Closure and convexity properties of closed relativistic strings

Giovanni Bellettini* Jens Hoppe† Matteo Novaga‡ Giandomenico Orlandi§

Abstract

We study various properties of closed relativistic strings. In particular, we characterize their closure under uniform convergence, extending a previous result by Y. Brenier on graph-like unbounded strings, and we discuss some related examples. Then we study the collapsing profile of convex planar strings which start with zero initial velocity, and we obtain a result analogous to the well-known theorem of Gage and Hamilton for the curvature flow of plane curves. We conclude the paper with the discussion of an example of weak Lipschitz evolution starting from the square in the plane.

keywords: String-theory, minimal surfaces, Minkowski space, geometric evolutions.

1 Introduction

Whereas string-theory in flat Minkowski space, as viewed by physicists, is thought to be completely understood on the classical non-interacting level, some of its aspects, such as singularity formation, have to be considered more or less open problems from the mathematical point of view. Despite the vast amount of solid work, in particular in the context of cosmic strings where numerous examples have been treated explicitely, it nevertheless seemed to us of some interest to attempt formulating a few aspects of the theory in a rigorous mathematical way.

The subject of this paper are time-like minimal surfaces in the $(1 + n)$-dimensional flat Minkowski space, namely those surfaces (called relativistic strings) which are critical points for the 2-dimensional Minkowski area. We recall (see for instance [23, Chapter 6]) that the Minkowski area $\mathcal{S}(X)$ of a time-like map $X : [0, T] \times [0, L] \to \mathbb{R}^{1+n}$ of class $\mathcal{C}^1$ is given by

$$\mathcal{S}(X) = \int_{[0,T]\times[0,L]} \sqrt{\langle X_t, X_x\rangle_m^2 - \langle X_t, X_t\rangle_m \langle X_x, X_x\rangle_m}\; dtdx, \tag{1.1}$$

where $\langle \cdot, \cdot\rangle_m$ denotes the Minkowskian scalar product in $\mathbb{R}^{1+n}$ associated with the metric tensor $\mathrm{diag}(-1, +1, \ldots, +1)$. In the sequel we always assume that X has the form

$$X(t, x) := (t, \gamma(t, x)), \qquad (t, x) \in [0, T] \times [0, L], \tag{1.2}$$

*Dipartimento di Matematica, Università di Roma Tor Vergata, via della Ricerca Scientifica, Roma 00133, Italy, and Laboratori Nazionali di Frascati dell'INFN, via E. Fermi, 40, Frascati (Roma) 00044, Italy. e-mail: Giovanni.Bellettini@lnf.infn.it

†KTH Mathematik, SE-100 44 Stockholm. e-mail: hoppe@kth.se

‡Dipartimento di Matematica Pura e Applicata, Università di Padova, via Trieste 63, Padova 35121, Italy. e-mail: novaga@math.unipd.it

§Dipartimento di Informatica, Università di Verona, strada le Grazie 15, Verona 37134, Italy. email: giandomenico.orlandi@univr.it

and that $\gamma(t,\cdot)$ is closed. Provided that all quantities involved are sufficiently smooth and up to a suitable reparametrization, it is well-known (see for instance [22, 15, 23]) that if X is a critical point of $\mathcal{S}$ then

$$\begin{cases} \gamma_{tt} = \gamma_{xx}, \\ \langle \gamma_t, \gamma_x \rangle = 0, \\ |\gamma_t|^2 + |\gamma_x|^2 = 1, \end{cases} \tag{1.3}$$

see also Section 2 for the details. Critical points of $\mathcal{S}$ have been considered by Born and Infeld [7] (in the case of graphs), and analyzed later on by various authors (see [20], [15] and references therein). A large number of explicit solutions to (1.3), possibly with singularities in the image of the parametrization (such as cusps, for instance, where the above regularity condition fails) is known, see for instance [3, Chapter 4], [22].

The nonlinear constraint in (1.3) is not closed under uniform convergence. Indeed, many examples in the physical literature [8, 22, 20, 21] show that the limit of a convergent sequence of relativistic strings is not, in general, a relativistic string, thus leading to the concept of wiggly string. A natural question is then to characterize the closure of relativistic strings. This issue is discussed in the physical literature, see for instance [22]. To our knowledge, a complete answer is provided by Y. Brenier in [6], in the case of strings which are entire graphs. In this paper we obtain the same result in the case of closed strings (see Theorem 3.1). Roughly speaking, the nonlinear constraint is convexified (compare (2.23) and (3.2)), and limit solutions have in general only Lipschitz regularity. Motivated by an example described by Neu in [20] and by Theorem 3.1, in Section 4 we discuss various examples. In particular, and as already observed by Neu when $n = 2$, we show how additional small oscillations superimposed on the initial datum can prevent the limit solution to collapse to a point (see Examples 4.2 and 4.3).

Related mathematical questions on problem (1.3) concern the qualitative properties of solutions for special initial data, and their asymptotic shape near a singularity time, for instance near a collapse. This latter problem is, in turn, intimately related to the existence of weak global solutions, to be defined also after the onset of a singularity. In this paper we begin a preliminary discussion on this subject. More precisely, in Section 5 we address the study of the convexity preserving properties of the solutions of (1.3), when $n = 2$, and their asymptotic profile near a collapsing time. In Proposition 5.4 we show that a relativistic string which is smooth and convex and has zero initial velocity, remains convex for subsequent times, and shrinks to a point while its shape approaches a round circle. This result is analogous to the one proven by Gage and Hamilton in [11] for curvature flow of plane curves, and the one proven in [17] for the hyperbolic curvature flow (non relativistic case). However, differently from the parabolic case (see [11, 13]), here the collapsing singularity is nongeneric, the generic singularity being the formation of a cusp, as discussed in [3, 9]. Adopting as a definition of weak solution the one given by D'Alembert formula for the linear wave system in (1.3), it follows that after the collapse the solution restarts, and the motion is continued in a periodic way. This is in accordance with the conservative character of the wave system in (1.3).

D'Alembert formula can still provide a possibile definition of weak solutions for Lipschitz immersions. In Example 5.6 we study the solution corresponding to a homotetically shrinking square. In this case it turns out that the conservation law (2.16) below is valid only in special interval of times. The same example shows that, differently to the case of smooth strings, for

Lipschitz strings the collapsing profile is not necessarily circular.

2 Notation and preliminary observations

For $n \geq 2$ we denote by $\mathbb{R}^{1+n}$ the $(1+n)$-dimensional Minkowski space, which is endowed with the metric tensor $\mathrm{diag}(-1, +1, \ldots, +1)$. We indicate by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the euclidean scalar product and norm in $\mathbb{R}^n$, respectively. Given $T > 0$ and $L > 0$, the Minkowski area $\mathcal{S}(X)$ of a time-like map $X : [0, T] \times [0, L] \to \mathbb{R}^{1+n}$ of class $\mathcal{C}^1$ is defined in (1.1), where $X = X(t, x)$, $X_t := \partial_t X$ and $X_x := \partial_x X$. Note that (1.1) is well defined if X is only Lipschitz continuous.

2.1 Assumptions on γ

As already said in the Introduction, we will assume that X has the form (1.2), where $\gamma \in \mathcal{C}^1([0, T] \times [0, L]; \mathbb{R}^n)$ satisfies the periodicity conditions

$$\gamma(\cdot, 0) = \gamma(\cdot, L), \qquad \gamma_x(\cdot, 0) = \gamma_x(\cdot, L). \tag{2.1}$$

When necessary, the map γ will be periodically extended with respect to x on the whole of $[0, T] \times \mathbb{R}$; we still denote by $\gamma \in \mathcal{C}^1([0, T] \times \mathbb{R})$ such an extension.

