Comparison principles for subelliptic equations of Monge-Ampère type

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Sunto. – *W*e present two comparison principles for viscosity sub- and supersolutions of Monge-Ampère-type equations associated to a family of vector fields. In particular, we obtain the uniqueness of a viscosity solution to the Dirichlet problem for the equation of prescribed horizontal Gauss curvature in a Carnot group.

Introduction

We consider fully nonlinear partial differential equations of the form

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0, \text{ in } \Omega, \tag{0.1}$$

where $\Omega \subseteq I\!R^n$ is open and bounded, $D_{\mathcal{X}}u$ denotes the gradient of u with respect to a given family of $C^{1,1}$ vector fields $X_1, ..., X_m$, $D_{\mathcal{X}}u := (X_1u, ..., X_mu)$, $D^2_{\mathcal{X}}u$ denotes the symmetrized Hessian matrix of u with respect to the same vector fields

$$(D^2_{\mathcal{X}}u)_{ij} := (X_iX_ju + X_jX_iu)/2,$$

and H is a given Hamiltonian, at least continuous and nondecreasing in u. Our main examples are the vector fields that generate the homogeneous Carnot groups [4, 7], and in that case $D_{\mathcal{X}}u$ and $D^2_{\mathcal{X}}u$ are called, respectively, the horizontal gradient and the horizontal Hessian.

A theory of fully nonlinear subelliptic equations was started recently by Bieske [5, 6] and Manfredi [17, 3], and Monge-Ampère equations of the form (0.1) with $H = f(x)$ are listed among the main examples. For such equations on the Heisenberg group Gutierrez and Montanari [12] proved, among other things, a comparison principle among smooth sub- and supersolutions (see also [11] for related results). An example that motivates the dependence on the gradient $D_{\mathcal{X}}u$ in H is the prescribed horizontal Gauss curvature equation in Carnot groups, as defined by Danielli, Garofalo and Nhieu [10],

$$-\det(D^2_{\mathcal{X}}u) + k(x)\left(1 + |D_{\mathcal{X}}u|^2\right)^{\frac{m+2}{2}} = 0, \text{ in } \Omega, \tag{0.2}$$

for a given continuous $k : \overline{\Omega} \to]0, +\infty[$.

In this paper we begin a study of the subelliptic Monge-Ampère-type equations (0.1) within the theory of viscosity solutions. We present two comparison results that extend to the subelliptic setting a theorem of H. Ishii and P.-L. Lions for euclidean Monge-Ampère equations [13] (i.e., the case when the vector fields are the canonical basis of $\mathbf{R}^n$). For the large literature on this case we refer to the recent surveys [8, 19] and the references therein. The new difficulties we encounter are three.

1. The PDE (0.1) is degenerate elliptic only on functions that are convex with respect to the vector fields $X_1, ..., X_m$, briefly *$\mathcal{X}$-convex*. Following Lu, Manfredi, and Stroffolini [15] such a function is an u.s.c. $u : \overline{\Omega} \to \mathbf{R}$ such that $-D^2_{\mathcal{X}} u \leq 0$ in Ω in viscosity sense, that is,

$$D^2_{\mathcal{X}}\varphi(x) \geq 0 \quad \forall\, \varphi \in C^2(\Omega), x \in \text{argmax}(u - \varphi). \tag{0.3}$$

We refer to the survey in [7] for the recent literature on the notions of convexity in Carnot groups. Since $\mathcal{X}$-convex functions are not Lipschitz continuous, in general, we get better results in Carnot groups, where they are Lipschitz with respect to the intrinsic metric [15, 10, 16, 18, 14].

2. The operator in (0.1) does not satisfy in general the standard structure conditions in viscosity theory. Therefore we consider equations of the form

$$-\log\det(D^2_{\mathcal{X}} u) + K(x, u, Du, D^2u) = 0, \text{ in } \Omega, \tag{0.4}$$

that verify the Lipschitz-type condition with respect to x of [9] for *uniformly $\mathcal{X}$-convex* subsolutions. Our first main results states the comparison among semicontinuous sub- and supersolutions of this equation provided that either K is strictly increasing in u or that the subsolution is strict. Here K is any degenerate elliptic operator satisfying the structure conditions of [9].

