CONVERGENCE OF EQUILIBRIA OF THREE-DIMENSIONAL THIN ELASTIC BEAMS

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Abstract. A convergence result is proved for the equilibrium configurations of a three-dimensional thin elastic beam, as the diameter h of the cross-section goes to zero. More precisely, we show that stationary points of the nonlinear elastic functional E^h, whose energies (per unit cross-section) are bounded by Ch^2, converge to stationary points of the Γ-limit of E^h/h^2. This corresponds to a nonlinear one-dimensional model for inextensible rods, describing bending and torsion effects. The proof is based on the rigidity estimate for low-energy deformations by Friesecke, James, and Müller [4] and on a compensated compactness argument in a singular geometry. In addition, possible concentration effects of the strain are controlled by a careful truncation argument.

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1. Introduction and main result

In this paper we extend our previous work with M.G. Schultz on the convergence of equilibria of planar thin elastic beams (see [10]) to the case of three-dimensional thin beams.

To set the stage let $h > 0$ and let S be a bounded open connected subset of $\mathbb{R}^2$ with Lipschitz boundary. We consider a thin beam whose reference configuration is given by the open set $\Omega_h = (0,L)\times hS$. Given any deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$, we define the elastic energy (per unit cross-section) associated to v as

$$E^h(v) := \frac{1}{h^2}\int_{\Omega_h} W(\nabla v)\,dz.$$

The stored-energy density function $W : \mathbb{M}^{3\times 3} \to [0,+\infty]$ is assumed to satisfy the following conditions:

(h1) frame indifference: $W(RF) = W(F)$ for every $R \in SO(3)$ and $F \in \mathbb{M}^{3\times 3}$;

(h2) $W = 0$ on $SO(3)$;

(h3) $W(F) \geq c\,\mathrm{dist}^2(F, SO(3))$, $c > 0$, for every $F \in \mathbb{M}^{3\times 3}$;

(h4) W is of class C^2 in a neighbourhood of $SO(3)$.

Here $SO(3)$ denotes the group of proper rotations. The frame indifference implies that there exists a function $\tilde{W}$ defined on symmetric matrices such that $W(\nabla v) = \tilde{W}((\nabla v)^T\nabla v)$; i.e., the elastic energy depends only on the pull-back metric of v.

To discuss the limiting behaviour of E^h, as $h \to 0$, it is convenient to rescale to a fixed domain $\Omega = (0,L)\times S$ by the change of variables

$$z = (x_1, hx_2, hx_3) \quad \text{and} \quad y(x) = v(z(x)).$$

With the notation

$$\nabla_h y = (\partial_1 y | \tfrac{1}{h}\partial_2 y | \tfrac{1}{h}\partial_3 y)$$

we can write the elastic energy as

$$E^h(v) = I^h(y) := \int_\Omega W(\nabla_h y)\, dx.$$

Without loss of generality we can assume that $\mathcal{L}^2(S) = 1$ and that the segment $(0,L)\times\{0\}\times\{0\}$ is a line of centroids for the beam; i.e.,

$$\int_S x_2\, dx_2 dx_3 = \int_S x_3\, dx_2 dx_3 = \int_S x_2 x_3\, dx_2 dx_3 = 0. \tag{1.1}$$

Under the previous assumptions it is possible to identify a complete hierarchy of limiting rod theories, depending on the scaling of I^h, by means of Γ-convergence. More precisely, for every $\beta \geq 0$ we have

$$\frac{1}{h^\beta} I^h \xrightarrow{\Gamma} I_\beta, \tag{1.2}$$

where, according to β, the functional I_β describes a different elastic model for rods. The Γ-convergence for $\beta = 0$ was proved by Acerbi, Buttazzo, and Percivale in [1], leading to a *nonlinear string model*. The scaling $\beta = 2$, which corresponds to a *nonlinear rod model*, has been studied in [8] and independently by Pantz in [12]. The result for $\beta = 4$ has been proved in [9], while for the other scalings Γ-convergence can be easily derived from [8] and [9].

The Γ-convergence results (1.2) guarantee that if $(y^{(h)})$ is a compact sequence of minimizers of I^h (with respect to some boundary conditions or body forces) such that $I^h(y^{(h)}) \leq Ch^\beta$, then, up to subsequences, $(y^{(h)})$ converges to a minimizer of I_β (for a comprehensive introduction to Γ-convergence we refer to [2]).

In this paper we deal with the problem of the convergence of *equilibria* in the scaling $\beta = 2$. In this case the natural class of admissible functions for the limit problem turns out to be

$$\begin{aligned} \mathcal{A} := \big\{ (y, d_2, d_3) \in W^{2,2}((0,L);\mathbb{R}^3)\times W^{1,2}((0,L);\mathbb{R}^3)\times W^{1,2}((0,L);\mathbb{R}^3) : \\ R := (y'|d_2|d_3) \in SO(3) \text{ a.e. in } (0,L) \big\}. \end{aligned} \tag{1.3}$$

On this class the Γ-limit functional is given by

$$I_2(y, d_2, d_3) := \frac{1}{2}\int_0^L Q_1(R^T R')\, dx_1, \tag{1.4}$$

where $R := (y'|d_2|d_3)$. The density Q_1 is a quadratic form on the space $\mathbb{M}^{3\times 3}_{skw}$ of skew-symmetric matrices, defined as

$$Q_1(A) := \min_{\alpha \in W^{1,2}(S;\mathbb{R}^3)} \int_S Q_3\Big(x_2 A e_2 + x_3 A e_3 \Big|\, \partial_2 \alpha \,\Big|\, \partial_3 \alpha\Big)\, dx_2 dx_3 \tag{1.5}$$

for every $A \in \mathbb{M}^{3\times 3}_{skw}$, where Q_3 is the quadratic form $Q_3(F) := \mathcal{L}F : F$ and $\mathcal{L}$ is the linear map on $\mathbb{M}^{3\times 3}$ given by $\mathcal{L} := D^2 W(Id)$.

In this limit model the function y represents the deformation of the mid-fiber of the rod, which has to be isometric because of the constraint $|y'| = 1$. The two Cosserat vectors d_2 and d_3 determine the rotation undergone by the cross-section of the rod at each point of the mid-fiber. We remark also that, as R belongs to $SO(3)$ a.e., the matrix $R^T R'$ is skew-symmetric. Moreover, the entries $(R^T R')_{1j}$ for $j = 2, 3$ are related to the curvature of the deformed mid-fiber, while $(R^T R')_{23}$ is related to the torsion of the mid-fiber and to the twist of the cross-section, after the deformation. Finally, the solutions to (1.5) with A replaced by $R^T(x_1)R'(x_1)$ describe the warping of the cross-section with respect to the normal plane (see [8]).

If, in addition, W is isotropic and S is a disc, then the quadratic form Q_1 can be explicitly computed and reduces to

$$Q_1(A) := \frac{1}{2\pi}\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}(A_{12}^2 + A_{13}^2) + \frac{\mu}{2\pi}A_{23}^2,$$

where λ and μ are the Lamé coefficients of the rod (see [8, Remark 3.5]).

We assume the beam to be subject to a body force of density $h^2 g$, with $g \in L^2((0, L); \mathbb{R}^3)$; thus, we consider the functionals

$$J^h(y) = \int_\Omega (W(\nabla_h y) - h^2 g(x_1) \cdot y)\, dx. \tag{1.6}$$

The corresponding Γ-limit at scale h^2 is then given by

$$J_2(y, d_2, d_3) = I_2(y, d_2, d_3) - \int_0^L g \cdot y\, dx_1 \tag{1.7}$$

if $(y, d_2, d_3) \in \mathcal{A}$, while J_2 takes the value $+\infty$ if $(y, d_2, d_3) \notin \mathcal{A}$ (here we took the liberty to identify maps on Ω which are independent of x_2, x_3 with maps on $(0, L)$). It is convenient to fix one end of the rod by requiring, e.g., $y(0) = 0$ and $d_k(0) = e_k$ for $k = 2, 3$, where $\{e_1, e_2, e_3\}$ denotes the canonical basis in $\mathbb{R}^3$.

We are now in a position to state the main theorem of the paper.

Theorem 1.1. *Assume that (h1)–(h4) are satisfied and that W is differentiable with globally Lipschitz derivative DW. Let $g \in L^2((0, L); \mathbb{R}^3)$. Let $(y^{(h)})$ be a sequence of stationary points of J^h, subject to the boundary condition $y^{(h)}(0, x_2, x_3) = (0, hx_2, hx_3)$ at $x_1 = 0$ and to natural boundary conditions on the remaining boundaries. Assume further that there exists a constant $C > 0$ such that*

$$\int_\Omega W(\nabla_h y^{(h)})\, dx \le Ch^2 \tag{1.8}$$

for every h. Then, up to subsequences,

$$y^{(h)} \to \bar{y} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^3), \tag{1.9}$$

$$\tfrac{1}{h}\partial_k y^{(h)} \to \bar{d}_k \quad \text{in } L^2(\Omega; \mathbb{R}^3), \quad k = 2, 3, \tag{1.10}$$

where $(\bar{y}, \bar{d}_2, \bar{d}_3) \in \mathcal{A}$ is a stationary point of

$$J_2(y, d_2, d_3) = \frac{1}{2}\int_0^L Q_1(R^T R')\, dx_1 - \int_0^L g \cdot y\, dx_1$$

with respect to the boundary conditions $y(0) = 0$, $d_k(0) = e_k$ for $k = 2, 3$, and natural boundary conditions at $x_1 = L$.

Remark 1.2. An easy application of the Poincaré inequality shows that the estimate (1.8) holds automatically for minimizers.

Remark 1.3. In [7] Mielke used a centre manifold approach to compare solutions in a thin strip to a $1d$ problem. His approach gives a comparison already for finite h, but it requires that the nonlinear strain $(\nabla_h y)^T \nabla_h y$ is close to the identity in $C^{0,\alpha}$ (and applied forces g cannot be easily included).

