\,.$$

Moreover by the continuity of the traces

$$\int_{\partial\Omega_D} |v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1} \le C\|v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon)\,.$$

Considering that on $\partial\Omega_D$ we have $v_\varepsilon(x) = x + \varepsilon g(x)$ we can write

$$\int_{\partial\Omega_D} |x - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1} \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2\int_{\partial\Omega_D} |g|^2\,d\mathcal{H}^{n-1}\,. \tag{3.13}$$

Let $\mathbf{K}$ be the closed cone generated by $SO(n) - I$, which is the union of the cone generated by $SO(n) - I$ and of the space of antisymmetric matrices. Therefore, $\dim(\ker(F)) < n-1$ if $F \in \mathbf{K}$ and $F \neq 0$. Let $S := \partial\Omega_D$. As S is contained in the Lipschitz manifold $\partial\Omega$ and $\mathcal{H}^{n-1}(S) > 0$, we have $\mathcal{H}^{n-1}(S_0) > 0$. This implies that $\dim(\mathrm{aff}(S_0)) \ge n-1$ and thus condition (3.8) is satisfied. Using Lemma 3.3 and the previous inequality we obtain

$$|I - R_\varepsilon|^2 \le C|I - R_\varepsilon|^2_S = C\int_{\partial\Omega_D} |x - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1}$$

and thus by (3.13)

$$\int_\Omega |I - R_\varepsilon|^2\,dx \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2\int_{\partial\Omega_D} |g|^2\,d\mathcal{H}^{n-1}\,. \tag{3.14}$$

By (3.12) and (3.14) we have easily

$$\int_\Omega |\nabla v_\varepsilon - I|^2\,dx \le C\varepsilon^2\mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2\int_{\partial\Omega_D} |g|^2\,d\mathcal{H}^{n-1}\,.$$

Substituting $\nabla v_\varepsilon = I + \varepsilon\nabla u_\varepsilon$ in the previous inequality we get (3.11). ∎

Proof of Proposition 2.3. Using Proposition 3.4 we have

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\,dx \le C\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) + C\int_{\partial\Omega_D} |g|^2\,d\mathcal{H}^{n-1}\,.$$

Hence we can write

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\,dx \le C(\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) + \mathcal{L}(u_{\varepsilon_j}) + 1)\,,$$

and by the Poincaré and the Holder inequality it follows that

$$\|u_{\varepsilon_j}\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C + C\|u_{\varepsilon_j}\|_{H^1(\Omega,\mathbf{R}^n)}\,,$$

which gives the boundedness of u_{ε_j} in $H^1(\Omega,\mathbf{R}^n)$. ■

Finally we remark that for $n = 2$ and for a sequence $u_{\varepsilon_j} \in H^1_0(\Omega,\mathbf{R}^2)$ we can prove the compactness result in a more elementary way without using Lemma 3.2. Indeed for every ε_j let $R_{\varepsilon_j}\colon \Omega \to SO(2)$ be given by Lemma 3.1. Define $M_{\varepsilon_j} = (R_{\varepsilon_j} - I)/\varepsilon_j$. Then, substituting $\nabla v_{\varepsilon_j} = I + \varepsilon_j \nabla u_{\varepsilon_j}$ in (3.3), we get

$$\int_\Omega |\nabla u_{\varepsilon_j} - M_{\varepsilon_j}|^2\,dx \le C_1\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j})\,.$$

Note that M_{ε_j} has the form

$$M_{\varepsilon_j} = \begin{pmatrix} a_{\varepsilon_j} & -b_{\varepsilon_j} \\ b_{\varepsilon_j} & a_{\varepsilon_j} \end{pmatrix}$$

for some real functions a_{ε_j} and b_{ε_j}. Denote the components of u by u^i. By a linear combination we obtain

$$\int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2\,dx = \int_\Omega |(\nabla_1 u^1_{\varepsilon_j} - a_{\varepsilon_j}) - (\nabla_2 u^2_{\varepsilon_j} - a_{\varepsilon_j})|^2\,dx \le C_2\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j})\,,$$

$$\int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2\,dx \int_\Omega |(\nabla_2 u^1_{\varepsilon_j} + b_{\varepsilon_j}) + (\nabla_1 u^2_{\varepsilon_j} - b_{\varepsilon_j})|^2\,dx \le C_3\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j})\,.$$

Moreover, being $n = 2$, we can write

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\,dx = \int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2\,dx + \int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2\,dx + 2\int_\Omega \det\nabla u_{\varepsilon_j}\,dx\,.$$

As $u_{\varepsilon_j} \in H^1_0(\Omega,\mathbf{R}^2)$ we have (see e.g. [2])

$$\int_\Omega \det\nabla u_{\varepsilon_j}\,dx = 0\,.$$

Then by the previous inequalities we get

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\,dx = \int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2\,dx + \int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2\,dx \le C\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j})$$

and thus u_{ε_j} is bounded in $H^1_0(\Omega,\mathbf{R}^2)$.

