MULTISCALE SINGULAR PERTURBATIONS AND HOMOGENIZATION OF OPTIMAL CONTROL PROBLEMS

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The paper is devoted to singular perturbation problems with a finite number of scales where both the dynamics and the costs may oscillate. Under some coercivity assumptions on the Hamiltonian, we prove that the value functions converge locally uniformly to the solution of an *effective* Cauchy problem for a limit Hamilton-Jacobi equation and that the effective operators preserve several properties of the starting ones; under some additional hypotheses, their explicit formulas are exhibited. In some special cases we also describe the effective dynamics and costs of the limiting control problem. An important application is the homogenization of Hamilton-Jacobi equations with a finite number of scales and a coercive Hamiltonian.

Keywords: Singular perturbations, viscosity solutions, Hamilton–Jacobi equations, dimension reduction, iterated homogenization, control systems in oscillating media, multiscale problems, oscillating costs.

MSC 2000. 49L25, 93C70, 35B25, 35B27, 49N70, 91A23.

Introduction

The controlled system

$$\begin{aligned} \dot{x}_s &= f(x_s, y_s, z_s, \alpha_s), \quad x_0 = x, \\ \varepsilon \dot{y}_s &= g(x_s, y_s, z_s, \alpha_s), \quad y_0 = y, \\ \varepsilon^2 \dot{z}_s &= r(x_s, y_s, z_s, \alpha_s), \quad z_0 = z, \end{aligned}$$

where α is the control and $\varepsilon > 0$ a small parameter, is a model of systems whose state variables evolve on three different time scales. Consider the cost functional

$$P^\varepsilon(t, x, y, z, \alpha_.) := \int_0^t l(x_s, y_s, z_s, \alpha_s)\, ds + h(x_t, y_t, z_t)$$

and the value function

$$u^\varepsilon(t, x, y, z) := \inf_{\alpha_.} P^\varepsilon(t, x, y, z, \alpha_.).$$

The analysis of the convergence of u^ε as $\varepsilon \to 0$ gives informations on the optimization problem after a sufficiently large time, namely, when the fast variables y and z have reached their regime behaviour.

For two time scales, i.e. $r \equiv 0$, the problem has a large mathematical and engineering literature, see the books by Kokotović et al.[1] and Bensoussan,[2] the references therein, and the more recent contributions by Artstein and Gaitsgory[3] and the authors.[4–7] For more than two time scales Gaitsgory and Nguyen[8] extended the method of limit occupational measures. In this paper we follow a method based on the Hamilton–Jacobi–Bellman equation satisfied by the value function, that is

$$\partial_t u^\varepsilon + \max_\alpha \left\{ -f \cdot D_x u^\varepsilon - g \cdot \frac{D_y u^\varepsilon}{\varepsilon} - r \cdot \frac{D_z u^\varepsilon}{\varepsilon^2} - l \right\} = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p.$$

It is based on the theory of viscosity solutions (for an overview, see the book[9]) and was used for two-scale problems by the first and the second named authors.[4,5] They also developed it further to stochastic systems and differential games.[6] In a companion paper[7] the authors extended the method to stochastic problems with three and n scales; in those cases the value functions solve some $2nd$ order degenerate parabolic equation. In the present paper, we show how to apply our method[7] to deterministic control problems and $1st$ order H-J-B equations with a finite number of scales.

An important advantage of our PDE approach is that it applies naturally in the generality of *differential games*, namely, problems governed by two conflicting players, α and β, acting on the system

$$\begin{aligned} \dot{x}_s &= f(x_s, y_s, z_s, \alpha_s, \beta_s), \\ \varepsilon \dot{y}_s &= g(x_s, y_s, z_s, \alpha_s, \beta_s), \\ \varepsilon^2 \dot{z}_s &= r(x_s, y_s, z_s, \alpha_s, \beta_s). \end{aligned}$$

If the game is 0-sum, i.e., the second player's goal is the maximization of the cost P^ε, the (lower) value function of the game satisfies the Cauchy problem for the Isaacs PDE

$$\begin{cases} \partial_t u^\varepsilon + \min\limits_{\beta} \max\limits_{\alpha} \left\{ -f \cdot D_x u^\varepsilon - g \cdot \frac{D_y u^\varepsilon}{\varepsilon} - r \cdot \frac{D_z u^\varepsilon}{\varepsilon^2} - l \right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p, \\ u^\varepsilon(0,x,y,z) = h(x,y,z). \end{cases}$$

Throughout this paper, in fact, we will never assume the Hamiltonian be of the Bellman form, that is, convex in the gradient variables. Therefore, our results apply, for instance, to the robust optimal control of systems with bounded unknown disturbances. Our goal is proving that the value functions u^ε converge locally uniformly to the viscosity solution of a new Cauchy problem (called the *effective* problem)

$$\begin{cases} \partial_t u + \overline{H}(x, D_x u) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ u(0,x) = \overline{h}(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{$\overline{\mathrm{HJ}}$}$$

For the two-scale case, the effective Hamiltonian $\overline{H}$ and the effective initial condition $\overline{h}$ are obtained, respectively, as the *ergodic constant* of a stationary problem and by the time-asymptotic limit of the solution to a related Cauchy problem. For the multiscale case, this construction must be done iteratively. Moreover, owing to this procedure and to the coercivity assumption, the effective (and the *intermediate*) operators inherit several properties of the starting ones, in particular those ensuring the Comparison Principle (we refer the reader to[7] for the case of intermediate or effective operators lacking these properties). An interesting issue is to represent the effective solution u as the value function of some new control problem. In some special cases, we shall prove that the effective PDE is associated to a limit control problem whose dynamics and effective costs can be described explicitely. An example of Hamilton-Jacobi equation that fits within our theory is

$$\partial_t u^\varepsilon + F\left(x, y, z, D_x u^\varepsilon\right) + \varphi_1(x,y,z) \frac{|D_y u^\varepsilon|}{\varepsilon} + \varphi_2(x,y,z) \frac{|D_z u^\varepsilon|}{\varepsilon^2} = 0 \tag{1}$$

in $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, where F is a standard Bellman-Isaacs operator, and, for some $\nu > 0$, $\varphi_1 \geq \nu$ and $\varphi_2 \geq \nu$. In this special case we can compute the effective Cauchy problem, which is

$$\begin{cases} \partial_t u + \max_{y,z} F(x,y,z,D_x u) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ u(0,x) = \min_{y,z} h(x,y,z) & \text{on } \mathbb{R}^n. \end{cases} \tag{$\overline{\text{HJ}}$}$$

An important byproduct of the previous theory is the homogenization of systems in highly heterogeneous media with more than two space scales. Now the system is

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s, \beta_s\right)$$

and the cost

$$P^\varepsilon(t,x,\alpha,\beta) := \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s, \beta_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}, \frac{x_t}{\varepsilon^2}\right).$$

In this case, the value function v^ε satisfies (in the viscosity sense) the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon + \min\limits_{\beta} \max\limits_{\alpha} \left\{-f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \alpha, \beta\right) \cdot D_x v^\varepsilon - l(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \alpha, \beta)\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T) \times \mathbb{R}^n, \\ v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right). \end{cases}$$

Here the oscillations are in space. By setting $y = x/\varepsilon, z = x/\varepsilon^2$ this problem can be written as a singular perturbation one. Motivated by this, we will call throughout the paper x the *macroscopic* variables, y the *mesoscopic* ones, and z the *microscopic* variables. Under an assumption of coercivity of the Hamiltonian in the gradient variables we prove that the solution v^ε converge locally uniformly to the solution of an effective problem ($\overline{\text{HJ}}$), for a suitable construction of the effective operator and initial data. Our result applies to the homogenization of the *eikonal equation*

$$\partial_t v^\varepsilon + \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)|D_x v^\varepsilon| = l\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \tag{2}$$

with $\varphi \geq \nu > 0$.

In the framework of viscosity solutions, the study of the two-scale homogenization, initiated by Lions, Papanicolaou, Varadhan[10] and improved by Evans,[11,12] has been extended to related questions: see, e.g., Capuzzo–Dolcetta and Ishii[13] for the rate of convergence, Horie and Ishii[14] and the first author[15,16] for periodic homogenization in perforated domains, Ishii,[17] Arisawa,[18] and Birindelli, Wigniolle[19] for non–periodic homogenization, Rezakhanlou and Tarver,[20] Souganidis,[21] Lions, Souganidis[22,23] for stochastic homogenization, the book,[9] Artstein, Gaitsgory,[3] the first two authors,[4] and the references therein for singular perturbations in optimal control.

Let us stress that all the aforementioned papers consider only two scales and that, as far as we know, fully nonlinear problems with multiple scales have been attacked for the first time in our paper;[7] in fact, iterated homogenization was addressed only in the variational setting, starting with the pioneering work of Bensoussan, J.L. Lions and Papanicolaou[24] for linear equations and, afterwards, for semilinear equations, using the Γ–convergence approach[25,26] (see also and references therein) or G–convergence techniques.[27–29]

The plan of the paper is as follows. The standing assumptions are listed in Section 1. Section 2 recalls the notions of ergodicity and stabilization for a Hamiltonian. Section 3 is devoted to the regular perturbations of two-scale problems because they are of independent interest and for later use. We address the multiscale singular perturbations and the multiscale homogenization respectively in Section 4 and in Section 5. Some examples arising from deterministic optimal control theory and differential games are collected in Section 6. One of them replaces the coercivity of the Hamiltonian with a non-resonance condition and allows to show that exchanging the roles of ε and ε^2 may produce a different effective PDE.