In the case of planar thin beams the Euler-Lagrange equations corresponding to the limit functional J_2 can be expressed in terms of a single ODE in the variable θ, describing the angle of the tangent vector to the deformed mid-fiber with respect to a fixed direction. One of the major differences in the case of three-dimensional thin beams is that the limiting Euler-Lagrange equations involve both a linear system of PDEs in the cross-section and a system of ODEs in terms of the bending moments of the rod (see Section 2). This requires an extra work in all the derivation argument.

However, the main ingredients of the proof of Theorem 1.1 remain basically the same as in the planar case discussed in [10]. First the quantitative rigidity estimate in [4] is used to define suitable strain-like and stress-like variables $G^{(h)}$ and $E^{(h)}$, which are almost curl-free and divergence-free (see Steps 2 and 3). Then we can argue in the spirit of the theory of compensated compactness, developed by Murat and Tartar [11, 14, 15], to obtain strong compactness of the stress $E^{(h)}$. This then allows us to pass to the limit in the Euler-Lagrange equations (see Step 7).

To rule out possible concentration effects of the strain a careful truncation argument for gradients in thin domains is employed (see Lemma 4.3). We emphasize that in the planar case this result can be proved using a simple extension argument by successive reflection, while in the $3d$ case an appropriate choice of the extension operator is needed.

2. Preliminary results

The aim of this section is to derive the Euler-Lagrange equations for the functional J_2 introduced in the previous section.

We begin by collecting some properties of the minimum problem (1.5) defining the limit density Q_1. Using Korn's inequality and the direct method of the calculus of variations it is easy to see that problem (1.5) has a solution. Moreover, there exists a unique minimizer belonging to the class

$$\mathcal{B} := \Big\{ \alpha \in W^{1,2}(S) : \int_S \alpha \, dx_2 dx_3 = \int_S \partial_2 \alpha \, dx_2 dx_3 = \int_S \partial_3 \alpha \, dx_2 dx_3 = 0 \Big\}$$

(see [8, Remark 3.4]). The Euler-Lagrange equations for problem (1.5) are computed in the next lemma.

Lemma 2.1. *Let $A \in \mathbb{M}^{3\times 3}_{skw}$ and let $F_A : W^{1,2}(S;\mathbb{R}^3) \to [0,+\infty)$ be the functional defined by*

$$F_A(\alpha) := \int_S Q_3 \Big(x_2 A e_2 + x_3 A e_3 \Big| \, \partial_2 \alpha \, \Big| \, \partial_3 \alpha \Big) \, dx_2 dx_3 \tag{2.1}$$

for every $\alpha \in W^{1,2}(S;\mathbb{R}^3)$. Then a function $\alpha \in \mathcal{B}$ is the minimizer of F_A if and only if the function $E : S \to \mathbb{M}^{3\times 3}$ given by

$$E := \mathcal{L}\Big(x_2 A e_2 + x_3 A e_3 \Big| \, \partial_2 \alpha \, \Big| \, \partial_3 \alpha \Big)$$

satisfies (in a weak sense) the boundary value problem

$$\begin{cases} \operatorname{div}_{x_2,x_3}(Ee_2\,|\,Ee_3) = 0 & \text{in } S, \\ (Ee_2\,|\,Ee_3)\,\nu_{\partial S} = 0 & \text{on } \partial S, \end{cases} \tag{2.2}$$

where $\nu_{\partial S}$ is the outer unit normal to ∂S. Moreover, the minimizer depends linearly on the entries of A.

Proof. As F_A is a convex functional, a function $\alpha \in \mathcal{B}$ minimizes F_A if and only if it satisfies

$$\int_S \mathcal{L}\Big(x_2 Ae_2 + x_3 Ae_3 \Big|\, \partial_2\alpha \,\Big|\, \partial_3\alpha\Big) : \Big(0\,|\,\partial_2\beta\,|\,\partial_3\beta\Big)\,dx_2dx_3 = 0$$

for every $\beta \in W^{1,2}(S;\mathbb{R}^3)$, which is equivalent to (2.2). The linear dependence of α on the entries of A follows directly from the equation (2.2). □

Remark 2.2. Let $(y,d_2,d_3) \in \mathcal{A}$, let $R := (y'\,|\,d_2\,|\,d_3)$, and let $A := R^TR'$. For every $x_1 \in (0,L)$ let $\alpha(x_1,\cdot) \in \mathcal{B}$ be the minimizer of (1.5) with A replaced by $A(x_1)$. Since $\alpha(x_1,\cdot)$ depends linearly on $A(x_1)$ by Lemma 2.1 and $A \in L^2((0,L);\mathbb{M}^{3\times 3}_{skw})$, we conclude that $\alpha \in L^2(\Omega;\mathbb{R}^3)$ and $\partial_k\alpha \in L^2(\Omega;\mathbb{R}^3)$ for $k = 2,3$.

The next lemma is concerned with the derivation of the Euler-Lagrange equations for the functional J_2. The stationary condition for a triple $(y,d_2,d_3) \in \mathcal{A}$ satisfying the boundary conditions will be expressed in terms of the bending moments $\tilde{E}$ and $\hat{E}$ defined below. Let $E : \Omega \to \mathbb{M}^{3\times 3}$ be the *stress* corresponding to the deformation (y,d_2,d_3), defined by

$$E(x) := \mathcal{L}\Big(x_2A(x_1)e_2 + x_3A(x_1)e_3 \Big|\, \partial_2\alpha(x) \,\Big|\, \partial_3\alpha(x)\Big), \tag{2.3}$$

where $A := R^TR'$, $R := (y'\,|\,d_2\,|\,d_3)$, and $\alpha \in L^2((0,L);\mathbb{R}^3)$ is such that $\alpha(x_1,\cdot) \in \mathcal{B}$ solves (1.5), with A replaced by $A(x_1)$, for a.e. $x_1 \in (0,L)$. We call the *bending moments* associated with the deformation (y,d_2,d_3) the functions $\tilde{E} : (0,L) \to \mathbb{M}^{3\times 3}$ and $\hat{E} : (0,L) \to \mathbb{M}^{3\times 3}$ given by

$$\tilde{E}(x_1) := \int_S x_2E(x)\,dx_2dx_3, \qquad \hat{E}(x_1) := \int_S x_3E(x)\,dx_2dx_3$$

for every $x_1 \in (0,L)$.

Lemma 2.3. *Let $(y,d_2,d_3) \in \mathcal{A}$ be such that $y(0) = 0$ and $d_k(0) = e_k$ for $k = 2,3$. Then (y,d_2,d_3) is a stationary point of J_2 with respect to the boundary conditions $y(0) = 0$ and $d_k(0) = e_k$ for $k = 2,3$ (and natural boundary conditions at $x_1 = L$) if and only if the following system of equations is satisfied:*

$$\begin{cases} \tilde{E}'_{11} = A_{13}(\hat{E}_{21} - \tilde{E}_{31}) - A_{23}\hat{E}_{11} - R^T\tilde{g}\cdot e_2, \\ \hat{E}'_{11} = -A_{12}(\hat{E}_{21} - \tilde{E}_{31}) + A_{23}\tilde{E}_{11} - R^T\tilde{g}\cdot e_3, \\ \hat{E}'_{21} - \tilde{E}'_{31} = A_{12}\hat{E}_{11} - A_{13}\tilde{E}_{11}, \\ \tilde{E}_{11}(L) = \hat{E}_{11}(L) = 0, \quad \hat{E}_{21}(L) - \tilde{E}_{31}(L) = 0, \end{cases} \tag{2.4}$$

where

$$\tilde{g}(x_1) := \int_L^{x_1} g(t)\,dt$$

for every $x_1 \in (0,L)$.

Remark 2.4. If W is isotropic, that is,

$$W(F) = W(FR) \quad \text{for every } F \in \mathbb{M}^{3\times 3},\ R \in SO(3),$$

then the linear operator $\mathcal{L}$ associated with the second derivatives of W at the identity reduces to

$$\mathcal{L}F = 2\mu\,\mathrm{sym}\,F + \lambda(\mathrm{tr}F)Id,$$

where λ and μ are the Lamé coefficients of the rod.

If we assume in addition that the cross-section S is a disc, then the minimizer $\alpha \in \mathcal{B}$ of (1.5) can be explicitly computed and, in terms of the entries of the matrix $A = R^T R'$, it is given by

$$\begin{aligned}\alpha = &-\tfrac{1}{4}\tfrac{\lambda}{\lambda+\mu}(x_2^2 A_{12} - x_3^2 A_{12} + 2x_2x_3A_{13})e_2\\ &-\tfrac{1}{4}\tfrac{\lambda}{\lambda+\mu}(-x_2^2 A_{13} + x_3^2 A_{13} + 2x_2x_3A_{12})e_3,\end{aligned}$$

(see [8, Remark 3.5]). In this case the stress is equal to

$$E = \begin{pmatrix} \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}(x_2A_{12} + x_3A_{13}) & \frac{1}{2}x_3A_{23} & -\frac{1}{2}x_2A_{23}\\ \frac{1}{2}x_3A_{23} & 0 & 0\\ -\frac{1}{2}x_2A_{23} & 0 & 0 \end{pmatrix},$$

while the bending moments are

$$\begin{aligned}\tilde{E} &= \tfrac{1}{4\pi}\tfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}A_{12}e_1\otimes e_1 - \tfrac{1}{8\pi}A_{23}(e_1\otimes e_3 + e_3\otimes e_1),\\ \hat{E} &= \tfrac{1}{4\pi}\tfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}A_{12}e_1\otimes e_1 + \tfrac{1}{8\pi}A_{23}(e_1\otimes e_2 + e_2\otimes e_1).\end{aligned}$$

Proof of Lemma 2.3. Let $R := (y' \,|\, d_2 \,|\, d_3)$ and let $A := R^T R'$. It is convenient to consider J_2 as a functional defined on the class

$$\mathcal{R} := \{P \in W^{1,2}((0,L);\mathbb{M}^{3\times 3}) :\ P \in SO(3) \text{ a.e. in } (0,L),\ P(0) = Id\},$$

whose tangent space at R is given by all functions of the form RB with $B \in W^{1,2}((0,L);\mathbb{M}^{3\times 3}_{skw})$ and $B(0) = 0$.

