ON A FREQUENCY FUNCTION APPROACH TO THE UNIQUE CONTINUATION PRINCIPLE

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Abstract. In this survey we discuss the frequency function method so as to study the problem of unique continuation for elliptic partial differential equations. The methods used in the note were mainly introduced by Garofalo and Lin.

1. Introduction

Let G be an open connected subset of $\mathbb{R}^n$, $n \geq 2$. We consider the problem of unique continuation for both the solutions to the Laplace equation and to equation

$$\Delta u = b(x) \cdot \nabla u, \tag{1.1}$$

where the drift coefficients, $\{b_i(x)\}_{i=1}^n$, are continuous and bounded in G. The classical unique continuation principle for the latter equation can be formulated as follows

(i) Let u_1 and u_2 be two solutions to (1.1) such that $u_1 = u_2$ in an open subset of G. Then $u_1 \equiv u_2$ in G.
(ii) Let u be a solution to (1.1) such that $u = 0$ in an open subset of G. Then $u \equiv 0$ in G.

The latter formulation is equivalent to the following: (ii') Let u be a solution to (1.1) and consider two open concentric balls $B_r \subset \overline{B}_R \subset G$ such that $u = 0$ on B_r, then $u \equiv 0$ in B_R.

Instead of using Carleman's method to deal with the unique continuaton, we follow the method introduced by Garofalo and Lin in [6] and [7], see also Fabes et al. [4]. Their method is based on the ingenious analysis of (a modification of) Almgren's frequency function, see [1], which, in turn, leads to monotonicity formulas and doubling inequalities. The main result in [7] is the unique continuation principle for the solutions to the equation

$$-\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + V(x)u = 0, \tag{1.2}$$

where $A(x) = (a_{ij}(x))_{i,j=1}^n$ is a real symmetric matrix-valued function satisfying the uniform ellipticity condition and it is Lipschitz continuous. The lower order terms, the drift coefficient $b(x)$ and the potential

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$V(x)$, are even allowed to have singularities. The reader should consult (1.4)–(1.6) in [7] for the exact structure conditions of b and V.

In the present survey, our goal is to provide a clear user's guide-type presentation on this topic, and we do not attempt to deal with the most general case (1.2). For such a treatise, the reader should consult more advanced papers [6] and [7], and a paper [19] by Tao and Zhang.

Our proofs are by contradiction, which makes it possible to use Poincaré's inequality in certain phases of the proof. By this observation we are able to obtain more straightforward treatment for the classical proof, however our method is indirect.

In outline, a brief discussion on the Rellich–Necas identity, as well as the notation, can be found in § 2. The unique continuation principle for the Laplace equation is covered in § 3, and for the solutions to (1.1) in § 4. We close this note by discussing possible generalizations to the nonlinear case in § 5, i.e., unique continuation principle for the p-Laplace equation,

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

where $1 < p < \infty$. Observe that in the case $p = 2$ we recover the Laplace equation. We do not claim that the frequency function method is a panacea for the unique continuation principle in this nonlinear case, but it seems to open new possibilities to study the problem.

We want to remark that the unique continuation for the solutions to (1.1) is interestingly entwined with the one for the p-Laplace equation (see § 5). Lastly, in contrast to the Laplace equation, equation (1.1) is more subtle and to reach the unique continuation principle for its solutions a great deal more analysis is required.

2. Preliminaries

Throughout the present note, G is open and connected subset of $\mathbb{R}^n$, $n \geq 2$. We use the notation $B_r = B(x, r)$ for concentric open balls of radii r centered at $x \in G$. Unless otherwise stated, the letter C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line. Moreover, $dx = dx_1 \ldots dx_n$ denotes the Lebesgue volume element in $\mathbb{R}^n$, whereas dS denotes the surface element. We denote by $|E|$ the n-dimensional Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^n$. Along ∂G, whenever G is smooth enough, is defined the outward pointing unit normal vector field at $x \in \partial G$ and is denoted by $\nu(x) = (\nu_1, \ldots, \nu_n)(x)$. We will also write $u_\nu = \nabla u \cdot \nu$ or $\partial u/\partial \nu$ for the directional derivative of u. We denote a tangential gradient by ∇_t.

