OPTIMAL CHANNEL NETWORKS, LANDSCAPE FUNCTION AND BRANCHED TRANSPORT

FILIPPO SANTAMBROGIO

ABSTRACT. Starting from transportation models for branching structures, we define a function that represents the elevation of the landscape in a river basin. This function is already well-known in the geophysical community but it is only considered under a very strong discretization. We generalize it to the continuous case and study its properties, providing several applications.

1. INTRODUCTION

Lots of branching structures transporting different kind of fluids, such as road systems, communication networks, river basins, blood vessels, leaves and trees and so on, may be easily thought of as coming from a variational principle. They appear when transport costs encourage joint transportation. Recently these problems received a lot of attention by mathematicians. This paper mainly discusses some features wich are crucial in river basins applications, but addresses also applications to other fields.

1.1. **Branching transport models by Gilbert and Xia.** A mathematical formalization for the branching transport problems is very classical and has been performed first for atomic measures and then generalized. We briefly present here the problem introduced by Gilbert in [11] and [12], where it is presented as an extension of Steiner's minimal length problems. The main applications that Gilbert referred to were in the field of communication networks. Given two atomic probability measures $\mu = \sum_{i=1}^{m} a_i \delta_{x_i}$ and $\nu = \sum_{j=1}^{n} b_j \delta_{y_j}$, consider

$$(P_G) \qquad \min \ E(G) := \sum_{h} w_h^{\alpha} \mathcal{H}^1(e_h), \qquad (1.1)$$

where the infimum is among all weighted oriented graphs $G = (e_h, \hat{e}_h, w_h)_h$ (where e_h are the edges, $\hat{e}_h$ represent their orientations and w_h the weights) satisfying Kirchhoff's Law: in each segment vertex which is not one of the x_i's or y_j's the total incoming mass equals the outcoming, while in each x_i we have

$$a_i + \text{ incoming mass } = \text{ outcoming mass}$$

and, conversely, in each y_j we have

$$\text{incoming mass } = \text{ outcoming mass} + b_j.$$

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These conditions correspond exactly to the well known Kirchhoff Law for electric circuits. The orientations $\hat{e}_h$ do not appear in the energy E but appear in fact in Kirchhoff constraints. The exponent α is a fixed parameter $0 < \alpha < 1$ so that the function $t \mapsto t^\alpha$ is concave and subadditive. In this way larger links bringing the mass from μ to ν are preferred to several smaller links transporting the same total mass. It is not difficult to check that the energy of any finite graph may be improved if we remove cycles from the graph. In this way we can minimize among finite graphs which are actually trees. This implies a bound on the number of edges and hence ensures a suitable compactness which is enough to prove existence of a minimizer.

More recently Xia, in [20], has proposed a new formalization leading to generalizations of this problem to arbitrary probability measures μ and ν. In this case the interest of the author of [20] is to view this problem as an extension of Monge-Kantorovich optimal transport theory (see [19]). In fact Steiner and Monge's problems represent the limit cases $\alpha = 0$ and $\alpha = 1$, respectively.

1.2. **Landscape equilibrium and OCNs in geophysics.** It is interesting to see how people working in geophysics arrive to very similar problems in the study of river basins. There is a wide literature on this and a quite comprehensive reference is [17]. The specific subject dealt with by this paper is developed both in [17] and in [2] (this last paper being our main reference, but a short previous summary of these ideas can be found in [1] as well). While studying the configuration of a river basin, the main objects are two: the landscape elevation, which is a function z giving the altitude of any point of the region we are considering, and a river network N, which is the datum of all the streams that concur to bring water (which falls on the region as rain) to a single point (where a lake is supposed to be present). A first link between both objects is the fact that at any point the direction followed by water is the direction of steepest descent of z. Hence, once we know z we are able to deduce N and to compute the multiplicity $\theta(x)$ at any point x, that is the quantity of water passing through x while following the steepest descent lines of z. At first the interest is towards an evolution model, which allows z and N (and hence θ) to depend on time as well. The evolution of z is ruled by an erosion equation of the form

$$\frac{\partial z}{\partial t} = -\theta |\nabla z|^2 + c, \tag{1.2}$$

where ∇z is the spatial gradient of z and c is a positive constant. The idea is that the erosion effect increases both with the quantity of water and with the slope. The constant c is called uplift and takes care of the fact that all the material brought down by erosion in the end is uniformly redistributed from below in the whole region as a geomorphological effect. Equation (1.2) is in fact a simplified version of other more general evolution equations involving higher order terms. The following phenomenon concerning solutions of (1.2) can be empirically observed: approximately, up to a certain time scale

both z and θ (i.e. N) move, in a very strong erosional evolution; then, up to a larger time scale the network is almost constant, letting $\theta(x,t) = \theta(x)$ depend on the position only, and the landscape function evolves without changing its lines of maximal slope; finally there is a much larger time scale such that z approximatively agrees with a landscape equilibrium, i.e. a stationary solution of (1.2). We are interested in studying landscape equilibria. In this case the steepest descent condition, that we can read as "∇z follows the direction of the network", is completed by a second one which we get by imposing $\partial z/\partial t = 0$ in (1.2). This leads to $|\nabla z| = c^{1/2}\theta^{-1/2}$ and this last condition is called slope-discharge relation. It is explicitly suggested in [2] that in (1.2) one could change the exponents of θ and $|\nabla z|$ (preserving anyway the increasing behavior with respect to both variables), thus obtaining different slope-discharge relationships. In general we get $|\nabla z| = c\,\theta^{\alpha-1}$ and the physically interesting case is when the exponent α is very close to $1/2$.

To find landscape equilibria a discretization is performed in [2] and a regular square grid is used. Functions defined on the pixels of the grid and vanishing at a given point x_0 representing the outlet are considered, as well as networks composed by edges of the grid, directed from every point to one of the neighbors.

- As we already mentioned, the conditions on the direction of the water allow to reconstruct a network from a function. In fact, given a function z with no local minima other than x_0, one can always follow the maximal slope paths of z.
- These are obtained by linking any point x of the grid to a point which realizes the minimum of z among the neighbors of z. Notice in particular that these paths are only composed by edges following the two main directions of the grid.
- In this way a network $N = N(z)$ can be deduced from z.
- On the other hand, the slope-discharge condition allows to reconstruct a function from a network N, provided it is tree-shaped.
- In order to make this reconstruction, first compute the multiplicities of the points of the network: at a point x its multiplicity $\theta(x)$ is the number of points which find x on their way to the outlet (this works under the assumption that the quantity of rain falling down at any pixel is the same, i.e. rain falls uniformly on the grid). See also Figure 1.2, where the multiplicity of a point x_i is computed as the number of points in the area A_i.
- Then set $z(x_0) = 0$ and for any other point x consider the only path on N linking x_0 to x. Set $z(x) = \sum_i \theta(x_i)^{\alpha-1}$, where the x_i's are the points on the path. In Figure 1.2 the path linking x_0 to x is shown.
- In this way we get a function $z = z(N)$.

In general it will not be true that a function $z(N)$ has maximal slope in the direction of the network N. Finding a landscape equilibrium means exactly satisfying both conditions at a time, through a fixed point problem. The
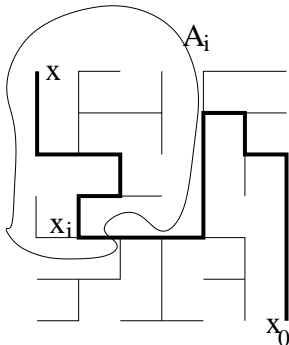
FIGURE 1. The path from x to x_0 and the multiplicity of x_i

algorithm starts from a tree-shaped network N, builds the function $z(N)$, and then the new network $N' = N(z(N))$. If $N' = N$, then the landscape function $z = z(N)$ is a landscape equilibrium.

The important idea presented in [2] is the relation between landscape equilibria and Optimal Channel Networks (OCNs in literature, see for instance [18], [16] and [13]). An OCN is a network N minimizing a certain dissipated energy. The dissipated energy in a system satisfying the slope-discharge relation is the total potential energy that water loses on the network. For each pixel we have a quantity of water θ which falls down towards the next pixel and its elevation decreases by a quantity which is proportional to $|\nabla z|$ and hence to $\theta^{\alpha-1}$. Hence, the total energy loss is given by $\sum_i \theta(x_i)\theta(x_i)^{\alpha-1} = \sum_i \theta(x_i)^{\alpha}$. It is clear that this energy is the same as in (1.1) (no length of segments is involved because in a regular grid they all have the same, given, length). What is proven in [2] is that, if N is an OCN minimizing this energy, then the landscape function $z = z(N)$ reconstructed from it is in fact an equilibrium. This actually means that not only the slope of z in the direction of the network is given by $\theta^{\alpha-1}$, which is true by construction, but also that this direction is the direction of maximal slope.