Let then $B \in W^{1,2}((0,L);\mathbb{M}^{3\times 3}_{skw})$ with $B(0) = 0$. In order to compute the Gâteaux differential of J_2 at R in the tangent direction given by RB, we consider a smooth curve $\gamma : [0,1] \to \mathcal{R}$ such that $\gamma(0) = R$ and $\dot\gamma(0) = RB$ (where the dot denotes derivative with respect to the variable $\varepsilon \in [0,1]$). Then we have

$$J_2(\gamma(\varepsilon)) = \frac{1}{2}\int_0^L Q_1(\gamma(\varepsilon)^T\gamma(\varepsilon)')\,dx_1 + \int_0^L \tilde{g}\cdot\gamma(\varepsilon)e_1\,dx_1, \tag{2.5}$$

where the prime denotes derivative with respect to $x_1 \in [0,L]$. Now, let $\beta^\varepsilon \in L^2(\Omega;\mathbb{R}^3)$ be such that $\beta^\varepsilon(x_1,\cdot) \in \mathcal{B}$ is the solution to the problem (1.5) with A replaced by $\gamma(\varepsilon)^T\gamma(\varepsilon)'$ for a.e. $x_1 \in (0,L)$. Then

$$\begin{aligned}&\frac{1}{2}\int_0^L Q_1(\gamma(\varepsilon)^T\gamma(\varepsilon)')\,dx_1\\ &= \frac{1}{2}\int_\Omega Q_3\Big(x_2\,\gamma(\varepsilon)^T\gamma(\varepsilon)'e_2 + x_3\,\gamma(\varepsilon)^T\gamma(\varepsilon)'e_3\Big|\,\partial_2\beta^\varepsilon\,\Big|\,\partial_3\beta^\varepsilon\Big)\,dx.\end{aligned}$$

Differentiating equation (2.5) at $\varepsilon = 0$ and taking into account the previous formula, we obtain

$$
\begin{aligned}
& dJ_2(R)[RB] \\
& \quad = \int_\Omega E : \Big(x_2(AB - BA + B')e_2 + x_3(AB - BA + B')e_3 \,\Big|\, \partial_2\beta \,\Big|\, \partial_3\beta \Big)\, dx \\
& \qquad + \int_0^L R^T\tilde{g}\cdot Be_1\, dx_1,
\end{aligned}
$$

where E is the stress defined in (2.3) and $\beta \in L^2(\Omega;\mathbb{R}^3)$ is such that $\beta(x_1,\cdot) \in \mathcal{B}$ is the solution to the problem (1.5) with A replaced by $B^TR' + R^TB'$ for a.e. $x_1 \in (0,L)$. Here we used the fact that by Lemma 2.1 the function β^ε depends linearly on the entries of $\gamma(\varepsilon)^T\gamma(\varepsilon)'$.

By (2.2) the vectorfield Ee_2, Ee_3 is divergence free in the variables x_2, x_3, hence

$$
\int_S (Ee_2\cdot\partial_2\beta + Ee_3\cdot\partial_3\beta)\, dx_2dx_3 = 0.
$$

Thus the differential of J_2 reduces to

$$
\begin{aligned}
dJ_2(R)[RB] \;=\; & \int_\Omega Ee_1\cdot(x_2(AB - BA + B')e_2 + x_3(AB - BA + B')e_3)\, dx \\
& + \int_0^L R^T\tilde{g}\cdot Be_1\, dx_1.
\end{aligned}
$$

Integration with respect to x_2, x_3 in the first term on the right-hand side yields

$$
\begin{aligned}
& dJ_2(R)[RB] = \int_0^L (\tilde{E}e_1\cdot B'e_2 + \hat{E}e_1\cdot B'e_3)\, dx_1 \\
& + \int_0^L (\tilde{E}e_1\cdot(AB - BA)e_2 + \hat{E}e_1\cdot(AB - BA)e_3)\, dx_1 + \int_0^L R^T\tilde{g}\cdot Be_1\, dx_1. \qquad (2.6)
\end{aligned}
$$

As A is skew-symmetric, we have that for any $F \in \mathbb{M}^{3\times 3}$ and for $k = 2,3$

$$
Fe_1\cdot(AB - BA)e_k = -AFe_1\cdot Be_k - \sum_{j\neq k} A_{jk}Fe_1\cdot Be_j.
$$

Using (2.6) and the previous formula, it is easy to check that the condition

$$
dJ_2(R)[RB] = 0 \quad \text{for every } B \in W^{1,2}((0,L);\mathbb{M}^{3\times 3}_{skw}) \text{ with } B(0) = 0
$$

is equivalent to the following three equations:

$$
\begin{aligned}
& \int_0^L \big(\phi'\,\tilde{E}_{11} + \phi\, A_{13}(\hat{E}_{21} - \tilde{E}_{31}) - \phi\, A_{23}\hat{E}_{11} - \phi\, R^T\tilde{g}\cdot e_2\big)\, dx_1 = 0, \\
& \int_0^L \big(\phi'\,\hat{E}_{11} - \phi\, A_{12}(\hat{E}_{21} - \tilde{E}_{31}) + \phi\, A_{23}\tilde{E}_{11} - \phi\, R^T\tilde{g}\cdot e_3\big)\, dx_1 = 0, \qquad (2.7) \\
& \int_0^L \big(\phi'(\hat{E}_{21} - \tilde{E}_{31}) + \phi\, A_{12}\hat{E}_{11} - \phi\, A_{13}\tilde{E}_{11}\big)\, dx_1 = 0
\end{aligned}
$$

for every $\phi \in W^{1,2}(0,L)$ with $\phi(0) = 0$. By integration by parts the previous equations are equivalent to system (2.4). $\square$

3. Proof of Theorem 1.1

Let $(y^{(h)})$ be a sequence of stationary points of J^h; i.e., suppose that the following condition is satisfied:

$$\int_\Omega \Big(DW(\nabla_h y^{(h)}) : \nabla_h \psi - h^2 g \cdot \psi \Big)\, dx = 0 \tag{3.1}$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\psi(0,x_2,x_3)=0$ for $(x_2,x_3)\in S$. Assume that (1.8) holds true.

The proof is split into several steps.

Step 1. Decomposition of the deformation gradients in rotation and strain.
By Proposition 4.1 there exists a sequence $(R^{(h)}) \subset C^\infty((0,L);\mathbb{M}^{3\times 3})$ such that $R^{(h)}(x_1)\in SO(3)$ for every $x_1\in(0,L)$ and

$$\|\nabla_h y^{(h)} - R^{(h)}\|_{L^2} \le Ch, \tag{3.2}$$

$$\|(R^{(h)})'\|_{L^2} + h\|(R^{(h)})''\|_{L^2} \le C, \tag{3.3}$$

$$|R^{(h)}(0) - Id| \le C\sqrt{h}. \tag{3.4}$$

By (3.3), up to subsequences, $R^{(h)}$ converge to some R weakly in $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$, hence uniformly in $L^\infty((0,L);\mathbb{M}^{3\times 3})$. Thus $R(x_1)\in SO(3)$ for every $x_1\in(0,L)$. From inequality (3.2) it follows that

$$\nabla_h y^{(h)} \to R \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}).$$

In particular, we have that $\partial_k y^{(h)} \to 0$ for $k=2,3$ and thus

$$\nabla y^{(h)} \to Re_1 \otimes e_1 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{3.5}$$

As $|y^{(h)}(0,x_2,x_3)| \le Ch \to 0$, we deduce from the Poincaré inequality that $y^{(h)}$ converge to some $\bar y$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and that $\bar y$ satisfies

$$\partial_1 \bar y = Re_1, \quad \partial_2 \bar y = \partial_3 \bar y = 0 \quad \text{a.e. in } \Omega.$$

Therefore, setting $\bar d_k := Re_k$ for $k=2,3$, we have that $(\bar y, \bar d_2, \bar d_3)\in\mathcal{A}$, and the convergence properties (1.9) and (1.10) are proved. Moreover, the boundary conditions at $x_1=0$ follow from (3.4) and the uniform convergence of $R^{(h)}$.

Let $G^{(h)}:\Omega\to\mathbb{M}^{3\times 3}$ be the function

$$G^{(h)} := \tfrac{1}{h}((R^{(h)})^T \nabla_h y^{(h)} - Id).$$

As the functions $G^{(h)}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$ by (3.2), we can assume, up to extracting a subsequence, that

$$G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}) \tag{3.6}$$

for some $G\in L^2(\Omega;\mathbb{M}^{3\times 3})$. Moreover, from the definition of $G^{(h)}$ it follows immediately that the deformation gradients can be decomposed as

$$\nabla_h y^{(h)} = R^{(h)}(Id + hG^{(h)}). \tag{3.7}$$

Step 2. Consequence of compatibility for the strain.
The decomposition (3.7) suggests that, roughly speaking, the strains $G^{(h)}$ have the structure of a scaled gradient, up to the factor $(R^{(h)})^T$. This implies that the limit strain G has to satisfy some compatibility constraints. In order to deduce these

conditions we introduce a sequence of auxiliary deformations $z^{(h)} : \Omega \to \mathbb{R}^3$ defined by

$$z^{(h)}(x) := \tfrac{1}{h} y^{(h)}(x) - \tfrac{1}{h} \int_0^{x_1} R^{(h)}(s) e_1 \, ds - x_2 R^{(h)}(x_1) e_2 - x_3 R^{(h)}(x_1) e_3. \quad (3.8)$$

Using (3.7) we obtain

$$\begin{aligned} \nabla_h z^{(h)} &= \tfrac{1}{h}(\nabla_h y^{(h)} - R^{(h)}) - x_2 (R^{(h)})' e_2 \otimes e_1 - x_3 (R^{(h)})' e_3 \otimes e_1 \\ &= R^{(h)}(G^{(h)} - x_2 A^{(h)} e_2 \otimes e_1 - x_3 A^{(h)} e_3 \otimes e_1), \end{aligned} \quad (3.9)$$

where $A^{(h)} := (R^{(h)})^T (R^{(h)})'$. Since $R^{(h)} \rightharpoonup R$ in $W^{1,2}((0,L);\mathbb{M}^{3\times 3})$, we have

$$A^{(h)} \rightharpoonup A := R^T R' \quad \text{weakly in } L^2((0,L);\mathbb{M}^{3\times 3}). \quad (3.10)$$