Definition 2.1. *We say that γ is regular if $\gamma_x(t, x) \neq 0$ for any $(t, x) \in [0, T] \times [0, L]$.*

Let $\gamma \in \mathcal{C}^1([0, T] \times [0, L]; \mathbb{R}^n)$ be regular; if there exist a bounded closed interval $I \subset \mathbb{R}$ and a map $\mathrm{r} \in \mathcal{C}^1([0, T] \times I; [0, L])$ such that $\mathrm{r}(t, \cdot)$ is strictly monotone, then the map $(t, \sigma) \in [0, T] \times I \to \gamma(t, \mathrm{r}(t, \sigma))$ is said a reparametrization of γ.
The normal velocity vector is given by $\gamma_t^\perp$, where $^\perp$ denotes the orthogonal projection onto the normal space, so that

$$\gamma_t^\perp = \gamma_t - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|} \rangle \frac{\gamma_x}{|\gamma_x|}. \tag{2.2}$$

Definition 2.2. *Let $\gamma \in \mathcal{C}^1([0, T] \times [0, L]; \mathbb{R}^n)$. We say that γ is strictly admissible if*

$$|\gamma_t^\perp|^2 < 1 \qquad \text{in } [0, T] \times [0, L]. \tag{2.3}$$

2.2 The lagrangian $\mathcal{L}$

Under assumptions (1.2) and (2.3) we have

$$\sqrt{\langle X_t, X_x \rangle_m^2 - \langle X_t, X_t \rangle_m \langle X_x, X_x \rangle_m} = \sqrt{\langle \gamma_t, \gamma_x \rangle^2 + |\gamma_x|^2 (1 - |\gamma_t|^2)},$$

and

$$\mathcal{S}(X) = \int_{[0,T] \times [0,L]} \mathcal{L}(\gamma_t, \gamma_x) \, dt dx. \tag{2.4}$$

Here the function $\mathcal{L} : \mathrm{dom}(\mathcal{L}) = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \ \langle \xi, \eta \rangle^2 \geq |\eta|^2 (|\xi|^2 - 1)\} \to [0, +\infty)$ is defined as

$$\mathcal{L}(\xi, \eta) := \sqrt{\langle \xi, \eta \rangle^2 + |\eta|^2 (1 - |\xi|^2)}, \qquad (\xi, \eta) \in \mathrm{dom}(\mathcal{L}).$$

Observe that $(\xi, \eta) \in \mathrm{dom}(\mathcal{L})$ implies $(\xi, \alpha\eta) \in \mathrm{dom}(\mathcal{L})$ for any $\alpha \in \mathbb{R}$, and

$$\mathcal{L}(\xi, \alpha\eta) = |\alpha| \mathcal{L}(\xi, \eta), \qquad \alpha \in \mathbb{R}, \ (\xi, \eta) \in \mathrm{dom}(\mathcal{L}). \tag{2.5}$$

Hence if $\widetilde{\gamma}$ is a reparametrization of the regular curve γ then the righe hand side of (2.4) remains unchanged.
Note also that

$$(\xi,\eta) \in \mathrm{dom}(\mathcal{L}), \ \langle \xi,\eta\rangle = 0, \quad |\xi|^2 + |\eta|^2 = 1 \qquad \Rightarrow \qquad \mathcal{L}(\xi,\eta) = |\eta|^2. \tag{2.6}$$

Definition 2.3. *Let $I \subset \mathbb{R}$ be a bounded closed interval, and let $\gamma \in \mathcal{C}^1([0,T] \times I; \mathbb{R}^n)$ be a regular map. We say that γ is parametrized in orthogonal way if*

$$\langle \gamma_t, \gamma_x\rangle = 0 \qquad \text{in } [0,T] \times I. \tag{2.7}$$

Remark 2.4. Any regular map $\gamma \in \mathcal{C}^1([0,T] \times [0,L]; \mathbb{R}^n)$ can be parametrized in orthogonal way (see for instance [1, Theorem 8]) in $[0,T] \times [0,L]$. Indeed, it is enough to consider the map $(t,x) \in [0,T] \times [0,L] \to \gamma(t, \mathrm{r}(t,x))$, where $\mathrm{r} \in \mathcal{C}^1([0,T] \times [0,L]; [0,L])$ satisfies the linear transport equation $\mathrm{r}_t = -\frac{\langle \gamma_t, \gamma_x\rangle}{|\gamma_x|^2} \mathrm{r}_x$. It follows that the parametrization becomes unique once we fix $\mathrm{r}(0,\cdot)$.

Notation: in what follows we use the symbol E with the following meaning. Let $\gamma \in \mathcal{C}^1([0,T] \times [0,L])$ be a regular strictly admissible map. First we reparametrize $\gamma(0,\cdot)$ on the interval $[0,E]$ in such a way that $|\gamma_x(0,\cdot)|^2 = 1 - |\gamma_t(0,\cdot)|^2$. Next, recalling Remark 2.4, we further uniquely reparametrize the map γ in an orthogonal way in the parameters space $[0,T] \times [0,E]$. We therefore achieve, at the same time, the two conditions

$$|\gamma_t(0,x)|^2 + |\gamma_x(0,x)|^2 = 1, \qquad x \in [0,E], \tag{2.8}$$

$$\langle \gamma_t(t,x), \gamma_x(t,x)\rangle = 0, \qquad (t,x) \in [0,T] \times [0,E]. \tag{2.9}$$

2.3 First variation of $\mathcal{S}$

We recall that $X \in \mathcal{C}^1([0,T] \times [0,L]; \mathbb{R}^{1+n})$ is called a critical point of $\mathcal{S}$ if

$$\frac{d}{d\lambda} \mathcal{S}(X + \lambda\Phi)_{|\lambda=0} = 0, \qquad \Phi \in \mathcal{C}^1\left([0,T] \times [0,L]; \mathbb{R}^{1+n}\right).$$

The first variation of $\mathcal{S}$ is a classical computation (see for instance [23, Section 6.5]).

Lemma 2.5. *Let $X \in \mathcal{C}^1([0,T]\times[0,L];\mathbb{R}^{1+n})$ be a critical point of $\mathcal{S}$ of the form (1.2), with γ satisfying the periodicity condition (2.1), regular and strictly admissible. Then*

$$\int_{[0,T]\times[0,L]} \frac{1}{\mathcal{L}(\gamma_t,\gamma_x)}\Big(|\gamma_x|^2\psi_t - \langle\gamma_t,\gamma_x\rangle\psi_x\Big)\, dtdx = 0, \qquad \psi \in \mathcal{C}^\infty([0,T]\times[0,L]), \tag{2.10}$$

$$\begin{aligned}\int_{[0,T]\times[0,L]} \frac{1}{\mathcal{L}(\gamma_t,\gamma_x)}\Big[&\langle\gamma_t,\gamma_x\rangle\Big(\langle\gamma_x,\phi_t\rangle + \langle\gamma_t,\phi_x\rangle\Big) \\ &- (|\gamma_t|^2-1)\langle\gamma_x,\phi_x\rangle - |\gamma_x|^2\langle\gamma_t,\phi_t\rangle\Big]\, dtdx = 0, \end{aligned} \qquad \phi \in \mathcal{C}^\infty([0,T]\times[0,L];\mathbb{R}^n), \tag{2.11}$$

$$\gamma_t^\perp = 0 \qquad \text{on } \{0\}\times[0,L] \quad \text{and on} \quad \{T\}\times[0,L]. \tag{2.12}$$

If $\gamma \in \mathcal{C}^2([0,T]\times[0,L];\mathbb{R}^n)$ then

$$-\left(\frac{|\gamma_x|^2}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_t + \left(\frac{\langle\gamma_t,\gamma_x\rangle}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_x = 0, \tag{2.13}$$

$$\frac{|\gamma_x|^2\gamma_{tt} + \big(|\gamma_t|^2-1\big)\gamma_{xx} - 2\langle\gamma_t,\gamma_x\rangle\gamma_{xt}}{\mathcal{L}(\gamma_t,\gamma_x)} + \left[\left(\frac{|\gamma_t|^2-1}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_x - \left(\frac{\langle\gamma_t,\gamma_x\rangle}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_t\right]\gamma_x = 0 \tag{2.14}$$

in $[0,T]\times[0,L]$.