Notice that the problems studied in [2] and in the other papers on the subject have undergone a very strong discretization. In fact they correspond to solve (P_G) where μ is a discretization on a regular grid of the Lebesgue measure and $\nu = \delta_{x_0}$, but with the extra constraint that only edges e_h which are given with the grid are allowed. Compared to continuous models there is a loss of rotational invariance, a fixed scale effect due to the mesh, and several questions concerning the river basin may lose their meaning (for instance questions about the interfaces between two separated parts of the basin and points where the water takes two different directions, or most regularity issues). On the other hand, a continuous counterpart for the landscape function could not be simply a regular solution of (1.2) or of its statical version, as for C^1 functions steepest descent curves are well-defined, but they never merge and therefore do not give raise to a positive multiplicity θ (except for the case $d = 1$, see [2]).

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1.3. **A landscape function appearing for derivative purposes.** We will briefly see here another aspect of branching transport problems such as (P_G) where a function similar to the landscape function appears.

Definition 1. We define irrigation cost of a finite atomic measure $\mu \in \mathcal{P}(\Omega)$ the minimum of problem (P_G) for $\nu = \delta_0$. This quantity is denoted by $X_\alpha(\mu)$.

A variational analysis of this functional yields the following.

Theorem 1.1. *Suppose $\mu = \sum_{i=1}^m a_i \delta_{x_i}$ with $a_i > 0$ (so that the finite set $K = \{x_i : i = 1, \ldots, n\}$ is actually the support of μ) and that μ_1 is another probability measure concentrated on K with $\mu_1 = \sum_{i=1}^m b_i \delta_{x_i}$. Then we have*

$$X_\alpha(\mu_1) \leq X_\alpha(\mu) + \alpha \sum_{i=1}^m z(x_i)(b_i - a_i),$$

where the function z is defined in this way: take an optimal graph G for the problem (P_G) for the measures μ and δ_0; this graph is a tree; for any x_i define

$$z(x_i) = \sum_{h \in H(i)} w_h^{\alpha-1} \mathcal{H}^1(e_h),$$

where $H(i)$ denotes the sets of indices of the edges of the unique path from 0 to x_i.

Proof. We will build a new oriented graph which is acceptable for Problem (P_G) when irrigating μ_1 starting from δ_0. This graph will be built by using the same edges $(e_h)_h$ as in G but changing the weights w_h's. We define the new weights w_h' by

$$w_h' = w_h + \sum_{i : h \in H(i)} (b_i - a_i).$$

It is easy to check that this new graph satisfies the constraints, and so we get

$$X_\alpha(\mu_1) \leq \sum_h (w_h')^\alpha \mathcal{H}^1(e_h) \leq X_\alpha(\mu) + \alpha \sum_h w_h^{\alpha-1} \sum_{i : h \in H(i)} (b_i - a_i),$$

where the last inequality is obtained by concavity of $t \mapsto t^\alpha$. By changing the order in performing the sums we easily get the thesis. $\qquad\square$

Remark 1. The link between this function z and the one used in geophysics is straightforward: to compute a value $z(x)$ in fact what we do is integrating the multiplicity of the graph along the river from 0 up to x. See Figure 1.3 and compare with Figure 1.2: in this case there are in general many more degrees of freedom, the multiplicity of the represented point x_i is the total mass of the region A_i and the geometry of points and edges is not prescribed.
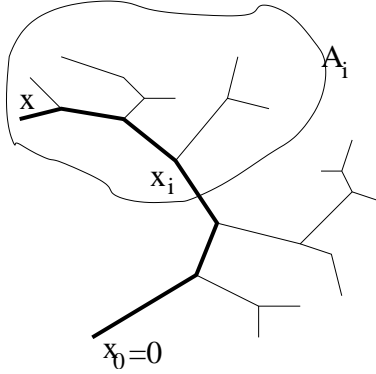
FIGURE 2. The path from x to x_0 and the multiplicity of x_i

Remark 2. As a consequence of Theorem 1.1, if we set $\mu_\varepsilon = \mu + \varepsilon(\mu_1 - \mu)$, we get the following inequality:

$$\limsup_{\varepsilon \to 0^+} \frac{X_\alpha(\mu_\varepsilon) - X_\alpha(\mu)}{\varepsilon} \leq \int z \, d(\mu_1 - \mu).$$

This inequality gives information on the derivative of the functional X_α and this fact is very useful in variational problems of the following kind:

$$(P_+) \qquad\qquad \min \quad X_\alpha(\mu) + F(\mu),$$

where F may be any functional whose derivative is known. We will show later an example and briefly explain the interest of these problems.

Remark 3. The result of Theorem 1.1 has been established under no constraints on the direction of the edges, i.e. in the setting of problem (P_G). It is easy to reproduce them in the case of grid-constrained OCNs, as in the proof there is no need to change the edges of the graph. Hence it is a result which is valid also in the setting of [2].

1.4. **Goals of the paper.** The main goal of this paper is to define a landscape function in the continuous case and analyze its properties. We will use the recent developments about these irrigation problems concerning arbitrary probability measures (and not only atomic ones) that we mentioned before. We will consider the irrigation of an arbitrary measure on a domain Ω starting from a single source δ_0. Here the main problem is that the optimal structures which arise are not necessarily trees in the sense that there may be points which are reached by several curves. We will anyway propose a landscape function z and check that it is well-defined. Then we will prove that it shares all the properties that we had in the discrete case, in particular the fact that on a point x_0 of the irrigation network it has maximal slope in the direction of the network itself and that this slope is given by $\theta^{\alpha-1}$, where θ is the multiplicity of the network at x_0. Moreover we will prove that even in the continuous case an inequality on the derivative of the energy X_α

involving the landscape function is available and finally we will give some continuity and semicontinuity results.

The interest of generalizing the concept of landscape function to the irrigation of arbitrary probability measures does not have only variational applications. In river basins applications, in fact, it is natural to consider directly a configuration where the starting measure is the Lebesgue measure instead of considering a grid discretization. Moreover, getting rid of the discretization will also add isotropy and other features to the models in [2] and [17].

The main results leading to the definition of a landscape function and to the study of its properties will be presented from Section 3 on. Section 2 is in fact devoted to a short summary of the different models and of the main features that will be used later.

2. Branched transport structures: models and tools

Recently, several models have been developed to study branching transport structures. At a first sight they are very different, but many equivalences can be proved. On one hand there are the Eulerian approaches, first by Gilbert and then by Xia, and on the other there are Lagrangian, time-dependent approaches mainly presented in [15] and [4]. The Eulerian models are the most similar to the one used by the geophysical community, and the generalization by Xia will be explained here a little more. The Lagrangian ones, on the other hand, are the most useful to deal with a possible landscape function thanks to the form of the functionals involved.

2.1. **Relaxed minimization by Xia and its minimal value.** Let us briefly see how Xia extended the Gilbert approach. The key point is that the constraint on the incoming and outcoming masses in each vertex (Kirchhoff Law) may be easily written as $\nabla \cdot \lambda_G = \mu - \nu$, where $\lambda_G = \sum_h w_h [[e_h]]$ is a vector measure ($[[e]]$ being the integration measure on the segment e following its direction: a measure which is absolutely continuous w.r.t. $\mathcal{H}^1$ with density $\hat{e}$ on e and 0 elsewhere). This consideration led Xia in [20] to extend the problem by relaxation to generic probabilities μ and ν. The problems becomes

$$(P_X) \qquad\qquad \min \bar{E}(\lambda) : \ \nabla \cdot \lambda = \mu - \nu$$

where $\bar{E}(\lambda) := \inf \liminf_n E(\lambda_{G_n})$ where the infimum is over all possible sequences of finite graphs $(G_n)_n$ such that the corresponding vector measures λ_{G_n} converge to λ.

It is possible to prove that, when μ and ν are both actually atomic measures, we retrieve the problem by Gilbert. This can be done by means of necessary optimality conditions: if we minimize over vector measures Xia's functional, proving that a minimizer must necessarily be a finite graph (see [21] and [5]). Hence Xia's formulation is an extension of Gilbert's one.

The minimum value of (P_X), which obviously depends on μ and ν, will be denoted by $d_\alpha(\mu, \nu)$. It is very important to understand when this minimal value is finite. It is proven in [20] that, when α is sufficiently close to 1, namely $\alpha > 1 - 1/d$, then this minimum is finite for any pair (μ, ν). Moreover the following uniform estimate (see [20]) holds

$$d_\alpha(\mu, \nu) \leq C_{\alpha,d} \operatorname{diam}(\Omega).$$

To deal with local perturbations, it is not difficult to deduce a sharper estimate, namely

$$d_\alpha(\mu, \nu) \leq C_{\alpha,d} \, \delta^\alpha \operatorname{diam}(\omega), \tag{2.1}$$

whenever $\mu - \nu = \delta(\mu' - \nu')$ and μ' and ν' are probability measures on $\omega \subset \Omega$. When α is below the threshold $1 - 1/d$ there are pairs of measures which are not linkable by a finite energy configuration. The possibility of reaching by finite energy a certain measure is somehow linked to its "dimension" (see [8]).