We shall make use of the following Rellich–Necas type identity. To the best of our knowledge, this formula was first employed by Payne and Weinberger in [16], and it is a variant of a formula due to Rellich [18] and Necas. We also refer to Jerison–Kenig [12]. A Rellich–Necas type formula appears, e.g., in Pucci–Serrin [17], Garofalo–Lewis [5], and Lewis–Vogel [13].

Lemma 2.1. *Let $u \in C^2(G) \cap C^1(\overline{G})$. The following formula is valid*

$$\begin{aligned} -\int_G (2(x\cdot\nabla u)\Delta u + (n-2)u\Delta u)\, dx \\ = \int_{\partial G} \big(|\nabla u|^2(x\cdot\nu) - 2(x\cdot\nabla u)u_\nu - (n-2)uu_\nu\big)\, dS. \end{aligned} \tag{2.2}$$

In particular, if u is harmonic in G then (2.2) *reduces to the following formula*

$$\int_{\partial G} \big(|\nabla u|^2(x\cdot\nu) - 2(x\cdot\nabla u)u_\nu - (n-2)uu_\nu\big)\, dS = 0 \tag{2.3}$$

Proof. The proof follows from the following divergence identity which stems from Noether's theorem; observe ([16, eq. 3.2], and also [12, p. 204]), by a direct calculation, that

$$\begin{aligned} \nabla\cdot \big(|\nabla u|^2 x - 2(x\cdot\nabla u)\nabla u - (n-2)u\nabla u\big) \\ = -2(x\cdot\nabla u)\Delta u - (n-2)u\Delta u. \end{aligned} \tag{2.4}$$

Then integrating over G and applying the Gauss theorem we arrive at (2.2). Equation (2.3) follows from (2.2) simply by setting $\Delta u = 0$. $\square$

Remark 2.5. We remark that using the fact that

$$|\nabla u|^2 = |\nabla_t u|^2 + |u_\nu|^2$$

and denoting $\alpha(x) = x - (x\cdot\nu)\nu$ we may rewrite (2.3) as follows

$$\begin{aligned} \int_{\partial G} \big(\left(|\nabla_t u|^2 - |u_\nu|^2\right)(x\cdot\nu) + 2(\alpha(x)\cdot\nabla u)u_\nu \\ - (n-2)uu_\nu\big)\, dS = 0, \end{aligned}$$

which is just equation (2) in Jerison–Kenig [12].

For harmonic functions and for each $B_r \subset G$, $x \in \partial B_r$, ν is the outward pointing unit normal at x, we may extract from (2.3), or from (2.4), the following

$$\begin{aligned} r\int_{\partial B_r} |\nabla u|^2\, dS &= 2r\int_{\partial B_r} |u_\nu|^2\, dS + (n-2)\int_{\partial B_r} uu_\nu\, dS \\ &= 2r\int_{\partial B_r} |u_\nu|^2\, dS + (n-2)\int_{B_r} |\nabla u|^2\, dx. \end{aligned} \tag{2.6}$$

Equivalently, (2.6) may be stated as a Hardt–Lin [10, Lemma 4.1] type monotonicity identity

$$\frac{d}{dr}\left(r^{2-n}\int_{B_r} |\nabla u|^2\, dx\right) = 2r^{2-n}\int_{\partial B_r} |u_\nu|^2\, dS.$$

We shall also need the following Poincaré type inequality, consult Giusti [8] for the proof. Suppose $u \in W^{1,2}(B_r)$ and let $Z = \{x \in B_r : u(x) = 0\}$. If there exists a constant $0 < \gamma < 1$ such that $|Z| \geq \gamma |B_r|$, then there exists a constant C_p, depending on n and γ, such that

$$\int_{B_r} u^2\,dx \leq C_p r^2 \int_{B_r} |\nabla u|^2\,dx. \tag{2.7}$$

3. Unique continuation: Laplace equation

Almgren's [1] insight was that for a harmonic function u the function

$$F(r) = \frac{r\int_{B_r} |\nabla u|^2\,dx}{\int_{\partial B_r} u^2\,dS}, \tag{3.1}$$

called the *frequency function*, is monotonically non-decreasing as a function of r. He observed, moreover, that by employing this property one is able to deduce the unique continuation principle for the solutions to the Laplace equation. See [1] for more properties of the frequency function.

In what follows, we denote the numerator by $rD(r)$ and the denominator by $I(r)$. The following is, of course, well-known but we treat it here since the proof is rather short and simple. Let us demonstrate how the result is reached.