1. Standing assumptions

We consider Bellman–Isaacs Hamiltonians

$$H(x, y, p_x, p_y) := \min_{\beta \in B} \max_{\alpha \in A} L^{\alpha,\beta}(x, y, p_x, p_y),$$

for the family of linear operators

$$L^{\alpha,\beta}(x, y, p_x, p_y) := -p_x \cdot f(x, y, \alpha, \beta) - p_y \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta).$$

The following assumptions will hold in Sections 2 and 3:

- The control sets A and B are compact metric spaces.
- The functions f, g and l are bounded continuous functions in $\mathbb{R}^n \times \mathbb{R}^m \times A \times B$ with values, respectively, in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}$.
- The drift vectors f and g are Lipschitz continuous in (x, y), uniformly in (α, β).
- The running cost l is uniformly continuous in (x, y), uniformly in (α, β).

– The initial condition h is bounded and uniformly continuous.
– The functions f, g, h and l are $\mathbb{Z}^m$-periodic in the fast variables y.
– H is coercive in p_y: there exist $\nu, C \in \mathbb{R}^+$ such that:

$$H(x, y, p_x, p_y) \geq \nu|p_y| - C\,(1 + |p_x|) \qquad \text{for every } x, y, p_x, p_y.$$

Let us observe that the last assumption holds provided that:

$$B_m(0, \nu) \subset \overline{\text{conv}}\{g(x, y, \alpha, \beta) \mid \alpha \in A\} \qquad \forall x, y, \beta$$

where $B_m(0, \nu)$ is the ball centered in 0 with radius ν in the space $\mathbb{R}^m$.

In the deterministic control theory, this relation entails a strong form of small-time controllability of the deterministic fast subsystem, i.e. any two points can be reached from one another by the player acting on α, whatever the second player does, and within a time proportional to the distance between the points.

We introduce the *recession function* (or homogeneous part) of H in p_y by

$$H'(x, y, p_y) := \min_{\beta \in B} \max_{\alpha \in A}\{-p_y \cdot g(x, y, \alpha, \beta)\}.$$

We note that H' is positively 1-homogeneous in p_y, namely $H'(x, y, \lambda p_y) = \lambda H'(x, y, p_y)$ for $\lambda \geq 0$ and that, for every $\overline{x}, \overline{p}_x \in \mathbb{R}^n$, there is a constant C so that

$$|H(x, y, p_x, p_y) - H'(x, y, p_y)| \leq C \quad \forall\,(y, p_y) \in \mathbb{R}^m \times \mathbb{R}^m, \tag{3}$$

for every (x, p_x) in a neighborhood of $(\overline{x}, \overline{p}_x)$.

In the case of three scales, treated in Section 4.1, the Hamiltonian depends also on z and p_z. Then the linear operators $L^{\alpha,\beta}$ have the additional term $-p_z \cdot r(x, y, z, \alpha, \beta)$, whereas $f, g,$ and l may depend on z as well. We make the same assumptions on the dependence of the data from z as from y, namely, periodicity, Lipschitz continuity of f, g, r, and uniform continuity of l. The obvious analogous assumptions are made in the general case of $j + 1$ scales studied in Section 4.2.

2. Ergodicity, stabilization and the effective problem

The aim of this Section is to recall from[6,7] the notions of ergodicity and of stabilization that are crucial in the definition of the effective problem ($\overline{\text{HJ}}$). We establish some properties of the effective operators and in some cases we provide their explicit formulas.

2.1. *Ergodicity and the effective Hamiltonian*

This Subsection is devoted to recall the definition of ergodicity introduced in.[5] For $(\overline{x}, \overline{p}_x)$ fixed, by the standard viscosity solution theory, the *cell δ-problem*

$$\delta w_\delta + H(\overline{x}, y, \overline{p}_x, D_y w_\delta) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic}, \tag{CP$_\delta$}$$

has a unique solution. We denote the solution by $w_\delta(y; \overline{x}, \overline{p}_x)$ so as to display its dependence on the frozen slow variables. We say that the Hamiltonian is *ergodic* in the fast variable at $(\overline{x}, \overline{p}_x)$ if

$$\delta w_\delta(y; \overline{x}, \overline{p}_x) \to \text{const} \qquad \text{as } \delta \to 0, \text{ uniformly in } y.$$

In this case, we define

$$\overline{H}(\overline{x}, \overline{p}_x) := -\lim_{\delta \to 0} \delta w_\delta(y; \overline{x}, \overline{p}_x);$$

the function $\overline{H}$ is called *effective Hamiltonian.* We say that H is ergodic if it is ergodic at every $(\overline{x}, \overline{p}_x)$. In the next Proposition we collect some properties of $\overline{H}$ and, in some special cases, also its explicit formula.

Proposition 2.1. *Under the standing assumptions there holds*

(a) *the Hamiltonian H is ergodic.*
(b) *$\overline{H}$ is regular: there are $C \in \mathbb{R}$ and a modulus of continuity ω such that:*

$$|\overline{H}(x_1, p) - \overline{H}(x_2, p)| \le C|x_1 - x_2|(1 + |p|) + \omega(|x_1 - x_2|) \ \forall x_i, p \in \mathbb{R}^n;$$
$$|\overline{H}(x, p_1) - \overline{H}(x, p_2)| \le C(|p_1 - p_2|) \qquad \forall x, p_i \in \mathbb{R}^n;$$

in particular, the Comparison Principle holds for the effective problem $(\overline{\text{HJ}})$.
(c) *if*

$$H(\overline{x}, y, \overline{p}_x, p_y) \ge H(\overline{x}, y, \overline{p}_x, 0) \qquad \forall y, p_y \in \mathbb{R}^m, \tag{4}$$

then $\overline{H}$ has the explicit formula:

$$\overline{H}(\overline{x}, \overline{p}_x) = \max_y H(\overline{x}, y, \overline{p}_x, 0).$$

Proof The proofs of (a) and (b) are slight adaptations of the arguments used in [5, Proposition 9], [4, Proposition 12] and [12, Lemma 2.2] so we omit them.

(c) The Comparison Principle for the cell δ-problem (CP$_\delta$) entails: $\delta w_\delta \ge -\sup_y H(\overline{x}, y, \overline{p}_x, 0)$; as $\delta \to 0$, we infer:

$$\overline{H}(\overline{x}, \overline{p}_x) \le \sup_y H(\overline{x}, y, \overline{p}_x, 0).$$

In order to prove the reverse inequality, we shall argue by contradiction, assuming: $\overline{H}(\overline{x},\overline{p}_x) < H(\overline{x},y,\overline{p}_x,0)$ in a open set U. Therefore, the cell δ-problem reads

$$\delta w_\delta + H_0(y, D_y w_\delta) + H(\overline{x}, y, \overline{p}_x, 0) = 0$$

where $H_0(y,q) := H(\overline{x},y,\overline{p}_x,q) - H(\overline{x},y,\overline{p}_x,0)$. The ergodicity of H and the relation (4) entail

$$0 \le \overline{H}(\overline{x},\overline{p}_x) - H(\overline{x},y,\overline{p}_x,0) + O(\delta) \qquad \text{in} U.$$

As $\delta \to 0$, we obtain the desired contradiction. □

Remark 2.1. Let us observe that condition (4) is satisfied if the control α splits into (α_1, α_2), where α_i belongs to the compact A_i $(i = 1, 2)$, the drift f and the running cost l do not depend on α_2, while the drift $g = g(x, y, \alpha_2, \beta)$ fulfills:

$$B_m(0,\nu) \subset \overline{\text{conv}}\{g(x,y,\alpha_2,\beta) \mid \alpha_2 \in A_2\} \qquad \forall x, y, \beta.$$

2.2. *Stabilization and the effective initial data*

The stabilization to a constant for degenerate eqs. was introduced by the first two authors.[5] For $\overline{x}$ fixed, the *cell Cauchy problem* for the homogeneous Hamiltonian H'

$$\partial_t w + H'(\overline{x}, y, D_y w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad w(0,y) = h(\overline{x},y) \quad \text{on } \mathbb{R}^m \tag{CP$'$}$$

has a unique bounded viscosity solution $w(t, y; \overline{x})$. Observe that by the positive homogeneity of H', the constants $\|h(\overline{x},\cdot)\|_\infty$ and -$\|h(\overline{x},\cdot)\|_\infty$ are respectively a super- and a subsolution. Furthermore, the Comparison Principle yields the uniform bound: $\|w(t,\cdot)\|_\infty \le \|h(\overline{x},\cdot)\|_\infty$ for all $t \ge 0$.