Using these two facts, together with (3.6), we conclude that

$$\nabla_h z^{(h)} \rightharpoonup R(G - x_2 A e_2 \otimes e_1 - x_3 A e_3 \otimes e_1) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \quad (3.11)$$

As $|z^{(h)}(0,x_2,x_3)| \le C\sqrt{h}$ by (3.4), we deduce from the Poincaré inequality that $z^{(h)}$ converge to some z weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. Moreover, the limit function z satisfies

$$R^T \partial_1 z = G e_1 - x_2 A e_2 - x_3 A e_3, \quad \partial_2 z = \partial_3 z = 0 \quad \text{a.e. in } \Omega. \quad (3.12)$$

In particular, z does not depend on x_2, x_3 and, thus, by the first equality in (3.12) Ge_1 is an affine function of x_2, x_3. If we denote by $\bar{G}$ the zeroth moment of G defined by

$$\bar{G}(x_1) := \int_S G(x) \, dx_2 dx_3, \qquad x_1 \in (0,L),$$

then it follows immediately from (1.1) and (3.12) that

$$\bar{G} e_1 = R^T z'. \quad (3.13)$$

To identify the second and third column of $G^{(h)}$ it is convenient to define $\alpha^{(h)} : \Omega \to \mathbb{R}^3$ as

$$\alpha^{(h)} := \tfrac{1}{h}(R^{(h)})^T z^{(h)} - \int_S \tfrac{1}{h}(R^{(h)})^T z^{(h)} \, dx_2 dx_3.$$

From (3.11) and the uniform convergence of $R^{(h)}$ it follows that

$$\partial_k \alpha^{(h)} \rightharpoonup G e_k \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3) \quad (3.14)$$

for $k = 2,3$. By the Poincaré inequality on the cross-section S, there exists a constant $C > 0$ such that for a.e. $x_1 \in (0,L)$

$$\|\alpha^{(h)}(x_1,\cdot)\|^2_{L^2(S)} \le C \|\partial_2 \alpha^{(h)}(x_1,\cdot)\|^2_{L^2(S)} + C \|\partial_3 \alpha^{(h)}(x_1,\cdot)\|^2_{L^2(S)}.$$

Integrating with respect to x_1, we deduce by (3.14) that $\alpha^{(h)} \rightharpoonup \alpha$ weakly in $L^2(\Omega;\mathbb{R}^3)$, where α satisfies $\alpha \in L^2(\Omega;\mathbb{R}^3)$, $\partial_k \alpha \in L^2(\Omega;\mathbb{R}^3)$ for $k = 2,3$, and $G e_k = \partial_k \alpha$ for $k = 2,3$. In particular, the function

$$\beta(x) := \alpha(x) - x_2 \int_S \partial_2 \alpha \, dx_2 dx_3 - x_3 \int_S \partial_3 \alpha \, dx_2 dx_3$$

satisfies $\beta \in L^2(\Omega;\mathbb{R}^3)$, $\partial_k \beta \in L^2(\Omega;\mathbb{R}^3)$ for $k = 2,3$, $\beta(x_1,\cdot) \in \mathcal{B}$ for a.e. $x_1 \in (0,L)$, and

$$G e_k - \bar{G} e_k = \partial_k \beta \quad \text{for } k = 2,3. \quad (3.15)$$

Step 3. Consequences of the Euler-Lagrange equations.
Let $E^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ be the scaled stress defined by

$$E^{(h)} := \tfrac{1}{h} DW(Id + hG^{(h)}). \tag{3.16}$$

Since DW is Lipschitz continuous and the $G^{(h)}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, the functions $E^{(h)}$ are also bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$. In fact, by Proposition 4.2 we have that

$$E^{(h)} \rightharpoonup E := \mathcal{L}G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{3.17}$$

We note in particular that E is symmetric, as $\mathcal{L}F = (\mathcal{L}F)^T$ for every $F \in \mathbb{M}^{3\times 3}$. Note also that $\mathcal{L}F = \mathcal{L}(\operatorname{sym} F)$ for every $F \in \mathbb{M}^{3\times 3}$.

By the decomposition (3.7) and by frame indifference we obtain that

$$DW(\nabla_h y^{(h)}) = R^{(h)} DW(Id + hG^{(h)}) = hR^{(h)} E^{(h)}.$$

Using this identity we can write the Euler-Lagrange equations (3.1) in terms of the stresses $E^{(h)}$. More precisely, we have

$$\int_\Omega (R^{(h)} E^{(h)} : \nabla_h \psi - hg \cdot \psi)\, dx = 0 \tag{3.18}$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $\psi = 0$ on $\{x_1 = 0\}$. Multiplying (3.18) by h and passing to the limit as $h \to 0$, we get

$$\int_\Omega (REe_2 \cdot \partial_2 \psi + REe_3 \cdot \partial_3 \psi)\, dx = 0. \tag{3.19}$$

As R is a pointwise rotation depending only on x_1, the previous equation yields

$$\begin{cases} \operatorname{div}_{x_2,x_3}(Ee_2 \,|\, Ee_3) = 0 & \text{in } S, \\ (Ee_2 \,|\, Ee_3)\, \nu_{\partial S} = 0 & \text{on } \partial S \end{cases} \tag{3.20}$$

for a.e. $x_1 \in (0,L)$. This implies in particular that for a.e. $x_1 \in (0,L)$

$$\int_S Ee_k \, dx_2 dx_3 = 0 \quad \text{for } k = 2,3. \tag{3.21}$$

Step 4. Symmetry properties of $E^{(h)}$.
From the frame indifference of W it follows that the matrix $DW(F)F^T$ is symmetric. Applying this with $F = Id + hG^{(h)}$, we obtain that

$$E^{(h)} - (E^{(h)})^T = -h(E^{(h)}(G^{(h)})^T - G^{(h)}(E^{(h)})^T). \tag{3.22}$$

As $E^{(h)}$ and $G^{(h)}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, we deduce in particular the estimate

$$\|E^{(h)} - (E^{(h)})^T\|_{L^1} \le Ch. \tag{3.23}$$

Step 5. Moments of the Euler-Lagrange equations.
Let us introduce the zeroth and first moments of the stress $E^{(h)}$, defined by

$$\bar{E}^{(h)}(x_1) := \int_S E^{(h)}(x)\, dx_2 dx_3,$$

$$\tilde{E}^{(h)}(x_1) := \int_S x_2 E^{(h)}(x)\, dx_2 dx_3, \qquad \hat{E}^{(h)}(x_1) := \int_S x_3 E^{(h)}(x)\, dx_2 dx_3$$

for every $x_1 \in (0,L)$. We shall derive the Euler-Lagrange equations satisfied by the moments.

Let $\varphi \in C^\infty([0,L];\mathbb{R}^3)$ be such that $\varphi(0)=0$. Using φ as test function in the Euler-Lagrange equation (3.18), we obtain

$$\int_\Omega (R^{(h)}E^{(h)}e_1 \cdot \varphi' - hg\cdot\varphi)\,dx = 0.$$

Integrating first with respect to the variables of S and taking into account that $R^{(h)}$, φ, and g depend only on the x_1 variable, we can rewrite the previous equality as

$$\int_0^L (R^{(h)}\bar{E}^{(h)}e_1 \cdot \varphi' - hg\cdot\varphi)\,dx_1 = 0.$$

Since this equation holds for every $\varphi \in C^\infty([0,L];\mathbb{R}^3)$ with $\varphi(0)=0$, we deduce that

$$\bar{E}^{(h)}e_1 = -h(R^{(h)})^T\tilde{g} \quad \text{a.e. in } (0,L), \tag{3.24}$$

where $\tilde{g}$ is the primitive of g defined in (2.6). In particular, passing to the limit, we obtain

$$\bar{E}e_1 = 0 \quad \text{a.e. in } (0,L). \tag{3.25}$$

Together with (3.21), this implies that $\bar{E}=0$ a.e. in $(0,L)$. As $E=\mathcal{L}G$, we obtain that $E=\mathcal{L}(G-\bar{G})$ and by (3.12), (3.13), and (3.15) we conclude that

$$E = \mathcal{L}\Big(x_2Ae_2 + x_3Ae_3 \,|\, \partial_2\beta \,|\, \partial_3\beta\Big). \tag{3.26}$$

Equation (3.20) and Lemma 2.1 guarantee that $\beta(x_1,\cdot)$ is a solution to the problem (1.5) defining $Q_1(A(x_1))$, for a.e. $x_1\in(0,L)$.

As for the first moments, let $\varphi \in C^\infty([0,L];\mathbb{R}^3)$ be such that $\varphi(0)=0$. Using $\psi(x) := x_2\varphi(x_1)$ as test functions in (3.18), we obtain

$$\int_\Omega (x_2R^{(h)}E^{(h)}e_1\cdot\varphi' + \tfrac{1}{h}R^{(h)}E^{(h)}e_2\cdot\varphi - hx_2g\cdot\varphi)\,dx = 0.$$

Integrating first with respect to x_2, x_3 and using (1.1), the equation reduces to

$$\int_0^L (R^{(h)}\tilde{E}^{(h)}e_1\cdot\varphi' + \tfrac{1}{h}R^{(h)}\bar{E}^{(h)}e_2\cdot\varphi)\,dx_1 = 0. \tag{3.27}$$

In particular, if we choose φ of the form $\varphi = \phi\, R^{(h)}e_1$ with $\phi\in C^\infty([0,L])$ and $\phi(0)=0$, we obtain

$$\int_0^L (\phi'\,\tilde{E}^{(h)}_{11} + \phi\,\tilde{E}^{(h)}e_1\cdot A^{(h)}e_1 + \phi\,\tfrac{1}{h}\bar{E}^{(h)}_{12})\,dx_1 = 0. \tag{3.28}$$

From the estimate (3.23) and the identity (3.24) it follows that the term $\frac{1}{h}\bar{E}^{(h)}_{12}$ is bounded in $L^1(0,L)$. Since $A^{(h)}$ and $\tilde{E}^{(h)}$ are bounded in $L^2((0,L);\mathbb{M}^{3\times3})$, the product $\tilde{E}^{(h)}e_1\cdot A^{(h)}e_1$ is also bounded in $L^1(0,L)$. Therefore, equation (3.28) implies that

$$\|\partial_1\tilde{E}^{(h)}_{11}\|_{L^1} \le C, \quad \tilde{E}^{(h)}_{11}(L) = 0, \tag{3.29}$$

hence the sequence $\tilde{E}^{(h)}_{11}$ is strongly compact in $L^p(0,L)$ for every $p<\infty$.