The following example shows that, if other potential wells are present, with the same value of the energy, we might lose compactness of solutions.

Example 3.5 *Let $\Omega = (-1,1)\times(-1,1)$, $\ell = 1$ and $w \in H^1_0(\Omega, \mathbf{R}^2)$ defined as $w^1(x_1,x_2) = -\max\{|x_1|, |x_2|\} + 1$ and $w^2(x_1,x_2) = 0$. Let $\varepsilon_j \to 0$, $w_{\varepsilon_j}(x) = w(x)/\varepsilon_j$ and $v_{\varepsilon_j}(x) = x + \varepsilon_j w_{\varepsilon_j}(x)$. Then $\nabla v_{\varepsilon_j} = I + \varepsilon_j \nabla w_{\varepsilon_j} = I + \nabla w$ does not depend on ε_j and takes only four values, denoted by $F_1, \dots, F_4$. Let $E_i = \frac{1}{2}(F_i^T F_i - I)$, for $i = 1, \dots, 4$. Let V be the function satisfying conditions (b) and (c) and such that $V(x,0) = V(x,E_i) = 0$ for $i = 1,\dots,4$. Then*

$$\begin{aligned}\inf\{\mathcal{G}_{\varepsilon_j}(u) \,:\, u \in H^1_0(\Omega)\} &\leq \frac{1}{\varepsilon_j^2}\int_\Omega V(x, \tfrac{1}{2}(\nabla v_{\varepsilon_j}^T \nabla v_{\varepsilon_j} - I))\,dx - \int_\Omega w_{\varepsilon_j}\,dx \\ &= -\frac{1}{\varepsilon_j}\|w\|_{L^1(\Omega,\mathbf{R}^n)}\,.\end{aligned}$$

If u_{ε_j} is a sequence satisfying (2.13) then

$$-\frac{1}{\varepsilon_j}\|w\|_{L^1(\Omega,\mathbf{R}^n)} + o(1) \geq \mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \geq -\,\|u_{\varepsilon_j}\|_{L^1(\Omega,\mathbf{R}^n)}\,,$$

hence $\|u_{\varepsilon_j}\|_{L^1(\Omega,\mathbf{R}^2)}$ diverges.

4 Γ-convergence

For $x \in \Omega$ and $E \in \mathbf{M}^{n\times n}_{sym}$ let $|E|_{\mathsf{A}(x)}$ be the norm defined by

$$|E|_{\mathsf{A}(x)} = \Big\{\frac{1}{2}\mathsf{A}(x)[E,E]\Big\}^{\frac{1}{2}} = \Big\{\frac{1}{2}\partial^2_E V(x,0)[E,E]\Big\}^{\frac{1}{2}}\,. \tag{4.1}$$

Note that by (2.5) and (2.8) we have

$$\alpha|E|^2 \leq |E|^2_{\mathsf{A}(x)} \leq \gamma|E|^2\,. \tag{4.2}$$

If $\Phi : \Omega \to \mathbf{M}^{n\times n}_{sym}$ is a measurable map, the function $x \mapsto |\Phi(x)|_{\mathsf{A}(x)}$ is denoted by $|\Phi|_{\mathsf{A}}$.

Let us fix a sequence $\varepsilon_j \to 0$. By Proposition 2.3 the functionals $\mathcal{G}_{\varepsilon_j}$ are equicoercive in $H^1_{g,\partial\Omega_D}$ and by Proposition 8.10 in [3] we can characterize the Γ-limit in the weak topology of $H^1_{g,\partial\Omega_D}$ in terms of weakly converging sequences. In particular we have

$$\mathcal{G}'(u) := \Gamma\text{-}\liminf_{\varepsilon_j\to 0} \mathcal{G}_\varepsilon(u) = \inf\{\liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(u_j) \,:\, \text{for } u_j \rightharpoonup u \text{ in } H^1_{g,\partial\Omega_D}\}\,,$$

$$\mathcal{G}''(u) := \Gamma\text{-}\limsup_{\varepsilon_j\to 0} \mathcal{G}_\varepsilon(u) = \inf\{\limsup_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(u_j) \,:\, \text{for } u_j \rightharpoonup u \text{ in } H^1_{g,\partial\Omega_D}\}\,.$$

We will prove that for every function $u \in H^1_{g,\partial\Omega_D}$ we have $\mathcal{G}''(u) \leq \mathcal{G}(u) \leq \mathcal{G}'(u)$, from which Proposition 2.4 follows.