We say that the pair (H, h) is *stabilizing* (to a constant) at $\overline{x}$ if

$$w(t, y; \overline{x}) \to \text{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y. \tag{5}$$

In this case, we define

$$\overline{h}(\overline{x}) := \lim_{t\to+\infty} w(t, y; \overline{x}). \tag{6}$$

We say that the pair (H, h) is stabilizing if it is stabilizing at every $\overline{x} \in \mathbb{R}^n$. The function $\overline{h}$ is called the *effective initial data*.

Proposition 2.2. *Under the standing assumptions, the pair* (H, h) *is stabilizing. Moreover, the effective initial datum* $\overline{h}$ *is continuous and has the form:*

$$\overline{h}(\overline{x}) = \min_y h(\overline{x}, y).$$

The proof is a slight adaptation of the arguments used in [5, Proposition 10] and in [4, Theorem 8] and we shall omit it.

3. Regular perturbation of singular perturbation problems

This Section is devoted to a convergence result for the regular perturbation of a singular perturbation problem

$$\begin{cases} \partial_t u^\varepsilon + H^\varepsilon\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \\ u^\varepsilon(0, x, y) = h^\varepsilon(x, y) & \text{on } \mathbb{R}^n \times \mathbb{R}^m. \end{cases} \tag{HJ$^\varepsilon$}$$

Regular perturbation means that $H^\varepsilon \to H$ and $h^\varepsilon \to h$ as $\varepsilon \to 0$ uniformly on all compact sets and that H, h and every H^ε, h^ε satisfy the standard assumptions of Sec. 1. For example, we have a regular perturbation when the control sets A and B are independent of ε and the functions f^ε, g^ε and l^ε converge locally uniformly to f, g and l. We suppose also that

$$|H^\varepsilon(x, y, 0, 0)| \leq C \qquad \forall (x, y), \tag{7}$$

for some constant C independent of ε small (this assumption is satisfied for instance if the running costs l^ε are equibounded). Let us note that the problem (HJ$^\varepsilon$) has a unique bounded solution (that is also periodic in y) and fulfills the Comparison Principle.[4,5]

The next result and the arguments of its proof will be used extensively in the next Sections.

Theorem 3.1. *Assume that* H^ε *and* h^ε *converge respectively to* H *and to* h *uniformly on the compact sets and that the equiboundedness condition* (7) *holds. Then,* u^ε *converges uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n$ *to the unique viscosity solution of* $(\overline{\text{HJ}})$ *where the effective Hamiltonian* $\overline{H}$ *and the effective initial datum* $\overline{h}$ *are defined respectively in Subsec. 2.1 and 2.2.*

Proof The proof of this Theorem relies on [7, Corollary 1] (see also [5, Theorem 1]) and on the ergodicity and stabilization results stated in Proposition 2.1 and in Proposition 2.2. For the sake of completeness, let us sketch the main features of the argument. The family $\{u^\varepsilon\}$ is equibounded; indeed, the Comparison Principle gives: $\|u^\varepsilon(t,\cdot)\|_\infty \le \sup_\varepsilon \|h^\varepsilon\|_\infty + Ct$. We can therefore define the upper semilimit $\overline{u}$ of u^ε as follows

$$\overline{u}(t,x) := \limsup_{\varepsilon\to 0,\ (t',x')\to(t,x)} \sup_y u^\varepsilon(t',x',y) \quad \text{if } t>0,$$
$$\overline{u}(0,x) := \limsup_{(t',x')\to(0,x),\ t'>0} \overline{u}(t',x') \quad \text{if } t=0.$$

We define analogously the lower semilimit $\underline{u}$ by replacing limsup with liminf and sup with inf. The two-steps definition of the semilimit for $t=0$ is needed to avoid a possible initial layer.

Let us notice that, under our hypotheses, the effective problem $(\overline{\mathrm{HJ}})$ satisfies the Comparison Principle and admits exactly one bounded solution u. If $\overline{u}$ and $\underline{u}$ are respectively a super- and a subsolution to $(\overline{\mathrm{HJ}})$ then the proof is accomplished. Actually, the Comparison Principle ensures: $\overline{u} \le u \le \underline{u}$. On the other hand the reverse inequality $\underline{u} \le \overline{u}$ is always true. Whence, we have $\overline{u} = \underline{u} = u$ and, by standard arguments, we deduce: $u^\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

Let us now ascertain that $\overline{u}$ is a subsolution to $(\overline{\mathrm{HJ}})$; being similar, the other proof is omitted. We proceed by contradiction assuming that there are a point $(\overline{t},\overline{x}) \in (0,T)\times\mathbb{R}^n$ and a smooth test function φ such that: $\overline{u}(\overline{t},\overline{x}) = \varphi(\overline{t},\overline{x})$, $(\overline{t},\overline{x})$ is a strict maximum point of $\overline{u}-\varphi$ and there holds

$$\partial_t\varphi(\overline{t},\overline{x}) + \overline{H}(\overline{x}, D_x\varphi(\overline{t},\overline{x})) \ge 3\eta$$

for some $\eta>0$. For every $r>0$, we define

$$H_r^\varepsilon(y,p_y) := \min\{H^\varepsilon(x,y,D_x\varphi(t,x),p_y) \mid |t-\overline{t}|\le r,\ |x-\overline{x}|\le r\}.$$

We put $\overline{H} := \overline{H}(\overline{x},\overline{p}_x)$ with $\overline{p}_x = D_x\varphi(\overline{t},\overline{x})$ and we fix $r_0>0$ so that

$$|\partial_t\varphi(t,x) - \partial_t\varphi(\overline{t},\overline{x})| \le \eta \qquad \text{as } |t-\overline{t}|<r_0,\ |x-\overline{x}|\le r_0.$$

Now we want to prove that, for every $r>0$ small enough, there is a parameter $\varepsilon'>0$ and an equibounded family of functions $\{\chi^\varepsilon \mid 0<\varepsilon<\varepsilon'\}$ (called *approximated correctors*) so that

$$H_r^\varepsilon(y,D_y\chi^\varepsilon) \ge \overline{H} - 2\eta \quad \text{in } \mathbb{R}^m. \tag{8}$$

To this aim, taking into account the ergodicity of H, we fix a parameter $\delta>0$ so that the solution w_δ to the cell δ-problem (CP_δ) fulfills:

$$\|\delta w_\delta + \overline{H}\|_\infty \le \eta.$$

Since $H_r^\varepsilon(y,p_y) \to H(\overline{x},y,\overline{p}_x,p_y)$ as $(\varepsilon,r)\to(0,0)$ uniformly on the compact sets, the stability property entails that the solution $w_{\delta,r}^\varepsilon$ of

$$\delta w_{\delta,r}^\varepsilon + H_r^\varepsilon(y,D_y w_{\delta,r}^\varepsilon) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_{\delta,r}^\varepsilon \text{ periodic},$$

converges uniformly to w_δ as $(\varepsilon, r) \to (0,0)$. In particular, for $\varepsilon' > 0$ and $0 < r' < \min\{r_0, \overline{t}\}$, we get

$$\|\delta w^\varepsilon_{\delta,r} + \overline{H}\|_\infty \leq 2\eta \qquad \text{when } 0 < \varepsilon < \varepsilon' \text{and } 0 < r < r'.$$

The function $\chi^\varepsilon = w^\varepsilon_{\delta,r}$ is a supersolution of (8). Moreover, by the Comparison Principle, the family $\{\chi^\varepsilon\}$ is equibounded: $\|\chi^\varepsilon\|_\infty \leq \delta^{-1}\sup\{|H^\varepsilon_r(y,0,0)| \mid y \in \mathbb{R}^m,\ 0 < \varepsilon < \varepsilon'\}$. Hence, our claim is proved.

We consider the perturbed test function

$$\psi^\varepsilon(t,x,y) := \varphi(t,x) + \varepsilon\chi^\varepsilon(y).$$

In the cylinder $Q_r =]\overline{t}-r, \overline{t}+r[\times B_r(\overline{x}) \times \mathbb{R}^m$, ψ^ε is a supersolution of $(\mathrm{HJ}^\varepsilon)$ (see[5] for the rigourous proof). Since $\{\psi^\varepsilon\}$ converges uniformly to φ on $\overline{Q_r}$, we obtain

$$\limsup_{\varepsilon\to 0,\ t'\to t,\ x'\to x}\ \sup_y (u^\varepsilon - \psi^\varepsilon)(t',x',y) = \overline{u}(t,x) - \varphi(t,x).$$

But $(\overline{t}, \overline{x})$ is a strict maximum point of $\overline{u} - \varphi$, so the above relaxed upper limit is negative on ∂Q_r. By compactness, one can find $\eta' > 0$ so that $u^\varepsilon - \psi^\varepsilon \leq -\eta'$ on ∂Q_r for ε small. Since ψ^ε is a supersolution in Q_r, we deduce from the Comparison Principle that $\psi^\varepsilon \geq u^\varepsilon + \eta'$ in Q_r for ε small. Taking the upper semi-limit, we get $\varphi \geq \overline{u} + \eta'$ in $(\overline{t}-r, \overline{t}+r) \times B(\overline{x}, r)$. This is impossible, for $\varphi(\overline{t},\overline{x}) = \overline{u}(\overline{t},\overline{x})$. Thus, we have reached the desired contradiction.