Analogously, one can show that

$$\int_0^L (R^{(h)}\hat{E}^{(h)}e_1\cdot\varphi' + \tfrac{1}{h}R^{(h)}\bar{E}^{(h)}e_3\cdot\varphi)\,dx_1 = 0 \tag{3.30}$$

for every $\varphi \in C^\infty([0,L];\mathbb{R}^3)$ such that $\varphi(0)=0$. Choosing the test function φ of the form $\varphi = \phi\, R^{(h)} e_1$ with $\phi \in C^\infty([0,L])$ and $\phi(0)=0$, one obtains

$$\int_0^L \big(\phi' \hat{E}^{(h)}_{11} - \phi\, \hat{E}^{(h)} e_1 \cdot A^{(h)} e_1 + \phi\, \tfrac{1}{h} \bar{E}^{(h)}_{13}\big)\, dx_1 = 0 \tag{3.31}$$

for every $\phi \in C^\infty([0,L])$ with $\phi(0)=0$. From this equation one can deduce as before that

$$\|\partial_1 \hat{E}^{(h)}_{11}\|_{L^1} \le C, \quad \hat{E}^{(h)}_{11}(L) = 0, \tag{3.32}$$

hence the sequence $\hat{E}^{(h)}_{11}$ is strongly compact in $L^p(0,L)$ for every $p<\infty$.

Finally, let us consider $\phi\, R^{(h)} e_2$ and $\phi\, R^{(h)} e_3$ as test functions in (3.30) and (3.27), respectively, with $\phi \in C^\infty([0,L])$ with $\phi(0)=0$. Taking the difference of the two equations we obtain

$$\begin{aligned} \int_0^L \phi' \,(\hat{E}^{(h)}_{21} - \tilde{E}^{(h)}_{31})\, dx_1 - \int_0^L \phi\, (\hat{E}^{(h)} e_1 \cdot A^{(h)} e_2 - \tilde{E}^{(h)} e_1 \cdot A^{(h)} e_3)\, dx_1 \\ + \int_0^L \phi\, \tfrac{1}{h} (\bar{E}^{(h)}_{23} - \bar{E}^{(h)}_{32})\, dx_1 = 0. \end{aligned} \tag{3.33}$$

As $A^{(h)}$ and $E^{(h)}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, the term $(A^{(h)}\hat{E}^{(h)})_{21} - (A^{(h)}\tilde{E}^{(h)})_{31}$ is bounded in $L^1(0,L)$. The difference $\frac{1}{h}(\bar{E}^{(h)}_{23} - \bar{E}^{(h)}_{32})$ is also bounded in $L^1(0,L)$ by (3.23). Therefore, we deduce from equation (3.33) that

$$\|\partial_1 (\hat{E}^{(h)}_{21} - \tilde{E}^{(h)}_{31})\|_{L^1} \le C, \quad \hat{E}^{(h)}_{21}(L) - \tilde{E}^{(h)}_{31}(L) = 0, \tag{3.34}$$

hence the sequence $\hat{E}^{(h)}_{21} - \tilde{E}^{(h)}_{31}$ is strongly compact in $L^p(0,L)$ for every $p<\infty$.

Step 6. Convergence of the energy by the div-curl lemma.

The strong compactness of the sequences $(\tilde{E}^{(h)}_{11})$, $(\hat{E}^{(h)}_{11})$, and $(\hat{E}^{(h)}_{21} - \hat{E}^{(h)}_{31})$ allows us to pass to the limit in the energy integral

$$\frac{1}{h^2} \int_\Omega DW(Id + hG^{(h)}) : hG^{(h)}\, dx = \int_\Omega E^{(h)} : G^{(h)}\, dx.$$

This can be done by exploiting the div-curl structure of the product $E^{(h)} : G^{(h)}$; indeed, the Euler-Lagrange equation (3.18) asserts that the scaled divergence of $R^{(h)} E^{(h)}$ is infinitesimal in $L^2(\Omega;\mathbb{R}^3)$ as $h \to 0$, while the decomposition (3.9) guarantees that the matrix $R^{(h)} G^{(h)}$ has basically the structure of a scaled gradient.

Let us fix $\varphi \in C^\infty(0,L)$ with $\varphi(0)=0$. Using formula (3.9) we have

$$\begin{aligned} \int_\Omega \varphi E^{(h)} : G^{(h)}\, dx &= \int_\Omega \varphi R^{(h)} E^{(h)} : R^{(h)} G^{(h)}\, dx \\ = \int_\Omega \varphi R^{(h)} E^{(h)} : \nabla_h z^{(h)}\, dx &+ \int_\Omega \varphi E^{(h)} e_1 \cdot (x_2 A^{(h)} e_2 + x_3 A^{(h)} e_3)\, dx. \end{aligned} \tag{3.35}$$

Concerning the first term on the right-hand side, the Euler-Lagrange equation (3.18) yields

$$\int_\Omega \varphi R^{(h)} E^{(h)} : \nabla_h z^{(h)}\, dx = h \int_\Omega \varphi g \cdot z^{(h)}\, dx - \int_\Omega \varphi' R^{(h)} E^{(h)} e_1 \cdot z^{(h)}\, dx.$$

Since $z^{(h)} \to z$ strongly in $L^2(\Omega;\mathbb{R}^3)$ and $R^{(h)}E^{(h)} \rightharpoonup RE$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$, we can pass to the limit in the formula above and we get

$$\lim_{h\to 0}\int_\Omega \varphi R^{(h)}E^{(h)} : \nabla_h z^{(h)}\,dx = -\int_\Omega \varphi' REe_1\cdot z\,dx.$$

Taking into account the fact that z is independent of x_2, x_3 and using the identity (3.25), we have

$$\int_\Omega \varphi' REe_1\cdot z\,dx = \int_0^L \varphi' R\bar{E}e_1\cdot z\,dx_1 = 0,$$

hence

$$\lim_{h\to 0}\int_\Omega \varphi R^{(h)}E^{(h)} : \nabla_h z^{(h)}\,dx = 0. \tag{3.36}$$

As for the last term in (3.35), integrating first with respect to the cross-section variables we have

$$\begin{aligned}&\int_\Omega \varphi E^{(h)}e_1\cdot(x_2A^{(h)}e_2 + x_3A^{(h)}e_3)\,dx\\ &= \int_0^L \varphi(\tilde{E}^{(h)}_{11}A^{(h)}_{12} + \hat{E}^{(h)}_{11}A^{(h)}_{13})\,dx_1 + \int_0^L \varphi(\hat{E}^{(h)}_{21} - \tilde{E}^{(h)}_{31})A^{(h)}_{23}\,dx_1.\end{aligned}$$

As $\tilde{E}^{(h)}_{11}$, $\hat{E}^{(h)}_{11}$, and $\hat{E}^{(h)}_{21} - \tilde{E}^{(h)}_{31}$ are strongly compact in $L^2(0,L)$ by Step 5, we can pass to the limit and we obtain

$$\begin{aligned}&\lim_{h\to 0}\int_\Omega \varphi E^{(h)}e_1\cdot(x_2A^{(h)}e_2 + x_3A^{(h)}e_3)\,dx\\ &= \int_0^L \varphi(\tilde{E}_{11}A_{12} + \hat{E}_{11}A_{13})\,dx_1 + \int_0^L \varphi(\hat{E}_{21} - \tilde{E}_{31})A_{23}\,dx_1 \qquad (3.37)\\ &= \int_\Omega \varphi Ee_1\cdot(x_2Ae_2 + x_3Ae_3)\,dx.\end{aligned}$$

Now from the first equality in (3.12) it follows that

$$\int_\Omega \varphi Ee_1\cdot(x_2Ae_2 + x_3Ae_3)\,dx = \int_\Omega \varphi Ee_1\cdot Ge_1\,dx - \int_\Omega \varphi Ee_1\cdot R^T z'\,dx. \tag{3.38}$$

Since R^Tz' does not depend on x_2, x_3, identity (3.25) implies

$$\int_\Omega \varphi Ee_1\cdot R^Tz'\,dx = \int_0^L \varphi \bar{E}e_1\cdot R^Tz'\,dx_1 = 0.$$

Thus, equality (3.38) reduces to

$$\int_\Omega \varphi Ee_1\cdot(x_2Ae_2 + x_3Ae_3)\,dx = \int_\Omega \varphi Ee_1\cdot Ge_1\,dx. \tag{3.39}$$

Combining together (3.35)–(3.37) and (3.39), we conclude that

$$\lim_{h\to 0}\int_\Omega \varphi E^{(h)} : G^{(h)}\,dx = \int_\Omega \varphi Ee_1\cdot Ge_1\,dx \tag{3.40}$$

By (3.20) the matrix $(Ee_2\,|\,Ee_3)$ is divergence free in S with zero normal component on ∂S for a.e. $x_1 \in (0,L)$, while $(Ge_2\,|\,Ge_3)$ is a gradient by (3.15). As the test function φ depends only on the variable x_1, the divergence theorem yields

$$\int_\Omega \varphi(Ee_2\cdot Ge_2 + Ee_3\cdot Ge_3)\,dx = 0,$$

hence

$$\int_\Omega \varphi E : G\,dx = \int_\Omega \varphi E e_1 \cdot G e_1\,dx. \tag{3.41}$$

By (3.40) and (3.41) we finally obtain the convergence of the energies

$$\lim_{h\to 0}\int_\Omega \varphi E^{(h)} : G^{(h)}\,dx = \int_\Omega \varphi E : G\,dx \tag{3.42}$$

for every $\varphi \in C_0^\infty(0,L)$.