Theorem 3.2. *Suppose $u \in C^2(G)$ and $\Delta u = 0$ in G. If there is an open set $D \subset G$ such that $u = 0$ in D, then $u \equiv 0$ in G.*

Proof. We prove the following from which the claim follows easily: Assume $0 < r_1 < r_2$ and $B_{r_1} \subset \overline{B}_{r_2} \subset G$. If $u(x) = 0$ in B_{r_1}, then $u(x) = 0$ in B_{r_2}. To prove this, we assume, on the contrary, that there exists $x_0 \in G$ so that $u(x) = 0$ in $B_{r_1}(x_0)$ but u is not identically zero in $B_{r_2}(x_0)$. It will be shown below that function $I(r)$ is non-decreasing. Then $I(r_2) > 0$ and there is a number $r_0 \in [r_1, r_2]$ such that $I(r_0) = 0$, but $I(s) > 0$ for $s > r_0$. We thus consider an interval $[s, r_2]$, where $s > r_0$.

Let us start by proving a Harnack type inequality for $I(r)$. Since

$$I'(r) = \frac{n-1}{r} I(r) + 2\int_{\partial B_r} u u_\nu\,dS, \tag{3.3}$$

it follows from the following Gauss–Green identity,

$$\int_{B_r} |\nabla u|^2\,dx + \int_{B_r} u\Delta u\,dx = \int_{\partial B_r} u u_\nu\,dS,$$

that

$$\int_{\partial B_r} u u_\nu\,dS = \int_{B_r} |\nabla u|^2\,dx \geq 0.$$

Hence $r \mapsto I(r)$ is non-decreasing. We will consider the function $H(r) = \log I(r)$, which is also non-decreasing. The derivative of $H(r)$ is

$$H'(r) = \frac{n-1}{r} + \frac{2F(r)}{r}. \tag{3.4}$$

We use (3.4) to obtain an upper bound for the oscillation of $H(r)$ on $[s,t] \subset [s, r_2]$ as follows

$$\begin{aligned}
\operatorname*{osc}_{r\in[s,t]} H(r) &= \max_{r\in[s,t]} H(r) - \min_{r\in[s,t]} H(r) \\
&= H(t) - H(s) = \int_s^t H'(r)\, dr \\
&= \int_s^t \left(\frac{n-1}{r} + \frac{2F(r)}{r} \right) dr \\
&\le \left(n - 1 + 2 \sup_{r\in[s,t]} F(r) \right) \log\left(\frac{t}{s}\right).
\end{aligned} \tag{3.5}$$

From (3.5) it follows that

$$\frac{\max_{r\in[s,t]} I(r)}{\min_{r\in[s,t]} I(r)} \le \left(\frac{t}{s}\right)^{n-1+2\sup_{r\in[s,t]} F(r)},$$

which implies the following Harnack type inequality

$$\max_{r\in[s,t]} I(r) \le \left(\frac{t}{s}\right)^{n-1+2\sup_{r\in[s,t]} F(r)} \min_{r\in[s,t]} I(r). \tag{3.6}$$

The next step is to show that Almgren's frequency function is non-decreasing. The derivative of $F(r)$ is

$$F'(r) = \frac{D(r)I(r) + rD'(r)I(r) - rD(r)I'(r)}{I^2(r)}, \tag{3.7}$$

where $D'(r) = \int_{\partial B_r} |\nabla u|^2\, dS$. From the Rellich–Necas type identity (2.6) we obtain

$$\begin{aligned}
rD'(r)I(r) &= \left(\int_{\partial B_r} u^2\, dS \right) \left(r \int_{\partial B_r} |\nabla u|^2\, dS \right) \\
&= \left(\int_{\partial B_r} u^2\, dS \right) \left(2r \int_{\partial B_r} |u_\nu|^2\, dS + (n-2)D(r) \right) \\
&= 2r \left(\int_{\partial B_r} u^2\, dS \right) \left(\int_{\partial B_r} |u_\nu|^2\, dS \right) + (n-2)D(r)I(r).
\end{aligned} \tag{3.8}$$

Plugging (3.8) and (3.3) into (3.7) we arrive at

$$\begin{aligned}
I^2(r)F'(r) &= 2r \left(\int_{\partial B_r} u^2\, dS \right) \left(\int_{\partial B_r} |u_\nu|^2\, dS \right) - 2r \left(\int_{\partial B_r} u u_\nu\, dS \right)^2 \\
&\ge 2r \left(\int_{\partial B_r} u u_\nu\, dS \right)^2 - 2r \left(\int_{\partial B_r} u u_\nu\, dS \right)^2 = 0,
\end{aligned}$$

where we used Hölder's inequality. It follows that $F(r)$ is non-decreasing, and hence we may control the exponent in (3.6) from above.