Proposition 4.1 *For every $u \in H^1_{g,\partial\Omega_D}$ we have $\mathcal{G}''(u) \leq \mathcal{G}(u)$.*

Proof. Consider first the case $u \in W^{1,\infty}(\Omega, \mathbf{R}^n)$. By (2.10) it follows that for a.e. $x \in \Omega$

$$\lim_{\varepsilon_j\to 0} W_{\varepsilon_j}(x, \nabla u) = \frac{1}{2}\mathsf{A}(x)\,[e(u), e(u)]\,.$$

Using the upper bound (2.7) we deduce that $V_{\varepsilon_j}(x, \nabla u)$ is equi-bounded in $L^\infty(\Omega)$. Then taking the sequence $u_{\varepsilon_j} = u$, by dominated convergence it follows that

$$\limsup_{\varepsilon_j \to 0} \mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) = \lim_{\varepsilon_j \to 0} \int_\Omega V_{\varepsilon_j}(x, \nabla u)\, dx - \mathcal{L}(u) = \frac{1}{2} \int_\Omega \mathsf{A}(x)\, [e(u), e(u)]\, dx - \mathcal{L}(u)\,. \quad (4.3)$$

If $u \notin W^{1,\infty}(\Omega, \mathbf{R}^n)$ by the definition of $H^1_{g,\partial\Omega_D}$ there exists a sequence u_k in $W^{1,\infty}(\Omega, \mathbf{R}^n)$, which satisfy the boundary condition $u_k = g$ on $\partial\Omega_D$ and converge to u strongly in $H^1(\Omega, \mathbf{R}^n)$. Since, by (4.3), $\mathcal{G}''(u_k) \leq \mathcal{G}(u_k)$, the lower semicontinuity of the Γ-limsup and the continuity of $\mathcal{G}$ respect to strong convergence imply that

$$\mathcal{G}''(u) \leq \liminf_{k \to \infty} \mathcal{G}''(u_k) \leq \liminf_{k \to \infty} \mathcal{G}(u_k) = \mathcal{G}(u)$$

and the proof is concluded. ■

Lemma 4.2 *Let $\varepsilon_j \to 0$ be a decreasing sequence. For every $k \in \mathbf{N}$ there exist an increasing sequence of Caratheodory functions $V^k_j : \Omega \times \mathbf{M}^{n\times n}_{sym} \to [0, +\infty)$ and a measurable function $\mu^k : \Omega \to (0, +\infty)$ such that $V^k_j(x, \cdot)$ is convex for a.e. $x \in \Omega$ and satisfies*

$$V^k_j(x, E) \leq V(x, \varepsilon_j E)/\varepsilon_j^2 \qquad \forall E \in \mathbf{M}^{n\times n}_{sym}\,, \quad (4.4)$$

$$V^k_j(x, E) = \Big(1 - \frac{1}{k}\Big)\, |E|^2_{\mathsf{A}(x)} \qquad \text{for } |E|_{\mathsf{A}(x)} \leq \mu^k(x)/\varepsilon_j\,. \quad (4.5)$$

Proof. By Taylor's formula, from (2.4) and (4.2) it follows that for a.e. $x \in \Omega$ and every $k \in \mathbf{N}$ there exists $r^k(x) > 0$ such that

$$\Big(1 - \frac{1}{k}\Big) |E|^2_{\mathsf{A}(x)} \leq V(x, E) \quad \text{for } |E|_{\mathsf{A}(x)} \leq r^k(x)\,. \quad (4.6)$$

Let us consider the function $h^k : \Omega \times \mathbf{M}^{n\times n}_{sym} \longrightarrow \mathbf{R}$ defined by

$$h^k(x, E) = \begin{cases} (1 - \frac{1}{k}) |E|^2_{\mathsf{A}(x)} & \text{for } |E|_{\mathsf{A}(x)} \leq r^k(x)\,, \\ \\ \psi(\gamma^{-\frac{1}{2}} |E|_{\mathsf{A}(x)}) & \text{for } |E|_{\mathsf{A}(x)} > r^k(x)\,, \end{cases}$$

which is less than or equal to $V(x, E)$ by (4.6), (4.2), and (3.2).