Definition 2. For any measure $\mu \in \mathcal{P}(\Omega)$ we set $X_\alpha(\mu) = d_\alpha(\mu, \delta_0)$. This is an extension of Definition 1. Moreover, a measure μ is called $\alpha-$irrigable if $X_\alpha(\mu) < +\infty$.

In [20] it is proven that, for $\alpha > 1 - 1/d$, the quantity d_α defines a new distance over the space of probability measures $\mathcal{P}(\Omega)$, which induces the weak topology. Both the continuity with respect to this topology and the triangle inequality will be used in the sequel.

2.2. **Patterns and traffic plan models.** This subsection is an informal summary of the models in [15] and [4] and their properties. Languages and approaches have been sometimes simplified in view of the aim of this paper.

Let Ω be a fixed compact domain in $\mathbb{R}^d$. Let us denote by Γ the set of $1-$Lipschitz curves $\gamma : [0, +\infty[\to \Omega$ that are eventually constant. It means that, if we define the stopping time of a curve γ by

$$\sigma(\gamma) = \inf \left\{ s \, : \, \gamma \text{ is constant on } [s, +\infty[\right\},$$

these are curves with $\sigma(\gamma) < +\infty$. Let us also denote by Γ_{arc} the set of those curves in Γ which are parametrized by arc length and by Γ_{inj} the set of curves in Γ which are injective on $[0, \sigma(\gamma)[$. In the sequel we will often identify a curve with its image, in the sense that sometimes we will write γ instead of $\gamma([0, \sigma(\gamma)]) = \gamma([0, +\infty[)$.

Given a probability measure η on the space Γ, for any point $x \in \mathbb{R}^d$ the $\eta-$multiplicity of x is defined by

$$[x]_\eta := \eta \left\{ \gamma \in \Gamma \, : \, x \in \gamma([0, \sigma(\gamma)]) \right\}. \tag{2.2}$$

Then we can define $Z_\eta(\gamma) = \int_0^{\sigma(\gamma)} [\gamma(t)]_\eta^{\alpha-1} dt$ and $J(\eta) = \int_\Gamma Z_\eta \, d\eta$. Notice that, for simplicity, here Z_η is defined without the term $|\gamma'|(t)$ which appears in the original definition in [4]. It will be deduced later that minimizers are actually parametrized by arc length.

Finally, we consider the maps $\pi_0, \pi_\infty : \Gamma \to \Omega$, given by $\pi_0(\gamma) = \gamma(0)$, and $\pi_\infty(\gamma) = \gamma(\sigma(\gamma))$. The two image measures $(\pi_0)_\sharp \eta$ and $(\pi_\infty)_\sharp \eta$, which belong to $\mathcal{P}(\Omega)$, will be called the starting and the terminal measure of η, respectively. Following the notation of [4] we may define a traffic plan as a measure $\eta \in \mathcal{P}(\Gamma)$ such that $\int_\Gamma \sigma(\gamma)\eta(d\gamma) < +\infty$. We will also call pattern a traffic plan η such that $(\pi_0)_\sharp \eta = \delta_0$. In the case of a pattern the terminal measure will also be called the measure irrigated by η

The minimization problem proposed in [4] is

$$(P) \quad \min \quad J(\eta) \; : \; \eta \text{ is a traffic plan}, (\pi_\infty)_\sharp \eta = \mu, \; (\pi_0)_\sharp \eta = \nu,$$

where μ and ν are given measures in $\mathcal{P}(\Omega)$. As $[\gamma(t)]_\eta \leq 1$, we have $Z_\eta(\gamma) \geq \sigma(\gamma)$. Hence it is straightforward that any η such that $J(\eta) < +\infty$ is actually a traffic plan. A traffic plan η which minimizes J among the traffic plans with the same starting and terminal measures, with $J(\eta) < +\infty$, will be called an optimal traffic plan. In the case $\nu = \delta_0$ it will be called optimal pattern.

A useful tool developed in [4] (see also [3]) is the following: if η is concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$ then the following remarkable formula holds:

$$J(\eta) = \int_{\mathbb{R}^d} [x]_\eta^\alpha \, \mathcal{H}^1(dx). \tag{2.3}$$

This formula gives an evident link with Gilbert and Xia's models.

2.3. Useful tools: optimality conditions for (P).
Before presenting some consequences of optimality, let us deal with some concepts from [15] which are crucial in the case $\nu = \delta_0$.

For any $t \geq 0$ consider an equivalence relation on Γ given by "the two curves γ_1 and γ_2 are in relation at time t if they agree on the interval $[0, t]$", and denote the equivalence classes by $[\cdot]_t$, so that

$$[\gamma]_t = \{\tilde{\gamma} \; : \; \tilde{\gamma}(s) = \gamma(s) \text{ for any } s \leq t\}.$$

For notational simplicity, let us set $|\gamma|_{t,\eta} := \eta([\gamma]_t)$.

Definition 3. Given $\eta \in \mathcal{P}(\Gamma)$, a curve $\gamma \in \Gamma$ is said to be η−good if

$$\int_0^{\sigma(\gamma)} |\gamma|_{t,\eta}^{\alpha-1} dt < +\infty.$$

Here are now the most important results that can be found in [15], [4], [3], [5] and [14] or easily deduced from them.

(1) Problem (P) admits a solution, provided the infimum is finite (i.e. there is at least a solution with finite energy).

(2) If η is an optimal traffic plan, then η is concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$. In particular, we may apply formula (2.3) for J.

(3) Suppose that η is an optimal traffic plan, that two curves $\gamma_0, \gamma_1 \in \Gamma_{arc} \cap \Gamma_{inj}$ meet twice (i.e. $\gamma_0(s_0) = \gamma_1(s_1)$, $\gamma_0(t_0) = \gamma_1(t_1)$ and $s_i \neq t_i$) and that $[\gamma_0(t)]_\eta \geq c > 0$ for any $t \in [s_0, t_0]$. Then either both curves coincide in the trajectory between the two common points or

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we have $\int_{s_0}^{t_0}[\gamma_0(t)]_\eta^{\alpha-1} dt < \int_{s_1}^{t_1}[\gamma_1(t)]_\eta^{\alpha-1} dt$. In particular two different curves with multiplicities bounded from below cannot part and then meet again.

(4) If η is an optimal pattern (in particular $\nu = \delta_0$), then for $\eta-$a.e. curve γ and a.e. $t < \sigma(\gamma)$ we have $[\gamma(t)]_\eta = \eta([\gamma]_t)$. Roughly speaking this means that if all the mass starts from a common point then there is no parting-and-meeting-again-later (this is the single path property described in [5].

(5) As a consequence, any optimal pattern η is concentrated on the set of $\eta-$good curves, and any $\eta-$good curve γ belongs to $\Gamma_{arc} \cap \Gamma_{inj}$ and satisfies $[\gamma(t)]_\eta = \eta([\gamma]_t)$ for any $t < \sigma(\gamma)$.

(6) Last but not least $\min (P) = d_\alpha(\mu, \nu)$, which means that the minima of the Lagrangian and of the Eulerian model coincide.

Remark 4. Notice that an optimal traffic plan η is concentrated on the set of $\eta-$good curves, but this does not mean that this set is linked to the support of η. In fact any restriction of an $\eta-$good curve is itself an $\eta-$good curve and hence, for instance, in the discrete case, we have plenty of $\eta-$good curves but the support of η is finite. In particular the set of $\eta-$good curves may be very different from the set of fibers of a traffic plan that we find in [4] or [5] and does not depend on any parametrization, but it is more intrinsic.

3. A general development formula

In this section we will develop in a useful way the variation of the functional J when passing from a traffic plan η to a traffic plan η'. Formula 2.3 will be crucial.