To finish the proof, from (3.6) we obtain

$$I(t) = \max_{r\in[s,t]} I(r) \leq \left(\frac{t}{s}\right)^{n-1+2F(t)} I(s).$$

Since $I(s) \to 0$ as $s \to r_0$, it follows that $I(t) = 0$. This is a contradiction. □

We state the following immediate corollary (of the preceding proof) as it migth be of independent interest to the reader.

Corollary 3.9. *Let $u \in C^2(G)$ and $\Delta u = 0$ in G. Suppose that*

$$I(r) = \int_{\partial B_r} u^2\, dS > 0$$

at every $r \in (s,t)$, $B_s \subset \overline{B}_t \subset G$. Then the following Harnack type inequality is valid

$$\max_{r\in(s,t)} ⨍_{\partial B_r} u^2\, dS \leq \left(\frac{t}{s}\right)^{2F(t)} \min_{r\in(s,t)} ⨍_{\partial B_r} u^2\, dS. \tag{3.10}$$

Note that for (3.10) one needs to observe that $r^{1-n}I(r)$ is non-decreasing, and then the inequality follows immediately from (3.6).

We may also estimate how rapidly a harmonic function grows near a point where it vanishes. Namely, it is well-known but noteworthy that from the fact that $F(r)$ is non-decreasing it directly follows from (3.10) that for $0 < r < R$

$$\int_{\partial B_r} u^2\, dS \geq \gamma r^{\beta+n-1},$$

where $\gamma := I(R)R^{-\beta-n+1}$ and $\beta := 2F(R)$.

4. Unique continuation: $\Delta u = b(x)\cdot\nabla u$

We shall deal with the following modified version of Almgren's frequency function

$$F(r) = \frac{r\int_{\partial B_r} uu_\nu\, dx}{\int_{\partial B_r} u^2\, dS}, \tag{4.1}$$

and denote the numerator by $rH(r)$ and the denominator by $I(r)$. Of course, for harmonic functions (4.1) is equal to (3.1) thanks to the Gauss–Green identity. It is important to note that the frequency function defined in (4.1) is not necessarily non-negative for all radii $r > 0$.

One may easily check that the frequency function defined in (4.1), as well as in (3.1), is invariant under scaling in the following sense: Let

$\tau \in \mathbb{R}$, $\tau > 0$, and denote $v(x) = u(\tau x)$, where u is a solution to (1.1). Then

$$F^v(r) = F^u(\tau r)$$

for each $r > 0$, where $F^v(r)$ denotes the frequency function associated with function v.

The theorem we prove is the following. The proof is an extension of the harmonic case presented in the preceding section yet more subtle and demanding.

Theorem 4.2. *Suppose $u \in C^2(G)$ is a solution to*

$$\Delta u = b(x) \cdot \nabla u$$

in G, where the drift coefficients $\{b_i(x)\}_{i=1}^n$ are continuous and bounded in G. If there is an open set $D \subset G$ such that $u = 0$ in D, then $u \equiv 0$ in G.

As opposed to Almgren's frequency function, (3.1), frequency function for solutions to (1.1) as defined in (4.1) is not known to be non-decreasing in r. To overcome this, the key idea is to obtain the following inequality

$$F'(r) \geq -\frac{\alpha}{r}(F(r) + \beta), \tag{4.3}$$

where $0 < \alpha, \beta < \infty$ are not depending on r. Inequality (4.3) is obtained only for small values of r. Then setting $T(r) := F(r) + \beta$, and thus $T'(r) = F'(r)$, we may rewrite (4.3) as follows

$$\frac{d}{dr} \log T(r) \geq -\frac{\alpha}{r}. \tag{4.4}$$

From (4.4) one may deduce the following for each pair $r < \rho$

$$T(r) \leq \left(\frac{\rho}{r}\right)^{\alpha} T(\rho),$$

i.e.,

$$F(r) \leq \left(\frac{\rho}{r}\right)^{\alpha} F(\rho) + \beta \left(\left(\frac{\rho}{r}\right)^{\alpha} - 1 \right). \tag{4.5}$$

The detailed proof below is rather technical, but straightforward.