Step 7. Definition of the truncated deformations.

In order to pass to the limit in the Euler-Lagrange equations (3.28), (3.31), and (3.33), a strong L^2-compactness for the sequence $(E^{(h)})$ is required. If $hG^{(h)}$ converges to 0 uniformly, then by Taylor expansion one can replace $E^{(h)}$ by $\mathcal{L}G^{(h)}$ in (3.42). Using the fact that $E = \mathcal{L}G$ and $\mathcal{L}$ is positive definite on symmetric matrices, one can conclude strong convergence for $\operatorname{sym} G^{(h)}$ and hence of $E^{(h)}$, outside a neighbourhood of $x_1 = 0$ (see Step 7 of the proof of Theorem 1.1 in [10]).

To avoid the extra assumption $h\|G^{(h)}\|_\infty \to 0$, we introduce an auxiliary sequence of truncated deformations $u^{(h)}$, whose corresponding scaled strains $H^{(h)}$ satisfy $h\|H^{(h)}\|_\infty \to 0$ (see (3.51)). The main point will be then to show strong convergence of $\operatorname{sym} H^{(h)}$ (see Step 8). This will imply, as before, strong convergence of the corresponding truncated stress $F^{(h)}$ (outside a neighbourhood of $x_1 = 0$). To pass to the limit in the Euler-Lagrange equations and conclude the proof, we will then need to estimate the remainder term $E^{(h)} - F^{(h)}$. This will be done in Step 9, using again the div-curl lemma and exploiting our careful choice of the truncations.

To carry out this plan, we consider the functions $z^{(h)}$ defined in (3.8) and their rescalings $\check z^{(h)}(x) := z^{(h)}(x_1, \frac{x_2}{h}, \frac{x_3}{h})$. Applying Lemma 4.3 to $\check z^{(h)}$ with $a = h^{-5/8}$ and $b = h^{-7/8}$ and undoing the rescaling, we construct a new sequence of functions $w^{(h)} : \Omega \to \mathbb{R}^3$ with the following properties:

$$\|\nabla_h w^{(h)}\|_{L^\infty} \le \lambda_h, \tag{3.43}$$

$$\begin{aligned} \lambda_h^2 \mathcal{L}^3(N_h) &\le \frac{C}{\ln(1/h)}\int_\Omega |\nabla_h z^{(h)}|^2\,dx \\ &\le \frac{C}{\ln(1/h)}\int_\Omega (|G^{(h)}|^2 + |A^{(h)}|^2)\,dx, \end{aligned} \tag{3.44}$$

$$\|\nabla_h z^{(h)} - \nabla_h w^{(h)}\|_{L^2}^2 \le \frac{C}{\ln(1/h)}\int_\Omega |\nabla_h z^{(h)}|^2\,dx, \tag{3.45}$$

where $\lambda_h \in [h^{-5/8}, h^{-7/8}]$ and $N_h := \{x \in \Omega :\ z^{(h)}(x) \ne w^{(h)}(x)\}$. In particular we have

$$h^{1/2}\lambda_h \to \infty, \quad h\lambda_h \to 0, \quad \text{and} \quad \lambda_h^2\mathcal{L}^3(N_h) \to 0. \tag{3.46}$$

We can introduce now the sequence of approximated deformations $u^{(h)} : \Omega \to \mathbb{R}^3$, which are associated with the auxiliary functions $w^{(h)}$:

$$u^{(h)} := hw^{(h)} + \int_0^{x_1} R^{(h)}(s)e_1\,ds + hx_2R^{(h)}e_2 + hx_3R^{(h)}e_3.$$

Let $H^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ be the corresponding approximated strains defined by the relation

$$\nabla_h u^{(h)} = R^{(h)}(Id + hH^{(h)}),$$

and let $F^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ be the corresponding stresses defined as

$$F^{(h)} := \frac{1}{h} DW(Id + hH^{(h)}). \tag{3.47}$$

Using the definition of $u^{(h)}$ it is easy to see that

$$H^{(h)} = (R^{(h)})^T \nabla_h w^{(h)} + x_2 A^{(h)} e_2 \otimes e_1 + x_3 A^{(h)} e_3 \otimes e_1. \tag{3.48}$$

It follows from (3.45) that $\nabla_h w^{(h)}$ and $\nabla_h z^{(h)}$ have the same weak limit and hence by (3.11)

$$H^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}). \tag{3.49}$$

Step 8. L^∞-convergence of $hH^{(h)}$ and strong convergence of $\operatorname{sym} H^{(h)}$ *and* $F^{(h)}$*.* We recall the estimate

$$\sup |f - \bar{f}|^2 \le 2\|f\|_{L^2} \|f'\|_{L^2}, \quad \text{with } \bar{f} := \frac{1}{L}\int_0^L f\, dx. \tag{3.50}$$

As $(R^{(h)})'$ and $h(R^{(h)})''$ are bounded in $L^2(0,L)$ by (3.3), we deduce that $|(R^{(h)})'| \le Ch^{-1/2}$, and therefore $|A^{(h)}| \le Ch^{-1/2}$. This inequality and (3.43) imply that

$$h|H^{(h)}| \le Ch\lambda_h + Ch^{1/2} \to 0. \tag{3.51}$$

By Taylor expansion of DW around the identity matrix we have

$$F^{(h)} = \frac{1}{h} DW(Id + hH^{(h)}) = \mathcal{L} H^{(h)} + \frac{1}{h}\eta(hH^{(h)}), \tag{3.52}$$

where $|\eta(A)|/|A| \to 0$, as $|A| \to 0$. For every $t > 0$ let us define

$$\omega(t) := \sup\Big\{ \frac{|\eta(A)|}{|A|} : \ |A| \le t \Big\};$$

then, it is easy to see that $\omega(t) \to 0$, as $t \to 0^+$. The expansion (3.52) and the definition of ω yield

$$|\mathcal{L}H^{(h)} : H^{(h)} - F^{(h)} : H^{(h)}| \le \omega(h\|H^{(h)}\|_{L^\infty}) |H^{(h)}|^2.$$

Together with (3.51), we obtain for every $\varphi \in C^\infty([0,L])$

$$\int_\Omega \varphi \mathcal{L} H^{(h)} : H^{(h)}\, dx - \int_\Omega \varphi F^{(h)} : H^{(h)}\, dx \to 0. \tag{3.53}$$

We now claim that

$$\int_\Omega \varphi F^{(h)} : H^{(h)}\, dx - \int_\Omega \varphi E^{(h)} : G^{(h)}\, dx \to 0. \tag{3.54}$$

Combining together the convergence of energy (3.42), the weak convergence (3.49), and (3.53), this would imply that

$$\lim_{h\to 0} \int_\Omega \varphi \mathcal{L}(H^{(h)} - G) : (H^{(h)} - G)\, dx = 0 \tag{3.55}$$

for every $\varphi \in C^\infty([0,L])$ with $\varphi(0) = 0$. From the assumptions on W we infer that there exists a constant $C > 0$ such that

$$\mathcal{L}A : A \ge C|\operatorname{sym} A|^2$$

for every $A \in \mathbb{M}^{3\times 3}$. This inequality, together with (3.55), implies

$$\operatorname{sym}(H^{(h)} - G) \to 0 \qquad \text{strongly in } L^2((a,L)\times S; \mathbb{M}^{3\times 3}) \tag{3.56}$$

for every $a > 0$. Using again the Taylor expansion (3.52), we easily deduce that

$$F^{(h)} \to E \qquad \text{strongly in } L^2((a,L)\times S;\mathbb{M}^{3\times 3}). \tag{3.57}$$

In order to prove (3.54) we write the difference as

$$\int_\Omega \varphi E^{(h)} : (H^{(h)} - G^{(h)})\, dx + \int_\Omega \varphi (F^{(h)} - E^{(h)}) : H^{(h)}\, dx. \tag{3.58}$$

The first term can be controlled by the div-curl lemma; indeed, equalities (3.48) and (3.9) yield

$$R^{(h)}(H^{(h)} - G^{(h)}) = \nabla_h (w^{(h)} - z^{(h)}),$$

so that, by the Euler-Lagrange equation (3.18), we have

$$\begin{aligned}\int_\Omega \varphi E^{(h)} : (H^{(h)} - G^{(h)})\, dx &= \int_\Omega \varphi R^{(h)} E^{(h)} : \nabla_h (w^{(h)} - z^{(h)})\, dx \\ = h \int_\Omega \varphi g \cdot (w^{(h)} - z^{(h)})\, dx &- \int_\Omega \varphi' R^{(h)} E^{(h)} e_1 \cdot (w^{(h)} - z^{(h)})\, dx.\end{aligned}$$

Since the sequence $w^{(h)} - z^{(h)}$ converges to 0 strongly in $L^2(\Omega;\mathbb{R}^3)$ and $R^{(h)}E^{(h)}$ is bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, we conclude that

$$\lim_{h\to 0} \int_\Omega \varphi E^{(h)} : (H^{(h)} - G^{(h)})\, dx = 0.$$

To estimate the second integral in (3.58) we recall that $F^{(h)}$ and $E^{(h)}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$. Therefore, by Hölder inequality and by (3.51) we have

$$\int_\Omega |\varphi (F^{(h)} - E^{(h)}) : H^{(h)}|\, dx \le C\Big(\int_{N_h} |H^{(h)}|^2\, dx\Big)^{1/2} \le C[(\lambda_h^2 + h^{-1})\mathcal{L}^3(N_h)]^{1/2}.$$

As the right-hand side converges to zero by (3.46), this concludes the proof of the claim (3.54).

Step 9. Passage to the limit in the Euler-Lagrange equations.