Proof. Let $\phi \in \mathcal{C}^\infty([0,T]\times[0,L];\mathbb{R}^n)$. For $\lambda\in\mathbb{R}$ and $|\lambda|$ small enough we have that $\gamma+\lambda\phi$ is regular and strictly admissible. Then, being X critical for $\mathcal{S}$, and taking $\psi \in \mathcal{C}^\infty([0,T]\times[0,L])$, we have

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\mathcal{S}\left(X+\lambda(\psi,\phi)\right)_{|\lambda=0} \\ &= \int_{[0,T]\times[0,L]} \frac{d}{d\lambda}\Bigg(\langle(1+\lambda\psi_t,\gamma_t+\lambda\phi_t),(\lambda\psi_x,\gamma_x+\lambda\phi_x)\rangle_m^2 \\ &\quad - \langle(1+\lambda\psi_t,\gamma_t+\lambda\phi_t),(1+\lambda\psi_t,\gamma_t+\lambda\phi_t)\rangle_m\,\langle(\lambda\psi_x,\gamma_x+\lambda\phi_x),(\lambda\psi_x,\gamma_x+\lambda\phi_x)\rangle_m\Bigg)^{1/2}_{|\lambda=0} dtdx \\ &= \int_{[0,T]\times[0,L]} \frac{\big(\langle\gamma_t,\gamma_x\rangle(\langle\gamma_x,\phi_t\rangle+\langle\gamma_t,\phi_x\rangle-\psi_x) - \langle\gamma_x,\phi_x\rangle(|\gamma_t|^2-1) - |\gamma_x|^2(\langle\gamma_t,\phi_t\rangle-\psi_t)\big)}{\mathcal{L}(\gamma_t,\gamma_x)}\, dtdx, \end{aligned}$$

and (2.10) and (2.11) immediately follow.
Since (1.2) implies that $X(0,x)$ belongs to the hyperplane $\{(x^0,x^1,\dots,x^n)\in\mathbb{R}^{1+n} : x^0=0\}$ for any $x\in[0,L]$, once integrating by parts with respect to t, the boundary contributions from (2.11) give

$$\frac{1}{\mathcal{L}(\gamma_t,\gamma_x)}\left(-\gamma_x\langle\gamma_t,\gamma_x\rangle + \gamma_t|\gamma_x|^2\right) = 0 \qquad \text{on } \{0\}\times[0,L]. \tag{2.15}$$

Recalling the γ is regular and using the expression of v in (2.2), it follows that (2.15) is equivalent to (2.12). A similar condition can be obtained on $\{T\}\times[0,L]$.

Assume now that $\gamma \in \mathcal{C}^2([0,T]\times[0,L];\mathbb{R}^n)$. Then (2.13) follows from (2.10) by taking ψ with compact support in $(0,T)\times[0,L]$, and integrating by parts. Taking ϕ with compact support in $(0,T)\times(0,L)$, integrating by parts in (2.11) and using (2.13), it follows

$$-\left(\frac{\langle\gamma_t,\gamma_x\rangle}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_t\gamma_x - 2\frac{\langle\gamma_t,\gamma_x\rangle}{\mathcal{L}(\gamma_t,\gamma_x)}\gamma_{xt} + \left(\frac{|\gamma_t|^2-1}{\mathcal{L}(\gamma_t,\gamma_x)}\right)_x\gamma_x + \frac{|\gamma_t|^2-1}{\mathcal{L}(\gamma_t,\gamma_x)}\gamma_{xx} + \frac{|\gamma_x|^2}{\mathcal{L}(\gamma_t,\gamma_x)}\gamma_{tt} = 0,$$

which is (2.14). □

Note that by the positive one-homogeneity of $\mathcal{L}(\xi,\cdot)$ in (2.5) it follows that (2.10), (2.11), (2.13) and (2.14) are invariant under reparametrizations of γ with respect to x.

Remark 2.6. Under the assumptions of Lemma 2.5, integrating (2.13) on $[0,L]$ one obtains the conservation law

$$\frac{d}{dt}\int_{[0,L]}\frac{|\gamma_x|^2}{\mathcal{L}(\gamma_t,\gamma_x)}\,dx = 0, \qquad t\in(0,T). \tag{2.16}$$

This conservation law can be equivalently written on the image $\gamma(t,[0,L])$ as follows:

$$\frac{d}{dt}\int_{\gamma(t,[0,L])}\frac{\theta(t,x)}{\sqrt{1-|v(t,\cdot)|^2}}\,d\mathcal{H}^1 = 0, \tag{2.17}$$

where, given $t\in[0,T]$, $\theta(t,x)$ is the cardinality of the set $\gamma^{-1}(t,\gamma(t,x))$ (in particular, $\theta(t,x)=1$ if $\gamma(t,\cdot)$ is an embedding), $v := \gamma_t^{\perp}$, and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure in $\mathbb{R}^n$. Indeed

$$\begin{aligned}\int_{[0,L]}\frac{|\gamma_x|^2}{\mathcal{L}(\gamma_t,\gamma_x)}\,dx &= \int_{[0,L]}\frac{|\gamma_x|}{\sqrt{\langle\gamma_t,\frac{\gamma_x}{|\gamma_x|}\rangle^2+1-|\gamma_t|^2}}\,dx = \int_{[0,L]}\frac{|\gamma_x|}{\sqrt{1-|\gamma_t^{\perp}|^2}}\,dx\\ &= \int_{\gamma(t,[0,L])}\frac{\theta(t,x)}{\sqrt{1-|v(t,\cdot)|^2}}\,d\mathcal{H}^1 = 0,\end{aligned}$$

where the last equality follows from the area formula [2].