We now check that $\overline{u}$ satisfies the initial condition, that is $\overline{u} \leq \overline{h}$. Let w^ε_r be the unique solution of the following Cauchy problem

$$\begin{cases} \partial_t w^\varepsilon_r + H^{\varepsilon,\prime}_r(y, D_y w^\varepsilon_r) = 0 \text{ in } (0,+\infty)\times\mathbb{R}^m, \\ w^\varepsilon_r(0,y) = h^\varepsilon_r(y) \qquad\qquad \text{on } \mathbb{R}^m, \quad w^\varepsilon_r \text{ periodic in } y \end{cases}$$

where the Hamiltonian $H^{\varepsilon,\prime}_r$ and the initial datum h^ε_r are given by

$$H^{\varepsilon,\prime}_r(y,p_y) := \min\{H^{\varepsilon,\prime}(x,y,p_y) \mid |x-\overline{x}| \leq r\},$$
$$h^\varepsilon_r(y) := \max\{h^\varepsilon(x,y) \mid |x-\overline{x}| \leq r\}.$$

Let us claim that

$$\limsup_{r\to 0,\ \varepsilon\to 0,\ t\to\infty}\ \sup_y |w^\varepsilon_r(t,y) - \overline{h}(\overline{x})| = 0. \tag{9}$$

Fix $\eta > 0$. The stabilization ensures that the solution w to the cell Cauchy problem (CP') fulfills

$$\|w(T,\cdot) - \overline{h}(\overline{x})\|_\infty \leq \eta$$

for some $T > 0$. Letting $(\varepsilon, r) \to (0,0)$, we have $H_r^{\varepsilon,\prime} \to H'(\overline{x}, \cdot)$ and $h_r^\varepsilon \to h(\overline{x}, \cdot)$ uniformly on the compact sets; therefore, by the stability properties of viscosity solutions, we know that $w_r^\varepsilon \to w'$ locally uniformly. Whence, there are ε' and r' so that

$$\|w_r^\varepsilon(T, \cdot) - \overline{h}(\overline{x})\|_\infty \leq 2\eta \qquad \text{for all } 0 < \varepsilon < \varepsilon',\ 0 < r < r'.$$

Since $H_r^{\varepsilon,\prime}(\cdot, 0) \equiv 0$, the Comparison Principle entails that

$$\|w_r^\varepsilon(t, \cdot) - \overline{h}(\overline{x})\|_\infty \leq 2\eta \qquad \text{for all } t \geq T,\ 0 < \varepsilon < \varepsilon',\ 0 < r < r'.$$

This gives (9).

Let $r > 0$, $\varepsilon' > 0$ and $T > 0$ be such that the last inequality is satified. For $Q_r^+(\overline{x}) := (0, r) \times B_r(\overline{x}) \times \mathbb{R}^m$, we fix M so that $M \geq \|u^\varepsilon\|_{L^\infty(Q_r^+(\overline{x}))}$ for all $\varepsilon < \varepsilon'$ and we construct a bump function ψ_0 that is nonnegative, smooth, with $\psi_0(\overline{x}) = 0$ and $\psi_0 \geq 2M$ on $\partial B_r(\overline{x})$. Finally, we choose the constant $C > 0$ given by (3) so that

$$|H^\varepsilon(x, y, D_x\psi_0(x), p_y) - H^{\varepsilon,\prime}(x, y, p_y)| \leq C$$

for every (y, p_y), $x \in B_r(\overline{x})$, $0 < \varepsilon < \varepsilon'$. We introduce the function

$$\psi^\varepsilon(t, x, y) := w_r^\varepsilon(\varepsilon^{-1}t, y) + \psi_0(x) + Ct$$

and we observe that it is a supersolution of

$$\begin{aligned} &\partial_t\psi^\varepsilon + H^\varepsilon(x, y, D_x\psi^\varepsilon, \varepsilon^{-1}D_y\psi^\varepsilon) = 0 \quad \text{in } Q_r^+(\overline{x}) \\ &\psi^\varepsilon = h^\varepsilon \quad \text{on } \{0\} \times B_r(\overline{x}) \times \mathbb{R}^m, \quad \psi^\varepsilon = M \quad \text{on } [0, r) \times \partial B_r(\overline{x}) \times \mathbb{R}^m. \end{aligned}$$

By the Comparison Principle, we deduce that

$$u^\varepsilon(t, x, y) \leq \psi^\varepsilon(t, x, y) = w_r^\varepsilon(\varepsilon^{-1}t, y) + \psi_0(x) + Ct \qquad \text{in } Q_r^+(\overline{x}).$$

Taking the supremum over y and sending $\varepsilon \to 0$, we obtain the inequality

$$\overline{u}(t, x) \leq \overline{h}(\overline{x}) + 2\eta + \psi_0(x) + Ct \qquad \text{for all } t > 0,\ x \in B_r(\overline{x}).$$

Sending $t \to 0^+$, $x \to \overline{x}$, we get $\overline{u}(0, \overline{x}) \leq \overline{h}(\overline{x}) + \eta$. Taking into account the arbitrariness of η, one can easily accomplish the proof. □

Remark 3.1. Let us stress that the coercivity assumption has been used only for establishing the following properties: *i*) the starting Hamiltonian H is ergodic, *ii*) the pair (H, h) is stabilizing, *iii*) the effective Hamiltonian $\overline{H}$ is sufficiently regular to fulfill the Comparison Principle.

It is worth to recall that there exist non-coercive Hamiltonians that enjoy properties (*i*)–(*iii*) (e.g., see Sec. 6 below). It is obvious that Theorem 3.1 applies also to these Hamiltonians.

4. Singular perturbations with multiple scales

This Section is devoted to the study of singular perturbation problems having a finite number of scales. For the sake of simplicity, in the first Subsection we shall focus our attention on the three scales case, which is the simplest one, providing a detailed proof of our result. After, in the second Subsection we shall briefly give the result for a wider class of problems.

4.1. *The three scale case*

We consider the problems:

$$\begin{cases} \partial_t u^\varepsilon + H^\varepsilon\left(x, y, z, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, \frac{D_z u^\varepsilon}{\varepsilon^2}\right) = 0 & \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p \\ u^\varepsilon(0,x,y,z) = h^\varepsilon(x,y,z) & \text{on } \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p \end{cases} \tag{10}$$

where H^ε and h^ε are 1-periodic in y and z. Each variable corresponds to a certain scale of the problem: x is the *macroscopic* (or the *slow*) variable, y is the *mesoscopic* (or the not so fast variable) variable and z is the *microscopic* (or the *fast*) variable.

Roughly speaking, we shall attack this problem iteratively: by virtue of the different powers of ε, one first considers *both x and y* as slow variables, freezing them and homogenizing with respect to z and after, still with x frozen, one shall homogenizes with respect to y. In other words, in a first approximation, problem (10) is a singular perturbation problem only in the variable z; under adequate assumptions of ergodicity and stabilization with respect to z, we shall achieve a *mesoscopic* effective Hamiltonian H_1 and $u^\varepsilon(t,x,y,z)$ should converge to the solution $v^\varepsilon(t,x,y)$ of the mesoscopic problem

$$\begin{cases} \partial_t v^\varepsilon + H_1\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ v^\varepsilon(0,x,y) = h_1(x,y) & \text{on } \mathbb{R}^n\times\mathbb{R}^m. \end{cases}$$

This problem falls within the theory of Sec. 3 (see also[5]); v^ε will converge to the solution u of the limit problem $(\overline{\text{HJ}})$ provided that H_1 is ergodic and (H_1, h_1) is stabilizing. In conclusion, we expect that $u^\varepsilon(t,x,y,z)$ will converge to $u(t,x)$ where the effective quantities are defined inductively.

For the sake of simplicity, we shall assume as before that the operator H is given by

$$H(x,y,z,p_x,p_y,p_z) := \min_{\beta\in B}\max_{\alpha\in A} L^{\alpha,\beta}(x,y,z,p_x,p_y,p_z),$$

for the family of linear operators

$$L^{\alpha,\beta}(x,y,z,p_x,p_y,p_z) := -p_x \cdot f(x,y,z,\alpha,\beta) - p_y \cdot g(x,y,z,\alpha,\beta) \\ - p_z \cdot r(x,y,z,\alpha,\beta) - l(x,y,z,\alpha,\beta).$$

We shall require the following assumptions:

– $H^\varepsilon \to H$ and $h^\varepsilon \to h$ as $\varepsilon \to 0$ uniformly on the compact sets. We also suppose that H, h and every H^ε, h^ε satisfy the standard assumptions of Sec. 1, i.e. they are 1-periodic in (y,z) and H^ε, H are HJI operators with the regularity in the coefficients suitably extended to the additional variable z.
– The Hamiltonians are equibounded: $|H^\varepsilon(x,y,z,0,0,0)| \le C$, for every x,y,z and ε.
– *Microscopic coercivity* The Hamiltonian H is coercive in p_z: there are $\nu, C \in \mathbb{R}^+$, such that

$$H(x,y,z,p_x,p_y,p_z) \ge \nu|p_z| - C(1+|p_x|+|p_y|) \qquad \forall x,y,z,p_x,p_y,p_z.$$

