STABILITY OF ABSTRACT LINEAR SEMIGROUPS ARISING FROM HEAT CONDUCTION WITH MEMORY

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Abstract. We establish some decay properties of the semigroup generated by a linear integro-differential equation in a Hilbert space, which is an abstract version of the equation

$$u_t(t) - \beta\Delta u(t) - \int_0^\infty k(s)\Delta u(t-s)ds = 0$$

describing hereditary heat conduction.

1. Introduction

Let H be a real Hilbert space, and let A be a strictly positive linear operator on H with domain $\mathcal{D}(A) \Subset H$ (thus A has compact inverse). Given a piecewise-smooth decreasing summable function $\mu \not\equiv 0$ on $\mathbb{R}^+ = (0,\infty)$, and naming $V = \mathcal{D}(A^{1/2})$, we consider the L^2-weighted space $\mathcal{M} = L^2_\mu(\mathbb{R}^+; V)$ along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, that is, the linear operator

$$(T\eta)(s) = -\eta'(s), \qquad \mathcal{D}(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\},$$

where the *prime* stands for the distributional derivative with respect to $s \in \mathbb{R}^+$, and $\eta(0) = \lim_{s\to 0}\eta(s)$ in V. In this paper, we address the study of the asymptotic behavior of the linear evolution system in the unknowns $u(t) : [0,\infty) \to H$ and $\eta^t : [0,\infty) \to \mathcal{M}$

$$\begin{cases} \dot{u}(t) + \beta A u(t) + \displaystyle\int_0^\infty \mu(s) A\eta^t(s)ds = 0, & t > 0, \\ \dot{\eta}^t = T\eta^t + u(t), & t > 0, \\ u(0) = u_0, \\ \eta^0(s) = \eta_0(s). \end{cases} \tag{1.1}$$

Here, β is a nonnegative parameter, while $u_0 \in H$ and $\eta_0 \in \mathcal{M}$ are given initial data. Problem (1.1) is cast in the so-called *memory setting* (see [4]), since η accounts for the past values of the variable u, and the *memory kernel* μ measures how much the system is

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influenced by its past history. Indeed, putting

$$k(s) = \int_s^\infty \mu(y)dy, \tag{1.2}$$

system (1.1) can be shown to be a reformulation of the integro-differential equation

$$\dot{u}(t) + \beta Au(t) + \int_0^\infty k(s)Au(t-s)ds = 0, \qquad t > 0, \tag{1.3}$$

with the initial conditions

$$u(0) = u_0, \qquad u(t)_{|t<0} = \eta_0'(-t).$$

In fact, (1.1) is more general than (1.3), which requires more regularity on the initial datum η_0. We will return on this equivalence later in §3.

Remark 1.1. When $H = L^2(\Omega)$ and $A = -\Delta$ with Dirichlet boundary conditions, (1.3) describes the evolution of the temperature relative to the equilibrium value in a rigid isotropic homogeneous heat conductor occupying a bounded domain Ω, where the heat conduction law is of Coleman-Gurtin type [2] if $\beta > 0$, or of Gurtin-Pipkin type [13] if $\beta = 0$.

Problem (1.1) generates a linear contraction semigroup $S(t)$ acting on the Hilbert space $H \times \mathcal{M}$, whose decay properties as $t \to \infty$ constitute the object of our investigation. More precisely, for a given memory kernel μ, we are interested to study the stability and the exponential stability of $S(t)$. Well-posedness and asymptotic results for (1.1) or (1.3) have been established in several works (see e.g. [6, 8, 9, 10, 15, 16]; see also [21, 22] for a similar system arising in the theory of simple fluids of Boltzmann type). In particular, [9] shows that $S(t)$ is exponentially stable provided that μ satisfies the differential inequality

$$\mu'(s) + \delta\mu(s) \le 0, \qquad \forall s > 0, \tag{1.4}$$

for some $\delta > 0$. This sufficient condition, commonly exploited to obtain exponential decay of semigroups arising from problems with memory (see e.g. the book [14]), is rather restrictive. Indeed, the recent paper [1] shows that a necessary condition in order for exponential stability to hold for a similar problem arising from viscoelasticity is that there exists $C \ge 1$ and $\delta > 0$ such that

$$\mu(t+s) \le Ce^{-\delta t}\mu(s), \tag{1.5}$$

for every $t \ge 0$ and almost every $s > 0$. We will see in the next section that the same fact holds true also for (1.1). As noted in [1], it is not difficult to demonstrate that (1.5) is equivalent to (1.4) if $C = 1$. Nonetheless, when $C > 1$, the gap between the two conditions is huge. Therefore, one of the aims of the present work is to establish the exponential stability of $S(t)$ under much weaker conditions than (1.4). A similar attempt has turned out to be successful in the analysis of a linearly viscoelastic equation with memory [17]. Here, at least in the case when H is finite-dimensional, (1.5) will be proved to be sufficient as well, unless the system admits periodic orbits.

Plan of the paper. In the following §2, we show that, within a proper functional setting, (1.1) generates a strongly continuous linear semigroup of contractions. The relation between (1.1) and the corresponding equation (1.3) is discussed in some detail in §3. In §4, we dwell on the stability of the semigroup, showing that, when $\beta = 0$, periodic trajectories occur for some particular kernels, referred to as resonant. Finally, in §5 and §6, we analyze the exponential stability of $S(t)$ when H is finite-dimensional and infinite-dimensional, respectively.

Notation. We denote by $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_X$ the norm and the inner product on a given space X. In particular,

$$\|u\|_V = \|A^{1/2}u\|_H, \qquad \|\eta\|^2_{\mathcal{M}} = \int_0^\infty \mu(s)\|\eta(s)\|^2_V ds.$$

Beside $\mathcal{M}$, we shall also consider the L^2-weighted space $\mathcal{N} = L^2_\mu(\mathbb{R}^+; H)$. We name $\{\alpha_m\}$ and $\{e_m\}$, with $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ or $m \in \{1, \ldots, N\}$ (depending whether H is infinite-dimensional or N-dimensional), the increasing sequence of the eigenvalues of A, with $\alpha_1 > 0$, and the corresponding sequence of eigenvectors, respectively. If H is infinite-dimensional, $\alpha_m \to \infty$ as $m \to \infty$. Finally, we introduce the phase space $\mathcal{H} = H \times \mathcal{M}$.

2. The Semigroup

We preliminarily state in a precise way the assumptions on the memory kernel.

General assumptions on $\boldsymbol{\mu}$. Let $\mu : \mathbb{R}^+ \to [0, \infty)$ be a decreasing function such that

$$\kappa = \int_0^\infty \mu(s) ds < \infty.$$

We assume that there exists a strictly increasing sequence $\{s_n\}$, with $s_0 = 0$, either finite (possibly reduced to s_0 only) or converging to $s_\infty \in (0, \infty]$, such that μ has jumps at $s = s_n$, $n > 0$, and is absolutely continuous on each interval (s_{n-1}, s_n) and on the interval (s_∞, ∞), if defined.

Remark 2.1. Note that μ can be unbounded in a neighborhood of zero. Besides, μ' is defined almost everywhere.

In the course of the investigation, we will sometimes encounter the following particular kernels (cf. [1]).