For a suitable choice of $\mu^k(x) > 0$ the function

$$\phi^k(x, t) = \begin{cases} (1 - \frac{1}{k}) t^2 & \text{for } 0 \leq t \leq \mu^k(x)\,, \\ \\ 2(1 - \frac{1}{k}) \mu^k(x) t - (1 - \frac{1}{k}) (\mu^k(x))^2 & \text{for } t \geq \mu^k(x)\,, \end{cases}$$

is convex in t and satisfies $\phi^k(x, |E|_{\mathsf{A}(x)}) \leq h^k(x, E) \leq V(x, E)$. To conclude the proof it is enough to define $V^k_j(x, E) := \phi^k(x, \varepsilon_j |E|_{\mathsf{A}(x)})/\varepsilon_j^2$. From the special form of $\phi^k(x, \cdot)$ it is easy to see that $V^k_j(x, \cdot)$ is increasing with respect to j and that (4.5) holds, while (4.4) follows from the inequality $\phi^k(x, |E|_{\mathsf{A}(x)}) \leq V(x, E)$. ■

Lemma 4.3 *Let $g_j : \Omega \times \mathbf{R}^m \to [0, +\infty)$ be Caratheodory functions such that $g_j(x, \cdot)$ is convex. Let $g_j(x, \xi)$ be increasing in j and pointwise converging to a function $g(x, \xi)$. If w_j converges weakly to w in $L^1(\Omega, \mathbf{R}^m)$, then*

$$\int_\Omega g(x, w)\, dx \le \liminf_{j \to +\infty} \int_\Omega g_j(x, w_j)\, dx\,. \tag{4.7}$$

Proof. As $g_i(x, w_j) \le g_j(x, w_j)$ for $j \ge i$, by the lower semicontinuity of the functional $\int_\Omega g_i(x, v)\, dx$ we have

$$\int_\Omega g_i(x, w)\, dx \le \liminf_{j \to +\infty} \int_\Omega g_i(x, w_j)\, dx \le \liminf_{j \to +\infty} \int_\Omega g_j(x, w_j)\, dx\,,$$

which proves (4.7) for $i \to \infty$. ■

Proposition 4.4 *For every $u \in H^1_{g, \partial\Omega_D}$ and every sequence $u_j \in H^1_{g, \partial\Omega_D}$ weakly converging to u, we have the Γ-liminf inequality*

$$\frac{1}{2} \int_\Omega \mathsf{A}(x)\, [e(u), e(u)]\, dx \le \liminf_{\varepsilon_j \to 0} \mathcal{G}_{\varepsilon_j}(u_j)\,, \tag{4.8}$$

from which it follows that $\mathcal{G}(u) \le \mathcal{G}'(u)$.

Proof. For every $k \in \mathbf{N}$ let $V^k_j(x, E)$ be the sequence given by Lemma 4.2. Note that by (4.5) for every $E \in \mathbf{M}^{n \times n}_{sym}$ we have

$$\lim_{j \to +\infty} V^k_j(x, E) = \Big(1 - \frac{1}{k}\Big)\, |E|^2_{\mathsf{A}(x)}\,. \tag{4.9}$$

Then inequality (4.4) gives

$$\begin{aligned} W_{\varepsilon_j}(x, \nabla u_{\varepsilon_j}) &= \frac{1}{\varepsilon_j^2} V(x, \varepsilon_j e(u_{\varepsilon_j}) + \tfrac{1}{2}\varepsilon_j^2 \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j}) \\ &\ge V^k_j(x, e(u_{\varepsilon_j}) + \tfrac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j})\,. \end{aligned}$$