– *Mesoscopic coercivity* For some $\nu, C \in \mathbb{R}^+$, there holds

$$H(x,y,z,p_x,p_y,0) \ge \nu|p_y| - C(1+|p_x|) \qquad \forall x,y,z,p_x,p_y.$$

It is worth to observe that the first two assumptions ensure that problem (10) admits exactly one continuous bounded solution u^ε that is periodic in (y,z). For instance, the second assumption is guaranteed by the equiboundedness of the running costs while the last two assumptions are satisfied if there is $\nu > 0$ such that, for every (x,y,z,β), there holds

$$B_m(0,\nu) \subset \overline{\text{conv}}\{g(x,y,z,\alpha,\beta) \mid \alpha \in A\}, \tag{11}$$

$$B_p(0,\nu) \subset \overline{\text{conv}}\{r(x,y,z,\alpha,\beta) \mid \alpha \in A\}. \tag{12}$$

We denote by H' the recession function of H with respect to the variables (y,z):

$$H'(x,y,z,p_y,p_z) := \min_{\beta\in B}\max_{\alpha\in A}\left\{-p_y\cdot g(x,y,z,\alpha,\beta) - p_z\cdot r(x,y,z,\alpha,\beta)\right\}.$$

The function H' is positively 1-homogeneous in (p_y,p_z) and for every $(\overline{x},\overline{p}_x)$ there is a constant C such that

$$|H(x,y,z,p_x,p_y,p_z) - H'(x,y,z,p_y,p_z)| \le C \qquad \forall y,p_y \in \mathbb{R}^m,\ z,p_z \in \mathbb{R}^p \tag{13}$$

for every (x,p_x) in a neighborhood of $(\overline{x},\overline{p}_x)$. We introduce also the recession function with respect to z for x and y frozen:

$$H''(x,y,z,p_z) := H'(x,y,z,0,p_z) = \min_{\beta\in B}\max_{\alpha\in A}\left\{-p_z\cdot r(x,y,z,\alpha,\beta)\right\}.$$

We observe that H'' is positively 1-homogeneous in p_z and for every $(\overline{x}, \overline{y}, \overline{p}_x, \overline{p}_y)$ there is a constant C such that

$$|H(x, y, z, p_x, p_y, p_z) - H''(x, y, z, p_z)| \leq C \qquad \forall z, p_z \in \mathbb{R}^p \tag{14}$$

for every (x, y, p_x, p_y) in a neighborhood of $(\overline{x}, \overline{y}, \overline{p}_x, \overline{p}_y)$.

Let us now state some results on the construction of the effective Hamiltonian $\overline{H}$ and the effective initial datum $\overline{h}$. For some special cases, explicit formulas are also provided.

Proposition 4.1. *Under the above assumptions, we have:*

(a) The Hamiltonian H is ergodic in the microscopic variable z. The effective Hamitonian $H_1 = H_1(x, y, p_x, p_y)$ is regular: there are $C \in \mathbb{R}$ and a modulus of continuity ω such that:

$$\begin{aligned}|H_1(x_1, y_1, p, q) - H_1(x_2, y_2, p, q)| &\leq \omega(|x_1 - x_2| + |y_1 - y_2|)\\ &\quad + C(|x_1 - x_2| + |y_1 - y_2|)(1 + |p| + |q|)\end{aligned}$$

$$|H_1(x, y, p_1, q_1) - H_1(x, y, p_2, q_2)| \leq C(|p_1 - p_2| + |q_1 - q_2|)$$

for every $x_i, p_i \in \mathbb{R}^n$, $y_i, q_i \in \mathbb{R}^m$.

(b) H_1 is coercive in p_y; moreover, it is ergodic in y and its effective Hamiltonian $\overline{H}$ fulfills the Comparison Principle.

(c) Assume that for each $(\overline{y}, \overline{p}_y)$ there holds

$$H(\overline{x}, \overline{y}, z, \overline{p}_x, \overline{p}_y, p_z) - H(\overline{x}, \overline{y}, z, \overline{p}_x, \overline{p}_y, 0) \geq 0 \qquad \forall z, p_z \in \mathbb{R}^p, \tag{15}$$

and

$$H(\overline{x}, y, z, \overline{p}_x, p_y, 0) - H(\overline{x}, y, z, \overline{p}_x, 0, 0) \geq 0 \qquad \forall y, p_y \in \mathbb{R}^m, \quad \forall z \in \mathbb{R}^p. \tag{16}$$

Then, $\overline{H}(\overline{x}, \overline{p}_x)$ can be written as

$$\overline{H}(\overline{x}, \overline{p}_x) = \max_{y,z} H(\overline{x}, y, z, \overline{p}_x, 0, 0).$$

(d) [7, Lemma 2] The recession function H' is ergodic in the microscopic variable z; its effective Hamiltonian is the recession function H_1' of H_1.

Proof a) By virtue of the microscopic coercivity and Proposition 2.1-(a), H is ergodic in z with an effective Hamiltonian H_1. The regularity of H_1 is an immediate consequence of Proposition 2.1-(b) (with x and p_x replaced respectively by (x, y) and (p_x, p_y)).

b) In order to prove that H_1 is coercive in p_y, we first observe that the solution w_δ to the microscopic δ-cell problem satisfies: $\delta w_\delta \leq -\inf_z H(x,y,z,p_x,p_y,0)$. As $\delta \to 0$, we get:

$$\inf_z H(x,y,z,p_x,p_y,0) \leq H_1(x,y,p_x,p_y)$$

and, by the mesoscopic coercivity, we deduce that H_1 is coercive in p_y. Applying Proposition 2.1, one obtains the second part of the statement.

c) Proposition 2.1-(*c*) yields: $H_1(x,y,p_x,p_y) = \max_z H(x,y,z,p_x,p_y,0)$. For each $(\overline{x},\overline{p}_x)$, relation (16) ensures:

$$H_1(\overline{x},y,\overline{p}_x,p_y) - H_1(\overline{x},y,\overline{p}_x,0) \geq 0 \qquad \forall y, p_y \in \mathbb{R}^m.$$

Applying again Proposition 2.1-(*c*), we obtain the statement.

d) For the sake of completeness, let us recall the arguments of [7, Lemma 2]. Arguing as in (*a*), one can prove that $H_\lambda(x,y,z,p_y,p_z) := \lambda^{-1}H(x,y,z,0,\lambda p_y,\lambda p_z)$ is ergodic in z with the effective Hamiltonian $\overline{H}_\lambda(x,y,p_y) := \lambda^{-1}H_1(x,y,0,\lambda p_y)$. Since $H_\lambda \to H'$ as $\lambda \to +\infty$ uniformly (for x bounded), by the Comparison Principle on the cell δ-problem, one obtains that H' is ergodic in z with $H_1' := \lim_{\lambda\to+\infty} \lambda^{-1}H_1(x,y,0,\lambda p_y)$ as effective Hamiltonian.

Let us now check that H_1' is the recession function of H_1. We first note that the positive 1-homogeneity of H' entails the one of H_1'. By estimate (13), the Comparison Principle yields:

$$|H_1(x,y,p_x,p_y) - H_1'(x,y,p_y)| \leq C \qquad \forall y, p_y \in \mathbb{R}^m$$

for every (x,p_x) in a neighborhood of $(\overline{x},\overline{p}_x)$, namely H_1' is the recession function of H_1. □

Proposition 4.2. *Under our assumptions, the pair* (H'',h) *is stabilizing at* $(\overline{x},\overline{y})$ *to* $\min_z h(\overline{x},\overline{y},z) =: h_1(\overline{x},\overline{y})$. *Moreover, the pair* (H_1,h_1) *is stabilizing at* $\overline{x}$ *to* $\min_{y,z} h(\overline{x},y,z) =: \overline{h}(\overline{x})$.

The proof of this Proposition relies on the iterative application of Proposition 2.2 and we shall omit it.