- **Step kernels:** Kernels of the form

  $$\mu(s) = \sum_{n=1}^\infty \gamma_n \chi_{[s_{n-1}, s_n)}(s),$$

  where $\{\gamma_n\}$ is a strictly decreasing positive sequence (up to when, possibly, $\gamma_n = 0$ for some $n \in \mathbb{N}$). Note that

  $$\sum_{n=1}^\infty \gamma_n (s_n - s_{n-1}) = \kappa.$$

- ◇ **ℓ-paced kernels:** Step kernels for which there exists $\tau > 0$ and a strictly increasing sequence $\{k_n\}$ of natural numbers, with $k_0 = 0$, such that
$$s_n = \tau k_n.$$
Up to redefining $\{k_n\}$, it is clear that τ is not uniquely determined (for instance, any τ/p with $p \in \mathbb{N}$ will do). It is however clear there exists the largest possible one, called *pace* of the kernel and denoted by ℓ, which is the greatest common divisor of $\{\tau k_n\}$.
- ◇ **Resonant kernels:** ℓ-paced kernel for which the quantity
$$\Omega_m = \frac{\ell}{2\pi}\sqrt{\alpha_m \kappa}\,, \qquad m \in \mathbb{N},$$
belongs to $\mathbb{N}$ for at least one $m \in \mathbb{N}$, The reason of the word "resonant" will be clear in the sequel.

Introducing the pair $z = (u, \eta)$, we rewrite (1.1) as the Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{d}{dt} z(t) = \mathbb{L} z(t), & t > 0, \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where $z_0 = (u_0, \eta_0)$. The linear operator $\mathbb{L}$ is defined by

$$\mathbb{L}(u, \eta) = (-A(\beta u + \textstyle\int_0^\infty \mu(s)\eta(s)ds), T\eta + u),$$

with domain

$$\mathcal{D}(\mathbb{L}) = \{(u, \eta) \in \mathcal{H} : u \in V,\ \eta \in \mathcal{D}(T),\ \beta u + \textstyle\int_0^\infty \mu(s)\eta(s)ds \in \mathcal{D}(A)\}.$$

An application of the Lumer-Phillips Theorem [19] entails

Theorem 2.2. *Problem* (2.1) *generates a contraction semigroup $S(t) = e^{t\mathbb{L}}$ on $\mathcal{H}$, that is,*

$$\|S(t)z_0\|_{\mathcal{H}} \le \|z_0\|_{\mathcal{H}}, \qquad \forall z_0 \in \mathcal{H}.$$

Besides, for every $z_0 \in \mathcal{D}(\mathbb{L})$, the energy equality

$$\frac{d}{dt}\|S(t)z_0\|^2_{\mathcal{H}} + 2\beta\|u(t)\|^2_V - \int_0^\infty \mu'(s)\|\eta^t(s)\|^2_V ds + \mathbb{J}[\eta^t] = 0, \tag{2.2}$$

holds, with

$$\mathbb{J}[\eta^t] = \sum_n [\mu(s_n^-) - \mu(s_n^+)]\|\eta^t(s_n)\|^2_V,$$

where the sum includes the value $n = \infty$ if $s_\infty < \infty$. Concerning the second component of the solution $S(t)z_0$, we have the explicit representation formula

$$\eta^t(s) = \begin{cases} \int_0^s u(t-y)dy, & 0 < s \le t, \\ \eta_0(s-t) + \int_0^t u(t-y)dy, & s > t, \end{cases} \tag{2.3}$$

which is valid for every $z_0 \in \mathcal{H}$. We address the reader to [9, 12], where the above instances are discussed in more detail.

Our analysis will be focused on the following issues.

- **Stability of $S(t)$:** For every $z_0 \in \mathcal{H}$,
$$\lim_{t\to\infty} \|S(t)z_0\|_{\mathcal{H}} = 0.$$
- **Exponential stability of $S(t)$:** There exist $\omega > 0$ and $M \geq 1$ such that
$$\|S(t)z_0\|_{\mathcal{H}} \leq M e^{-\omega t}\|z_0\|_{\mathcal{H}}, \qquad \forall z_0 \in \mathcal{H}.$$

In order to investigate the stability of $S(t)$, we shall exploit the following abstract result (see [1, Lemma 4.1]).

Lemma 2.3. *Let $S(t)$ be a contraction semigroup on a Hilbert space $\mathcal{H}$, and let $\mathcal{V} \subset \mathcal{H}$ be a reflexive Banach space with continuous and dense embedding (but not necessarily compact). Suppose that for every $z_0 \in \mathcal{V}$ the following hold.*

(i) *$\|S(t)z_0\|_{\mathcal{H}} = \|z_0\|_{\mathcal{H}}$ for all $t > 0$ implies that $z_0 = 0$.*

(ii) *The set $\bigcup_{t\geq t_*} S(t)z_0$ is bounded in $\mathcal{V}$ and relatively compact in $\mathcal{H}$, for some $t_* \geq 0$.*

Then $S(t)$ is stable.

As far as exponential stability is concerned, we have

Lemma 2.4. *A contraction semigroup $S(t)$ on a real Hilbert space $\mathcal{H}$ is exponentially stable if and only if there exists $\varepsilon > 0$ such that*
$$\inf_{\lambda\in\mathbb{R}} \|(i\lambda - \mathbb{L})z_0\|_{\mathcal{H}} \geq \varepsilon\|z_0\|_{\mathcal{H}}, \qquad \forall z_0 \in \mathcal{D}(\mathbb{L}).$$
In the formula above, $\mathcal{H}$ and $\mathbb{L}$ are understood to be the complexifications of the original $\mathcal{H}$ and $\mathbb{L}$, respectively.

Lemma 2.4, contained in [9], is a slight modification of [3, Theorem 5.1.5] (see also [20]).

We conclude the section by proving the necessary condition for exponential decay anticipated in the Introduction.

Theorem 2.5. *If $S(t)$ is exponentially stable, then the kernel μ fulfills* (1.5).

Proof. Denote for simplicity $\mathcal{M}_1 = L^2_\mu(\mathbb{R}^+, \mathbb{R})$. Let $\zeta \in \mathcal{M}_1$, and choose $z_0 = (0, \zeta e_1)$. Then, the corresponding solution to (2.1) is given by $S(t)z_0 = (\varphi(t)e_1, \xi^t e_1)$, where
$$\begin{cases} \dot\varphi(t) + \beta\alpha_1\varphi(t) + \alpha_1 \displaystyle\int_0^\infty \mu(s)\xi^t(s)ds = 0, \qquad t > 0, \\ \dot\xi^t = T\xi^t + \varphi(t), \qquad t > 0, \\ \varphi(0) = 0, \\ \xi^0(s) = \zeta(s). \end{cases}$$
Here, T is the the infinitesimal generator of the right-translation semigroup $\Sigma(t)$ on $\mathcal{M}_1$, acting as
$$[\Sigma(t)\zeta](s) = \begin{cases} 0, & 0 < s \leq t, \\ \zeta(s-t), & s > t. \end{cases}$$
By the exponential decay assumption, we know that
$$\max\{|\varphi(t)|, \sqrt{\alpha_1}\,\|\xi^t\|_{\mathcal{M}_1}\} \leq \|S(t)z_0\|_{\mathcal{H}} \leq \sqrt{\alpha_1}\, M e^{-\omega t}\|\zeta\|_{\mathcal{M}_1}.$$