Theorem 3.1. *Let η and η' be probability measures on Γ and $\Delta\eta = \eta' - \eta$. Let us suppose that both η and $\Delta\eta$ are concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$ and $\int_\Gamma Z_\eta \, d|\Delta\eta| < +\infty$. Then*

$$J(\eta') \leq J(\eta) + \alpha \int_\Gamma Z_\eta \, d\Delta\eta - \alpha(1-\alpha) \int_{\mathbb{R}^d} [x]_{\Delta\eta}^2 \, \mathcal{H}^1(dx). \qquad (3.1)$$

Proof. Let us set $S_\eta = \{x \in \mathbb{R}^d : [x]_\eta > 0\}$ and analogously $S_{\eta'} = \{x \in \mathbb{R}^d : [x]_{\eta'} > 0\}$. First we prove that under the assumptions of this theorem $\mathcal{H}^1(S_{\eta'} \setminus S_\eta) = 0$. In fact, for any point $x \in S_{\eta'} \setminus S_\eta$ we have necessarily $[x]_\eta = 0$ and $[x]_{\Delta\eta} > 0$. Hence it is sufficient to prove that the integral of $[x]_{\Delta\eta}$ on this set vanishes to get the desired result. Yet we have

$$\int_{S_{\eta'}\setminus S_\eta}[x]_{\Delta\eta}\mathcal{H}^1(dx) = \int_{S_{\eta'}\setminus S_\eta}\mathcal{H}^1(dx)\int_\Gamma \Delta\eta(d\gamma)I_{x\in\gamma} = \int_\Gamma \Delta\eta(d\gamma)\mathcal{H}^1\big(\gamma\cap(S_{\eta'}\setminus S_\eta)\big),$$

where I denotes the function whose value is 1 if the condition in the index is satisfied, 0 otherwise (indicator function). The second assumption of the theorem implies that for $\Delta\eta-$a.e. curve γ the quantity $Z_\eta(\gamma)$ is finite and hence, for a.e. t, we have $\gamma(t) \in S_\eta$. Since γ is $1-$Lipschitz continuous,

this yields $\mathcal{H}^1(\gamma \setminus S_\eta) = 0$. Hence we have $\int_{S_{\eta'} \setminus S_\eta} [x]_{\Delta\eta} \mathcal{H}^1(dx) = 0$, which proves $\mathcal{H}^1(S_{\eta'} \setminus S_\eta) = 0$.

Now, as both η and η' are concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$, to evaluate J we can use the expression in (2.3) and get

$$
J(\eta') = \int_{S_\eta} \left([x]_\eta + [x]_{\Delta\eta} \right)^\alpha \mathcal{H}^1(dx)
$$

$$
\leq J(\eta) + \alpha \int_{S_\eta} [x]_\eta^{\alpha-1} [x]_{\Delta\eta} \mathcal{H}^1(dx) - \alpha(1-\alpha) \int_{S_\eta} [x]_{\Delta\eta}^2 \mathcal{H}^1(dx), \quad (3.2)
$$

where we have used the fact that $S_{\eta'} \subset S_\eta$ up to $\mathcal{H}^1-$negligible sets and the concavity inequalities

$$
(t+s)^\alpha \leq t^\alpha + \alpha t^{\alpha-1}s - \alpha(1-\alpha)(\max\{t,t+s\})^{\alpha-2}s^2 \leq t^\alpha + \alpha t^{\alpha-1}s - \alpha(1-\alpha)s^2
$$

(this last inequality being valid when both t and $t + s$ belong to $]0, 1]$).

Let us now work on the second term of the last sum we obtained. We have

$$
\int_{S_\eta} [x]_\eta^{\alpha-1} [x]_{\Delta\eta} \mathcal{H}^1(dx) = \int_{S_\eta} \mathcal{H}^1(dx) \int_\Gamma \Delta\eta(d\gamma)\, [x]_\eta^{\alpha-1} I_{x \in \gamma}.
$$

Here we want to change the order of integration and to do this we check what happens in absolute value:

$$
\int_{S_\eta} \mathcal{H}^1(dx) \int_\Gamma |\Delta\eta|(d\gamma)[x]_\eta^{\alpha-1} I_{x \in \gamma} = \int_\Gamma |\Delta\eta|(d\gamma) \int_{S_\eta} \mathcal{H}^1(dx)\, [x]_\eta^{\alpha-1} I_{x \in \gamma}
$$

$$
= \int_\Gamma |\Delta\eta|(d\gamma) \int_0^{\sigma(\gamma)} [\gamma(t)]_\eta^{\alpha-1} dt = \int_\Gamma Z_\eta d|\Delta\eta| < +\infty. \quad (3.3)
$$

In this series of equality, the first one is just changing the integration order, while the second relies on the fact that $|\Delta\eta|-$a.e. we have $\mathcal{H}^1(\gamma \setminus S_\eta) = 0$ and, γ being parametrized by arc length, the $\mathcal{H}^1-$integral on its image may become an integral in dt on $[0, \sigma(\gamma)]$. The finiteness of the last integral in (3.3) allows us to change the order of integration between $\Delta\eta$ and $\mathcal{H}^1$ and by analogous computations we get

$$
\int_{S_\eta} \mathcal{H}^1(dx) \int_\Gamma \Delta\eta(d\gamma)[x]_\eta^{\alpha-1} I_{x \in \gamma} = \int_\Gamma Z_\eta d\Delta\eta.
$$

Inserting this last equality in (3.2) gives the thesis. $\qquad\square$

4. EXISTENCE OF THE LANDSCAPE FUNCTION AND APPLICATIONS

In this section we come specifically back to Problem (P) for $\nu = \delta_0$. Even when not specifically stated, from now on η will be an optimal pattern irrigating an $\alpha-$irrigable measure μ.

4.1. **Well-definedness of the landscape function.** First a very elementary truncation lemma is needed. As it is just the formalization of a well-known principle (that a part of an optimal structure is itself optimal), it will not be proven here. It is in fact proven in [5] when stating the optimality of the connected components of a traffic plan in $\mathbb{R}^d \setminus \{x_0\}$.

Lemma 4.1. *If γ_0 is a curve such that $|\gamma|_{t_0,\eta} > 0$, set $x_0 = \gamma_0(t_0)$, $A = [\gamma_0]_{t_0}$, $\mu_A = (\pi_\infty)_\sharp (I_A \cdot \eta)$, $\mu' = \mu - \mu_A + \eta(A)\delta_{x_0}$, $\eta' = \eta - I_A \cdot \eta + \eta(A)\delta_{\bar{\gamma}_0}$, where the curve $\bar{\gamma}_0$ is the curve γ_0 stopped at time t_0. Then η' is an optimal pattern irrigating the measure μ'.*

Theorem 4.2. *If γ_0 and γ_1 are two $\eta-$good curves sharing the same endpoint $\bar{x}$, then $Z_\eta(\gamma_0) = Z_\eta(\gamma_1)$.*

Proof. If the two curves are identical the thesis is easily obtained. If they are not identical, then they must split at a certain time $\bar{t}$. It is possible that one of them stops at time $\bar{t}$, but not both, as in this case they would be identical. Then we can choose two times t_0 and t_1 with $|\gamma_i|_{t_i,\eta} > 0$ and $\bar{t} \leq t_i \leq \sigma(\gamma_i)$ for $i = 0, 1$ (if one of the two curves stops at time $\bar{t}$, say for instance $\sigma(\gamma_0) = \bar{t}$, then we are forced to choose $t_0 = \sigma(\gamma_0) = \bar{t}$ and we have $|\gamma_0|_{t_0,\eta} = |\gamma_1|_{\bar{t},\eta}$ and $|\gamma_1|_{\bar{t},\eta} > 0$ as a consequence of $\bar{t} < \sigma(\gamma_1)$). Figure 4.1 shows the two possible situations.
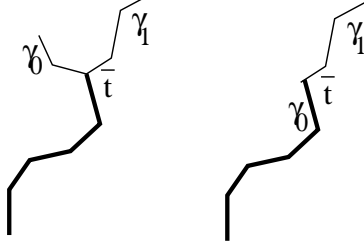

FIGURE 3. Both curves may go on after $\bar{t}$ or one may stop

Let us set $x_i = \gamma_i(t_i)$ and $l = |x_1 - x_0|$. Then we use the notations of the previous Lemma and we write

$$d_\alpha(\delta_0, \mu') \leq d_\alpha(\delta_0, \mu'') + d_\alpha(\mu', \mu''), \tag{4.1}$$

where $\mu'' = \mu - \mu_A + \eta(A)\delta_{x_1}$. Define $\eta'' = \eta - I_A \cdot \eta + \eta(A)\delta_{\bar{\gamma}_1}$, where the curve $\bar{\gamma}_1$ is the curve γ_1 stopped at time t_1. It is easy to check that $(\pi_\infty)_\sharp \eta'' = \mu''$ and then

$$d_\alpha(\delta_0, \mu'') \leq J(\eta'') \leq J(\eta') + \alpha \int_\Gamma Z_\eta \, d(\eta'' - \eta')$$

$$= J(\eta') + \alpha\eta(A) \left(\int_0^{t_1} |\gamma_1|_{t,\eta}^{\alpha-1} \, dt - \int_0^{t_0} |\gamma_0|_{t,\eta}^{\alpha-1} \right).$$

Here we have used Theorem 3.1 to estimate $J(\eta'')$. Actually by this theorem we should have had $Z_{\eta'}$ instead of Z_η. Yet we can interchange $Z_{\eta'}$ and Z_η because we have only replaced the measure η on A by a same amount of