Theorem 4.1. *Under the above assumptions, the solution* u^ε *to problem* (10) *converges uniformly on the compact subsets of* $(0,T)\times\mathbb{R}^n$ *to the unique viscosity solution of* ($\overline{\mathrm{HJ}}$) *where the effective Hamiltonian* $\overline{H}$ *and the effective initial datum* $\overline{h}$ *are defined respectively in Proposition 4.1 and 4.2.*

Proof The proof of this Theorem is based on [7, Theorem 2] and on the properties of ergodicity and stabilization established in Proposition 4.1 and 4.2. For the sake of completeness, let us just stress the crucial parts. We shall argue as in the proof of Theorem 3.1: we set

$$\overline{u}(t,x) := \limsup_{\varepsilon\to 0,\ (t',x')\to(t,x)} \sup_{y,z} u^\varepsilon(t',x',y,z) \quad \text{if } t>0,$$
$$\overline{u}(0,x) := \limsup_{(t',x')\to(0,x),\ t'>0} \overline{u}(t',x') \quad \text{if } t=0.$$

and we want to show that $\overline{u}$ is a subsolution to ($\overline{\text{HJ}}$). As before, it suffices to prove that, for every $r>0$ small enough, there is a parameter $\varepsilon'>0$ and an equibounded family of continuous correctors $\{\chi^\varepsilon \mid 0<\varepsilon<\varepsilon'\}$ so that

$$H^\varepsilon(x,y,z,D_x\varphi(t,x),D_y\chi^\varepsilon,\frac{D_z\chi^\varepsilon}{\varepsilon}) \geq \overline{H}-2\eta$$

in $Q_r(\bar t,\overline{x}) := (\bar t - r, \bar t + r)\times B_r(\overline{x})\times\mathbb{R}^m\times\mathbb{R}^p$, for every $\varepsilon<\varepsilon'$. To this aim, we consider the mesoscopic δ-cell problem

$$\delta w_\delta + H_1(\overline{x},y,\overline{p}_x,D_y w_\delta)=0 \tag{17}$$

with $\overline{p}_x := D_x\varphi(\bar t,\overline{x})$. For $\delta>0$ sufficiently small, the ergodicity of H_1 (established in Proposition 4.1-(b)) ensures

$$\|\delta w_\delta + \overline{H}\|_\infty \leq \eta \tag{18}$$

where $\overline{H}$ is defined as before. For every $\varepsilon>0$ and $r>0$, we consider the problem

$$\delta w^\varepsilon_{\delta,r} + H^\varepsilon_r(y,z,D_y w^\varepsilon_{\delta,r},\frac{D_z w^\varepsilon_{\delta,r}}{\varepsilon})=0$$

with $H^\varepsilon_r(y,z,p_y,p_z) := \min_{x\in B_r(\overline{x})} H^\varepsilon(x,y,z,D_x\varphi(x),p_y,p_z)$. We note that $H^\varepsilon_r(y,z,p_y,p_z)\to H(\overline{x},y,z,\overline{p}_x,p_y,p_z)$ locally uniformly as $(\varepsilon,r)\to(0,0)$ and that the limit Hamiltonian H is ergodic in z with effective Hamiltonian H_1. Hence, applying Theorem 3.1 for the stationary eq., we obtain that $\{w^\varepsilon_{\delta,r}\}$ uniformly converge to w_δ as $(\varepsilon,r)\to(0,0)$. By (18), we deduce that there are small ε' and r' so that

$$\|\delta w^\varepsilon_{\delta,r} + \overline{H}\|_\infty \leq 2\eta \qquad \text{for all } 0<\varepsilon<\varepsilon',\ 0<r<r'.$$

Finally, as in Theorem 3.1, it suffices to define $\chi^\varepsilon(y,z) := w^\varepsilon_{\delta,r}(y,z)$ (for $0<r<r'$ fixed).

Now, let us check that $\overline{u}(0,\overline{x}) \le \overline{h}(\overline{x})$. We introduce the following notations:

$$H_r^{\varepsilon,\prime}(y,z,p_y,p_z) := \min_{|x-\overline{x}|\le r} H^{\varepsilon,\prime}(x,y,z,p_y,p_z),$$
$$h_r^{\varepsilon}(y,z) := \max_{|x-\overline{x}|\le r} h^{\varepsilon}(x,y,z).$$

It can be easily checked that, as $(\varepsilon,r)\to(0,0)$, $H_r^{\varepsilon,\prime}$ and h_r^ε converge locally uniformly respectively to $H'(\overline{x},\cdot)$ and to $h(\overline{x},\cdot)$. Let w_r^ε be the unique solution of the Cauchy problem

$$\begin{aligned}&\partial_t w_r^\varepsilon + H_r^{\varepsilon,\prime}(y,z,D_y w_r^\varepsilon, \varepsilon^{-1}D_z w_r^\varepsilon) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m\times\mathbb{R}^p\\ &w_r^\varepsilon(0,y,z) = h_r^\varepsilon(y,z), \quad \text{on } \mathbb{R}^m\times\mathbb{R}^p, \quad w_r^\varepsilon \text{ periodic in } y \text{ and in } z.\end{aligned}$$

By Proposition 2.1-(d) and Proposition 4.2, the limit Hamiltonian $H'(\overline{x},\cdot)$ is ergodic with effective Hamiltonian H_1' and the pair (H'',h) stabilizes with respect to the microscopic variable z. Thus, Theorem 3.1 ensures that, as $(\varepsilon,r)\to(0,0)$, the solution w_r^ε converges locally uniformly to the one of the mesoscopic cell Cauchy problem

$$\begin{aligned}&\partial_t w + H_1'(\overline{x},y,D_y w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m,\\ &w(0,y) = h_1(\overline{x},y) \quad \text{on } \mathbb{R}^m, \quad w \text{ periodic in } y.\end{aligned}$$

Since (H_1',h_1) is stabilizing (still by Proposition 4.2), for every $\eta>0$, there exists $T>0$ such that $\|w(T,\cdot)-\overline{h}(\overline{x})\|_\infty \le \eta$. Hence, for every $\eta>0$ and for T sufficiently large, there exist ε' and r' so small that

$$\|w_r^\varepsilon(T,\cdot,\cdot)-\overline{h}(\overline{x})\|_\infty \le 2\eta \quad \text{for every } \varepsilon\le\varepsilon',\ r\le r'.$$

Therefore, by the Comparison Principle, we obtain:

$$\|w_r^\varepsilon(t,\cdot,\cdot)-\overline{h}(\overline{x})\|_\infty \le \eta \quad \text{for every } \varepsilon\le\varepsilon',\ r\le r',\ t\ge T$$

and we conclude as before. □

Remark 4.1. This Theorem also applies to non coercive Hamiltonian that are microscopically stabilizing and ergodic with a mesoscopic Hamiltonian that fulfills the Comparison Principle, stabilizes, and is ergodic with an effective Hamiltonian that also fulfills the Comparison Principle (e.g., see Sec. 6 below). We refer the reader to the paper[7] for the case when the Comparison Principle fails either for the mesoscopic Hamiltonian or for the effective one.

4.2. *The general case*

We consider the problem having $j+1$ scales:

$$\begin{cases} \partial_t u^\varepsilon + H^\varepsilon\left(x, y_1, \ldots, y_j, D_x u^\varepsilon, \varepsilon^{-1} D_{y_1} u^\varepsilon, \ldots, \varepsilon^{-j} D_{y_j} u^\varepsilon\right) = 0 \\ \qquad\qquad\qquad\qquad (0,T) \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_j} \\ u^\varepsilon(0, x, y_1, \ldots, y_j) = h^\varepsilon(x, y_1, \ldots, y_j) \qquad \mathbb{R}^n \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_j}. \end{cases} \tag{19}$$

We assume the following hypotheses:

- $H^\varepsilon \to H$ and $h^\varepsilon \to h$ locally uniformly as $\varepsilon \to 0$; these functions are periodic in $(y_1, \ldots, y_j)$;
- the initial data h^ε and h are BUC;
- the H^ε are equibounded: $|H^\varepsilon(x, y_1, \ldots, y_j, 0, 0, \ldots, 0)| \le C$;
- *iterated coercivity:* there are $\nu, C \in \mathbb{R}^+$, such that, for every $k = j, \ldots, 1$, there holds

$$H(x, y_1, \ldots, y_j, p_x, p_{y_1}, \ldots, p_{y_k}, 0, \ldots, 0) \ge \nu |p_{y_k}| - C\left(1 + |p_x| + \sum_{i=1}^{k-1} |p_{y_i}|\right)$$

  for every $x, y_1, \ldots, y_j, p_x, p_{y_1}, \ldots, p_{y_k}$;
- there exists a recession function $H^{\varepsilon,\prime} = H^{\varepsilon,\prime}(x, y_1, \ldots, y_j, p_{y_1}, \ldots, p_{y_j})$, positively 1-homogeneous in $(p_{y_1}, \ldots, p_{y_j})$, which satisfies, for some constant $C > 0$

$$\left|H^\varepsilon(x, y_1, \ldots, y_j, p_x, p_{y_1}, \ldots, p_{y_j}) - H^{\varepsilon,\prime}(x, y_1, \ldots, y_j, p_{y_1}, \ldots, p_{y_j})\right| \le C$$

  for every $y_i, p_{y_i} \in \mathbb{R}^{m_i}$ $(i = 1, \ldots, j)$, for every (x, p_x) in a neighborhood of $(\overline{x}, \overline{p}_x)$ and for every ε.

Let us observe that the equiboundedness of the running costs ensures the third assumptions; furthermore, the Hamiltonian

$$H(x, y_1, \ldots, y_j, p_x, p_{y_1}, \ldots, p_{y_j}) = \min_\beta \max_\alpha \left\{ -f \cdot p_x - \sum_{i=1}^{j} g_i \cdot p_{y_i} - l \right\}$$

is iteratively coercive whenever there holds

$$B_{m_i}(0, \nu) \subset \overline{\mathrm{conv}}\{g_i(x, y_1, \ldots, y_j, \alpha, \beta) \mid \alpha \in A\}$$

for every $x, y_1, \ldots, y_j, \beta$ $(i = 1, \ldots, j)$.

We write $H_j = H$ and $h_j = h$. For each $i = j, \ldots, 1$, H_i fulfills the Comparison Principle and is ergodic with respect to y_i. We denote by H_{i-1} its effective Hamiltonian. We set $\overline{H} := H_0$.