Corollary 2.7. *Assume that* $\gamma \in \mathcal{C}^1([0,T]\times[0,L];\mathbb{R}^n)$ *is regular, strictly admissible, and satisfies* (2.11)*. Define*

$$\rho(x) := \frac{|\gamma_x(0,x)|}{\sqrt{1-|\gamma_t(0,x)|^2}}, \qquad x \in [0,L].$$

If γ *is parametrized in an orthogonal way then*

(i) the conservation law (2.16) *strengthen into the pointwise conservation law*

$$\frac{|\gamma_x(t,x)|}{\sqrt{1-|\gamma_t(t,x)|^2}} = \rho(x), \qquad (t,x) \in [0,T]\times[0,L]; \tag{2.18}$$

(ii) the condition (2.11) *becomes*

$$\int_{[0,T]\times[0,L]} \langle \gamma_t, \phi_t\rangle \; dtdx = \int_{[0,T]\times[0,L]} \langle \frac{\gamma_x}{\rho}, \left(\frac{\phi}{\rho}\right)_x \rangle \; dtdx;$$

(iii) if we reparametrize $\gamma(0,\cdot)$ *on the interval* $[0,E]$ *so that* ρ *is constantly equal to* 1*, that is if* (2.8) *holds, then*

$$|\gamma_t|^2 + |\gamma_x|^2 = 1 \qquad \text{in } [0,T]\times[0,E], \tag{2.19}$$

and γ *becomes a* $\mathcal{C}^1$ *distributional solution of the wave linear system*

$$\gamma_{tt} = \gamma_{xx} \qquad \text{in } [0,T]\times[0,E]. \tag{2.20}$$

Proof. Using the orthogonality condition (2.7), equation (2.10) reduces to

$$\int_{[0,T]\times[0,L]} \frac{|\gamma_x|}{\sqrt{1-|\gamma_t|^2}} \; \psi_t \; dtdx = 0, \qquad \psi \in \mathcal{C}^\infty([0,T]\times[0,L]),$$

which implies (2.18).
From (2.11) and the fact that the parametrization of γ is orthogonal, it follows

$$\int_{[0,T]\times[0,L]} \left(\langle \frac{\gamma_x(1-|\gamma_t|^2)}{\mathcal{L}(\gamma_t,\gamma_x)}, \phi_x\rangle - \langle \frac{\gamma_t|\gamma_x|^2}{\mathcal{L}(\gamma_t,\gamma_x)}, \phi_t\rangle \right) \; dtdx = 0. \tag{2.21}$$

Orthogonality of the parametrization also implies that $\mathcal{L}(\gamma_t,\gamma_x) = |\gamma_x|\sqrt{1-|\gamma_t|^2}$, so that (2.21) becomes

$$\int_{[0,T]\times[0,L]} \left(\langle \frac{\gamma_x\sqrt{1-|\gamma_t|^2}}{|\gamma_x|}, \phi_x\rangle - \langle \frac{\gamma_t|\gamma_x|}{\sqrt{1-|\gamma_t|^2)}}, \phi_t\rangle \right) \; dtdx = 0,$$

which gives (ii).
Eventually, assertion (iii) follows directly from (i) and (ii). □

Remark 2.8. We point out that Corollary 2.7 (iii) shows that if the constraint $|\gamma_t|^2+|\gamma_x|^2 = 1$ is valid at the initial time $t = 0$, then it remains valid at subsequent times.

A number of solutions of (2.7), (2.19), (2.20) are known, see for instance [22, Section 6.2.4], [3, Chapter 4], [12], the simplest one being probably the following [20]. Let $n = 2$, $R > 0$ and $a(s) = b(s) := R(\cos \frac{s}{R}, \sin \frac{s}{R})$ for any $s \in \mathbb{R}$. The solution to (2.20) becomes

$$\gamma(t,x) = R\left(\cos\frac{x}{R}, \sin\frac{x}{R}\right)\cos\frac{t}{R}, \qquad (t,x) \in (-R\pi/2, R\pi/2) \times [0,E],$$

with $E = 2\pi R$. Note that at the singular times $t = \pm E/4$, the condition $\gamma_x(t,\cdot) \neq 0$ is not satisfied, and $\gamma(t,[0,E])$ reduces to a point.

2.4 Representation of the solutions and a concept of weak solution

Let $X \in \mathcal{C}^1([0,T] \times [0,E]; \mathbb{R}^{1+n})$ (resp. $X \in \mathcal{C}^2([0,T] \times [0,E]; \mathbb{R}^{1+n})$) be a critical point of $\mathcal{S}$ of the form (1.2), where $\gamma \in \mathcal{C}^1([0,T] \times [0,L]; \mathbb{R}^n)$ (resp. $\gamma \in \mathcal{C}^2([0,T] \times [0,L]; \mathbb{R}^n)$) is strictly admissible and regular. We have seen that there exists an orthogonal parametrization of γ satisfying (2.8), hence by Corollary 2.7 (iii) we have that γ becomes a distributional (resp. classical) solution to (2.20). Hence there exist E-periodic maps $a, b \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n)$ (resp. $\mathcal{C}^2(\mathbb{R}; \mathbb{R}^n)$) such that

$$\gamma(t,x) = \frac{1}{2}\left[a(x+t) + b(x-t)\right], \qquad (t,x) \in [0,T] \times [0,E], \tag{2.22}$$

$$|a'| = |b'| = 1 \qquad \text{in } \mathbb{R}. \tag{2.23}$$

Note that $\gamma_t(0,\cdot) = 0$ if and only if there exists $w \in \mathbb{R}^n$ such that $a = b + w$.

Remark 2.9. Since a and b are defined on the whole of $\mathbb{R}$, the right hand side of (2.22) can be considered as the definition of the map γ on the left hand side also for $(t,x) \in (\mathbb{R} \setminus [0,T]) \times [0,E]$. Namely, the right hand side of (2.22) provides a global in time $\mathcal{C}^1$ (resp. $\mathcal{C}^2$) weak solution, denoted by γ to (2.7), (2.19) and (2.20) defined for $(t,x) \in \mathbb{R} \times [0,E]$. In general it may happen that $\gamma_x(\bar{t},\bar{x}) = 0$ for some $(\bar{t},\bar{x}) \in (\mathbb{R} \setminus [0,T]) \times [0,E]$, since (2.23) does not prevent that $a'(\bar{x}+\bar{t}) = -b'(\bar{x}-\bar{t})$. Hence singularities in the image $\gamma(\bar{t},[0,E])$ (such as cusps, for instance) are in general expected, and may possibly persist in time (see Remark 5.3 below). We point out that such a weak solution could not coincide with the weak solution proposed in [4] when singularities are present. Another notion of weak solution to the lorentzian minimal surface equation in the case of graphs has been proposed in [6].

We conclude this section by observing that the time-slices $\gamma(t,\cdot)$ of a surface which is critical for $\mathcal{S}$ satisfy the geometric equation

$$\mathrm{a} = \left(1 - |v|^2\right)\kappa. \tag{2.24}$$

Here, if $\gamma \in \mathcal{C}^2([0,T] \times [0,L]; \mathbb{R}^n)$ is regular, $v = \gamma_t^\perp$ denotes the normal velocity vector, κ denotes the curvature vector and a the normal acceleration vector, respectively given[1] by

$$\kappa = \frac{\gamma_{xx}^\perp}{|\gamma_x|^2}, \qquad \mathrm{a} = \left(v_t - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \frac{v_x}{|\gamma_x|}\right)^\perp = \gamma_{tt}^\perp + \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \left(\frac{\gamma_{xx}}{|\gamma_x|^2} - 2\frac{\gamma_{xt}}{|\gamma_x|}\right)^\perp. \tag{2.25}$$

[1] When γ is an embedding, if we set $\Gamma(t) := \gamma(t,[0,L])$, then $\mathrm{a} = (\nabla \eta_{tt})^\perp$ on $\Gamma(t)$, where $\eta(t,z) := \mathrm{dist}(z,\Gamma(t))^2/2$ for $(t,z) \in [0,T] \times \mathbb{R}^n$. In the case $n = 2$ it holds $v = -d_t \nabla d$ and $\mathrm{a} = -d_{tt}\nabla d$, where d is the signed distance from $\Gamma(t)$.