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mass concentrated on $\bar{\gamma}_0$, and on $\gamma_0 \cup \gamma_1$ this does not affect multiplicities. As far as the second term of the sum in (4.1)is concerned it is easy to see that we have

$$d_\alpha(\mu', \mu'') \leq l\eta(A)^\alpha.$$

By inserting these estimates in (4.1) we get

$$\int_0^{t_0} |\gamma_0|^{\alpha-1} \, dt - \int_0^{t_1} |\gamma_1|^{\alpha-1} \, dt \leq \alpha^{-1} l\eta(A)^{\alpha-1}.$$

Now estimate the length l by

$$
\begin{aligned}
l &= |x_0 - x_1| \leq |x_0 - \bar{x}| + |\bar{x} - x_1| \leq (\sigma(\gamma_0) - t_0) + (\sigma(\gamma_1) - t_1) \\
&\leq \eta(A)^{1-\alpha} \int_{t_0}^{\sigma(\gamma_0)} |\gamma_0|_{t,\eta}^{\alpha-1} dt + \eta(B)^{1-\alpha} \int_{t_1}^{\sigma(\gamma_1)} |\gamma_1|_{t,\eta}^{\alpha-1} dt.
\end{aligned}
$$

Hence

$$\int_0^{t_0} |\gamma_0|^{\alpha-1} dt - \int_0^{t_1} |\gamma_1|^{\alpha-1} dt \leq \alpha^{-1} \left(\int_{t_0}^{\sigma(\gamma_0)} |\gamma_0|_{t,\eta}^{\alpha-1} dt + \frac{\eta(B)^{1-\alpha}}{\eta(A)^{1-\alpha}} \int_{t_1}^{\sigma(\gamma_1)} |\gamma_1|_{t,\eta}^{\alpha-1} dt \right).$$

Notice that we cannot have $|\gamma_i|_{\sigma(\gamma_i),\eta} > 0$ for both $i = 0, 1$, thanks to the no-loop property (property 3). So, if it is $|\gamma_1|_{\sigma(\gamma_1),\eta} = 0$, once we fix t_0 such that $\eta(A) > 0$, we can choose t_1 so that $\eta(B) \leq \eta(A)$ since $\eta(B) \to 0$ as $t_1 \to \sigma(\gamma_1)$. Otherwise, if $|\gamma_1|_{\sigma(\gamma_1),\eta} > 0$, we can choose directly $t_1 = \sigma(t_1)$. In both cases we have

$$\int_0^{t_0} |\gamma_0|^{\alpha-1} \, dt - \int_0^{t_1} |\gamma_1|^{\alpha-1} \, dt \leq \alpha^{-1} \left(\int_{t_0}^{\sigma(\gamma_0)} |\gamma_0|_{t,\eta}^{\alpha-1} dt + \int_{t_1}^{\sigma(\gamma_1)} |\gamma_1|_{t,\eta}^{\alpha-1} dt \right).$$
$$(4.2)$$

Then we let t_0 and t_1 tend to $\sigma(\gamma_0)$ and $\sigma(\gamma_1)$, according to the criteria for the choice of t_1 we have used so far, and we get at the limit

$$Z_\eta(\gamma_0) - Z_\eta(\gamma_1) \leq 0,$$

because the integral terms on the right hand side of (4.2) tend to zero as a consequence of the fact that γ_0 and γ_1 are both $\eta-$good curves. By interchanging the role of γ_0 and γ_1 the thesis is proven. $\square$

Corollary 4.3. *If two different $\eta-$good curves γ_0 and γ_1 meet at a certain point $x = \gamma_0(t_0) = \gamma_1(t_1)$, then $|\gamma_0|_{t_0,\eta} = |\gamma_1|_{t_1,\eta} = 0$.*

Proof. If one of the two multiplicities $|\gamma_i|_{t_i,\eta}$ were positive a strict inequality between $Z_\eta(\gamma_0)$ and $Z_\eta(\gamma_1)$ should hold. Yet equality has just been proven and this is a contradiction. $\square$

Corollary 4.4. *Any $\eta-$good curve γ is in fact injective on $[0, \sigma(\gamma)]$.*

Proof. the injectivity on $[0, \sigma(\gamma)[$ is already known. Hence, consider the case $\gamma(\sigma(\gamma)) = \gamma(t)$ for $t < \sigma(\gamma)$. This would imply $|\gamma|_{t,\eta} > 0$ but it is contradiction with Corollary 4.3, applied to the curve γ and to the curve $\bar{\gamma}$, which is γ stopped at time t. $\square$

Remark 5. The injectivity on $[0, \sigma(\gamma)]$ was already known for $\eta-$a.e. curve γ (see [4]). Yet, it was not possible to identify an explicit class of curves sharing this property. For the purposes of this paper it is important to switch from a generic "a.e." to the fact that this is true for $\eta-$good curves.

The result of Theorem 4.2 allows us to define a function on Ω by the values of Z_η.

Definition 4. We define the landscape function associated to the traffic plan η as the function z_η given by

$$z_\eta(x) = \begin{cases} Z_\eta(\gamma) & \text{if } \gamma \text{ is } \eta-\text{good and } x = \gamma(\sigma(\gamma)); \\ +\infty & \text{if no } \eta - \text{good curve ends at } x. \end{cases}$$

Remark 6. It was in fact possible to prove more easily that $\mu-$a.e. the value of z was well defined (in the sense that if on a non negligible set of points x we had two different values for Z_η we would have had the possibility to strictly improve the value of J). Yet, we do not want a function z which is defined a.e. but a pointwise defined value, to deal later with pointwise properties, being also concerned with negligible sets such as S_η.

Remark 7. Notice, as in Remark 4, that restrictions of $\eta-$good curves are still $\eta-$good and that this implies that if the landscape function is finite on a point x then it is finite also on the whole $\eta-$curve arriving up to x.

4.2. **Variational applications: the functional X_α.** Some consequences of the existence of the landscape function are presented here.

Corollary 4.5. *For the functional X_α we have the following representation formula $X_\alpha(\mu) = \int_\Omega z \, d\mu$, where $z = z_\eta$ is the landscape function associated to any optimal pattern η irrigating the measure μ.*

Proof. It is sufficient to take the formula $X_\alpha(\mu) = J(\eta) = \int_\Gamma Z_\eta d\eta$ and use the fact that $Z_\eta(\gamma)$ depends only on $\pi_\infty(\gamma)$ through $Z_\eta(\gamma) = z(\pi_\infty(\gamma))$ and get

$$X_\alpha(\mu) = \int_\Gamma Z_\eta d\eta = \int_\Omega z \, d((\pi_\infty)\sharp\eta) = \int_\Omega z \, d\mu.$$

$\square$

Corollary 4.6. *If μ is $\alpha-$irrigable, then any landscape function z is finite $\mu-$a.e.*

Proof. Corollary 4.5 yields $\int z \, d\mu = X_\alpha(\mu) < +\infty$ and from this the result is straightforward. $\square$

Remark 8. As the word "any" in the previous statement suggests, there is no uniqueness for the landscape function, and there is a landscape function for any optimal pattern.

Moreover, using Theorem 3.1 together with the existence of the landscape function, a derivative result extending the discrete case can be obtained. Notice that the following theorem will be useful also for other purposes, for instance when looking for continuity properties of the landscape function (see Section 6).

Theorem 4.7. *For a given function g on Ω, such that $||g||_{L^\infty(\mu)} \leq 1$ and such that $\int_\Omega g \, d\mu = 0$, set $\mu_1 = \mu(1 + g)$. Then*

$$X_\alpha(\mu_1) \leq X_\alpha(\mu) + \alpha \int_\Omega z(x)g(x)\mu(dx),$$

where the function $z = z_\eta$ is the landscape function according to an arbitrary optimal pattern η irrigating the measure μ.

Proof. We will consider a variation of η given by $\eta_1 = (1 + (g \circ \pi)) \cdot \eta$. Since $\pi_\sharp \eta_1 = (1 + g) \cdot \mu$, we have

$$X_\alpha(\mu_1) - X_\alpha(\mu) \leq J(\eta_1) - J(\eta).$$

We want to apply Theorem 3.1 to this situation, with $\Delta\eta = (g \circ \pi) \cdot \eta$. Since $\Delta\eta$ is absolutely continuous with respect to η with bounded density, it is straightforward that both the conditions required by the theorem ($\Delta\eta$ being concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$ and Z_η being $|\Delta\eta|$−integrable) are satisfied, so that one gets

$$J(\eta') \leq J(\eta) + \alpha \int_\Gamma Z_\eta d\Delta\eta.$$

Now use the fact that Z_η depends only on its terminal point and get

$$\int_\Gamma Z_\eta d\Delta\eta = \int_\Omega z \, d((\pi_\infty)\sharp\Delta\eta) = \int_\Omega zg d\mu.$$

Putting together all the results yields the thesis. □

A simple consequence of this theorem may be expressed in terms of derivatives.