Since $\nabla u_{\varepsilon_j} \rightharpoonup \nabla u$ in $L^2(\Omega, \mathbf{M}^{n \times n})$ we have that $\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j} \longrightarrow 0$ strongly in $L^1(\Omega, \mathbf{M}^{n \times n})$, hence $e(u_{\varepsilon_j}) + \frac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j} \rightharpoonup e(u)$ weakly in $L^1(\Omega, \mathbf{M}^{n \times n})$. Then by Lemma 4.3 and (4.9) for every $k \in \mathbf{N}$ we have

$$\begin{aligned} \liminf_{\varepsilon_j \to 0} \int_\Omega W_{\varepsilon_j}(x, \nabla u_{\varepsilon_j})\, dx &\ge \liminf_{j \to +\infty} \int_\Omega V^k_j(x, e(u_{\varepsilon_j}) + \tfrac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j})\, dx \\ &\ge \frac{1}{2} \int_\Omega \Big(1 - \frac{1}{k}\Big) \mathsf{A}(x)\, [e(u), e(u)]\, dx\,. \end{aligned} \tag{4.10}$$

Taking the supremum as $k \to \infty$ and considering the weak continuity of $\mathcal{L}$ we deduce inequality (4.8). ■

5 Convergence of minimizers

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. It is enough to prove the statement for evey sequence $\varepsilon_j \to 0$. Since $\mathcal{G}_{\varepsilon_j}(g) \leq C < +\infty$, we have $\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \leq C < +\infty$, then by Proposition 2.3 u_{ε_j} is equibounded in $H^1(\Omega, \mathbf{R}^n)$. Thus there exists a subsequence u_{ε_k} converging weakly to some limit $w \in H^1_{g,\partial\Omega_D}$. By Γ-convergence we know that w must be the minimizer u_0 of the limit functional $\mathcal{G}$ (see, e.g., [3], Corollary 7.17)

Finally, as the limit w depends neither on the subsequence w_{ε_k} nor on the sequence ε_j, the whole sequence w_ε converges weakly to u in $H^1_{g,\partial\Omega_D}$. ■

In the sequel we will assume that $V(x,E)$ satisfies conditions (a), (b), and (c'). It is not restrictive to assume that $1 < p < 2$. Let α be the constant appearing in (b). From (a), (b), and (c') it follows that there exists a nondecreasing, continuous function $\phi(t)$ of the form

$$\phi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq c\,, \\ \alpha c^2 & \text{for } c \leq t \leq d\,, \\ (\alpha c^2 d^{-p}) t^p & \text{for } d \leq t\,, \end{cases} \qquad \text{for } 0 < c < d\,,$$

such that $\phi(|E|) \leq V(x,E)$ for a.e. $x \in \Omega$. Consider the function $\psi_p(t)$ defined as

$$\psi_p(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq \mu\,, \\ a(t-b)^p & \text{for } \mu \leq t\,, \end{cases} \tag{5.1}$$

for $a = \alpha\, p^{-p}\, 2^p\, \mu^{2-p}$ and $b = (1 - \frac{p}{2})\mu$. It is not difficult to check that $\psi_p(t)$ is increasing, C^1, and convex. As $1 < p < 2$, we have

$$\lim_{\mu \to 0} \alpha\, p^{-p}\, 2^p\, \mu^{2-p} = 0\,, \tag{5.2}$$

thus for μ sufficiently small $\psi_p(t) \leq \phi(t)$ for every $t \geq 0$ and then $\psi_p(|E|) \leq V(x,E)$ for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$.

Lemma 5.1 *Let $\varepsilon_j \to 0$. For every $k \in \mathbf{N}$ there exists an increasing sequence of Caratheodory functions $V^k_j : \Omega \times \mathbf{M}^{n\times n}_{sym} \to [0, +\infty)$ and a measurable function $\mu^k : \Omega \to (0, +\infty)$ such that for a.e. $x \in \Omega$ the function $V^k_j(x,\cdot)^{\frac{1}{p}}$ is convex and (4.4) and (4.5) hold.*

Proof. We follow the proof of Lemma 4.2, with ψ replaced by ψ_p, and we consider the functions

$$\phi^k_p(x,t) = \begin{cases} (1 - \frac{1}{k}) t^2 & \text{for } 0 \leq t \leq \mu^k(x)\,, \\ \\ a(x)(t - b(x))^p & \text{for } t \geq \mu^k(x)\,. \end{cases}$$