Proof of Theorem 4.2. As in the proof of Theorem 3.2, the proof is by contradiction. Suppose, on the contrary, that there is an open set $D \subset G$ such that $u = 0$ in D, but u is not identically zero in G. Then it is possible to pick arbitrary small neighborhoods $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$, $\overline{B}_{r_1}(x_0)$, $\overline{B}_{r_2}(x_0) \subset G$, such that $u(x) = 0$ in $B_{r_1}(x_0)$ but u is not identically zero in $B_{r_2}(x_0)$. This can be shown by connecting a point $x_1 \in D$ to a point $x_2 \in G \setminus D$ such that $u(x_2) \neq 0$, by a rectifiable curve in G, taking a finite sub-cover of balls with arbitrary small radii, and by employing a well-known chaining argument. Observe further that radii r_1 and r_2, which are to be fixed later, can be chosen in such a way that there exists $0 < \gamma_0 < 1$ so that

$$\frac{|B_{r_1}(x_0)|}{|B_{r_2}(x_0)|} \geq \gamma_0.$$

This enables us to employ Poincaré's inequality.

In order to show that $I(r)$ is non-decreasing for small values of r, we start by showing that there exists $r_2 > 0$ such that $H(r) \geq 0$ for each $0 < r \leq r_2$. By the Poincaré inequality, (2.7), we get

$$\begin{aligned}\left| \int_{B_r} u \Delta u \, dx \right| &\leq \left(\int_{B_r} u^2 \, dx \right)^{1/2} \left(\int_{B_r} |b(x) \cdot \nabla u|^2 \, dx \right)^{1/2} \\ &\leq \sqrt{C_p} M \left(r^2 \int_{B_r} |\nabla u|^2 \, dx \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 \, dx \right)^{1/2} \\ &= \sqrt{C_p} M r \int_{B_r} |\nabla u|^2 \, dx,\end{aligned}$$

where $M := \|b\|_{L^\infty(G)} < \infty$ and C_p is the constant in the Poincaré inequality and here it depends on γ_0. We now select r_2 small enough so that $\sqrt{C_p} M r < 1/2$ for every $r \leq r_2$. Plugging the preceding estimate into the Gauss–Green formula we arrive at

$$\int_{B_r} |\nabla u|^2 \, dx \leq 2 \int_{\partial B_r} u u_\nu \, dS = 2H(r), \tag{4.6}$$

and hence $H(r) \geq 0$ for every $0 < r \leq r_2$. In addition, we easily see that $I(r)$ is non-decreasing on $(0, r_2)$ as

$$I'(r) = \frac{n-1}{r} I(r) + 2 \int_{\partial B_r} u u_\nu \, ds = \frac{n-1}{r} I(r) + 2H(r). \tag{4.7}$$

Since we know that $I(r_2) > 0$, there exists a radius $\tilde{r} \in [r_1, r_2]$ such that $I(\tilde{r}) = 0$, but $I(r) > 0$ for $r > \tilde{r}$. From here on out, we thus consider an interval $(\tilde{r}, r_2]$.

Let us examine the derivative of $F(r)$. We have

$$F'(r) = \frac{H(r)I(r) + rH'(r)I(r) - rH(r)I'(r)}{I^2(r)}. \tag{4.8}$$

On the other hand, we obtain again from the Gauss-Green formula that

$$H'(r) = \int_{\partial B_r} |\nabla u|^2 \, dS + \int_{\partial B_r} u \Delta u \, dS. \tag{4.9}$$

Plugging (4.9) into (4.8) and using (4.7) we have the following expression for the derivative of the frequency function

$$\begin{aligned}I^2(r)F'(r) = H(r)I(r) &+ rI(r) \int_{\partial B_r} |\nabla u|^2 \, dS + rI(r) \int_{\partial B_r} u \Delta u \, dS \\ &- (n-1)H(r)I(r) - 2rH^2(r).\end{aligned} \tag{4.10}$$

At this point, we distinguish the following two possibilities. This is one of the crucial points in the proof of this theorem, and is in many ways analogous to Case 1 and 2, i.e., (2.49) and (2.51) in Garofalo and Lin [7]. As it will become clear, out of the two cases (B) is much stronger.