3. To cover the case of H not strictly increasing in u, which is the most frequent in applications, we need to perturb a $\mathcal{X}$-convex subsolution to a uniformly $\mathcal{X}$-convex strict subsolution. In the case of vector fields that generate a Carnot group we adapt the method of [13] and [1] to get the following Comparison Principle, under essentially the same assumptions as the euclidean result of Ishii and Lions [13].

Theorem 0.1 *Assume $H : \Omega \times \mathbf{R} \times \mathbf{R}^m \to]0, +\infty[$ is continuous, nondecreasing in the second entry, and for all $R > 0$ there is L_R such that*

$$|H^{1/m}(x, r, q + q_1) - H^{1/m}(x, r, q)| \leq L_R|q_1| \quad \forall\, x \in \overline{\Omega}, |r| \leq R, |q| \leq R, |q_1| \leq 1. \tag{0.5}$$

Suppose the vector fields $X_1, ..., X_m$ are the generators of a Carnot group on $\mathbf{R}^n$. Let $u : \overline{\Omega} \to \mathbf{R}$ be a bounded, $\mathcal{X}$-convex, u.s.c. subsolution of (0.1) *and $v : \overline{\Omega} \to \mathbf{R}$ be a bounded l.s.c. supersolution of* (0.1)*. Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+. \tag{0.6}$$

In particular, there is at most one $\mathcal{X}$-convex viscosity solution of (0.1) *with prescribed continuous boundary data.*

Note that it applies to the prescribed horizontal Gauss curvature equation (0.2).

In Section 1 we state the Comparison Principle for the equation (0.4) with the main lemma needed for its proof. Section 2 is devoted to recalling the definition of generators of a Carnot group and stating a few facts about them. Finally, in Section 3 we outline the construction of the strict subsolution and the rest of the proof of Theorem 0.1. Our paper [2] contains the full proofs of these results, some extensions and variants, the existence of solutions to the Dirichlet problem via the Perron-Ishii method, and further examples and bibliography.

1. – Definitions and comparison with strict subsolutions

Let σ be the $n \times m$ matrix-valued function whose columns σ^j are the coefficients of the vector fields $X_1, ..., X_m$, $j = 1, \cdots, m$. We assume $\sigma_i^j = \sigma_{ij} \in C^{1,1}(\overline{\Omega})$ for all i, j. Observe that, for a smooth function u, $D_{\mathcal{X}} u(x) = \sigma(x)^T Du(x)$ and

$$D^2_{\mathcal{X}} u(x) = \sigma^T(x) D^2 u(x)\, \sigma(x) + Q(x, Du),\ Q_{ij}(x,p) := \left[D\sigma^j\, \sigma^i + D\sigma^i\, \sigma^j\right](x) \cdot \frac{p}{2}.$$

Therefore we rewrite (0.1) and (0.4) in the form $G(x, u, Du, D^2u) = 0$ with G proper in the sense of [9].

We say that a continuous function $F : \overline{\Omega} \times I\!\!R \times I\!\!R^n \times S^n \to I\!\!R$ *satisfies the structure conditions* (of viscosity theory) on a given set of admissible symmetric matrices $\mathcal{M} \subseteq S^n$ if it is nondecreasing in the second entry, nonincreasing in the last entry for matrices in $\mathcal{M}$, and for some modulus ω

$$F\left(y, r, \frac{x-y}{\epsilon}, Y\right) - F\left(x, r, \frac{x-y}{\epsilon}, X\right) \le \omega\left(|x-y|\left(1 + \frac{|x-y|}{\epsilon}\right)\right)$$

for all $\epsilon > 0$, $x, y \in \overline{\Omega}$, $r \in \mathbf{R}$, $X, Y \in \mathcal{M}$ satisfying