To show (2.24), observe that

$$(\gamma_t^\perp)_t{}^\perp = \gamma_{tt}^\perp - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \left(\frac{\gamma_x}{|\gamma_x|}\right)_t^\perp = \gamma_{tt}^\perp - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \frac{\gamma_{tx}^\perp}{|\gamma_x|},$$
$$(\gamma_t^\perp)_x{}^\perp = \gamma_{tx}^\perp - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \left(\frac{\gamma_x}{|\gamma_x|}\right)_x^\perp = \gamma_{tx}^\perp - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \frac{\gamma_{xx}^\perp}{|\gamma_x|},$$

so that

$$\mathrm{a} = \gamma_{tt}^\perp - 2\langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle \frac{\gamma_{tx}^\perp}{|\gamma_x|} - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle^2 \frac{\gamma_{xx}^\perp}{|\gamma_x|^2}, \tag{2.26}$$

and therefore if γ is parametrized in an orthogonal way, then $\mathrm{a} = (\gamma_t^\perp)_t^\perp$. Now, projecting both sides of (2.14) onto the normal space to $\gamma(t, \cdot)$ gives

$$\gamma_{tt}^\perp + \frac{|\gamma_t|^2 - 1}{|\gamma_x|^2}\gamma_{xx}^\perp - 2\frac{\langle \gamma_t, \gamma_x\rangle}{|\gamma_x|^2}\gamma_{xt}^\perp = 0. \tag{2.27}$$

Inserting (2.26) into (2.27) gives

$$\mathrm{a} = \frac{1 - |\gamma_t|^2 - \langle \gamma_t, \frac{\gamma_x}{|\gamma_x|}\rangle^2}{|\gamma_x|^2}\gamma_{xx}^\perp = \left(1 - |\gamma_t^\perp|^2\right) \frac{\gamma_{xx}^\perp}{|\gamma_x|^2} = (1 - |v|^2)\kappa.$$

3 Closure of solutions

The closure result is motivated by the example in [20] and is strictly related with the discussion in [22, Section 6.5.2] (see also the references therein). This result is similar to the one in [6], where maps which are graphs defined in the whole of $\mathbb{R} \times \mathbb{R}$ are considered.

Theorem 3.1. *Let $\{E_k\}$ be a sequence of positive numbers converging to $E \in [0, +\infty)$ as $k \to +\infty$. Let $\{\gamma_k\} \subset \mathcal{C}^1([0, T] \times [0, E_k]; \mathbb{R}^n)$ be a sequence of E_k-periodic regular strictly admissible orthogonally parametrized maps*

$$|\gamma_{kt}(0, x)|^2 + |\gamma_{kx}(0, x)|^2 = 1, \qquad x \in [0, E_k], \tag{3.1}$$

and solving the wave system (2.20)*. The following assertions hold.*

(i) if [2] *$\{\gamma_k\}$ converges to a map $\gamma \in \mathrm{Lip}([0, T] \times \mathbb{R}; \mathbb{R}^n)$ uniformly in $[0, T] \times \mathbb{R}$ as $k \to +\infty$, then there exist E-periodic maps $a, b \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}^n)$ with*

$$|a'| \leq 1, \qquad |b'| \leq 1 \qquad \text{a.e. in } \mathbb{R} \tag{3.2}$$

such that γ has the representation (2.22) *in $[0, T] \times \mathbb{R}$.*

(ii) If $\gamma \in \mathrm{Lip}([0, T] \times [0, E]; \mathbb{R}^n)$ can be represented as in (2.22) *where $a, b \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}^n)$ are E-periodic maps satisfying* (3.2)*, then there exists a sequence $\{\gamma_k\} \subset \mathcal{C}^2([0, T] \times [0, E]; \mathbb{R}^n)$ of E-periodic maps solving* (2.20)*,* (2.7)*,* (2.8) *in $[0, T] \times [0, E]$ such that $\{\gamma_k\}$ converges to γ uniformly in $[0, T] \times \mathbb{R}$.*

[2] following the convention described in Section 2.1, we consider here each γ_k as periodically extended with respect to x on the whole of $[0, T] \times \mathbb{R}$.

Figure 1: (a): the construction of $\bar{a}_k$ defined in (3.4) for a piecewise linear map a, in the (image of) the interval $[L_i, L_{i+1}]$; the slopes are $c_{i+1} \pm d_{i+1}$. (b): the smoothing of the corners in order to have $\bar{a}_k \in \mathcal{C}^2$, keeping the length constraint satisfied.

Proof. Let us prove (i). Let a_k, b_k, with $|a_k'| = |b_k'| = 1$, be such that (2.22) holds with E, γ, a, b replaced by E_k, γ_k, a_k, b_k, respectively. Then assertion (i) follows by recalling that, if $L > \sup_k E_k$, the set $\{u \in W^{1,\infty}(\mathbb{R};\mathbb{R}^n) : u \text{ is } L \text{ periodic}, |u'| \leq 1 \text{ a.e.}\}$ is the weak* closure of $\{u \in W^{1,\infty}(\mathbb{R};\mathbb{R}^n) : u \text{ is } L \text{ periodic}, |u'| = 1 \text{ a.e.}\}$, and in particular it is closed under the uniform convergence on the compact subsets.

Let us prove (ii). Given $a, b \in \mathrm{Lip}([0,E];\mathbb{R}^n)$ satisfying $|a'| \leq 1$ and $|b'| \leq 1$ almost everywhere, it is enough to find two sequences $\{a_k\}$, $\{b_k\} \subset \mathcal{C}^2([0,E];\mathbb{R}^n)$, with $|a_k'| = |b_k'| = 1$, uniformly converging to a, b, respectively, as $k \to \infty$. It is also sufficient to prove this assertion for a, b belonging to the dense (for the uniform convergence) class of piecewise linear immersions satisfying (3.2), since one then concludes for general a, b using a diagonal argument. We will show the assertion for the map a, the construction for b being similar. Let a be an E-periodic piecewise linear immersion satisfying (3.2), i.e. we identify the points $\{0\}$ and $\{E\}$ and there exist $m+2$ points $0 =: L_0 < L_1 < \cdots < L_{m+1} := E$ of the interval $[0,E]$ such that

$$a(x) = a(L_i) + (x - L_i)c_{i+1}, \qquad x \in [L_i, L_{i+1}], \quad i = 0, \dots, m,$$

with $c_{i+1} \in \mathbb{R}^n$, $|c_{i+1}| \leq 1$ for $i = 0, \dots, m$. Choose $d_{i+1} \in \mathbb{R}^n$ so that

$$\langle d_{i+1}, c_{i+1} \rangle = 0, \qquad |d_{i+1}|^2 = 1 - |c_{i+1}|^2, \qquad i = 0, \dots, m. \tag{3.3}$$

Fix $k \in \mathbb{N}$ even. For $i = 0, \dots, m$ we take a partition of $[L_i, L_{i+1}]$ into k subintervals of equal length: precisely, i.e., $j = 0, \dots, k$ set $L_i^j := L_i + \frac{j}{k}(L_{i+1} - L_i)$ (we write $L_i^0 = L_i$ and $L_i^k = L_{i+1}$). Define

$$\bar{a}_k(x) := a(x) + (-1)^j (x - L_i^j) d_{i+1}, \qquad L_i^j \leq x \leq L_i^{j+1}, \tag{3.4}$$

see Figure 1 (a). Since k is even, $\bar{a}_k \in \mathrm{Lip}([0,E];\mathbb{R}^n)$. Moreover from (3.3) it follows $|\bar{a}_k'(x)| = 1$ for any $x \in [0,E]$ out of a finite set depending on k. Eventually, by construction $|\bar{a}_k(x) - a(x)| \leq \frac{L}{k}$ for any $x \in [0,E]$, so that $\bar{a}_k \to a$ uniformly in $[0,E]$ as $k \to +\infty$. Once a similar construction for b (thus leading to the definition of $\{\bar{b}_k\}$) is made, let us consider the sequence $\{\gamma_k\}$ of maps defined as $\bar{\gamma}_k(t,x) := \frac{1}{2}\left[\bar{a}_k(x+t) + \bar{b}_k(x-t)\right]$ for any $(t,x) \in [0,T] \times [0,E]$. These maps belong to $\mathrm{Lip}([0,T] \times [0,E];\mathbb{R}^n)$, and must be regularized in order to avoid the presence of corners.