In particular, setting $\psi(t)=\int_0^t \varphi(y)dy$, we have

$$|\psi(t)| \leq \int_0^t |\varphi(y)|dy \leq \frac{M\sqrt{\alpha_1}}{\omega}\|\zeta\|_{\mathcal{M}_1}.$$

On account of (1.2) and (2.3),

$$\begin{aligned}
M^2 e^{-2\omega t}\|\zeta\|^2_{\mathcal{M}_1} \geq \|\xi^t\|^2_{\mathcal{M}_1} \\
&\geq \int_t^\infty \mu(s)|\zeta(s-t)+\psi(t)|^2 ds \\
&\geq \frac{1}{2}\int_t^\infty \mu(s)|\zeta(s-t)|^2 ds - |\psi(t)|^2 k(t) \\
&\geq \frac{1}{2}\|\Sigma(t)\zeta\|^2_{\mathcal{M}_1} - \frac{M^2\alpha_1}{\omega^2}k(t)\|\zeta\|^2_{\mathcal{M}_1}.
\end{aligned}$$

Thus, for every $\zeta \in \mathcal{M}_1$,

$$\|\Sigma(t)\zeta\|_{\mathcal{M}_1} \leq \Upsilon(t)\|\zeta\|_{\mathcal{M}_1},$$

having set

$$\Upsilon(t) = M\sqrt{2e^{-2\omega t} + \frac{2\alpha_1}{\omega^2}k(t)}\,.$$

Since $\Upsilon(t)\to 0$ as $t\to\infty$, by means of standard arguments of semigroup theory (see e.g. [19]), we conclude that $\Sigma(t)$ is exponentially stable on $\mathcal{M}_1$. In light of [1, Theorem 3.3], this is the same as postulating condition (1.5) to hold. □

3. On the Equivalence between (1.1) and (1.3)

Before proceeding with the stability analysis, we come back to the equivalence between system (1.1) and the integro-differential equation (1.3). We take here the occasion to clarify some details which have been overlooked in the review paper [12]. Indeed, in order to prove such an equivalence, we need to ask an additional condition on μ, when not compactly supported. Namely, there exists $K \geq 0$ such that

$$k(s) \leq K\sqrt{\mu(s)}\,, \tag{3.1}$$

with k given by (1.2), for all s large enough. If μ is compactly supported, that is,

$$S_\infty = \sup\{s\in\mathbb{R}^+ : \mu(s)>0\} < \infty,$$

then we immediately find the stronger relation

$$k(s) \leq \left(\int_s^\infty \sqrt{\mu(y)}\,dy\right)\sqrt{\mu(s)}. \tag{3.2}$$

Remark 3.1. The original model of hereditary heat conduction leading to (1.3) requires that the kernel k appearing in the equation be summable (see e.g. [8]). In terms of μ, it amounts to saying that

$$\int_0^\infty s\mu(s)ds < \infty.$$

Although this requirement is inessential in order to obtain existence and uniqueness results for (1.1), if we assume the summability of k, then (3.1) is automatically satisfied. Precisely, if μ is not compactly supported, denoting

$$r(s) = \frac{\mu(s)}{[k(s)]^2},$$

we claim that

$$\lim_{s\to\infty} r(s) = \infty,$$

which clearly implies (3.1). Indeed, for an arbitrary $\sigma > 0$, the tangent $\tau(s)$ to the curve $k(s)$ at $s = \sigma$ is given by

$$\tau(s) = -r(\sigma)[k(\sigma)]^2 s + k(\sigma) + r(\sigma)[k(\sigma)]^2\sigma.$$

Since k is convex, $\tau(s) \leq k(s)$. Therefore,

$$\lim_{\sigma\to\infty} \frac{1}{r(\sigma)} = \lim_{\sigma\to\infty} 2\int_\sigma^{\sigma+1/[r(\sigma)k(\sigma)]} \tau(s)ds \leq 2\limsup_{\sigma\to\infty}\int_\sigma^\infty k(s)ds = 0.$$

We are now in a position to establish our argument, under assumption (3.1). We first mention that, applying the Lumer-Phillips Theorem [19] in the weaker phase space $\mathcal{W} = \mathcal{D}(A^{-1/2}) \times \mathcal{N}$, it is easy to show that (1.1) generates a contraction semigroup $S_0(t)$ on $\mathcal{W}$, which clearly coincides with $S(t)$ when restricted to $\mathcal{H}$.

Let $u(t) \in C([0,\infty), H)$ be a solution to the integro-differential equation (1.3), with a prescribed initial datum $u(t)_{|t\leq 0}$. We require that $u_0 = u(0) \in H$, $\eta_0 \in \mathcal{M}$ and $\eta_0' \in \mathcal{N}$, where we set

$$\eta_0(s) = \int_0^s u(-y)dy.$$

We show that $u(t)$ is equal to the first component of $S(t)(u_0, \eta_0)$. As a byproduct, the solution to (1.3) is unique. To this aim, we multiply (1.3) by $A^{-1}w$, for a generic vector $w \in H$, to obtain

$$\langle A^{-1}\dot{u}(t), w\rangle_H + \beta\langle u(t), w\rangle_H + \int_0^\infty k(s)\langle u(t-s), w\rangle_H ds = 0.$$

Defining

$$\eta^t(s) = \int_{t-s}^t u(y)dy,$$

we have that

$$(\eta^t)'(s) = u(t-s).$$

Hence, both η^t and $(\eta^t)'$ belong to $\mathcal{N}$ for every $t \geq 0$ and $\eta^t(s) \to 0$ in H as $s \to 0$, which is the same as saying that $A^{-1/2}\eta^t \in \mathcal{D}(T)$. Consequently, arguing as in [12], the limit

$$\lim_{s\to S_\infty} \mu(s)\|\eta^t(s)\|_H^2$$

exists, and is equal to 0 if $S_\infty = \infty$ or $s_\infty < S_\infty < \infty$. Integrating by parts, and exploiting the continuity of $\eta^t(s)$ at $s = 0$, we get

$$\begin{aligned}\int_0^\infty k(s)\langle u(t-s), w\rangle_H ds &= \int_0^{S_\infty} k(s)\langle (\eta^t)'(s), w\rangle_H ds \\ &= \int_0^{S_\infty} \mu(s)\langle \eta^t(s), w\rangle_H ds + \lim_{s\to S_\infty} k(s)\langle \eta^t(s), w\rangle_H \\ &= \int_0^\infty \mu(s)\langle \eta^t(s), w\rangle_H ds.\end{aligned}$$

Indeed, from (3.1) (if $S_\infty = \infty$) or (3.2) (if $S_\infty < \infty$),

$$|k(s)\langle \eta^t(s), w\rangle_H| \le k(s)\|\eta^t(s)\|_H\|w\|_H \le \frac{k(s)}{\sqrt{\mu(s)}}\sqrt{\mu(s)}\,\|\eta^t(s)\|_H\|w\|_H \to 0,$$

as $s \to S_\infty$. Therefore,

$$\langle A^{-1}\dot u(t), w\rangle_H + \beta\langle u(t), w\rangle_H + \int_0^\infty \mu(s)\langle \eta^t(s), w\rangle_H ds = 0.$$

Since w is arbitrary and η^t fulfills (2.3) by construction (with η_0 as above), we conclude that $(u(t), \eta^t) = S_0(t)(u_0, \eta_0) = S(t)(u_0, \eta_0)$, owing to the fact that $(u_0, \eta_0) \in \mathcal{H}$.