As before, one can prove that the Hamiltonian H has a recession function H' which is the uniform limit on the compact sets of $H^{\varepsilon,\prime}$ as $\varepsilon \to 0$. Moreover, as in Proposition 4.1-(d), we obtain that H_i has a recession function H_i', that every H_i' is ergodic and that its effective Hamiltonian is H_{i-1}' (for every $i = j, \dots, 1$).

For $H_i'' := H_i'(x, y_1, \dots, y_i, 0, \dots, 0, p_i)$, the pair (H_i'', h_i) is stabilizing with respect to y_i at each point $(x, y_1, \dots, y_{i-1})$ (for $i = j, \dots, 1$). We denote by h_{i-1} its effective initial data and we put $\overline{h} = h_0$.

Theorem 4.2. *Under the above assumptions, u^ε converges uniformly on the compact subsets of $(0,T) \times \mathbb{R}^n$ to the unique viscosity solution of* $(\overline{\mathrm{HJ}})$.

We shall omit the proof if this Theorem: actually, it can be easily obtained by using iteratively the arguments followed in Theorem 4.1.

Remark 4.2. This result can be immediately extended to pde with non power-like scales:

$$\partial_t u^\varepsilon + H^\varepsilon(x, y_1, \dots, y_j, D_x u^\varepsilon, \varepsilon_1^{-1} D_{y_1} u^\varepsilon, \dots, \varepsilon_j^{-1} D_{y_j} u^\varepsilon) = 0,$$

with $\varepsilon_1 \to 0$ and $\varepsilon_i/\varepsilon_{i-1} \to 0$ ($i = 2, \dots, j$). The above eq. encompasses (19) for $\varepsilon_i = \varepsilon^i$.

5. Iterated homogenization for coercive equations

In this Section we address the study of the iterated homogenization of first order equations with multiple scales. For the sake of simplicity, we shall focus our attention on the three-scale case:

$$\begin{cases} \partial_t v^\varepsilon + F^\varepsilon\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D_x v^\varepsilon\right) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ v^\varepsilon(0,x) = h^\varepsilon(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) & \text{on } \mathbb{R}^n. \end{cases} \tag{20}$$

The operator $F^\varepsilon = F^\varepsilon(x,y,z,p)$ and the function $h^\varepsilon = h^\varepsilon(x,y,z)$ are periodic in y and in z; moreover, they are respectively a regular perturbation of F and h, namely, $F^\varepsilon \to F$ and $h^\varepsilon \to h$ locally uniformly as $\varepsilon \to 0$. We assume that F is a HJI operator

$$F(x,y,z,p_x) := \min_{\beta \in B} \max_{\alpha \in A} \{-p_x \cdot f(x,y,z,\alpha,\beta) - l(x,y,z,\alpha,\beta)\}$$

where the drift f and the costs l and h fulfill the requirements stated in Sec. 1. We assume also that F is coercive with respect to p: for $\nu, C \in \mathbb{R}^+$ there holds

$$F(x,y,z,p) \geq \nu|p| - C \qquad \forall x,y,z,p; \tag{21}$$

(for instance, this condition holds if $B_n(0,\nu) \subset \overline{\text{conv}}\{f(x,y,z,\alpha,\beta) \mid \alpha \in A\}$ for every (x,y,β)).

Let us emphasize that problem (20) encompasses the problem studied by Lions, Papanicolaou and Varadhan.[10] Actually, we extend the previous literature in two directions: we consider regular perturbations F^ε and h^ε of F and of h and, mainly, we address the three-scale problem both for the HJI eq. and for the initial condition.

Our purpose is to apply Theorem 4.1 by proving that (20) is a particular case of (10). To this aim, we introduce the shadow variables $y = x/\varepsilon$ and $z = x/\varepsilon^2$ and consider the solution $u^\varepsilon(t,x,y,z)$ of (10) with the Hamiltonian H given by

$$H(x,y,z,p_x,p_y,p_z) = F(x,y,z,p_x+p_y+p_z).$$

The Hamiltonian H clearly satisfies the assumptions of Sec. 4. By uniqueness, one sees immediately that

$$v^\varepsilon(t,x) = u^\varepsilon(t,x,x/\varepsilon,x/\varepsilon^2).$$

By the periodicity in y, Theorem 4.1 ensures that u^ε converges uniformly on compact subsets to the unique solution of ($\overline{\text{HJ}}$). Therefore, the following result holds:

Corollary 5.1. *Under the above assumptions, v^ε converges uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n$ to the unique viscosity solution of* ($\overline{\text{HJ}}$).

Remark 5.1. Arguing as in the Subsec. 4.2, one can easily extend this result to the homogenization with an arbitrary number of scales.

6. Examples

In this Section we discuss some examples arising in the optimal control theory and in deterministic games. For simplicity, we shall only address three-scale problems.

The first Subsection is devoted to a singular perturbation of a deterministic game; in some special cases, the effective problem is still a deterministic game and we shall provide the explicit formulas for the effective quantities (dynamics, pay-off, etc.). The second Subsection concerns the homogenization of a deterministic optimal control problem. In the third Subsection the coercivity of the Hamiltonian is replaced by a non-resonance condition introduced by Arisawa and Lions.[30] Here we show that the effective problem may change if the roles of ε and ε^2 are exchanged.

6.1. *Singular perturbation of a differential game*

Fix $T > 0$ and, for each $\varepsilon > 0$, consider the dynamics

$$\begin{aligned} \dot{x}_s &= f(x_s, y_s, z_s, \alpha_s, \beta_s), \qquad & x_0 = x, \\ \varepsilon \dot{y}_s &= g(x_s, y_s, z_s, \alpha_s, \beta_s), \qquad & y_0 = y, \\ \varepsilon^2 \dot{z}_s &= r(x_s, y_s, z_s, \alpha_s, \beta_s), \qquad & z_0 = z. \end{aligned}$$

for $0 \leq s \leq T$. The admissible controls α_s and β_s are measurable function with value respectively in the compact sets A and B. They are governed by two different players. We consider the cost functional

$$P^\varepsilon(t, x, y, z, \alpha, \beta) := \int_0^t l(x_s, y_s, z_s, \alpha_s, \beta_s)\, ds + h(x_t, y_t, z_t).$$

The goal of the first player controlling α is to minimize P^ε, wheras the second player wishes to maximize P^ε by controlling β.

Consider the *upper* value function

$$u^\varepsilon(t, x, y, z) := \sup_{\beta \in B(t)} \inf_{\alpha \in \mathcal{A}(t)} P^\varepsilon(t, x, y, z, \alpha, \beta[\alpha]),$$

where $\mathcal{A}(t)$ denotes the set of admissible controls of the first player in the interval $[0, t]$ and $B(t)$ denotes the set of admissible strategies of the second player in the same interval (i.e., nonanticipating maps from $\mathcal{A}(t)$ into the admissible controls of the second player; see[31] for the precise definition).

Under the assumptions of Sec. 1 the upper value function is the unique viscosity solution of the HJI eq. (10) with[31]

$$H^\varepsilon = H(x, y, z, p_x, p_y, p_z) = \max_{\alpha \in A} \min_{\beta \in B} \left\{ -p_x \cdot f - p_y \cdot g - p_z \cdot r - l \right\}.$$

Let us assume (11)–(12), so the Hamitonian H is microscopically and mesoscopically coercive. Suppose in addition that H has the properties (15)–(16). By Theorem 4.1, the upper value u^ε converge locally uniformly to the solution u to the effective problem

$$\begin{cases} \partial_t u + \max\limits_{y,z,\alpha} \min\limits_{\beta} \left\{ -D_x u \cdot f(x, y, z, \alpha, \beta) - l(x, y, z, \alpha, \beta) \right\} = 0, \\ u(0, x) = \min\limits_{y,z} \{ h(x, y, z) \}. \end{cases}$$

Using again the theory of Evans and Souganidis,[31] one can see that u is the upper value of the following effective deterministic differential game. The effective dynamics are

$$\dot{x}_s = f(x_s, y_s, z_s, \alpha_s, \beta_s), \qquad x_0 = x,$$

where y and z are new controls, while the effective cost is

$$P(t,x,y,z,\alpha,\beta) := \int_0^t l(x_s,y_s,z_s,\alpha_s,\beta_s)\,ds + \min_{y,z} h(x_t,y,z).$$

In the effective game the first player wants to minimize P by choosing the controls y, z, α, and the second player wants to maximize P by choosing β.