Given $k \in \mathbb{N}$, $k \geq 1$, let $\ell_k \in \left(0, \min_{i=0,\dots,m} \dfrac{L_{i+1} - L_i}{3k}\right)$ and fix $\eta \in (0, \ell_k/3)$. We apply Lemma 3.2 below with $\ell = \ell_k$ in the intervals $[L_i^j - \ell_k, L_i^j + \ell_k]$, identifying $\mathbb{R}^2$ with $\bar{a}_k(L_i^j) + \mathrm{span}\{c_{i+1} + d_{i+1}, c_{i+1} - d_{i+1}\}$ if $j \neq 0$, and with $\bar{a}_k(L_i) + \mathrm{span}\{c_{i+1} + d_{i+1} + c_i - d_i, c_{i+1} + d_{i+1} - c_i + d_i\}$

if $j = 0$ (see Figure 1 (b)). In both cases set $s := x - L_i^j$ and $\gamma_{ijk}(s) := \bar a_k(s + L_i^j) = \bar a_k(x)$. Let $\widetilde\gamma_{ijk}$ be the approximations of γ_{ijk} obtained by Lemma 3.2. The map

$$a_k(x) := \begin{cases} \widetilde\gamma_{ijk}(x - L_i^j) & \text{if } L_i^j - \ell_k \le x \le L_i^j + \ell_k \\ \bar a_k(x) & \text{otherwise in } [0, E] \end{cases}$$

is of class $\mathcal{C}^2$, E-periodic, and such that $|a_k'| = 1$ and $\|a_k - \bar a_k\|_{L^\infty([0,E])} \le L/k$. □

Lemma 3.2. *Let $\ell > 0$, $(\tau_1, \tau_2) \in \mathbb{R}^2$ be a unit vector such that $\tau_1, \tau_2 > 0$, and let $\gamma(s) := (s\tau_1, |s|\tau_2)$ for $-\ell \le s \le \ell$. For any $\eta \in (0, \ell/3)$ there exists $\widetilde\gamma \in \mathcal{C}^2([-\ell, \ell]; \mathbb{R}^2)$ such that $|\widetilde\gamma'| = 1$ in $[-\ell, \ell]$, $\widetilde\gamma(s) = \gamma(s)$ for $|s| \ge \frac{\ell}{2}$ and $\|\gamma - \widetilde\gamma\|_{L^\infty([-\ell,\ell])} \le \eta$.*

Proof. Consider without loss of generality $\ell < 1$, fix $\eta \in (0, \frac{\ell}{3})$ and let $0 < \alpha, \beta \le \eta/2$ be two parameters to be fixed later. Define the map $\gamma_{\alpha,\beta} \in \mathcal{C}^2([-\ell, \ell]; \mathbb{R}^2)$ as

$$\gamma_{\alpha,\beta}(y) := \begin{cases} (\tau_1 y, -\frac{\tau_2}{8\alpha^3} y^4 + \frac{3\tau_2}{4\alpha} y^2 + \frac{3\tau_2\alpha}{8}) & \text{if } |ty| \le \alpha \\ y(\tau_1, \tau_2) + \beta(y - \frac{\ell}{3})^3(\frac{\ell}{2} - y)^3(-\tau_2, \tau_1) & \text{if } \frac{\ell}{3} \le y \le \frac{\ell}{2} \\ (\tau_1 y, \tau_2 |y|) & \text{otherwise in } [-\ell, \ell], \end{cases}$$

see Figure 1 (b). The definition in $[-\alpha, \alpha]$ corresponds to the smoothened corners, while the definition in $[\ell/3, \ell/2]$ corresponds to the small "bump" out of the corners.
For α and β sufficiently small $|\gamma_{\alpha,\beta}(y) - \gamma(y)| \le \eta/2$. Moreover

$$\int_{-\alpha}^{\alpha} |\gamma_{\alpha,\beta}'| \, dy < 2\alpha, \qquad \int_{\ell/3}^{\ell/2} |\gamma_{\alpha,\beta}'| \, dy > \frac{\ell}{6},$$

and

$$\int_{\ell/3}^{\ell/2} |\gamma_{\alpha,\beta}'| \, dy \to \frac{\ell}{6} \qquad \text{as } \beta \to 0^+.$$

Hence there exist α and β such that

$$\int_{-\alpha}^{\alpha} |\gamma_{\alpha,\beta}'| \, dy + \int_{\ell/3}^{\ell/2} |\gamma_{\alpha,\beta}'| \, dy = 2\alpha + \frac{\ell}{6}$$

and in particular

$$\int_{-\ell/2}^{\ell/2} |\gamma_{\alpha,\beta}'| dy = \frac{\ell}{2} - \alpha + \frac{\ell}{3} - \alpha + 2\alpha + \frac{\ell}{6} = \ell \,. \tag{3.5}$$

Let $s = s(y)$ be the arc-length parameter for the curve $\gamma_{\alpha,\beta}$, and observe that $|s - y(s)| \le 2\alpha$. Set $\widetilde\gamma(s) := \gamma_{\alpha,\beta}(y(s))$. By (3.5) we deduce that $\widetilde\gamma(s) = \gamma(s)$ for $|s| \ge \ell/2$. Moreover,

$$\begin{aligned} |\widetilde\gamma(s) - \gamma(s)| &= |\gamma_{\alpha,\beta}(y(s)) - \gamma(s)| \le |\gamma_{\alpha,\beta}(y(s)) - \gamma(y(s))| + |\gamma(y(s)) - \gamma(s)| \\ &\le \frac{\eta}{2} + |y(s) - s| \le \frac{\eta}{2} + 2\alpha \le \eta \,. \end{aligned}$$

□

Remark 3.3. Let $\gamma \in \mathcal{C}^2([0,T] \times [0,L];\mathbb{R}^n)$ be a regular strictly admissible orthogonally parametrized map. Then (2.26) implies that

$$\mathrm{a} = \gamma_{tt}^{\perp}. \tag{3.6}$$

If γ in addition satisfies the wave system (2.20), so that the representation formula (2.22) holds, but assuming only $|a'| = |b'| \leq 1$ instead of (2.23), then being $\gamma_{tt}^{\perp} = \gamma_{xx}^{\perp}$, we have the identity

$$\mathrm{a} = (1-|v|^2)\kappa - (1-|v|^2)\kappa + \alpha\left(\gamma_{xx}^{\perp} - \gamma_{tt}^{\perp}\right) + \gamma_{tt}^{\perp}, \qquad \alpha \in \mathbb{R}.$$

Choosing $\alpha = 2(1-|v|^2)/(1-|v|^2+|\gamma_x|^2)$ we get, using $\kappa|\gamma_x|^2 = \gamma_{xx}^{\perp}$ and (3.6),

$$-\mathrm{a} + (1-|v|^2)\kappa = \frac{1-\phi^2}{1+\phi^2}\left(\mathrm{a} + (1-|v|^2)\kappa\right), \tag{3.7}$$

where $\phi := \frac{|\gamma_x|}{\sqrt{1-|\gamma_t|^2}}$. In analogy with the discussion in [20, Section 5], the left-hand side of (3.7) is the mean curvature of the surface, while the right-hand side can be interpreted as a sort of sectional curvature of the surface in the null direction, multiplied by the positive factor $\frac{1-\phi(t,x)^2}{1+\phi(t,x)^2}$.