Conversely, if $u_0 \in H$ and $\eta_0 \in \mathcal{M}$, with $\eta_0' \in \mathcal{N}$ and $\eta_0(s) \to 0$ in H as $s \to 0$, using the representation formula (2.3) and reversing the argument, we see that the first component $u(t)$ of $S(t)(u_0, \eta_0)$ is a solution to (1.3) with initial data $u(0) = u_0$ and $u(t)_{|t<0} = \eta_0'(-t)$. In summary, we established both the sought equivalence and an existence and uniqueness result for (1.3).

4. Stability

The preliminary question is whether for all kernels μ in the considered class the corresponding semigroup $S(t)$ is at least stable. The answer is negative.

Proposition 4.1. *If $\beta = 0$ and μ is a resonant kernel, then $S(t)$ admits periodic orbits. Thus, in particular, it is not stable.*

Proof. Due to our hypotheses, there exists $m \in \mathbb{N}$ for which $\nu = \sqrt{\alpha_m \kappa}$ fulfills

$$\frac{\ell\nu}{2\pi} \in \mathbb{N}.$$

We prove the claim by showing that the solution $(u(t), \eta^t)$ to (2.1) with initial data $z_0 = (u_0, \eta_0) \in \mathcal{D}(\mathbb{L})$ given by

$$u_0 = 0, \qquad \eta_0(s) = \frac{1}{\nu}[\cos\nu s - 1]e_m,$$

is

$$u(t) = [\sin\nu t]e_m, \qquad \eta^t(s) = \frac{1}{\nu}[\cos\nu(t-s) - \cos\nu t]e_m.$$

Indeed, $u(0) = u_0$, $\eta^0(s) = \eta_0(s)$ and, by a direct calculation,

$$\dot\eta^t(s) = -(\eta^t)'(s) + u(t).$$

Concerning the first equation of (2.1), we readily get the equality

$$\dot{u}(t) + \int_0^\infty \mu(s) A\eta^t(s) ds = \frac{\alpha_m}{\nu}\bigg(\int_0^\infty \mu(s)\cos\nu(t-s) ds\bigg) e_m = 0,$$

since $\cos\nu(t-s)$ is ℓ-periodic and μ is constant on each interval $[\ell k_{n-1}, \ell k_n)$. $\square$

Remark 4.2. A similar instance occurs for abstract semigroups arising in linear viscoelasticity, as shown in [1, Proposition 5.1]. We should remark that in [1] the resonance condition is stated incorrectly.

A positive result on stability is

Theorem 4.3. *Assume that*

$$\int_0^\infty s^2\mu(s) ds < \infty. \tag{4.1}$$

Then, $S(t)$ is stable if and only if

(i) *either $\beta > 0$; or*
(ii) *$\beta = 0$ and μ is not resonant.*

Before coming to the proof, some observations are in order.

Remark 4.4. Condition (4.1), already encountered in the papers [15, 21, 22] in connection with stability issues, is the same as requiring the integrability of $\int_s^\infty k(y)dy$. This is necessary if we want to include initial data η_0 of the form $\eta_0(s) = sv$, with $v \in V$. With reference to (1.3), it amounts to saying that we can consider constant initial data $u(t)_{|t<0} = v$.

Remark 4.5. Let us consider the following simple, albeit paradigmatic, example. Take $H = L^2(0,\pi)$, and let

$$A = -\frac{d^2}{dx^2}, \qquad \mathcal{D}(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

Then, $\{\alpha_m\} = \{m^2\}$. Setting

$$\Omega = \frac{\ell}{2\pi}\sqrt{\kappa}\,,$$

we see that $\Omega_m = m\Omega$. Hence, in this case, a kernel μ is resonant if and only if $\Omega \in \mathbb{Q}$.

Similar considerations can be made for other particular realizations of H and of the operator A, obtaining, from case to case, statements depending on the distribution of the eigenvalues of A (cf. [1]).

Proof of Theorem 4.3. The strategy of the proof is outlined in [1], although the arguments have to be adapted to our particular equation. We will reach the desired conclusion by means of some lemmata, whose proofs will be reported only when appreciable differences from [1] occur.

Lemma 4.6. *There exists a reflexive Banach space $\mathcal{V} \subset \mathcal{D}(\mathbb{L})$, with continuous and dense (but not compact) embedding into $\mathcal{H}$, such that, for every $z_0 \in \mathcal{V}$, the set $\bigcup_{t\geq 1} S(t)z_0$ is bounded in $\mathcal{V}$ and relatively compact in $\mathcal{H}$.*

Proof. Define

$$\mathcal{V} = \mathcal{D}(A) \times \big[L^2_\mu(\mathbb{R}^+; \mathcal{D}(A^{3/2})) \cap \mathcal{D}(T)\big],$$

endowed with the norm

$$\|(u,\eta)\|^2_{\mathcal{V}} = \|Au\|^2_H + \|A\eta\|^2_{\mathcal{M}} + \|T\eta\|^2_{\mathcal{M}}.$$

Let $z_0 = (u_0, \eta_0) \in \mathcal{V}$ be fixed. In this proof, c will stand for a generic constant depending only on z_0. Multiplying (2.1) by $(A^2u, A^2\eta)$ in $\mathcal{H}$, on account of (2.2), we immediately get the bound

$$\|Au(t)\|^2_H + \|A\eta^t\|^2_{\mathcal{M}} \leq c.$$

The computation is formal, but it can be rigorously justified either via semigroup arguments or within a suitable regularization scheme. Since $\eta_0 \in \mathcal{D}(T)$, by means of (2.3), we learn that

$$(\eta^t)'(s) = \begin{cases} u(t-s), & 0 < s \leq t, \\ \eta_0'(s-t), & s > t, \end{cases}$$

and we easily obtain the remaining control

$$\|T\eta^t\|_{\mathcal{M}} \leq c.$$

Hence, we have the boundedness of $\bigcup_{t\geq 0} S(t)z$ in $\mathcal{V}$. By virtue of a slight generalization of [18, Lemma 5.5], the required compactness in $\mathcal{H}$ would follow from the uniform control (as $t \geq 1$) of the *tails* of η^t

$$\lim_{x\to\infty} \left[\sup_{t\geq 1} \int_{(0,\frac{1}{x})\cup(x,\infty)} \mu(s)\|\eta^t(s)\|^2_V ds \right] = 0.$$

But the bound on $u(t)$ and (2.3) yield (defining $\eta_0(s) = 0$ if $s < 0$)

$$\|\eta^t(s)\|^2_V \leq cs^2 + 2\|\eta_0(s-t)\|^2_V,$$

which, using a simple argument devised in [11], allows us to say that the limit above is indeed equal to zero. We remark that (4.1) is needed to draw this conclusion. □

Till the end of the section, z_0 is a *generic* element of $\mathcal{D}(\mathbb{L})$. In correspondence of z_0, we set

$$U(t) = \int_0^t u(y)dy.$$

Note that $U(t)$ is constant if and only if $U(t) \equiv 0$ if and only if $u(t) \equiv 0$.