Note that $\phi^k_p(x,t)^{\frac{1}{p}}$ is convex for $a(x) = (1 - \frac{1}{k}) 2^p p^{-p} (\mu^k(x))^{2-p}$ and $b(x) = (1 - \frac{p}{2})\mu^k(x)$. By (5.2) for $\mu^k(x)$ sufficiently small we have that

$$\phi^k_p(x, |E|_{\mathsf{A}(x)}) \leq V(x,E)$$

for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$. Then the sequence defined by $V^k_j(x,E) := \phi^k_p(x, \varepsilon_j |E|_{\mathsf{A}(x)})/\varepsilon_j^2$ satisfies (4.4) and (4.5), is increasing with respect to j, and $V^k_j(x,\cdot)^{\frac{1}{p}}$ is convex for a.e. $x \in \Omega$.
■

Lemma 5.2 *Let $\Phi_n \rightharpoonup \Phi$ weakly in $L^1(\Omega, \mathbf{M}^{n\times n})$ such that $|\Phi_n|_{\mathsf{A}}$ converges to $|\Phi|_{\mathsf{A}}$ in measure. Then Φ_n converges to Φ in measure.*

Proof. By passing to a subsequence and to a suitable measurable subdomain, it is not restrictive to suppose that $\Phi(x) \neq 0$ for every $x \in \Omega$ and that $|\Phi_n|_{\mathsf{A}}$ converges to $|\Phi|_{\mathsf{A}}$ pointwise.

By (4.2) and by weak convergence we have

$$\int_\Omega \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n - \Phi\rangle_{\mathsf{A}}\, dx \ \longrightarrow 0\,, \tag{5.3}$$

where $\langle\cdot,\cdot\rangle_{\mathsf{A}}$ is the scalar product associated with the norm $|\cdot|_{\mathsf{A}}$, i.e.,

$$\langle \Psi_1, \Psi_2\rangle_{\mathsf{A}} = \frac{1}{2}\mathsf{A}(x)[\Psi_1(x), \Psi_2(x)]\,.$$

Moreover by the Schwarz inequality

$$\int_\Omega \left(\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n - \Phi\rangle_{\mathsf{A}}\right)^+ dx \le \int_\Omega \left(|\Phi_n|_{\mathsf{A}} - |\Phi|_{\mathsf{A}}\right)^+ \, dx\,.$$

As $|\Phi_n|_{\mathsf{A}}$ is equiintegrable and converges to $|\Phi|_{\mathsf{A}}$ in measure, it converges also in $L^1(\Omega)$. Thus

$$\int_\Omega \left(\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n - \Phi\rangle_{\mathsf{A}}\right)^+ dx \quad\longrightarrow 0\,.$$

Then by (5.3) $\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n - \Phi\rangle_{\mathsf{A}} \longrightarrow 0$ in $L^1(\Omega)$ and, up to a subsequence, it converges for a.e. $x \in \Omega$, hence $\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n\rangle_{\mathsf{A}} \longrightarrow |\Phi|_{\mathsf{A}}$ pointwise a.e. in Ω.

Considering the identity

$$|\Phi_n - \Phi|^2_{\mathsf{A}} = \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n - \Phi\rangle^2_{\mathsf{A}} + |\Phi_n|^2_{\mathsf{A}} - \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}\,,\, \Phi_n\rangle^2_{\mathsf{A}}\,,$$

we deduce that $|\Phi_n - \Phi|^2_{\mathsf{A}} \longrightarrow 0$ pointwise for a.e. $x \in \Omega$.

Since for every subsequence of Φ_n we can find a further subsequence converging pointwise to Φ, it follows that Φ_n converges to Φ in measure. ■

Proposition 5.3 *Let $\varepsilon_j \to 0^+$ and let $u_j \rightharpoonup u$ weakly in $H^1(\Omega, \mathbf{R}^n)$ such that*

$$\frac{1}{\varepsilon_j^2}\int_\Omega V(x, \varepsilon_j e(u_j) + \tfrac{1}{2}\varepsilon_j^2 C(u_j))\, dx \longrightarrow \int_\Omega |e(u)|^2_{\mathsf{A}}\, dx\,. \tag{5.4}$$

Then $u_j \longrightarrow u$ strongly in $W^{1,q}(\Omega, \mathbf{R}^n)$ for $1 \le q < 2$.