(A) $I(r)\int_{\partial B_r}|\nabla u|^2\,dx \le 4H^2(r)$;

(B) $I(r)\int_{\partial B_r}|\nabla u|^2\,dx > 4H^2(r)$.

Clearly either (A) or (B) holds true.

Suppose first that (A) is valid. We continue by estimating the terms on the right in (4.10). The third term can be estimated as follows using equation (1.1) and hypothesis (A)

$$\begin{aligned}\left|\int_{\partial B_r} u\Delta u\,dS\right| &\le \left|\int_{\partial B_r} u(b(x)\cdot\nabla u)\,dS\right| \le M\int_{\partial B_r}|u||\nabla u|\,dS\\ &\le M\left(\int_{\partial B_r}u^2\,dS\right)^{1/2}\left(\int_{\partial B_r}|\nabla u|^2\,dS\right)^{1/2}\\ &\le 2MH(r).\end{aligned}\tag{4.11}$$

We handle the second term on the right in (4.10) using Rellich–Necas type equation (2.2). We have

$$\begin{aligned} rI(r)\int_{\partial B_r}|\nabla u|^2\,dS &= 2rI(r)\int_{\partial B_r}|u_\nu|^2\,dS + (n-2)I(r)\int_{\partial B_r}uu_\nu\,dS\\ &\quad -2I(r)\int_{B_r}(x\cdot\nabla u)\Delta u\,dx-(n-2)I(r)\int_{B_r}u\Delta u\,dx.\end{aligned}\tag{4.12}$$

We note first that by using Hölder's inequality the first term on the right in (4.12) can be estimated as follows

$$I(r)\int_{\partial B_r}|u_\nu|^2\,dS \ge \left(\int_{\partial B_r}uu_\nu\,dS\right)^2 = H^2(r).\tag{4.13}$$

Then the last two terms in (4.12) can be controlled as follows. On one hand, we obtain

$$\begin{aligned}\left|\int_{B_r}(x\cdot\nabla u)\Delta u\,dx\right| &\le r\left(\int_{B_r}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{B_r}|\Delta u|^2\,dx\right)^{1/2}\\ &\le Mr\int_{B_r}|\nabla u|^2\,dx \le 2MrH(r)\end{aligned}\tag{4.14}$$

for every $\tilde r < r \le r_2$, where we used (1.1) and (4.6). On the other hand, we may estimate as above by using Poincaré inequality (2.7) and

(4.6)

$$\left|\int_{B_r} u\Delta u\,dx\right| \le \left(\int_{B_r} u^2\,dx\right)^{1/2}\left(\int_{B_r} |\Delta u|^2\,dx\right)^{1/2}$$
$$\le \sqrt{C_p}Mr\int_{B_r} |\nabla u|^2\,dx \le H(r). \qquad (4.15)$$

By first plugging (4.13), (4.14), and (4.15) into (4.12), and then by coupling (4.12) and (4.11) with (4.10), we may continue estimating (4.10) using again hypothesis (A) as follows

$$\begin{aligned} I^2(r)F'(r) \ge{}& H(r)I(r) + 2rH^2(r) + (n-2)H(r)I(r) - 4MrH(r)I(r) \\ & - (n-2)H(r)I(r) - 2MrH(r)I(r) - (n-1)H(r)I(r) \\ & - 2rH^2(r) \\ \ge{}& -(n-2)H(r)I(r) - 6MrH(r)I(r). \end{aligned}$$

From which we get an inequality of the form (4.3) for $\tilde{r} < r \le r_2$

$$F'(r) \ge -\frac{n-2}{r}F(r) - 6MF(r) \ge -\frac{\alpha}{r}F(r),$$

where $\alpha = n - 2 + 6Mr_2$.

Assume now that (B) holds true.