Lemma 4.7. *If the equality*

$$2\beta\|u(t)\|^2_V - \int_0^\infty \mu'(s)\|\eta^t(s)\|^2_V ds + \mathbb{J}[\eta^t] = 0, \qquad \forall t \geq 0, \tag{4.2}$$

implies that $U(t)$ is constant for every $z_0 \in \mathcal{D}(\mathbb{L})$, then $S(t)$ is stable. In particular, stability occurs if $\beta > 0$.

Proof. We show that Lemma 2.3 applies. Indeed, (ii) is verified from Lemma 4.6. To show (i), assume that $\|S(t)z_0\|_{\mathcal{H}}$ is constant for every $z_0 \in \mathcal{D}(\mathbb{L})$ (and thus, for every $z_0 \in \mathcal{V}$). Then, using the hypothesis and (2.2), we have that $u(t) = 0$ for every $t \geq 0$. Hence, by means of (2.3),

$$\|\eta_0\|^2_{\mathcal{M}} = \|z_0\|^2_{\mathcal{H}} = \|S(t)z_0\|^2_{\mathcal{H}} = \|\eta^t\|^2_{\mathcal{M}} = \int_0^\infty \mu(t+s)\|\eta_0(s)\|^2_V ds, \qquad \forall t \geq 0.$$

An application of the dominated convergence theorem shows that the last integral converges to zero as $t \to \infty$, so implying that $\eta_0 = 0$. □

We now investigate the more difficult case $\beta = 0$.

Lemma 4.8. *Let $\beta = 0$. If μ is not an ℓ-paced kernel, then $S(t)$ is stable.*

We omit the proof, since it is essentially the same as the one of [1, Theorem 4.9]. We just mention that the key argument is to show that, if (4.2) holds and μ is not ℓ-paced, then $U(t)$ is periodic with arbitrarily small periods, and so it is constant. The conclusion follows from Lemma 4.7.

In the last lemma, we deal with the remaining case of ℓ-paced kernels. This will complete the proof of Theorem 4.3. Although the subsequent proof parallels the one of [1, Proposition 5.2], we will write it down in full detail for two reasons. Firstly, here we considerably simplify the argument. Secondly, the conclusion that we reach is stronger than in [1].

Lemma 4.9. *Let $\beta = 0$, and let μ be an ℓ-paced kernel. Then, $S(t)$ is stable if and only if μ is not resonant.*

Proof. Let $z_0 \in \mathcal{D}(\mathbb{L})$, and assume that (4.2) holds. Since

$$\mu(s) = \sum_{n=1}^\infty \gamma_n \chi_{[\ell k_{n-1}, \ell k_n)}(s),$$

assumption (4.2) translates into

$$\eta^t(\ell k_n) = 0, \qquad \forall t \geq 0,$$

for every $n \in \mathbb{N}$ such that $\mu(\ell k_{n-1}) > 0$. Rewriting (2.3) in terms of U, we have

$$\eta^t(s) = \begin{cases} U(t) - U(t-s), & 0 < s \leq t, \\ U(t) + \eta_0(s-t), & s > t. \end{cases}$$

Hence,

$$U(t) = U(t - \ell k_n), \qquad \forall t \geq \ell k_n$$

and

$$\eta_0(s) = -U(\ell k_n - s), \qquad \forall s \in (0, \ell k_n].$$

The first equality says that $U(t)$ is ℓk_n-periodic, and so it is also ℓ-periodic, since the greatest common divisor of $\{k_n\}$ is equal to one. Extending $U(t)$ for all times by periodicity, from the second equality we learn that

$$\eta_0(s) = -U(-s) \qquad \Longrightarrow \qquad \eta^t(s) = U(t) - U(t-s),$$

for every s such that $\mu(s) > 0$, so that we obtain the equality

$$\int_0^\infty \mu(s) A\eta^t(s)ds = \kappa AU(t) - \int_0^\infty \mu(s) AU(t-s)ds = \kappa AU(t),$$

since $U(t-s)$ is ℓ-periodic and μ is constant on each interval $[\ell k_{n-1}, \ell k_n)$. We conclude that $U(t)$ satisfies the abstract wave equation

$$\ddot{U}(t) + \kappa AU(t) = 0.$$

For every m, the ℓ-periodic function $\gamma_m(t) = \langle U(t), e_m\rangle_H$ solves the ordinary differential equation

$$\ddot{\gamma}_m(t) + \kappa\alpha_m\gamma_m(t) = 0.$$

In particular, $\gamma_m(t)$ is τ_m-periodic, with $\tau_m = 2\pi/\sqrt{\kappa\alpha_m}$. Most important, τ_m is the *smallest* period of $\gamma_m(t)$, unless $\gamma_m(t)$ is identically zero, meaning that $\ell = p\tau_m$ for some $p \in \mathbb{N}$, in conflict with the assumption that μ is not resonant. Thus, $\gamma_m(t)$ must vanish for every m, which implies that $U(t) \equiv 0$. We are now in a position to apply Lemma 4.7 to infer the stability of $S(t)$. Conversely, if μ is resonant, we know from Proposition 4.1 that $S(t)$ is not stable. □

5. Exponential Stability: The Finite-Dimensional Case

We now dwell on the particular instance where $H = \mathbb{R}^N$. Accordingly, A is a strictly positive $(N \times N)$-matrix. In this case, although $\mathcal{H}$ remains infinite-dimensional, if the necessary condition (1.5) for exponential decay is satisfied, then $S(t)$ behaves exactly as a semigroup on a finite dimensional space, namely, stability implies exponential stability. Indeed, we have

Theorem 5.1. *Assume that μ satisfies* (1.5)*.*

(i) *If $\beta > 0$, then $S(t)$ is exponentially stable.*
(ii) *If $\beta = 0$, then $S(t)$ is exponentially stable, unless μ is resonant.*

Proof. We restrict ourselves to prove the case (ii), since (i) holds for a general Hilbert space H, not necessarily finite-dimensional (see the following Theorem 6.3). Throughout the proof, $|\cdot|$ will denote both the absolute value in $\mathbb{R}$ and the euclidean norm in $\mathbb{C}^N$.