We end this Subsection by giving a simple condition on the control system that implies (15)–(16). Suppose the controls of the first player are in the separated form $\alpha = \left(\alpha^S, \alpha^F, \alpha^V\right) \in A^S \times A^F \times A^V$, where α^S is the control of the slow variables x, α^F of the fast variables y, and α^V is the control of the very fast variables z. More precisely

$$f = f\left(x,y,z,\alpha^S,\beta\right), \qquad l = l\left(x,y,z,\alpha^S,\beta\right),$$
$$g = g\left(x,y,z,\alpha^F,\beta\right), \qquad r = r\left(x,y,z,\alpha^V,\beta\right).$$

In this case the conditions (11)–(12) become

$$B_m(0,\nu) \subset \overline{\text{conv}}\{g(x,y,z,\alpha^F,\beta) \mid \alpha^F \in A^F\},$$
$$B_p(0,\nu) \subset \overline{\text{conv}}\{r(x,y,z,\alpha^V,\beta) \mid \alpha^V \in A^V\}$$

for every x, y, z, β. Then it is easy to check that (15)–(16) hold, and the effective Hamiltonian is

$$H(x,p_x) = \max_{(y,z)\in[0.1]^2} \max_{\alpha^S\in A^S} \min_{\beta\in B} \left\{-p_x \cdot f\left(x,y,z,\alpha^S,\beta\right) - l\left(x,y,z,\alpha^S,\beta\right)\right\}.$$

Note that in the further special case

$$g = \varphi_1(x,y,z)\alpha^F, \quad A^F = B_m(0,1), \quad \varphi_1(x,y,z) \geq \nu > 0,$$
$$r = \varphi_2(x,y,z)\alpha^V, \quad A^V = B_p(0,1), \quad \varphi_2(x,y,z) \geq \nu > 0,$$

the PDE in (10) becomes the model problem (1) presented in the Introduction.

6.2. *Homogenization of a deterministic optimal control problem*

For each $\varepsilon > 0$, we consider dynamics having the form

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s\right), \qquad x_0 = x$$

for $0 \leq s \leq T$. The admissible controls $\alpha.$ are measurable function with value in the compact A. We denote by $\mathcal{A}(t)$ the set of admissible controls on the interval $(0,t)$. Our goal is to choose the control in order to minimize the payoff functional:

$$P^\varepsilon(t,x,\alpha) := \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s\right)\, ds + h\left(x_t, \frac{x_t}{\varepsilon}, \frac{x_t}{\varepsilon^2}\right).$$

By standard theory[9] the value function

$$v^\varepsilon(t,x) := \inf_{\alpha\in\mathcal{A}(t)} P^\varepsilon(t,x,y,\alpha),$$

is the unique viscosity solution of the HJ eq. (20), provided that f, l and h fulfill the assumptions of Sec. 1. Moreover, let us recall that the Hamitonian F is coercive provided that

$$B_n(0,\nu) \subset \overline{\text{conv}}\{f(x,y,z,\alpha) \mid \alpha \in A\} \qquad \text{for all } (x,y,z).$$

Under these assumptions, Corollary 5.1 ensures that v^ε converges locally uniformly to the solution of problem ($\overline{\text{HJ}}$). Here $\overline{h}(x) = \min_{y,z} h(x_t, y, z)$, but $\overline{H}$ does not have an explicit representation.

Note that the eikonal equation (2) is a special case of this example. It is enough to take

$$A = B_n(0,1), \qquad f(x,y,z,\alpha) = \varphi(x,y,z)\alpha, \qquad \varphi(x,y,z) \geq \nu > 0.$$

6.3. *Multiscale singular perturbation under a nonresonance condition*

This Subsection is devoted to the case of a nonresonance condition, introduced by Arisawa and Lions[30] (see also the first two authors[6]), that ensures the ergodicity for a class of non-coercive Hamiltonian with an effective operator that fulfills the Comparison Principle and can be written explicitly.

As a byproduct, we show that the roles of ε and ε^2 can not be exchanged in general. To this aim, we first consider a three scales perturbation problem that is nonresonant in the microscopic variable z and coercive in the mesoscopic variable y, and then a problem that is coercive in z and nonresonant in y. The two effective Hamiltonians are different.

<u>1st case</u>: For each $\varepsilon > 0$, consider the dynamics

$$\dot{x}_s = f(x_s, y_s, z_s, \alpha_s^S), \qquad \dot{y}_s = \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s^F), \qquad \dot{z}_s = \frac{1}{\varepsilon^2} r(x_s, y_s)$$

for $0 \leq s \leq T$ with initial conditions

$$x_0 = x, \qquad y_0 = y, \qquad z_0 = z.$$

The admissible control $\alpha = (\alpha^S, \alpha^F)$ splits in two control: one for the *slow* and one for the *fast* variable (the superscript recall this fact). Furthermore, α^S_s and α^F_s are measurable functions with value respectively in the compacts A^S and A^F. By the choice of α, one wants to minimize the *payoff* functional:

$$P^\varepsilon(t,x,y,z,\alpha) := \int_0^t l(x_s,y_s,z_s,\alpha^S_s)\,ds + h(x_t,y_t).$$

Then the value function u^ε solves the Cauchy problem:

$$\begin{cases} \partial_t u^\varepsilon + \max_{\alpha^S\in A^S}\{-D_x u^\varepsilon\cdot f(x,y,z,\alpha^S) - l(x,y,z,\alpha^S)\} + \\ \qquad\qquad \max_{\alpha^F\in A^F}\{-\frac{1}{\varepsilon}D_y u^\varepsilon\cdot g(x,y,\alpha^F)\} - \frac{1}{\varepsilon^2} r(x,y)\cdot D_z u^\varepsilon = 0 \\ u^\varepsilon(0,x,y,z) = h(x,y) \end{cases}$$

We require that the microscopic dynamic is nonresonant and that the mesoscopic one is coercive, namely:

$$r(x,y)\cdot k \neq 0\ \forall k\in\mathbb{Z}^p\setminus\{0\}, \quad B_m(0,\nu)\subset \overline{\mathrm{conv}}\{g(x,y,\alpha^F)\mid \alpha^F\in A^F\}$$

for every (x,y). The Hamiltonian H is ergodic in the fast variable[6,30] and the mesoscopic Hamiltonian H_1 has the form:

$$H_1(x,y,p_x,p_y) = \int_{(0,1)^p} \max_{\alpha^S\in A^S}\{-p_x\cdot f(x,y,z,\alpha^S) - l(x,y,z,\alpha^S)\}\,dz + \max_{\alpha^F\in A^F}\{-p_y\cdot g(x,y,\alpha^F)\}.$$

One can easily check that H_1 fulfills the Comparison Principle, is ergodic with respect to y and satisfies (4). Therefore Theorem 4.1 ensures that u^ε converge locally uniformly to the solution of problem ($\overline{\mathrm{HJ}}$) where the effective quantities are given by:

$$\overline{H}(x,p_x) = \max_y \int_{(0,1)^p} \max_{\alpha^S\in A^S}\{-p_x\cdot f(x,y,z,\alpha^S) - l(x,y,z,\alpha^S)\}\,dz$$

$$\overline{h}(x) = \min_y h(x,y).$$

<u>2^{nd} case</u>: For each $\varepsilon > 0$, consider the dynamics

$$\dot{x}_s = f(x_s,y_s,z_s,\alpha_s), \qquad \dot{y}_s = \frac{1}{\varepsilon}g(x_s), \qquad \dot{z}_s = \frac{1}{\varepsilon^2}r(x_s,y_s,z_s,\alpha_s)$$

for $0\le s\le T$ with initial conditions

$$x_0 = x, \qquad y_0 = y, \qquad z_0 = z.$$

The control α is a measurable function with value in the compact A. As before, by the choice of α, one wants to minimize the payoff:

$$P^\varepsilon(t,x,y,z,\alpha) := \int_0^t l(x_s,y_s,z_s,\alpha_s)\,ds + h(x_t,z_t).$$

The value function u^ε is the viscosity solution of the following Cauchy problem:

$$\begin{cases} \partial_t u^\varepsilon + \max\limits_{\alpha\in A}\{-D_x u^\varepsilon \cdot f(x,y,z,\alpha) - \frac{1}{\varepsilon^2} D_z u^\varepsilon \cdot r(x,y,z,\alpha) - l(x,y,z,\alpha)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{\varepsilon} D_y u^\varepsilon \cdot g(x) = 0 \\ u^\varepsilon(0,x,y,z) = h(x,z) \end{cases}$$

Assume that the dynamics are microscopically coercive and mesoscopically nonresonant:

$$g(x)\cdot k \neq 0 \quad \forall k \in \mathbb{Z}^m \setminus \{0\}, \qquad B_p(0,\nu) \subset \overline{\mathrm{conv}}\{r(x,y,z,\alpha) \mid \alpha \in A\}$$

for every (x,y,z). We require also that condition (15) is satisfied. Hence, H is microscopically ergodic and stabilizing and the mesoscopic quantities are given by:

$$\begin{aligned} H_1(x,y,p_x,p_y) &= \max_{\alpha,z}\{-p_x\cdot f(x,y,z,\alpha) - l(x,y,z,\alpha)\} - p_y\cdot g(x) \\ h_1(x,y) &= \min_z h(x,z). \end{aligned}$$

Arguing as before, one can prove that the H_1 is ergodic, stabilizes, and fulfills the Comparison Principle. Whence, by Theorem 4.1, the value function u^ε converge locally uniformly to the solution of $(\overline{\mathrm{HJ}})$ with:

$$\begin{aligned} \overline{H}(x,p_x) &= \int_{(0,1)^m} \max_{\alpha,z}\{-p_x\cdot f(x,y,z,\alpha) - l(x,y,z,\alpha)\}\,dy \\ \overline{h}(x) &= \min_z h(x,z). \end{aligned}$$

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