$$-\frac{3}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{1.1}$$

We say that $u : \overline{\Omega} \to \mathbf{R}$ u.s.c. is *uniformly $\mathcal{X}$-convex* if for some $\gamma > 0$

$$D^2_{\mathcal{X}} \varphi \ge \gamma I, \quad \forall\, \varphi \in C^2(\Omega), x \in \operatorname{argmax}(u - \varphi), \tag{1.2}$$

where I denotes the identity matrix. In other words, with the notations of [9], $(p, X) \in \mathcal{J}^{2,+}u(x)$ satisfies $\sigma^T(x) X\, \sigma(x) + Q(x, p) \ge \gamma I$, and this inequality defines the set of admissible matrices $\mathcal{M} = \mathcal{M}(p, \gamma)$.

The main ingredient for the results of this section is the following.

Lemma 1.1 *For each $\gamma > 0$ the function $F(x, p, X) := -\log\det(\sigma^T(x) X\, \sigma(x) + Q(x, p))$ satisfies the structure conditions on $\mathcal{M}(p, \gamma)$.*

The proof relies on a representation of F as a maximum of operators that satisfy the structure conditions, via the following formula, holding for $A \in S^m$, $A \ge \gamma I$,

$$\log\det(A) = \min\{m \log a - m + \operatorname{tr}(AM) : a > 0, M \in S^m, 0 \le M \le \frac{1}{\gamma} I, \det M = a^{-m}\}.$$

The next two Comparison Principles can now be proved by standard methods in viscosity theory [9], see [2] for the details. In the definition of *supersolution* v of (0.1) and (0.4) we restrict to $\mathcal{X}$-convex test functions. E.g., for (0.4) we require that

$$-\log\det(D^2_{\mathcal{X}} \varphi) + K(x, v, D\varphi, D^2\varphi) \ge 0 \quad \forall x \in \arg\min(v - \varphi),$$

for any $\varphi \in C^2(\Omega)$ such that $D^2_{\mathcal{X}} \varphi(x)$ is positive definite, cf. [13], Section V.3.

Theorem 1.1 *Assume* $K : \overline{\Omega} \times I\!R \times I\!R^n \times S^n \to I\!R$ *satisfies the structure conditions on* S^n. *Let* $u : \overline{\Omega} \to \mathbf{R}$ *be bounded, uniformly* $\mathcal{X}$*-convex, and for all open* Ω_1 *with* $\overline{\Omega}_1 \subseteq \Omega$ *there is* $\gamma_1 > 0$ *such that* u *is subsolution of*

$$-\log\det(D^2_{\mathcal{X}} u) + K(x, u, Du, D^2 u) \leq -\gamma_1, \ \ in\ \Omega_1. \tag{1.3}$$

Let $v : \overline{\Omega} \to \mathbf{R}$ *be a bounded l.s.c. supersolution of* (0.4). *Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Theorem 1.2 *The conclusion of the previous theorem remains true if* u *is a subsolution of* (0.4), *not necessarily strict, provided that, for some* $C > 0$,

$$K(x, r, p, X) - K(x, s, p, X) \geq C(r - s), \ -M \leq s \leq r \leq M, \ M := \max\{\|u\|_\infty, \|v\|_\infty\}.$$

Under this condition there is at most one uniformly $\mathcal{X}$*-convex viscosity solution of* (0.4) *with prescribed continuous boundary data.*

2. – Generators of Carnot groups

We begin with recalling some well-known definitions. We adopt the terminology and notations of the recent book [7]. Consider a group operation $\circ$ on $\mathbf{R}^n = \mathbf{R}^{n_1} \times ... \times \mathbf{R}^{n_r}$ with identity 0, such that $(x, y) \mapsto y^{-1} \circ x$ is smooth, and the dilation $\delta_\lambda : \mathbf{R}^n \to \mathbf{R}^n$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbf{R}^{n_i}.$$