Corollary 4.8. *Set $\mu_\varepsilon = \mu + \varepsilon g \cdot \mu$. Then the following derivative inequality holds:*

$$\limsup_{\varepsilon \to 0^+} \frac{X_\alpha(\mu + \varepsilon g \cdot \mu) - X_\alpha(\mu)}{\varepsilon} \leq \alpha \int_\Omega z(x)g(x)\mu(dx).$$

4.3. **Variational applications: minimizing** (P_+). As we said the last derivative inequality may be useful in variational problems involving X_α. For the sake of clearness we provide a short example.

Example 1. Let us consider the functional $F : \mathcal{P}(\Omega) \to [0, +\infty]$ given by

$$F(\mu) = \begin{cases} \int_\Omega u^2 \, d\mathcal{L}^d & \text{if } \mu = u \cdot \mathcal{L}^d, \\ +\infty & \text{if } \mu \text{ is not absolutely continuous.} \end{cases}$$

If we want to minimize (for $\alpha > 1 - 1/d$) the sum $X_\alpha(\mu) + F(\mu)$ over all probabilities μ on Ω we get as an optimality condition, by deriving and using Corollary 4.8,

$$\alpha z + 2u = const \text{ a.e. on } \{u > 0\}. \tag{4.3}$$

This implies several interesting properties. First of all we can notice that, both z and u being positive, they are also bounded. It was not *a priori* evident that $u \in L^\infty(\Omega)$, since the natural condition was $u \in L^2(\Omega)$. Since $z(x) \geq |x|$, this gives also an estimate on the support of μ. As the constant appearing in (4.3) could be uniformly estimated (it is sufficient to multiply (4.3) by u and integrate, thus obtaining $const = \alpha X_\alpha(\mu) + 2F(\mu) \leq 2\min(X_\alpha + F)$), this could also be used to prove an existence result for $\Omega = \mathbb{R}^d$, proving also that actually minimizers of $X_\alpha + F$ are supported in a given bounded ball. Moreover, Formula 4.3 yields some regularity result on u according to the results we will prove later on z.

Variational problems such as (P_+) have been first proposed in [15], where the authors suggested to consider problems involving both the irrigation pattern χ and the irrigated measure μ_χ, and are very similar to what considered in [6]. In fact in [6] a similar sum is minimized, but with a standard Wasserstein distance instead of the term $X_\alpha(\mu) = d_\alpha(\mu, \delta_0)$. Such a model was proposed to study urban planning problems, with μ standing for the population density in a region, exactly as (P_+) may be used in studying the shape of a leaf or a flower, represented by μ. In fact the minimization of a sum of an X_α term and a convex functional on μ could be an easy model taking into account that leaves want to be as spread as possible to catch sunlight but have to be irrigated starting from a single source. In the framework of [6] the key condition coming from optimality was $\psi + f'(u) = const$ and the landscape function z dealt with in this paper plays somehow the role of the Kantorovich potential ψ. Also Corollary 4.5 can be seen as a similarity between the landscape function and the Kantorovich potential (see [19]). Moreover, the Hölder continuity result at the end of this paper perfectly agrees with the fact that Kantorovich potentials (which correspond to $\alpha = 1$) are Lipschitz continuous.

5. Properties of the landscape function

5.1. Semicontinuity.

Lemma 5.1. *Given any $\eta \in \mathcal{P}(\Gamma)$, the function $Z_\eta : \Gamma \to \mathbb{R}$ is lower semicontinuous with respect to pointwise convergence.*

Proof. This result is almost implicitly proven both in [15] and in [4], but never explicitly stated. It is anyway proven that $x \mapsto [x]_\eta$ is upper semicontinuous, and hence $x \mapsto [x]_\eta^{\alpha-1}$ is l.s.c. Then, to prove $\liminf_n Z_\eta(\gamma_n) \geq Z_\eta(\gamma)$, fix a time $t_1 < \sigma(\gamma)$ and use $\liminf \sigma(\gamma_n) \geq \sigma(\gamma)$. Eventually we

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have $\sigma(\gamma_n) > t_1$ and, by Fatou's Lemma, we get

$$\liminf_n Z_\eta(\gamma_n) \geq \liminf_n \int_0^{t_1} [\gamma_n]_\eta^{\alpha-1} dt \geq \int_0^{t_1} [\gamma]_\eta^{\alpha-1} dt.$$

Passing to the limit as $t_1 \to \sigma(\gamma)$ gives the thesis. $\qquad\square$

Theorem 5.2. *The landscape function z is lower semi-continuous.*

Proof. Consider a sequence $x_n \to x$ and, correspondingly, some $\eta-$good curves γ_n such that $\pi_\infty(\gamma_n) = x_n$ and $z(x_n) = Z_\eta(\gamma_n)$. We may assume $\sup_n z(x_n) < +\infty$. Since it holds $\sigma(\gamma_n) \leq Z_\eta(\gamma_n) = z(x_n)$, we also have $\sup_n \sigma(\gamma_n) < +\infty$ and we can extract a subsequence (not relabeled) such that $\gamma_n \to \gamma$ uniformly. It is not difficult to prove that $\pi_\infty(\gamma) = x$. Thus, it is sufficient to use Lemma 5.1 to get $Z_\eta(\gamma) \leq \liminf_n Z_\eta(\gamma_n) = \liminf_n z(x_n)$. This implies that γ is an $\eta-$good curve and that $z(x) = Z_\eta(\gamma)$, which yields the thesis. $\qquad\square$

5.2. **Maximal slope in the network direction.** The next property that can be proven in general (i.e., under no extra assumption on α, Ω, μ ...) on the landscape function is the most important in view of its meaning in river basins applications. Our interest is in a continuous counterpart of the landscape function of [2]. What we actually need is a result concerning the fact that, on the points of the irrigation network S_η, the direction of maximal slope of z is exactly the direction of the network. If an $\eta-$good curve γ_0 is fixed, by the definition of z, for a.e. t_0 the derivative of z along the curve γ at the point $x_0 = \gamma_0(t_0)$ is exactly $|\gamma_0|_{t_0,\eta}^{\alpha-1}$. This is the reason why we prove the following result. Notice that, as we said, in this continuous case the function z cannot be expected to be very regular, and in fact the maximal slope result we are going to prove involves differentiability in a very pointwise way but very weak as well.

Theorem 5.3. *Let $x_0 = \gamma_0(t_0)$, where γ_0 is an $\eta-$good curve, t_0 a time with $t_0 \leq \sigma(\gamma_0)$ and $\theta_0 := |\gamma_0|_{t_0,\eta} > 0$. Then, for any $x \notin \gamma_0([0,t_0])$, we have*

$$z(x) \geq z(x_0) - \theta_0^{\alpha-1}|x - x_0| - o(|x - x_0|).$$

This corresponds to saying that the slope at x_0 in the direction of the network is actually the maximal slope at x_0.

Proof. Let us fix $x \notin \gamma_0([0,t_0])$ such that $z(x) < z(x_0)$. We may assume that $x = \gamma_x(t_x)$ for an $\eta-$good curve γ_x (otherwise $z(x) = +\infty$) and that the two curves γ_0 and γ_x get apart at a certain time $t_1(x) < t_0$ (the case $t_1(x) \geq t_0$ implies in fact $z(x) \geq z(x_0)$). By Lemma 5.4 (see below) we know that $t_1(x) \to t_0$ as $|x - x_0| \to 0$. Let us set $\theta(t) = |\gamma_0|_{t,\eta}$: for $t \in [t_1(x), t_0]$ we may write $\theta(t) \leq \theta_0(1 + \varepsilon_x)$, where ε_x is infinitesimal as $|x - x_0| \to 0$ as a consequence of $t_1(x) \to t_0$. We use again Lemma 4.1 and its notations. In particular $A = [\gamma_0]_{t_0}$ and $\theta_0 = \theta(t_0) = \eta(A)$. Let us also define, as in Theorem 4.2, $\mu'' = \mu - \mu_A + \eta(A)\delta_x$ and $\eta'' = \eta - I_A \cdot \eta + \eta(A)\delta_{\bar{\gamma}_x}$, where the curve $\bar{\gamma}_x$ is the curve γ_x stopped at time t_x, and it is easy to check that
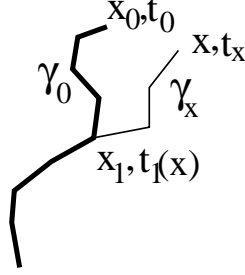
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FIGURE 4. Curves and points in the proof