Let us fix $\phi \in C^\infty([0,L])$ vanishing on an interval $(0,a)$. In order to pass to the limit in the Euler-Lagrange equations, we need to prove some preliminary convergence results. First of all we claim that

$$\lim_{h\to 0} \int_\Omega \phi\, x_k E^{(h)} e_1 \cdot A^{(h)} e_j\, dx = \int_\Omega \phi\, x_k E e_1 \cdot A e_j\, dx \tag{3.59}$$

for every $k = 2,3$ and every $j = 1,2,3$. Indeed,

$$\begin{aligned}\int_\Omega \phi\, x_k E^{(h)} e_1 \cdot A^{(h)} e_j\, dx \;&=\; \int_\Omega \phi\, x_k F^{(h)} e_1 \cdot A^{(h)} e_j\, dx \\ &\quad + \int_{N_h} \phi\, x_k (E^{(h)} - F^{(h)}) e_1 \cdot A^{(h)} e_j\, dx.\end{aligned} \tag{3.60}$$

By the strong convergence (3.57) we have

$$\int_\Omega \phi\, x_k F^{(h)} e_1 \cdot A^{(h)} e_j\, dx \to \int_\Omega \phi\, x_k E e_1 \cdot A e_j\, dx.$$

As for the last term in (3.60), using Hölder's inequality we obtain

$$\int_{N_h} |\phi\, x_k (E^{(h)} - F^{(h)}) e_1 \cdot A^{(h)} e_j|\, dx \le C\Big(\int_{N_h} |A^{(h)}|^2\, dx\Big)^{1/2}.$$

Since $|A^{(h)}| \leq Ch^{-1/2}$ and $h^{-1}\mathcal{L}^3(N_h) \to 0$ by (3.46), the previous estimate implies that the second integral on the right-hand side of (3.60) converges to 0. This concludes the proof of the claim (3.59).

Integrating first with respect to the variables of the cross-section in (3.59), we obtain for $k = 2, 3$ and $j = 1$

$$\lim_{h\to 0} \int_0^L \phi\, \tilde{E}^{(h)} e_1 \cdot A^{(h)} e_1\, dx_1 = \int_0^L \phi\, \tilde{E} e_1 \cdot A e_1\, dx_1, \tag{3.61}$$

$$\lim_{h\to 0} \int_0^L \phi\, \hat{E}^{(h)} e_1 \cdot A^{(h)} e_1\, dx_1 = \int_0^L \phi\, \hat{E} e_1 \cdot A e_1\, dx_1. \tag{3.62}$$

Arguing as in the proof of (3.59), it is easy to show that

$$\lim_{h\to 0} \int_0^L \phi\, x_k\, \mathrm{skew}\, (E^{(h)} e_1 \otimes A^{(h)} e_k)\, dx = \int_\Omega \phi\, x_k\, \mathrm{skew}\, (E e_1 \otimes A e_k)\, dx \tag{3.63}$$

for every $k = 2, 3$ and every $\phi \in C^\infty([0, L])$ vanishing on $(0, a)$.

In order to pass to the limit in the Euler-Lagrange equations (3.28) and (3.31), it remains to study the convergence of the terms

$$\int_0^L \phi\, \tfrac{1}{h} \bar{E}^{(h)}_{1k}\, dx_1$$

for $k = 2, 3$. We first decompose the integral as

$$\int_0^L \phi\, \tfrac{1}{h} \bar{E}^{(h)}_{1k}\, dx_1 = \int_0^L \phi\, \tfrac{1}{h} \bar{E}^{(h)}_{k1}\, dx_1 + 2 \int_\Omega \phi\, \tfrac{1}{h} \mathrm{skew}\, (E^{(h)})_{1k}\, dx. \tag{3.64}$$

By (3.24) we immediately deduce that

$$\lim_{h\to 0} \int_0^L \phi\, \tfrac{1}{h} \bar{E}^{(h)}_{k1}\, dx_1 = -\int_0^L \phi\, R^T \tilde{g} \cdot e_k\, dx_1. \tag{3.65}$$

As for the second integral on the right-hand side of (3.64), it follows from (3.22) that $\frac{1}{h}\mathrm{skew}\, E^{(h)} = -\mathrm{skew}\, (E^{(h)}(G^{(h)})^T)$, so that equality (3.9) yields

$$\begin{aligned} \tfrac{1}{h}\mathrm{skew}\, (E^{(h)}) \quad &= \quad -\mathrm{skew}\, (E^{(h)}(\nabla_h z^{(h)})^T R^{(h)}) - x_2\, \mathrm{skew}\, (E^{(h)} e_1 \otimes A^{(h)} e_2) \\ &\quad\; -x_3\, \mathrm{skew}\, (E^{(h)} e_1 \otimes A^{(h)} e_3). \end{aligned}$$

Since $\mathrm{skew}\, A = \mathrm{skew}\, (RAR^T)$ for every $A \in \mathbb{M}^{3\times 3}$ and every $R \in SO(3)$, we have that

$$\mathrm{skew}\, (E^{(h)}(\nabla_h z^{(h)})^T R^{(h)}) = \mathrm{skew}\, (R^{(h)} E^{(h)} (\nabla_h z^{(h)})^T).$$

This identity, together with the Euler-Lagrange equation (3.18) and the strong convergence of $z^{(h)}$, implies that

$$\begin{aligned} &\lim_{h\to 0} \int_\Omega \mathrm{skew}\, (E^{(h)}(\nabla_h z^{(h)})^T R^{(h)}) \phi\, dx \\ &= \int_\Omega \mathrm{skew}\, (R E e_1 \otimes z) \phi'\, dx = \int_0^L \mathrm{skew}\, (R \bar{E} e_1 \otimes z) \phi'\, dx = 0, \end{aligned}$$

where we have used the fact that z and R are independent of x_2, x_3 and that $\bar{E} e_1 = 0$ by (3.25). Combining this equality with (3.63), we conclude that

$$\lim_{h\to 0} \int_0^L \phi\, \tfrac{1}{h} \mathrm{skew}\, \bar{E}^{(h)}\, dx_1 = -\int_0^L \phi\, \mathrm{skew}\, (\tilde{E} e_1 \otimes A e_2 + \hat{E} e_1 \otimes A e_3)\, dx_1 \tag{3.66}$$

for every $\phi \in C^\infty([0,L])$ vanishing on $(0,a)$.

By (3.65) and (3.66) we finally obtain that

$$\lim_{h\to 0}\int_0^L \phi\,\tfrac{1}{h}\bar{E}^{(h)}_{1k}\,dx_1$$
$$= -2\int_0^L \phi\,\mathrm{skew}\,(\tilde{E}e_1\otimes Ae_2+\hat{E}e_1\otimes Ae_3)_{1k}\,dx_1-\int_0^L \phi\,R^T\tilde{g}\cdot e_k\,dx_1.$$

Together with (3.61) and (3.62), this shows that we can pass to the limit in (3.28) and (3.31). Thus, we obtain the equations

$$\int_0^L (\phi'\,\tilde{E}_{11}+\phi\,A_{13}(\hat{E}_{21}-\tilde{E}_{31})-\phi\,A_{23}\hat{E}_{11}-\phi\,R^T\tilde{g}\cdot e_2)\,dx_1=0 \tag{3.67}$$

and

$$\int_0^L (\phi'\,\hat{E}_{11}-\phi\,A_{12}(\hat{E}_{21}-\tilde{E}_{31})+\phi\,A_{23}\tilde{E}_{11}-\phi\,R^T\tilde{g}\cdot e_3)\,dx_1=0 \tag{3.68}$$

for every $\phi \in C^\infty([0,L])$ vanishing on $(0,a)$.

Analogously, by (3.59) we deduce

$$\lim_{h\to 0}\int_0^L \phi\,(\hat{E}^{(h)}e_1\cdot A^{(h)}e_2-\tilde{E}^{(h)}e_1\cdot A^{(h)}e_3)\,dx_1$$
$$=\int_0^L \phi\,(\hat{E}e_1\cdot Ae_2-\tilde{E}e_1\cdot Ae_3)\,dx_1,$$

while by (3.66) we have

$$\lim_{h\to 0}\int_0^L \phi\,\tfrac{1}{h}(\bar{E}^{(h)}_{23}-\bar{E}^{(h)}_{32})\,dx_1=-\int_0^L \phi\,(A_{32}\tilde{E}_{21}-A_{23}\hat{E}_{31})\,dx_1.$$

Combining these two properties, we can pass to the limit also in the equation (3.33) and we obtain

$$\int_0^L (\phi'\,(\hat{E}_{21}-\tilde{E}_{31})+\phi\,A_{12}\hat{E}_{11}-\phi\,A_{13}\tilde{E}_{11})\,dx_1=0 \tag{3.69}$$

for every $\phi \in C^\infty([0,L])$ vanishing on $(0,a)$.

By approximation it is easy to see that the limiting equations (3.67), (3.68), and (3.69) hold for every $\phi \in C^\infty([0,L])$ with $\phi(0)=0$.

Finally, taking into account (3.26) and integrating by parts, one can check that conditions (3.67)–(3.69) coincide with the Euler-Lagrange equations (2.4) for J_2. □

4. Truncation and compactness

In this section we collect some auxiliary results which were used in the proof of Theorem 1.1.

The first proposition contains an approximation result by means of smooth rotations for sequences of deformations with elastic energy of order h^2. This is the point where the rigidity lemma by Friesecke, James, and Müller (see [4, Theorem 3.1]) is used in a crucial way.