Proof. For every k let $V_j^k(x, E)$ be the sequence given by Lemma 5.1. Denote $e(u_j) + \frac{1}{2}\varepsilon_j C(u_j)$ by Φ_j. By (4.4) for every k and every j we have

$$V_j^k(x, \Phi_j) \le \frac{1}{\varepsilon_j^2} V(x, \varepsilon_j e(u_j) + \tfrac{1}{2}\varepsilon_j^2 C(u_j))\,. \tag{5.5}$$

By (5.4) it follows that, for every k, $V_j^k(x,\Phi_j)^{\frac{1}{p}}$ is bounded in $L^p(\Omega)$ uniformly with respect to j and k. Being $p>1$, by a diagonal argument there exists a sequence $j_m \to \infty$ such that for every k

$$V_{j_m}^k(x,\Phi_{j_m})^{\frac{1}{p}} \rightharpoonup w^k \qquad \text{weakly in } L^p(\Omega)\,, \tag{5.6}$$

for a suitable function $w^k \in L^p(\Omega)$. Moreover by the weak convergence of u_j it follows that Φ_j converges weakly to $e(u)$ in $L^1(\Omega,\mathbf{M}^{n\times n})$. Since the functions $V_j^k(x,\xi)^{\frac{1}{p}}$ are convex in ξ, then by Lemma 4.3 and by (4.5), for every Borel set $B\subset\Omega$ we have

$$\Big(1-\frac{1}{k}\Big)^{\frac{1}{p}}\int_B |e(u)|_{\mathsf{A}}^{\frac{2}{p}}\,dx \le \liminf_{m\to\infty}\int_B V_{j_m}^k(x,\Phi_{j_m})^{\frac{1}{p}}\,dx = \int_B w^k\,dx\,.$$

Thus

$$w^k \ge \Big(1-\frac{1}{k}\Big)^{\frac{1}{p}}|e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{a.e. in } \Omega\,. \tag{5.7}$$

Moreover, by the weak lower semicontinuity of the norm, from (5.4), (5.5), and (5.6) it follows that

$$\int_\Omega (w^k)^p\,dx \le \int_\Omega |e(u)|_{\mathsf{A}}^2\,dx\,. \tag{5.8}$$

Being $p>1$, there exists $w\in L^p(\Omega)$ and a subsequence of w^k which converges weakly to w in $L^p(\Omega)$. Then passing to the limit in (5.7) we get

$$(w)^p \ge |e(u)|_{\mathsf{A}}^2$$

for a.e. $x\in\Omega$, and by (5.8) we have

$$\int_\Omega w^p\,dx \le \int_\Omega |e(u)|_{\mathsf{A}}^2\,dx\,.$$

These inequalities imply that $w=|e(u)|_{\mathsf{A}}^{\frac{2}{p}}$. Being the limit independent of the subsequence, we have proved for whole sequence w^k that

$$w^k \rightharpoonup |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{weakly in } L^p(\Omega). \tag{5.9}$$

Let $\mu^k(x)$ be the functions defined in Lemma 5.1. As $\mu^k>0$ a.e. in Ω, there exists a decreasing sequence of constants η_k such that

$$\operatorname{meas}(\{x\in\Omega:\mu^k(x)<\eta^k\}) \le \frac{1}{k}\,. \tag{5.10}$$

Considering that $V_{j_m}^k(x,\Phi_{j_m})^{\frac{1}{p}}$ is bounded in $L^p(\Omega)$ uniformly with respect to m and k, and hence we can use a metric equivalent to the weak topology, by (5.6) and (5.9) we can extract a subsequence i_k of j_m such that, writing for simplicity ε_k instead of ε_{i_k}, we have

$$\frac{\eta^k}{\varepsilon_k} > k \qquad \text{and} \qquad V_{i_k}^k(x,\Phi_{i_k})^{\frac{1}{p}} \rightharpoonup |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \quad \text{weakly in } L^p(\Omega)\,. \tag{5.11}$$

Then by (5.4) and (5.5) we have

$$\limsup_{k\to\infty}\int_\Omega V_{i_k}^k(x,\Phi_{i_k}) \le \int_\Omega |e(u)|_{\mathsf{A}}^2\,dx\,.$$

By the uniform convexity of the $L^p(\Omega)$ space this implies that

$$V_{i_k}^k(x,\Phi_{i_k})^{\frac{1}{p}} \longrightarrow |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{strongly in } L^p(\Omega)\,.$$

Then we have

$$V_{i_k}^k(x,\Phi_{i_k}) \longrightarrow |e(u)|_{\mathsf{A}}^2$$

strongly in $L^1(\Omega)$ and a.e. in Ω.