We estimate the third term on the right in (4.10) as follows using equation (1.1) and the Cauchy inequality with $\varepsilon = 1/(2M) > 0$

$$\left|\int_{\partial B_r} u\Delta u\,dS\right| \le \left|\int_{\partial B_r} u(b(x)\cdot\nabla u)\,dS\right| \le M\int_{\partial B_r} |u||\nabla u|\,dS$$
$$\le 2M^2\int_{\partial B_r} u^2\,dS + \frac{1}{2}\int_{\partial B_r} |\nabla u|^2\,dS. \qquad (4.16)$$

Then we estimate in (4.10) using first hypothesis (B) and then (4.16) as follows

$$\begin{aligned} I^2(r)F'(r) ={}& H(r)I(r) + rI(r)\int_{\partial B_r} |\nabla u|^2\,dS + rI(r)\int_{\partial B_r} u\Delta u\,dS \\ & - (n-1)H(r)I(r) - 2rH^2(r) \\ \ge{}& 2rH^2(r) + \frac{1}{2}rI(r)\int_{\partial B_r} |\nabla u|^2\,dS - 2M^2rI^2(r) \\ & - \frac{1}{2}rI(r)\int_{\partial B_r} |\nabla u|^2\,dS - (n-1)H(r)I(r) - 2rH^2(r) \\ \ge{}& -2M^2rI^2(r) - (n-1)H(r)I(r). \end{aligned}$$

This implies an inequality of the required form for $\tilde{r} < r \le r_2$

$$F'(r) \ge -\frac{n-1}{r}\left(F(r) + \frac{2(Mr_2)^2}{n-1}\right).$$

In conclusion, cases both (A) and (B) lead to an inequality of the form (4.3).

We may proceed as in the proof of Theorem 3.2. In a similar fashion, we obtain a Harnack type inequality as in (3.6), i.e.,

$$I(t) \leq \left(\frac{t}{s}\right)^{n-1+2\sup_{r\in[s,t]} F(r)} I(s)$$

for $[s,t] \subset (\tilde{r}, r_2]$, where using (4.5) we may estimate

$$\sup_{r\in[s,t]} F(r) \leq \left(\frac{r_2}{\tilde{r}}\right)^{\alpha} F(r_2) + \beta\left(\left(\frac{r_2}{\tilde{r}}\right)^{\alpha} - 1\right). \tag{4.17}$$

Since $I(s) \to 0$ as $s \to \tilde{r}$, it follows that $I(t) = 0$. This is a contradiction. $\square$

We remark that by using the frequency function it is possible to obtain a representation formula for $I(r)$. More precisely, the fact that

$$\frac{\tilde{I}'(r)}{\tilde{I}(r)} = \frac{2}{r} F(r),$$

where $\tilde{I}(r) = r^{1-n} I(r)$, implies the following

$$\int_{\partial B_r} u^2\, dS = \gamma \exp\left(-2\int_r^R F(t)\frac{dt}{t}\right) r^{n-1} \tag{4.18}$$

for $0 < r < R$, where $\gamma := I(R)$. Equation (4.18) enables to derive a priori lower bounds for $I(r)$ provided that an estimate of the form (4.17) is available for the frequency function $F(r)$. Note, however, that the method in the present paper is by contradiction, and hence we are not able to apply directly (4.17). A posteriori, it is known that an estimate like (4.17) is valid for the solutions to (1.1), see [7, 19].

5. Nonlinear generalizations

Consider the p-Laplace equation in G

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0, \qquad 1 < p < \infty. \tag{5.1}$$

For $p = 2$ we recover the Laplace equation $\Delta u = 0$. We refer the reader to, e.g., Heinonen et al. [11] and Lindqvist [14] for a detailed study of the p-Laplace equation and various properties of its solutions. The problem of unique continuation, both (i) and (ii), is still, to the best of our knowledge, an open problem, except for the linear case $p = 2$. The planar case for (ii) has been solved by Manfredi in [15], see also Bojarski and Iwaniec [2], as they have observed that the complex gradient of a solution to (5.1) is quasiregular.

In addition to unique continuation, a long-standing open problem is to find a frequency function associated with solutions to (5.1).