$(\pi_\infty)_\sharp \eta'' = \mu''$. Then, by the optimality of η' (we recall, according to the notations of Lemma 4.1 $\mu' = \mu - \mu_A + \eta(A)\delta_{x_0}$ and $\eta' = \eta - I_A \cdot \eta + \eta(A)\delta_{\bar{\gamma}_0}$, where $\bar{\gamma}_0$ is the curve γ_0 stopped at time t_0, see also Figure 5.2), we have

$$J(\eta') = X_\alpha(\mu') \le X_\alpha(\mu'') + d_\alpha(\mu'', \mu') \le J(\eta'') + |x - x_0|\theta_0^\alpha. \qquad (5.1)$$

We want to compare $J(\eta')$ and $J(\eta'')$ and to do this here we need a more refined estimate than what we could find by using Theorem 3.1. As $\eta' - \eta'' = \theta_0(\delta_{\bar{\gamma}_0} - \delta_{\bar{\gamma}_x})$, we have in particular $[y]_{\eta''} = [y]_{\eta'} + \theta_0(I_{y \in \bar{\gamma}_x} - I_{y \in \bar{\gamma}_0})$. By using Formula (2.3) we get the following:

$$J(\eta'') - J(\eta') = \int_{\bar{\gamma}_x \setminus \bar{\gamma}_0} \big(([y]_{\eta'} + \theta_0)^\alpha - [y]_{\eta'}^\alpha\big)d\mathcal{H}^1 - \int_{\bar{\gamma}_0 \setminus \bar{\gamma}_x} \big([y]_{\eta'}^\alpha - ([y]_{\eta'} - \theta_0)^\alpha\big)d\mathcal{H}^1.$$

It is not difficult to check that, for $y \in \bar{\gamma}_x \cup \bar{\gamma}_0$, it holds $[y]_{\eta'} = [y]_\eta$, as we have replaced the part of η concentrated on A by an equal amount of mass on $\bar{\gamma}_0$. Hence we may estimate (rewriting the integrals w.r.t. $\mathcal{H}^1$ as integrals in dt)

$$J(\eta'') - J(\eta') \le \alpha \int_{t_1(x)}^{t_x} |\gamma_x|_{t,\eta}^{\alpha-1}\theta_0 \, dt - \int_{t_1(x)}^{t_0} \big(\theta(t)^\alpha - (\theta(t) - \theta_0)^\alpha\big) \, dt.$$

Since the function $s \mapsto s^\alpha - (s - \theta_0)^\alpha$ is decreasing and $\theta(t) \le (1 + \varepsilon_x)\theta_0$, we get $\theta(t)^\alpha - (\theta(t) - \theta_0)^\alpha \ge \theta_0^\alpha\big((1 + \varepsilon_x)^\alpha - \varepsilon_x^\alpha\big)$. Hence we get

$$J(\eta'') - J(\eta') \le \alpha(z(x) - z(x_1)) - |t_0 - t_1(x)|\theta_0^\alpha\big((1 + \varepsilon_x)^\alpha - \varepsilon_x^\alpha\big),$$

where $x_1 = \gamma_0(t_1(x)) = \gamma_x(t_1(x))$. Write $(1 + \varepsilon_x)^\alpha - \varepsilon_x^\alpha = (1 + \varepsilon_x')^{-1}$ and $\varepsilon_x' > 0$ is infinitesimal as $x \to x_0$. From $\theta_0^{\alpha-1} \ge (\theta(t))^{\alpha-1}$ we get $|t_0 - t_1(x)|\theta_0^\alpha \ge \theta_0(z(x_0) - z(x_1))$. Now notice that, for $|x - x_0|$ sufficiently small, the inequality $\alpha < (1 + \varepsilon_x')^{-1}$ is satisfied, and hence

$$J(\eta'') \le J(\eta') + (1 + \varepsilon_x')^{-1}\theta_0(z(x) - z(x_0).$$

If we finally insert it into (5.1) we finally get

$$z(x) - z(x_0) \ge -\theta_0^{\alpha-1}|x - x_0|(1 + \varepsilon_x'), \quad \square.$$

Lemma 5.4. *According to the notations of Theorem 5.3, when $x \to x_0$ and $z(x) \le z(x_0)$, the parting time $t_1(x)$ tends to t_0.*

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Proof. Suppose, by contradiction, that there exists a sequence $x_k \to x_0$ such that $\lim_k t_1(x_k) = \bar{t} < t_0$ and $z(x_k) \leq z(x_0)$. Since γ_0 is injective (Corollary 4.4), we may infer the existence of a positive quantity δ such that $|\gamma_0(t_1(x_k)) - x_0| \geq \delta$ (otherwise there would be a time $t \leq \bar{t} < t_0$ with $\gamma_0(t) = x_0$). For any k consider an η-good curve γ_k such that $x_k = \gamma_k(t_k)$. First notice that, at least for k large enough, thanks to $|\gamma_k(t_1(x_k)) - x_k| = |\gamma_0(t_1(x_k)) - x_k| \to |\gamma_0(\bar{t}) - x_0| \geq \delta$, we have $t_k > \bar{t} + \delta/2$. Then let us consider the points $\gamma_k(\bar{t}+\delta/2)$: this collection of points must in fact be finite, otherwise we would have $|\gamma_k|_{\bar{t}+\delta/2,\eta} \to 0$ and hence $z(x_k) \geq |\gamma_k|_{\bar{t}+\delta/2,\eta}^{\alpha-1}|t_k - (\bar{t} + \delta/2)| \to +\infty$ because $|t_k - (\bar{t} + \delta/2)| \geq |x_k - \gamma_0(\bar{t})| - \delta/2 \geq \delta/2$. This is in contradiction with $z(x_k) \leq z(x_0)$ and then we may suppose, up to subsequences, that $\gamma_k(\bar{t} + \delta/2) = \bar{x}$ (for a point $\bar{x}$ which does not belong to the image of γ_0, otherwise we would contradict Property 3) and that γ_k uniformly converges to a curve γ. At the limit we should get a curve γ passing through $\gamma_0(\bar{t})$, $\bar{x}$ and x_0, i.e. we have created a loop because γ_0 does not pass through $\bar{x}$ (see Figure 5.2 as well).
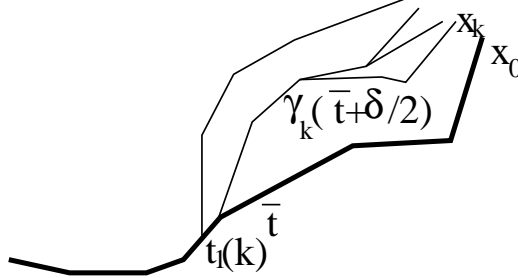

FIGURE 5. A sequence of curves creating a loop at the limit

From $z(x_k) \leq z(x_0)$ we can infer by semicontinuity (Lemma 5.1) that γ is an $\eta-$good curve and hence this loop is against Corollary 4.3. $\square$

6. HÖLDER CONTINUITY UNDER EXTRA ASSUMPTIONS

Here we will be able to prove some extra regularity properties on z, but we have to add some assumptions. The most important ones are on α ($\alpha > 1 - 1/d$ is required) and on the irrigated measure μ (a lower bound on its density is supposed).

6.1. **Campanato spaces by medians.** We will here give a simple variant of a well-known result by Campanato (see [7]) about an integral characterization of Hölder continuous functions.

Definition 5. Given a measurable function u on a domain U we call median of u in U any number m which satisfies the following equivalent conditions:

- $|\{x \in U : u(x) > m\}| \leq \frac{1}{2}|U|$ and $|\{x \in U : u(x) < m\}| \leq \frac{1}{2}|U|$;
- there exists a measurable subset $A \subset \{x \in U : u(x) = m\}$ such that $|\{x \in U : u(x) > m\} \cup A| = \frac{1}{2}|U|$;
- the function $t \mapsto \int_U |u(x) - t|dx$ achieves its minimum at $t = m$.

The sets of medians of u in U is an interval of $\mathbb{R}$; the middle point of this interval is called central median of u in U

Definition 6. If A is a given positive number, a domain $\Omega \subset \mathbb{R}^d$ is said to be of type A if it holds $|\Omega_{x_0,r}| \geq Ar^d$ for any $r \in [0, \operatorname{diam}\Omega]$, where $\Omega_{x_0,r} = \Omega \cap B(x_0, r)$.

Lemma 6.1. *If Ω is a domain of type A and u is a function in $L^1(\Omega)$ such that*

$$\int_{\Omega_{x_0,r}} |u - \tilde{u}_{x_0,r}|dx \leq Cr^{d+\beta},$$

for a finite constant C and any $r \in [0, \operatorname{diam}\Omega]$, where $\tilde{u}_{x_0,r}$ is the central median of u on $\Omega_{x_0,r}$, then u admits a representative which is Hölder continuous of exponent β.