Assume that μ is not resonant. Arguing by contradiction, we suppose that $S(t)$ is not exponentially stable. Then, exploiting Lemma 2.4, we can find sequences $\lambda_n \in \mathbb{R}$, $u_n \in \mathbb{C}^N$ and $\eta_n \in \mathcal{M}_N = L^2_\mu(\mathbb{R}^+, \mathbb{C}^N)$, with $\eta_n' \in \mathcal{M}_N$ and $\eta_n(0) = 0$, such that

$$|u_n|^2 + \|\eta_n\|^2_{\mathcal{M}_N} = 1 \tag{5.1}$$

and

$$\begin{cases} i\lambda_n u_n + \displaystyle\int_0^\infty \mu(s) A\eta_n(s)ds = a_n, \\ i\lambda_n\eta_n(s) - u_n + \eta_n'(s) = b_n(s), \end{cases} \tag{5.2}$$

for some $a_n \to 0$ in $\mathbb{C}^N$ and $b_n \to 0$ in $\mathcal{M}_N$. The second equation of (5.2) can be integrated, to obtain

$$\eta_n(s) = \frac{1 - e^{-i\lambda_n s}}{i\lambda_n}\, u_n + \Gamma_n(s, \lambda_n), \tag{5.3}$$

having defined, for $\lambda \in \mathbb{R}$,

$$\Gamma_n(s,\lambda) = e^{-i\lambda s}\int_0^s e^{i\lambda\sigma} b_n(\sigma)d\sigma.$$

In the limit, the expression still makes sense if $\lambda_n = 0$. We now claim that

$$\lim_{n\to\infty} \|\Gamma_n(\cdot,\lambda)\|_{\mathcal{M}_N} = 0, \tag{5.4}$$

uniformly as $\lambda \in \mathbb{R}$. Indeed,

$$|\Gamma_n(s,\lambda)| \leq \int_0^s |b_n(\sigma)|d\sigma,$$

and therefore,

$$\|\Gamma_n(\cdot,\lambda)\|^2_{\mathcal{M}_N} \leq \int_0^\infty \mu(s)\bigg(\int_0^s |b_n(\sigma)|d\sigma\bigg)^2 ds = \int_0^\infty \bigg(\int_0^s \sqrt{\mu(s)}\,|b_n(\sigma)|d\sigma\bigg)^2 ds.$$

By virtue of (1.5),

$$F(s) = \int_0^s \sqrt{\mu(s)}\,|b_n(\sigma)|d\sigma \leq \sqrt{C}\int_0^s e^{-\frac{\delta}{2}(s-\sigma)}\sqrt{\mu(\sigma)}\,|b_n(\sigma)|d\sigma = \sqrt{C}\,(f * \sqrt{\mu}\,|b_n|)(s),$$

where we set $f(s) = e^{-\delta s/2}$ and $*$ denotes the convolution on $\mathbb{R}^+$. Using a standard measure theoretical result,

$$\|F\|_{L^2(\mathbb{R}^+)} \leq \sqrt{C}\,\|f\|_{L^1(\mathbb{R}^+)}\|\sqrt{\mu}\,b_n\|_{L^2(\mathbb{R}^+)} = \frac{2\sqrt{C}}{\delta}\|b_n\|_{\mathcal{M}_N},$$

and, consequently,

$$\|\Gamma_n(\cdot,\lambda)\|_{\mathcal{M}_N} \leq \frac{2\sqrt{C}}{\delta}\|b_n\|_{\mathcal{M}_N} \to 0,$$

as $n \to \infty$.

The next step is to show that λ_n is bounded. If not so, passing to a subsequence, $|\lambda_n| \to \infty$. Then, collecting (5.3) and (5.4), we readily see that $\eta_n \to 0$ in $\mathcal{M}_N$, and the first equation of (5.2) yields $i\lambda_n u_n \to 0$, which forces the convergence $u_n \to 0$, against (5.1).

Knowing that u_n and λ_n are bounded, up to subsequences, we may assume that $\lambda_n \to \lambda$ and $u_n \to u$. Thus, from (5.3) and (5.4),

$$\eta_n(s) \to \eta(s) = \frac{1-e^{-i\lambda s}}{i\lambda}\,u$$

in $\mathcal{M}_N$ and, in light of (5.1),

$$|u|^2 + \|\eta\|^2_{\mathcal{M}_N} = |u|^2\bigg(1 + \int_0^\infty \frac{2-2\cos\lambda s}{\lambda^2}\mu(s)ds\bigg) = 1 \qquad \Longrightarrow \qquad u \neq 0.$$

Exploiting the first equation of (5.2), we also obtain that

$$i\lambda u + \int_0^\infty \mu(s)A\eta(s)ds = 0. \tag{5.5}$$

We can exclude the value $\lambda = 0$ since, in that case, $\eta(s) = su$, and (5.5) entails $Au = 0$. As $\lambda \neq 0$, substituting the expression of $\eta(s)$ in (5.5), we end up with the equation

$$[\lambda^2 + (\hat{\mu}(\lambda) - \kappa)A]u = 0, \tag{5.6}$$

where

$$\hat{\mu}(\lambda) = \int_0^\infty e^{-i\lambda s}\mu(s)ds \tag{5.7}$$

is the *half Fourier transform* of μ. We conclude that u solves (5.6) if and only if $u = e_m$ for some $m \in \{1, \dots, N\}$. Thus, (5.6) leads to the equality

$$\lambda^2 - \kappa\alpha_m = -\alpha_m\hat{\mu}(\lambda). \tag{5.8}$$

Since $\alpha_m > 0$, it must be

$$\Im\hat{\mu}(\lambda) = -\int_0^\infty \mu(s)\sin\lambda s\, ds = 0,$$

which is possible if and only if $\mu(s)$ assumes the same value (up to a nullset) on each interval I_p of the form

$$I_p = [\tfrac{2p\pi}{|\lambda|}, \tfrac{2(p+1)\pi}{|\lambda|}], \qquad p \in \mathbb{N}.$$

This implies that also $\Re\hat{\mu}(\lambda) = 0$ (so that $\hat{\mu}(\lambda) = 0$), and μ is ℓ-paced with pace

$$\ell = \frac{2\pi q}{|\lambda|},$$

for some $q \in \mathbb{N}$. A further use of (5.8) then yields $\Omega_m = q$, which violates the assumption that μ is not resonant. $\square$

6. Exponential Stability: The Infinite-Dimensional Case

When H is infinite-dimensional, the picture is completely different. Indeed, it is no longer true that, under (1.5), stability implies exponential stability. We will show this fact for the particular example given by Remark 4.5, although analogous considerations apply to more general situations. To this end, we first need the following result on the half Fourier transform (5.7) of a step kernel (see [1, Lemma 6.2] for a proof).

Lemma 6.1. *Let μ be a non-resonant step kernel and define*

$$c_m = m\hat{\mu}(m\sqrt{\kappa}\,).$$

Then, $c_m \neq 0$ for every $m \in \mathbb{N}$, and there is a sequence $\{m_j\}$ such that $c_{m_j} \to 0$.