If δ_λ is an automorphism of the group $(\mathbf{R}^n, \circ)$ for all $\lambda > 0$, $(\mathbf{R}^n, \circ, \delta_\lambda)$ is a homogeneous Lie group on $\mathbf{R}^n$. We say that $m = n_1$ smooth vector fields $X_1, ..., X_m$ on $\mathbf{R}^n$ generate $(\mathbf{R}^n, \circ, \delta_\lambda)$, and that this is a (homogeneous) Carnot group, if $X_1, ..., X_m$ are invariant with respect to the left translations on $\mathbf{R}^n$ $\tau_\alpha(x) := \alpha \circ x$ for all $\alpha \in \mathbf{R}^n$, $X_i(0) = \partial/\partial x_i$, $i = 1, ..., m$, and the rank of the Lie algebra generated by $X_1, ..., X_m$ is n at every point $x \in \mathbf{R}^n$. We refer, e.g., to [4, 7] for the connections of this definition with the classical one in the context of abstract Lie groups and for the properties of the generators. We will use only the following property, and refer to Remark 1.4.6, p. 59 of [7] for more precise informations.

Proposition 2.1 *If* $X_1, ..., X_m$ *are generators of a Carnot group, then*

$$X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} \sigma_{ij}(x) \frac{\partial}{\partial x_i}$$

with $\sigma_{ij}(x) = \sigma_{ij}(x_1, ..., x_{i-1})$ *homogeneous polynomials of a degree* $\leq n - m$.

For generators of Carnot groups the Lipschitz continuity of $\mathcal{X}$-convex functions with respect to the intrinsic metric and bounds on the horizontal gradient in the sense of distributions were studied in [15, 10, 16, 18, 14]. We deduce the following gradient bound in viscosity sense.

Proposition 2.2 *Let* u *be convex in* Ω *with respect to the generators of a Carnot group. Then, for every open* Ω_1 *with* $\overline{\Omega}_1 \subseteq \Omega$, *there exists a constant* C *such that* u *is a viscosity subsolution of* $|\sigma^T(x)\, Du| \leq C$ *in* Ω_1.

3. – Outline of proof of the Comparison Principle

In this section we outline the proof of Theorem 0.1. The definition of viscosity supersolution of (0.1) uses only $\mathcal{X}$-convex test functions as in Section 1 and in [13]. Given the $\mathcal{X}$-convex subsolution u we consider

$$u_{\epsilon,\mu}(x) := u(x) + \epsilon e^{\mu \frac{\sum_{i=1}^{m} |x_i|^2}{2}},$$

for positive ϵ, μ. A calculation using Proposition 2.1 shows that $u_{\epsilon,\mu}$ is uniformly $\mathcal{X}$-convex with $\gamma = \epsilon\mu$.

Lemma 3.1 *For any open Ω_1 with $\overline{\Omega}_1 \subseteq \Omega$ there are positive constants $\overline{\mu}$, independent of ϵ, and γ_2 such that, for $\mu \geq \overline{\mu}$, $u_{\epsilon,\mu}$ is a subsolution of*

$$-\det{}^{1/m}(D^2_{\mathcal{X}} u) + H^{1/m}(x, u, D_{\mathcal{X}} u) \leq -\gamma_2, \ \ in\ \Omega_1.$$

The proof of the lemma relies on the Minkowski inequality for $\det^{1/m}(A+B)$ with A, B positive definite, and the identity $\det(I + qq^T) = 1 + |q|^2$ for any column vector $q \in \mathbf{R}^m$. Moreover, by the boundedness of u and $D_{\mathcal{X}} u$, Proposition 2.2, we can assume the Lipschitz property (0.5) with a uniform constant L_R. The rest of the proof goes along the lines of [1].

Next, we exploit again the boundedness of $D_{\mathcal{X}} u$ in viscosity sense to see that $u_{\epsilon,\mu}$ satisfies (1.3) for $K = \log H$ and a suitable $\gamma_1 > 0$. Then Theorem 1.1 applies and gives $\sup_{\Omega}(u_{\epsilon,\mu} - v) \leq \max_{\partial\Omega}(u_{\epsilon,\mu} - v)^+$. Letting $\epsilon \to 0$ gives the conclusion.

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