Proof. This is nothing but the fact that Campanato spaces may be built by using medians instead of average values. See the proof of Theorem 1.2 at page 70 in [10] and adapt it. In fact it is easy to see that for each point x_0 the value $\tilde{u}_{x_0,r}$ converges as $r \to 0$ to a value $\tilde{u}(x_0)$ and that

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C|x - y|^\beta,$$

exactly as in the proof we mentioned. What we need to prove is that $\tilde{u}(x) = u(x)$ a.e.. This can be obtained in this way: let us denote the average value of u on $\Omega_{x_0,r}$ by $\bar{u}_{x_0,r}$. Then

$$|\bar{u}_{x_0,r} - \tilde{u}_{x_0,r}| \leq |\Omega_{x_0,r}|^{-1}\int_{\Omega_{x_0,r}} |u(x) - \tilde{u}_{x_0,r}|dx \leq |\Omega_{x_0,r}|^{-1}\int_{\Omega_{x_0,r}} |u(x) - \bar{u}_{x_0,r}|dx,$$

where the second inequality has been established as a consequence of the minimality property of the median. As at Lebesgue points the last expression tends to zero, this implies that the average $\bar{u}_{x_0,r}$ and the median $\tilde{u}_{x_0,r}$ share the same limit a.e. On the same points we also have $\bar{u}_{x_0,r} \to u(x_0)$, and this proves $\tilde{u}(x_0) = u(x_0)$ a.e. $\qquad\square$

6.2. Hölder continuity of the landscape function.

Theorem 6.2. *Suppose that Ω is a domain of type A for $A > 0$, that $\alpha > 1 - 1/d$ and that $\mu \in \mathcal{P}(\Omega)$ is a probability measure such that the density of its absolutely continuous part is bounded from below by a positive constant. Then any landscape function z has a representative $\tilde{z}$ which is Hölder continuous of exponent $\beta = d(\alpha - (1 - 1/d))$.*

Proof. Let us fix a measure μ_1 and apply Theorem 4.7 to it and μ. By using the triangle inequality for d_α, we get

$$-d_\alpha(\mu, \mu_1) \leq X_\alpha(\mu_1) - X_\alpha(\mu) \leq \alpha \int_\Omega z\, d(\mu_1 - \mu), \qquad (6.1)$$

provided μ_1 is a measure of the form allowed in Theorem 4.7, i.e. $\mu_1 << \mu$ with bounded density. From (6.1) we get

$$\alpha \int_\Omega z \, d(\mu - \mu_1) \le d_\alpha(\mu, \mu_1). \tag{6.2}$$

Suppose that μ has an absolutely continuous part with density everywhere larger than $\lambda_0 > 0$ and choose

$$\mu_1 = \mu - \lambda_0 I_A \cdot \mathcal{L}^d + \lambda_0 I_B \cdot \mathcal{L}^d,$$

where A and B are two measurable subsets of $\Omega_{x_0, \varepsilon}$ with $|A| = |B|$, $A \cup B = \Omega_{x_0, \varepsilon}$ and $A \subset \{z \ge m\}$ and $B \subset \{z \le m\}$ and m is the central median value for z in $\Omega_{x_0, \varepsilon}$. By construction μ_1 is a probability measure to which the estimate of Theorem 4.7 may be applied. With this choice of μ and μ_1 we get

$$\int_\Omega z \, d(\mu - \mu_1) = \int_A z(x) \, \lambda_0 \, dx - \int_B z(x) \, \lambda_0 \, dx$$

$$= \lambda_0 \left(\int_A (z(x) - m) dx - \int_B (z(x) - m) dx \right) = \lambda_0 \int_{\Omega_{x_0, \varepsilon}} |z(x) - m| dx.$$

Putting into (6.2)

$$\int_{\Omega_{x_0, \varepsilon}} |z(x) - m| dx \le (\alpha \lambda_0)^{-1} d_\alpha(\mu, \mu_1).$$

To estimate $d_\alpha(\mu, \mu_1)$ use (2.1) and get

$$\int_{\Omega_{x_0, \varepsilon}} |z(x) - m| dx \le \frac{C_{\alpha, d}}{\lambda_0^{1-\alpha}} \varepsilon^{1+\alpha d}.$$

Since $1 + \alpha d = d + \beta$, Lemma 6.1 may be applied. $\qquad \square$

An important consequence of this fact is the following:

Corollary 6.3. *Under the same assumptions on Ω, α and μ of Theorem 6.2, the inequality*

$$X_\alpha(\mu_1) \le X_\alpha(\mu) + \int_\Omega \tilde{z} d(\mu_1 - \mu)$$

holds for any measure $\mu_1 \in \mathcal{P}(\Omega)$.

Proof. The inequality holds for μ_1 of the form $\mu_1 = (1 + g) \cdot \mu$ with $g \in L^\infty$, but any measure $\mu_1 \in \mathcal{P}(\Omega)$ may be approximated by these kind of measures. Since $\tilde{z}$ is continuous, at both the sides of the inequalities we have quantities which are continuous with respect to weak convergence in the variable μ_1. This allows to conclude that the same inequality is valid for any μ_1. $\qquad \square$

Even if we have proven that the landscape function z equals a.e. a function which is Hölder continuous, this is not enough. In fact, this result does not provide information on the behavior of z on negligible sets. Yet, the pointwise values of z on S_η are of particular interest (as in last Section),

and S_η is one-dimensional and thus negligible. This is why the next step will be proving that z and $\tilde{z}$ actually agree everywhere.

Theorem 6.4. *Let m_ε denote the central median of z in the ball $B(x_0, \varepsilon)$. Under the same assumptions of Theorem 6.2 one has $m_\varepsilon \to z(x_0)$ as $\varepsilon \to 0$. Consequently, we have $\tilde{z}(x_0) = z(x_0)$.*

Proof. By the semicontinuity of z it is easy to get $\liminf_{\varepsilon \to 0} m_\varepsilon \geq z(x_0)$, hence only an estimate from above for m_ε is needed. Let us now consider a ball $B(x_0, \varepsilon)$ and a set $A_\varepsilon \subset B(x_0, \varepsilon) \cap \{z \geq m_\varepsilon\}$ such that $|A_\varepsilon| = |B(x_0, \varepsilon)|/2$. Then set $\Gamma_\varepsilon = \{\gamma \in \Gamma : (\pi_\infty)(\gamma) \in A_\varepsilon\}$, $\mu_\varepsilon = \mu + \mu(A_\varepsilon)\delta_{x_0} - I_{A_\varepsilon} \cdot \mu$, and $\eta_\varepsilon = \eta + \eta(\Gamma_\varepsilon)\delta_{\gamma_0} - I_{\Gamma_\varepsilon} \cdot \eta$, where γ_0 is an $\eta-$good curve stopping at x_0. Theorem 3.1 can be applied to η and η_ε and hence we have

$$
\begin{aligned}
J(\eta_\varepsilon) \;\leq\; & J(\eta) + \alpha\left(\eta(\Gamma_\varepsilon)Z_\eta(\gamma_0) - \int_{\Gamma_\varepsilon} Z_\eta d\eta\right) \\
=\; & J(\eta) + \alpha\mu(A_\varepsilon)z(x_0) - \alpha\int_{A_\varepsilon} z(x)\mu(dx) \leq J(\eta) + \alpha\mu(A_\varepsilon)(z(x_0) - m_\varepsilon).
\end{aligned}
$$

Hence we have

$$
X_\alpha(\mu) \leq X_\alpha(\mu_\varepsilon) + C\varepsilon\mu(A_\varepsilon)^\alpha \leq X_\alpha(\mu) + \alpha\mu(A_\varepsilon)(z(x_0) - m_\varepsilon) + C\varepsilon\mu(A_\varepsilon)^\alpha.
$$

This implies

$$
m_\varepsilon - z(x_0) \leq C\varepsilon\mu(A_\varepsilon)^{\alpha-1} \leq C\varepsilon^{1+d(\alpha-1)}.
$$

Since the exponent $1 + d(\alpha-1)$ is larger than 0 we get $\limsup_{\varepsilon \to 0} m_\varepsilon \leq z(x_0)$. To get the second part of the thesis, just use $\tilde{z}(x_0) = \lim_{\varepsilon \to 0} m_\varepsilon$. $\qquad\square$

Remark 9. The landscape function z is in general never Lipschitz continuous (not even locally), as on the set S_η it has slopes given by $\theta^{\alpha-1}$. This means that, if we have arbitrarily small values of θ, we cannot have a Lipschitz constant for z. Yet estimates of the kind $\theta \geq c > 0$ would imply $\mathcal{H}^1(S_\eta) < +\infty$ and no measure whose support is not one-dimensional may be irrigated by a set of finite length (or locally finite length).

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F. Santambrogio, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, `f.santambrogio@sns.it`