Proposition 4.1. *Let $(u^{(h)}) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence such that*

$$F^{(h)}(u^{(h)}) := \int_\Omega \operatorname{dist}^2(\nabla_h u^{(h)}, SO(3))\,dx \le Ch^2$$

for every $h > 0$. Then there exists an associated sequence $(R^{(h)}) \subset C^\infty((0,L);\mathbb{M}^{3\times 3})$ such that

$$R^{(h)}(x_1) \in SO(3) \quad \text{for every } x_1 \in (0,L), \tag{4.1}$$

$$\|\nabla_h u^{(h)} - R^{(h)}\|_{L^2} \le Ch, \tag{4.2}$$

$$\|(R^{(h)})'\|_{L^2} + h\,\|(R^{(h)})''\|_{L^2} \le C \tag{4.3}$$

for every $h > 0$. If, in addition, $u^{(h)}(0,x_2,x_3) = (0,hx_2,hx_3)$, then

$$|R^{(h)}(0) - Id| \le C\sqrt{h}. \tag{4.4}$$

Proof. The argument follows closely the proof of [10, Proposition 4.1]. For every $h > 0$ the set Ω_h can be partitioned in cylinders of the form $I_h \times hS$, where I_h is an interval of length comparable to h. Applying the rigidity estimate [4, Theorem 3.1] in each such cylinder, one first construct a sequence $(Q^{(h)})$ of piecewise constant rotations satisfying (4.2) and a difference quotient variant of (4.3). As the mollifications $\tilde{Q}^{(h)}$ of $Q^{(h)}$ at scale h are uniformly close to $Q^{(h)}$, it is possible to project $\tilde{Q}^{(h)}$ back on $SO(3)$; this provides the sequence $(R^{(h)})$. For the details we refer to [10]. □

The next proposition allows to identify the weak limit of the sequence of stresses $(E^{(h)})$, once the weak limit of the strains $(G^{(h)})$ is known. For the proof, which is based on Taylor expansion, we refer to [10, Proposition 4.2].

Proposition 4.2. *Assume that the energy density W is differentiable and its derivative DW is Lipschitz continuous. Assume moreover that DW is differentiable at the identity. Suppose that*

$$G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3})$$

and define the rescaled stresses as in (3.16) by

$$E^{(h)} := \frac{1}{h} DW(Id + hG^{(h)}).$$

Then

$$E^{(h)} \rightharpoonup E := \mathcal{L}G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}), \tag{4.5}$$

where $\mathcal{L} := D^2W(Id)$.

We conclude this section with the truncation lemma used in the proof of Theorem 1.1. This a variant for thin domains of the standard results on the truncations of gradients (see, e.g., [3]).

Lemma 4.3. *There exists a constant $C > 0$ with the following property: for every $h > 0$, every $b > a > 0$ and every $u \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ there exist $\lambda \in [a,b]$ and a* *function* $v \in W^{1,\infty}(\Omega_h;\mathbb{R}^3)$ *such that*

$$\|\nabla v\|_{L^\infty} \leq \lambda, \tag{4.6}$$

$$\lambda^2 \mathcal{L}^3(\{x \in \Omega_h : \ u(x) \neq v(x)\}) \leq \frac{C}{\ln(b/a)} \int_{\{x\in\Omega_h: \ |\nabla u(x)|>\lambda\}} |\nabla u|^2 \, dx, \tag{4.7}$$

$$\|\nabla u - \nabla v\|_{L^2}^2 \leq \frac{C}{\ln(b/a)} \int_{\{x\in\Omega_h: \ |\nabla u(x)|>\lambda\}} |\nabla u|^2 \, dx. \tag{4.8}$$

Proof. Let Q be a square containing S. Without loss of generality we can assume that $Q = (0, M)^2$. Let

$$V := \Big\{ v \in L^2(S;\mathbb{R}^3) : \ \bar{v} := \int_S v \, dx_2 dx_3 = 0 \Big\}.$$

Then there exists a linear extension operator $\tilde{\mathcal{E}} : V \to \{v \in L^2(Q;\mathbb{R}^3) : \operatorname{supp} v \subset\subset Q\}$ such that $\tilde{\mathcal{E}}(v) \in W^{1,2}(Q;\mathbb{R}^3)$ for every $v \in V \cap W^{1,2}(S;\mathbb{R}^3)$ and for some constant $C > 0$ there holds

$$\|\tilde{\mathcal{E}}(v)\|_{L^2(Q)} \leq C\|v\|_{L^2(S)} \quad \text{for every } v \in V, \tag{4.9}$$

$$\|\nabla_{x_2,x_3}\tilde{\mathcal{E}}(v)\|_{L^2(Q)} \leq C\|\nabla_{x_2,x_3} v\|_{L^2(S)} \quad \text{for every } v \in V \cap W^{1,2}(S;\mathbb{R}^3) \tag{4.10}$$

(see, e.g., [13]). We can extend $\tilde{\mathcal{E}}$ to the whole space $L^2(S;\mathbb{R}^3)$ by considering the operator $\mathcal{E} : L^2(S;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^3)$ defined by

$$\mathcal{E}(v) := \tilde{\mathcal{E}}(v - \bar{v}) + \bar{v} \quad \text{for every } v \in L^2(S;\mathbb{R}^3).$$

It is easy to see that, if $v \in W^{1,2}(S;\mathbb{R}^3)$, then $\mathcal{E}(v) - \bar{v} \in W_0^{1,2}(Q;\mathbb{R}^3)$. Moreover, it follows immediately from (4.9) and (4.10) that there exists a constant C such that

$$\|\mathcal{E}(v)\|_{L^2(Q)} \leq C\|v\|_{L^2(S)} \quad \text{for every } v \in L^2(S;\mathbb{R}^3), \tag{4.11}$$

$$\|\nabla_{x_2,x_3}\mathcal{E}(v)\|_{L^2(Q)} \leq C\|\nabla_{x_2,x_3} v\|_{L^2(S)} \quad \text{for every } v \in W^{1,2}(S;\mathbb{R}^3). \tag{4.12}$$

Let $h > 0$ and let $\mathcal{E}_h : L^2(hS;\mathbb{R}^3) \to L^2(hQ;\mathbb{R}^3)$ be the extension operator obtained by scaling $\mathcal{E}$. Then, inequalities (4.11) and (4.12) imply that

$$\|\mathcal{E}_h(v)\|_{L^2(hQ)} \leq C\|v\|_{L^2(hS)} \quad \text{for every } v \in L^2(hS;\mathbb{R}^3), \tag{4.13}$$

$$\|\nabla_{x_2,x_3}\mathcal{E}_h(v)\|_{L^2(hQ)} \leq C\|\nabla_{x_2,x_3} v\|_{L^2(hS)} \quad \text{for every } v \in W^{1,2}(hS;\mathbb{R}^3), \tag{4.14}$$

where the constant C is independent of h.

Now, let $u \in W^{1,2}(\Omega_h;\mathbb{R}^3)$. First of all we can extend u to the set $U_h := (0, L) \times hQ$ by defining

$$\tilde{u}(x_1, \cdot) := \mathcal{E}_h(u(x_1, \cdot))$$

for a.e. $x_1 \in (0, L)$. By (4.13) and (4.14) we deduce that

$$\|\tilde{u}\|_{L^2(U_h)} \leq C\|u\|_{L^2(\Omega_h)}, \tag{4.15}$$

$$\|\nabla_{x_2,x_3}\tilde{u}\|_{L^2(U_h)} \leq C\|\nabla_{x_2,x_3} u\|_{L^2(\Omega_h)}. \tag{4.16}$$

As $\mathcal{E}_h$ is a linear operator, we have that $\partial_1\tilde{u}(x_1, \cdot) = \mathcal{E}_h(\partial_1 u(x_1, \cdot))$ for a.e. $x_1 \in (0, L)$, and thus, by (4.13)

$$\|\partial_1\tilde{u}\|_{L^2(U_h)} \leq C\|\partial_1 u\|_{L^2(\Omega_h)}. \tag{4.17}$$

As $\tilde{u}$ is constant on $(0,L)\times h\partial Q$, we can extend $\tilde{u}$ by successive reflection to the set $U := (0,L)\times Q$. By [10, Lemma 4.3] there exist $\lambda \in [a,b]$ and $w \in W^{1,\infty}(U;\mathbb{R}^3)$ such that

$$\|\nabla w\|_{L^\infty(U)} \leq \lambda \tag{4.18}$$

and

$$\lambda^2 \mathcal{L}^3(\{x \in U : \ \tilde{u}(x) \neq w(x)\}) \leq \frac{C}{\ln(b/a)} \int_U |\nabla \tilde{u}|^2\, dx. \tag{4.19}$$

Let N_h be the largest integer such that $h(N_h+1) \leq 1$. For $i,j = 0,\dots,N_h$ let $Q_{h,ij}$ be the square $(ihM, jhM) + hQ$, let $S_{h,ij} := (0,L)\times Q_{h,ij}$, and let

$$R_h := U \setminus \bigcup_{0\leq i,j\leq N_h} S_{h,ij}.$$

Since

$$\sum_{0\leq i,j\leq N_h} \mathcal{L}^3(\{\tilde{u} \neq w\} \cap S_{h,ij}) \leq \mathcal{L}^3(\{\tilde{u} \neq w\}),$$

there exists some indeces i_0, j_0 such that

$$\begin{aligned} \lambda^2 \mathcal{L}^3(\{\tilde{u} \neq w\} \cap S_{h,i_0j_0}) &\leq \frac{1}{(N_h+1)^2} \lambda^2 \mathcal{L}^3(\{\tilde{u} \neq w\}) \\ &\leq \frac{C}{(N_h+1)^2} \frac{1}{\ln(b/a)} \int_U |\nabla \tilde{u}|^2\, dx. \end{aligned} \tag{4.20}$$

Let $v : \Omega_h \to \mathbb{R}^3$ be the function defined by

$$v(x) := w(x_1, i_0hM + (-1)^{i_0}x_2, j_0hM + (-1)^{j_0}x_3) \qquad \text{for every } x \in \Omega_h.$$

It is clear that $v \in W^{1,\infty}(\Omega_h;\mathbb{R}^3)$ and that it satisfies (4.6) by (4.18). Moreover, since $\tilde{u}$ coincides with u in Ω_h and it has been extended to U by reflection, we have

$$\{x \in \Omega_h : \ u(x) \neq v(x)\} \subset \{x \in S_{h,i_0j_0} : \ u(x) \neq w(x)\} \tag{4.21}$$

and

$$\int_U |\nabla \tilde{u}|^2\, dx \leq (N_h+2)^2 \int_{U_h} |\nabla \tilde{u}|^2\, dx \leq C(N_h+2)^2 \int_{\Omega_h} |\nabla u|^2\, dx, \tag{4.22}$$

where the last inequality follows from (4.16) and (4.17). Now assertion (4.7) follows from (4.20)–(4.22).

Finally, inequality (4.8) is a standard consequence of (4.7). This concludes the proof of Lemma 4.3. □

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