4 Some examples

In view of Theorem 3.1, we are interested in understanding the structure of the uniform limits of $\mathcal{C}^2$ critical points of the functional $\mathcal{S}$ of the form (1.2). The following example shows that such limits cannot satisfy, in general, any kind of partial differential equation.

Example 4.1. Let $A \in \mathcal{C}^2(\mathbb{R};\mathbb{R}^n)$ be an L-periodic map satisfying $|A'| = 1$. Let also $\varepsilon \in (0,1)$ be such that $L/\varepsilon \in \mathbb{N}$, and define $B_\varepsilon : \mathbb{R} \to \mathbb{R}$ as

$$B_\varepsilon(s) := \varepsilon A\left(\frac{s}{\varepsilon}\right), \qquad s \in \mathbb{R}.$$

Let $E = 2L$. Define

$$\gamma_\varepsilon(t,x) := \frac{1}{2}\left[2A\left(\frac{x+t}{2}\right) + B_\varepsilon(x-t)\right], \qquad (t,x) \in \mathbb{R} \times [0,E].$$

Then $\gamma_\varepsilon \in \mathcal{C}^2(\mathbb{R} \times [0,2L];\mathbb{R}^n)$, it satisfies $\langle \gamma_{\varepsilon t}, \gamma_{\varepsilon x}\rangle = 0$, $|\gamma_{\varepsilon t}|^2 + |\gamma_{\varepsilon x}|^2 = 1$ and it is a global in time solution of (2.20). The maps $\gamma_\varepsilon(t,x)$ converge, as $\varepsilon \to 0^+$, to $A((x+t)/2)$ uniformly on the compact subsets of $\mathbb{R} \times [0,E]$, and $A((x+t)/2)$ is a reparametrization of $A(x/2)$, $x \in [0,E]$. In particular, if $\gamma_0 \in \mathcal{C}^2([0,L];\mathbb{R}^n)$ is a closed regular curve, the curve $\gamma(t,x) := \gamma_0(x)$ (the image of which is the "cylinder" $\mathbb{R} \times \gamma_0([0,L])$) is a local uniform limit of a sequence corresponding to $\mathcal{C}^2$-critical points of the functional $\mathcal{S}$.

The next example should be compared with the example given by Neu in [20], and with the one in [6, Section 1].

Example 4.2. Let $n = 2$ and $a(s) := (\cos s, \sin s)$ for any $s \in \mathbb{R}$. We want to approximate uniformly the pair $(a(s), a(s))$ with pairs which have approximately the form $(a(s), a(s) + \frac{\varepsilon}{2}a(s/\varepsilon))$. Since we want to keep the constraints in (2.23), and in addition we want to control the periods, we need to make suitable reparametrizations. The conclusion of the example will be that there exists $\alpha > 1$ (see (4.6) below) such that the map

$$\gamma(t, x) = \frac{1}{2}\left[\alpha a\left(\frac{x+t}{\alpha}\right) + a\left(\frac{x-t}{\alpha}\right)\right] \tag{4.1}$$

can be obtained as local uniform limit of (the second components, see (1.2)) a sequence of $\mathcal{C}^2$ critical points of $\mathcal{S}$. In particular, the presence of $\alpha > 1$ prevents $\gamma(t, x)$ to vanish, since (4.1) implies

$$|\gamma(t, x)|^2 = \frac{1}{4}\left(1 + \alpha^2 + 2\alpha\cos(2t/\alpha)\right) \geq \frac{(1-\alpha)^2}{4} > 0.$$

We begin by introducing the function $s_\varepsilon : \mathbb{R} \to \mathbb{R}$, having a 2π-periodic derivative, as follows: for any $x \in \mathbb{R}$ we set

$$s_\varepsilon(x) := \int_0^x \sqrt{\left[\sin\sigma + \frac{1}{2}\sin(\sigma/\varepsilon)\right]^2 + \left[\cos\sigma + \frac{1}{2}\cos(\sigma/\varepsilon)\right]^2}\, d\sigma = \int_0^x \sqrt{\frac{5}{4} + \cos\left(\frac{\sigma}{\varepsilon} - \sigma\right)}\, d\sigma.$$

Note that $s'_\varepsilon \geq \frac{1}{2}$ everywhere. Denote by $x_\varepsilon : \mathbb{R} \to \mathbb{R}$ the inverse of s_ε. Then we define

$$b_\varepsilon(s) := a(x_\varepsilon(s)) + \frac{\varepsilon}{2}a\left(\frac{x_\varepsilon(s)}{\varepsilon}\right), \qquad s \in \mathbb{R}, \tag{4.2}$$

and note that b_ε is $\ell_\varepsilon := s_\varepsilon(2\pi)$-periodic, and

$$|b'_\varepsilon(s)| = |x'_\varepsilon(s)|\left|a'(x_\varepsilon(s)) + \frac{1}{2}a'(x_\varepsilon(s)/\varepsilon)\right| = 1, \qquad s \in \mathbb{R}. \tag{4.3}$$

The period ℓ_ε is larger than the period of a, due to the presence of the additional oscillations.

Let also

$$a_\varepsilon(s) := \frac{\ell_\varepsilon}{2\pi} a\left(\frac{2\pi s}{\ell_\varepsilon}\right), \qquad s \in \mathbb{R}.$$

The map a_ε has the same period as b_ε and satisfies

$$|a'_\varepsilon| = 1. \tag{4.4}$$

Define

$$\gamma_\varepsilon(t,x) := \frac{1}{2}\left[a_\varepsilon(x+t) + b_\varepsilon(x-t)\right], \qquad (t,x) \in \mathbb{R}\times[0,\ell_\varepsilon].$$

Then, thanks to (4.3), (4.4) we have

$$\langle \gamma_{\varepsilon t}, \gamma_{\varepsilon x}\rangle = 0, \qquad |\gamma_{\varepsilon t}|^2 + |\gamma_{\varepsilon x}|^2 = 1$$

and the wave system (2.20). Now we claim that there exists $\alpha > 1$ such that for any $s \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} a_\varepsilon(s) = \alpha a\left(\frac{s}{\alpha}\right), \qquad \lim_{\varepsilon\to 0} b_\varepsilon(s) = a\left(\frac{s}{\alpha}\right).$$

To prove the claim, let $\phi(p) := \sqrt{\frac{5}{4} + p}$ for any $p \in \mathbb{R}$, and observe that

$$\lim_{\varepsilon\to 0} s_\varepsilon(x) = \frac{x}{2\pi}\int_0^{2\pi} \phi(\cos\sigma)\, d\sigma. \tag{4.5}$$