Proposition 6.2. *If μ is a step kernel, then the semigroup $S(t)$ of Remark 4.5 is never exponentially stable.*

Proof. We assume that μ is not resonant, otherwise $S(t)$ is not even stable. Given $m \in \mathbb{N}$ and $\zeta_m = (0, m^{-1}e_m) \in \mathcal{H}$, we look for $\rho_m \in \mathbb{C}$ and $\psi_m \in L^2_\mu(\mathbb{R}^+, \mathbb{C})$, with $\psi'_m \in L^2_\mu(\mathbb{R}^+, \mathbb{C})$ and $\psi_m(0) = 0$, such that the vector $z_m = (\rho_m e_m, \psi_m e_m) \in \mathcal{D}(\mathbb{L})$ solves the (complex) equation

$$(im\sqrt{\kappa} - \mathbb{L})z_m = \zeta_m.$$

Note that

$$\|\zeta_m\|_{\mathcal{H}} = \sqrt{\kappa}$$

and

$$\|z_m\|_{\mathcal{H}} \geq \|\rho_m e_m\|_H = |\rho_m|.$$

Thus, in light of Lemma 2.4, we will reach the conclusion by showing that $|\rho_{m_j}| \to \infty$ for some subsequence $\{m_j\}$. Indeed, written in components, the equality above reads

$$\begin{cases} im\sqrt{\kappa}\,\rho_m + m^2\displaystyle\int_0^\infty \mu(s)\psi_m(s)ds = 0, \\ im\sqrt{\kappa}\,\psi_m(s) + \psi_m'(s) - \rho_m = \dfrac{1}{m}. \end{cases}$$

Integrating the second equation, with the condition $\psi_m(0) = 0$, we obtain

$$\psi_m(s) = \frac{1}{im^2\sqrt{\kappa}}(1 + m\rho_m)(1 - e^{-im\sqrt{\kappa}\,s}),$$

and substituting $\psi_m(s)$ into the first equation, we find the relation

$$mc_m\rho_m = m\kappa - c_m.$$

We now exploit Lemma 6.1. Since $c_m \neq 0$,

$$\rho_m = \frac{\kappa}{c_m} - \frac{1}{m}$$

and the corresponding $\psi_m(s)$ solves the system. Besides, $|\rho_{m_j}| \to \infty$ for some $\{m_j\}$. $\square$

In order to provide sufficient conditions for exponential stability to hold, we first need a definition borrowed from [17]. Introducing the Borel probability measure m_μ on $\mathbb{R}^+$ as

$$m_\mu(\mathcal{P}) = \frac{1}{\kappa}\int_{\mathcal{P}} \mu(s)ds, \qquad \mathcal{P} \subset \mathbb{R}^+,$$

the *flatness set* of μ is

$$\mathcal{F}_\mu = \{s \in \mathbb{R}^+ : \mu(s) > 0 \text{ and } \mu'(s) = 0\},$$

while the *flatness rate* of μ is

$$\mathcal{R}_\mu = m_\mu(\mathcal{F}_\mu).$$

The main result of the present section reads as follows.

Theorem 6.3. *Let* (1.5) *hold. If* $\beta = 0$, *assume in addition that* $\mathcal{R}_\mu < 1/2$. *Then,* $S(t)$ *is exponentially stable.*

Remark 6.4. Here, we are not making any assumption on the dimension of H. Besides, as it will be clear from the subsequent proofs, the theorem holds without the requirement that A has compact inverse. Indeed, only the continuous inclusion $V \subset H$ is needed. For further use, let us denote by $\alpha > 0$ the corresponding Poincaré constant, that is,

$$\alpha\|v\|_H^2 \leq \|v\|_V^2, \qquad \forall v \in V.$$

Remark 6.5. The interesting question of what happens when H is infinite-dimensional, $\beta = 0$ and $\mathcal{R}_\mu \in [\frac{1}{2}, 1)$ is not covered by our theory, and remains open. This problem, already encountered in [17], seems to be particularly hard to solve. Even more so, after Theorem 5.1, we cannot hope to have any numerical evidence either.

We shall split the proof in two cases, according to the value of β (strictly positive or zero). The following simple result, a very particular instance of the renowned Datko's theorem [5], will be extremely useful to tackle the more difficult case $\beta = 0$. A three-line proof is given.

Lemma 6.6. *Assume that there exists $c \geq 0$ such that*

$$\int_0^\infty \|S(t)z_0\|_{\mathcal{H}}^2 dt \leq c,$$

for any $z_0 \in \mathcal{D}(\mathbb{L})$ with $\|z_0\|_{\mathcal{H}} = 1$. Then $S(t)$ is exponentially stable.

Proof. Due to (2.2), for every $z_0 \in \mathcal{D}(\mathbb{L})$ with $\|z_0\|_{\mathcal{H}} = 1$ we have

$$\|S(\tau)z_0\|_{\mathcal{H}}^2 \leq \frac{1}{\tau} \int_0^\tau \|S(t)z_0\|_{\mathcal{H}}^2 dt \leq \frac{c}{\tau}.$$

By density, we conclude that the norm of $S(t)$ decays to zero. □

In the sequel, for $z_0 = (u_0, \eta_0) \in \mathcal{D}(\mathbb{L})$, we denote for simplicity

$$\mathcal{E}(t) = \|S(t)z_0\|_{\mathcal{H}}^2.$$

As before, we set

$$U(t) = \int_0^t u(y)dy,$$

and we take k as in (1.2).

Proof of Theorem 6.3 [case $\boldsymbol{\beta = 0}$]. Given $z_0 \in \mathcal{D}(\mathbb{L})$ and $\nu \in (0, 1)$, we consider the functionals

$$\begin{aligned}
\Phi(t) &= -\frac{1}{\kappa} \int_0^\infty \varphi_\nu(s) \langle u(t), \eta^t(s) \rangle_H ds, \\
\Psi_1(t) &= \frac{1}{2} \langle U(t), u(t) \rangle_H, \\
\Psi_2(t) &= \frac{1}{2} \int_0^\infty k(s) \|\eta^t(s) - U(t)\|_V^2 ds,
\end{aligned}$$

where

$$\varphi_\nu(s) = \mu(s_\nu)\chi_{(0,s_\nu]}(s) + \mu(s)\chi_{(s_\nu,\infty]}(s),$$

for some fixed $s_\nu \in (0, s_1)$ such that $\int_0^{s_\nu} \mu(s)ds \leq \nu/2$, is introduced (in place of μ) to handle the (possible) case when μ is singular in the origin (see [17]).

Lemma 6.7. *There exists $K > 0$ such that*

$$\|U(t)\|_H^2 \leq K\big[\Psi_2(t) + \|\eta^t\|_{\mathcal{M}}^2\big]. \tag{6.1}$$

Proof. We have

$$\begin{aligned}\alpha\Big(\int_0^\infty k(s)ds\Big)\|U(t)\|_H^2 &\leq \int_0^\infty k(s)\|U(t)\|_V^2 ds \\ &\leq 4\Psi_2(t) + 2\int_0^\infty k(s)\|\eta^t(s)\|_V^2 ds \\ &\leq 4\Psi_2(t) + \frac{2C}{\delta}\|\eta^t\|_{\mathcal{M}}^2.\end{aligned}$$

Here, we used the representation formula (2.3) written in terms of U along with the inequality

$$k(s) \leq \frac{C}{\delta}\mu(s), \tag{6.2}$$

which is a straightforward consequence of (1.5). □

By means of direct calculations, we have

Lemma 6.8. *The functional*

$$\Psi(t) = 2\Psi_1(t) + \Psi_2(t)$$

fulfills the differential equality

$$\frac{d}{dt}\Psi(t) = -\frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \|u(t)\|_H^2. \tag{6.3}$$