In [9] the authors of the present paper deal with the problem of unique continuation by studying a certain generalization of Almgren's frequency function for the p-Laplacian. By this approach some partial results on the unique continuation problem in both cases (i) and (ii) were obtained. Two possible nonlinear generalizations for the frequency function defined in [9] were as follows

$$F_p(r) = \frac{r^{p-1} \int_{B_r} |\nabla u|^p \, dx}{\int_{\partial B_r} |u|^p \, dS}, \tag{5.2}$$

and a slight modification of (5.2)

$$\widetilde{F}_p(r) = \frac{r \int_{B_r} |\nabla u|^p \, dx}{\int_{\partial B_r} |u|^p \, dS}. \tag{5.3}$$

As for the frequency functions defined in (3.1) and (4.1), it is easy to check that $F_p(r)$ satisfies the following scaling property for each $\tau \in \mathbb{R}$, $\tau > 0$,

$$F_p^v(r) = F_p^u(\tau r),$$

where u is a solution to (5.1) and $v(x) = u(\tau x)$. The scaling property for the frequency function defined in (5.3) is slightly different and can be stated as follows

$$\widetilde{F}_p^v(r) = \tau^{p-2} \widetilde{F}_p^u(\tau r).$$

The results obtained in [9] were the following.

Theorem 5.4. *Suppose $u \in W^{1,p}_{\mathrm{loc}}(G) \cap C^2(G)$ is a solution to the p-Laplace equation in G. Consider an affine function*

$$L(x) = l(x) + l_0,$$

where $l_0 \in \mathbb{R}$ and

$$l(x) = \sum_{i=1}^n \alpha_i x_i$$

is not identically zero. Then if $u(x) = L(x)$ in $B_r \subset G$, $u(x) = L(x)$ for every $x \in G$.

Remark 5.5. It can be shown that the difference $u - L$ satisfies a uniformly elliptic equation in divergence form with constant principal part coefficients, see equation (3.2) in [9]. It is standard, see e.g. [20, Theorem 8.1, pp. 145–146], that there exists a linear transformation of coordinates of the form

$$\xi_i = \sum_{j=1}^n c_{ij} x_j, \quad i = 1, \dots, n,$$

with nonsingular matrix $[c_{ij}]$, in such a way that equation (3.2) in [9] can be reduced, in terms of the new coordinates $\xi_1, \xi_2, \dots, \xi_n$, to the canonical form (1.1). Hence, in regard to nonlinear generalizations, it is of interest to study the unique continuation principle for the solutions to (1.1).

The preceding theorem could be also stated as follows. Suppose $u, v \in W^{1,p}_{\mathrm{loc}}(G) \cap C^2(G)$ are two solutions to the p-Laplace equation in G. Assume further that $\nabla v \neq 0$ in G. Then if $u(x) = v(x)$ in $B_r \subset G$, $u(x) = v(x)$ for every $x \in G$.

Theorem 5.6. *Suppose $u \in C^1(G)$. Assume further that there exist two concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$ such that the frequency function $F_p(r)$ is defined, i.e., $I(r) > 0$ for every $r \in (r_b, R_b]$, and moreover, $\|F_p\|_{L^\infty((r_b,R_b])} < \infty$. Then there exists some $r^\star \in (r_b, R_b]$ such that*

$$\int_{\partial B_{r_1}} |u|^p \, dS \leq 4 \int_{\partial B_{r_2}} |u|^p \, dS, \tag{5.7}$$

for every $r_1, r_2 \in (r_b, r^\star]$. In particular, the following weak doubling property is valid

$$\int_{\partial B_{r^\star}} |u|^p \, dS \leq 4 \int_{\partial B_r} |u|^p \, dS, \tag{5.8}$$

for every $r \in (r_b, r^\star]$.

In the following we formulate a partial result on the unique continuation problem for the p-Laplace equation. It says that the local boundedness of the frequency function implies the unique continuation principle. In this respect the situation is similar to the linear case $p = 2$, and we thus generalize this phenomenon to every $1 < p < \infty$.

Theorem 5.9. *Suppose u is a solution to the p-Laplace equation in G. Consider arbitrary concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$. Assume the following: whenever $I(r) > 0$ for every $r \in (r_b, R_b]$, then $\|F_p\|_{L^\infty((r_b,R_b])} < \infty$. Then the following unique continuation principle follows: If u vanishes on some open ball in G, then u is identically zero in G.*

It remains an open problem whether the frequency function $F_p(r)$ is locally bounded for the solutions to the p-Laplace equation. Local boundedness combined with the method of the present paper would solve the unique continuation problem for equation (5.1).

Remark 5.10. The corresponding divergence identity (2.4) for solutions to the p-Laplace equation is available, as well as the corresponding Rellich–Necas type formula, see e.g. [3, Noether's theorem] and [10, Lemma 4.1], respectively.

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