Now we can prove that $|e(u_{i_k})|_{\mathsf{A}}$ converges in measure to $|e(u)|_{\mathsf{A}}$. Indeed, for every $\delta > 0$ the set $\{|\,|e(u_{i_k})|_{\mathsf{A}} - |e(u)|_{\mathsf{A}}\,| > \delta\}$ is contained in

$$\Big\{\, \big||e(u_{i_k})|_{\mathsf{A}} - |e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}}\big| > \tfrac{\delta}{2}\Big\} \cup \Big\{\, \big||e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} - |e(u)|_{\mathsf{A}}\big| > \tfrac{\delta}{2}\Big\}\,. \tag{5.12}$$

The first set is contained in $\{|\frac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} > \frac{\delta}{2}\}$, whose measure tends to zero since $\varepsilon_k C(u_{i_k}) \longrightarrow 0$ in $L^1(\Omega, \mathbf{M}^{n\times n})$. Note that for $x \in \{\mu^k(x) > \eta^k\}$ if

$$|e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} < k$$

then by (4.5) and (5.11) we have

$$V_{i_k}^k(x,\Phi_{i_k}) = \frac{k-1}{k}|e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}}^2\,.$$

Then the second set in (5.12) is contained in

$$\{\mu^k(x) < \eta^k\} \cup \{|e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} > k\} \cup \{|(\tfrac{k}{k-1}V_{i_k}^k(x,\Phi_{i_k}))^{\frac{1}{2}} - |e(u)|_{\mathsf{A}}| > \tfrac{\delta}{2}\}\,.$$

The measure of all these sets tends to zero as $k \to +\infty$. The first one by (5.10), the second one since $|e(u_{i_k}) + \frac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}}$ is equibounded in $L^1(\Omega)$, and the third one because $(\frac{k}{k-1}V_{i_k}^k(x,\Phi_{u_k}))^{\frac{1}{2}} \longrightarrow |e(u)|_{\mathsf{A}}$ pointwise. This concludes the proof of the convergence in measure of $|e(u_{i_k})|_{\mathsf{A}}$ to $|e(u)|_{\mathsf{A}}$.

Then by Lemma 5.2 it follows that $e(u_{i_k})$ converges in measure to $e(u)$. As $e(u_{i_k})$ is bounded in $L^2(\Omega, \mathbf{M}^{n\times n})$, we deduce that $e(u_{i_k})$ converges strongly to $e(u)$ in $L^q(\Omega, \mathbf{M}^{n\times n})$ for $1 \leq q < 2$. Since the limit does not depend on the subsequence we have that $e(u_j)$ converges strongly to $e(u)$ in $L^q(\Omega, \mathbf{M}^{n\times n})$.

By the Korn inequality (see e.g. [6]) there exists a constant C_q such that

$$\int_\Omega |\nabla(u-u_j)|^q\,dx \leq C_q \int_\Omega |e(u-u_j)|^q\,dx + C_q \int_\Omega |u-u_j|^q\,dx\,.$$

As $e(u_j)$ converges strongly to $e(u)$ in $L^q(\Omega, \mathbf{M}^{n\times n})$ and u_j converges strongly to u in $L^q(\Omega, \mathbf{R}^n)$ by the Rellich theorem, we deduce that u_j converges to u in the strong topology of $W^{1,q}(\Omega, \mathbf{R}^n)$. ■

Proof of Theorem 2.2. Let $\varepsilon_j \to 0$. By Proposition 2.3 u_{ε_j} converges weakly to u in $H^1(\Omega, \mathbf{R}^n)$ and by Γ-convergence we have $\mathcal{G}(u_{\varepsilon_j}) \longrightarrow \mathcal{G}(u)$ (see e.g. [3] Corollary 7.17). By weak continuity we have $\mathcal{L}(u_{\varepsilon_j}) \longrightarrow \mathcal{L}(u)$, so that (5.4) holds. The conclusion follows from Proposition 5.3. ■

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References

[1] R. A. Adams: *Sobolev spaces*, Academic Press, New York (1975)

[2] P. G. Ciarlet: *Three dimensional elasticity*, North Holland, Amsterdam (1988)

[3] G. Dal Maso: *An Introduction to* Γ*-convergence*, Birkhäuser, Boston (1993)

[4] G. Friesecke, R. D. James, S. Müller: Rigorous derivation of nonlinear plate theory and geometric rigidity, *C. R. Acad. Sci. I* (to appear)

[5] M. E. Gurtin: *An introduction to continuum mechanics* Academic Press, New York (1981)

[6] R. Temam: *Problèmes mathématiques en plasticité*, Gauthier-Villars, Paris (1983)