Concerning $\Phi(t)$, denoting, for $\mathcal{P} \subset \mathbb{R}^+$,

$$\Gamma_{\mathcal{P}}^+[\eta^t] = \int_{\mathcal{P}} \mu(s)\|\eta^t(s)\|_V^2 ds, \qquad \Gamma_{\mathcal{P}}^-[\eta^t] = \int_{\mathbb{R}^+\setminus\mathcal{P}} \mu(s)\|\eta^t(s)\|_V^2 ds,$$

and arguing as in [17, Lemma 4.1], we can prove

Lemma 6.9. *For any $\nu \in (0,1)$, there exists $c_\nu > 0$, depending only on ν, such that, for every measurable set $\mathcal{P} \subset \mathbb{R}^+$,*

$$\begin{aligned}\frac{d}{dt}\Phi(t) \leq -(1-\nu)\|u(t)\|^2 + (1+\nu)m_\mu(\mathcal{P})\Gamma_{\mathcal{P}}^+[\eta^t] + c_\nu\Gamma_{\mathcal{P}}^-[\eta^t] \\ - c_\nu\Big(\int_0^\infty \mu'(s)\|\eta^t(s)\|_V^2 ds - \mathbb{J}[\eta^t]\Big).\end{aligned} \tag{6.4}$$

Lemma 6.10. *There exist a measurable set $\mathcal{P} \subset \mathbb{R}^+$ and constants $\nu \in (0,1)$, $a < 1$, $M > 0$ and $\varepsilon > 0$ such that the functional*

$$\mathcal{L}(t) = M\mathcal{E}(t) + \Phi(t) + a\Psi(t)$$

satisfies

$$\frac{d}{dt}\mathcal{L}(t) + \varepsilon\mathcal{E}(t) \leq 0. \tag{6.5}$$

Proof. Since $\mathcal{R}_\mu < 1/2$, we can choose $n \in \mathbb{N}$ large enough such that, setting

$$\mathcal{P} = \{s \in \mathbb{R}^+ : n\mu'(s) + \mu(s) > 0\},$$

we have that $m_\mu(\mathcal{P}) < 1/2$. Indeed, the sets $\mathcal{P}_j = \{s \in \mathbb{R}^+ : j\mu'(s) + \mu(s) > 0\}$ are nested with respect to $j \in \mathbb{N}$ and fulfill $\bigcap_{j=1}^\infty \mathcal{P}_j = \mathcal{F}_\mu$, up to a set of null measure. Then, taking

$$a = \frac{1}{2} + \hat{\mu}(\mathcal{P}) < 1,$$

and collecting (6.3) and (6.4), we obtain

$$\begin{aligned}\frac{d}{dt}[\Phi(t) + a\Psi(t)] \leq -(1-a-\nu)\|u(t)\|^2 - \frac{1}{2}(1-a-\nu)\Gamma^+_{\mathcal{P}}[\eta^t] + c_\nu \Gamma^-_{\mathcal{P}}[\eta^t] \\ - c_\nu\bigg(\int_0^\infty \mu'(s)\|\eta^t(s)\|^2_V ds - \mathbb{J}[\eta^t]\bigg).\end{aligned}$$

Hence, recalling that

$$\|\eta^t\|^2_{\mathcal{M}} = \Gamma^+_{\mathcal{P}}[\eta^t] + \Gamma^-_{\mathcal{P}}[\eta^t],$$

provided that we fix ν small enough, we end up with the differential inequality

$$\frac{d}{dt}[\Phi(t) + a\Psi(t)] + \varepsilon\mathcal{E}(t) \leq c\Gamma^-_{\mathcal{P}}[\eta^t] - c\Big(\int_0^\infty \mu'(s)\|\eta^t(s)\|^2_V ds - \mathbb{J}[\eta^t]\Big),$$

for some $\varepsilon > 0$ and some $c > 0$. Finally, setting $M \geq c + nc$, on account of (2.2), we are led to

$$\frac{d}{dt}\mathcal{L}(t) + \varepsilon\mathcal{E}(t) \leq c\Gamma^-_{\mathcal{P}}[\eta^t] + nc\int_0^\infty \mu'(s)\|\eta^t(s)\|^2_V ds \leq 0,$$

since $n\mu'(s) + \mu(s) \leq 0$, for almost every $s \in \mathbb{R}^+ \setminus \mathcal{P}$. □

Remark 6.11. It is clear that, up to choosing $M > 0$ large enough, we also have that

$$M\mathcal{E}(t) + \Phi(t) \geq 0.$$

Besides, exploiting (6.2), it is apparent that

$$\mathcal{L}(0) \leq Q\mathcal{E}(0),$$

for some $Q > 0$.

We are now in a position to conclude the proof of Theorem 6.3. For $z_0 \in \mathcal{D}(\mathbb{L})$, with $\|z_0\|_{\mathcal{H}} = 1$, we integrate (6.5) on $[0, \tau]$. In view of the remark above, and since

$$\|u(\tau)\|^2_H + \|\eta^\tau\|^2_{\mathcal{M}} \leq \mathcal{E}(0) = 1,$$

this entails

$$a\Psi_2(\tau) + \varepsilon\int_0^\tau \mathcal{E}(t)dt \leq Q\mathcal{E}(0) - a\langle U(\tau), u(\tau)\rangle_H \leq Q + a\|U(\tau)\|_H.$$

On the other hand, from Lemma 6.7,

$$a\|U(\tau)\|_H \leq \frac{aK}{4} + \frac{a}{K}\|U(\tau)\|^2_H \leq 1 + \frac{K}{4} + a\Psi_2(\tau).$$

Therefore,

$$\varepsilon\int_0^\tau \mathcal{E}(t)dt \leq Q + 1 + \frac{K}{4}.$$

From the arbitrariness of $\tau > 0$, the exponential stability of $S(t)$ follows by an application of Lemma 6.6.

Proof of Theorem 6.3 [case $\beta > 0$]. For $z_0 \in \mathcal{D}(\mathbb{L})$, we introduce the functional

$$\Upsilon(t) = \int_0^\infty k(s)\|\eta^t(s)\|_V^2 ds,$$

which, in light of (6.2), satisfies the differential inequality

$$\begin{aligned}\frac{d}{dt}\Upsilon(t) &= -\|\eta^t\|_{\mathcal{M}}^2 + 2\int_0^\infty k(s)\langle \eta^t(s), u(t)\rangle_V ds \\ &\leq -\|\eta^t\|_{\mathcal{M}}^2 + \frac{2C}{\delta}\|u(t)\|_V \int_0^\infty \mu(s)\|\eta^t(s)\|_V ds \\ &\leq -\frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + c\|u(t)\|_V^2,\end{aligned}$$

for some $c > 0$. Defining then

$$\mathcal{L}_1(t) = M\mathcal{E}(t) + \Upsilon(t),$$

for some $M > 0$ large enough, it is clear from (2.2) that the differential inequality

$$\frac{d}{dt}\mathcal{L}_1(t) + \varepsilon\mathcal{E}(t) \leq 0$$

holds for some $\varepsilon > 0$. Owing to (6.2), $\mathcal{L}_1(t)$ controls and is controlled by $\mathcal{E}(t)$. Hence, the claim is a direct consequence of the Gronwall lemma.

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