Indeed, $s_\varepsilon(x) = \int_0^x \phi(\cos(\sigma/\varepsilon - \sigma))\, d\sigma$, and the change of variable $y = \sigma/\varepsilon - \sigma$ gives

$$s_\varepsilon(x) = \frac{x}{x(\frac{1}{\varepsilon}-1)}\int_0^{x(\frac{1}{\varepsilon}-1)} \phi(\cos y)\, dy.$$

Hence $s_\varepsilon(x)$ equals x times the mean value of the 2π-periodic function $\phi(\cos y)$ in the interval $[0, x(\frac{1}{\varepsilon}-1)]$, which converges to the mean value of $\phi(\cos y)$ on $[0,2\pi]$, and this proves (4.5).
Define

$$\alpha := \frac{1}{2\pi}\lim_{\varepsilon\to 0} s_\varepsilon(2\pi) = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{5}{4}+\sigma}\, d\sigma > 1, \tag{4.6}$$

so that $\lim_{\varepsilon\to 0} s_\varepsilon(x) = \alpha x$, hence $\lim_{\varepsilon\to 0} x_\varepsilon(s) = s/\alpha$, and the claim follows.
Then

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon(t,x) = \frac{1}{2}\left[\alpha a\left(\frac{x+t}{\alpha}\right) + a\left(\frac{x-t}{\alpha}\right)\right] =: \gamma(t,x)$$

uniformly for (t,x) in the compact subsets of $\mathbb{R}\times\mathbb{R}$.
The limit curve γ is such that $X(t,x) := (t,\gamma(t,x))$ is not a critical point of $\mathcal{S}$; it is interesting to observe, as remarked in [20], that the additional oscillations "desingularize" the limit, in the sense that the image of the map γ has not anymore any singular point.

The last example is similar to Example 4.2, but in $n = 3$ dimensions; here the situation is simpler, since the analog of the arc-length reparametrization in (4.2) is automatically satisfied.

Example 4.3. Assume $n = 3$. Consider cylindrical coordinates in $\mathbb{R}^3$ and set, for $s \in \mathbb{R}$,

$$e_r := (\cos s, \sin s, 0), \qquad e_s := (-\sin s, \cos s, 0)\,, \qquad e_z := (0, 0, 1).$$

Let $\alpha, \beta \in (-1, 1)$ be such that $\alpha^2 + \beta^2 = 1$, $n \in \mathbb{N}$, and define the 2π-periodic maps $a, b_n : \mathbb{R} \to \mathbb{R}^3$ as

$$\begin{aligned} a(s) :=& e_s, \\ b_n(s) :=& \alpha e_s + \beta \left(e_s \sin(ns) \frac{n}{n^2 - 1} - e_r \cos(ns) \frac{1}{n^2 - 1} + e_z \cos(ns) \frac{1}{n} \right). \end{aligned}$$

A direct computation gives

$$\begin{aligned} b_n'(s) =& -\alpha e_r - \beta e_r \sin(ns) \frac{n}{n^2 - 1} - \beta e_s \cos(ns) \frac{1}{n^2 - 1} \\ &+ \beta e_s \cos(ns) \frac{1}{n^2 - 1} + \beta e_r \sin(ns) \frac{n}{n^2 - 1} - \beta e_z \sin(ns) \\ =& -\alpha e_r + \beta e_s \cos(ns) - \beta e_z \sin(ns). \end{aligned}$$

so that

$$|b_n'(s)|^2 = \alpha^2 + \beta^2 = 1, \qquad |a'(s)|^2 = 1, \qquad s \in \mathbb{R}.$$

Moreover

$$\lim_{n \to +\infty} b_n(s) = \alpha e_s = \alpha a(s) =: b(s)$$

uniformly in $\mathbb{R}$.
Define

$$\gamma_n(t, x) := \frac{1}{2} \left[a(x + t) + b_n(x - t) \right], \qquad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Then γ_n satisfy $\langle \gamma_{nt}, \gamma_{nx} \rangle = 0$, $|\gamma_{nt}|^2 + |\gamma_{nx}|^2 = 1$, and (2.20). Moreover

$$\lim_{n \to +\infty} \gamma_n(t, x) = \frac{1}{2} \left[a(x + t) + \alpha a(x - t) \right] =: \gamma(t, x)$$

uniformly in on the compact subsets of $\mathbb{R} \times \mathbb{R}$. Also in this example $\gamma(t, x)$ cannot vanish, since

$$|\gamma(t, x)|^2 = \frac{1}{4} \left[1 + \alpha^2 + 2\alpha \cos(2t) \right] = \frac{1}{4} \left[(1 + \alpha)^2 \cos^2 t + (1 - \alpha)^2 \sin^2 t \right] \geq \frac{(1 - \alpha)^2}{4} > 0.$$

Observe that letting $a(s) = (-\sin(s + 2\phi), \cos(s + 2\phi), 0)$ for $\phi \in (0, \pi)$, we have for the resulting γ

$$|\gamma(t, x)|^2 = \frac{1}{4} \left[(1 + \alpha)^2 \cos^2(t + \phi) + (1 - \alpha)^2 \sin^2(t + \phi) \right],$$

and again $|\gamma(t, x)| \geq (1 - \alpha)/2$.

It would be interesting to understand whether there are connections between the examples considered in this section and the results of [10].

5 Evolution of $\mathcal{C}^2$ uniformly convex curves with $\gamma_t(0,\cdot) = 0$

Let $\bar{t} > 0$ and let $\gamma \in C^2([0,\bar{t}) \times [0,E];\mathbb{R}^n)$ be a solution of (2.7), (2.19) and (2.20). In particular, there exist E-periodic maps $a, b \in C^2(\mathbb{R};\mathbb{R}^n)$ such that $\gamma(t,x) = \frac{1}{2}[a(x+t) + b(x-t)]$ for any $(t,x) \in [0,\bar{t}) \times [0,E]$. Therefore, recalling the discussion in Remark 2.9, γ can be extended to a global solution $\gamma \in \mathcal{C}^2(\mathbb{R} \times [0,E];\mathbb{R}^n)$. Adopting this definition of global solution, we show in this section that initial convex curves may shrink to a point, and then continue the motion in a periodic way.

Definition 5.1. *Let $\bar{t} > 0$ and $p \in \mathbb{R}^n$. We say that $\bar{t}$ is a collapsing time, and that γ has a collapsing singularity at $\bar{t}$ with p as collapsing point, if $\gamma(\bar{t},x) = p$ for any $x \in [0,E]$.*

At the collapsing time we have

$$0 = \gamma_x(\bar{t},x) = \frac{1}{2}\left[a'(x+\bar{t}) + b'(x-\bar{t})\right], \qquad x \in [0,E]. \tag{5.1}$$

Let us now assume $n = 2$, $\gamma_t(0,\cdot) = 0$, so that we can choose $a = b \in \mathcal{C}^2(\mathbb{R};\mathbb{R}^2)$, with a the arc-length parametrization, on $[0,E]$, of a closed uniformly convex curve of class $\mathcal{C}^2$. Since the initial curve is uniformly convex, for any $x \in [0,E]$ there exists a unique $\mathrm{t}(x) \in (0,E/2)$ such that

$$\gamma_x(\mathrm{t}(x),x) = \frac{1}{2}\left[a'(x+\mathrm{t}(x)) + a'(x-\mathrm{t}(x))\right] = 0, \tag{5.2}$$

and the function t belongs to $\mathcal{C}^1([0,E];(0,E/2))$. Moreover, if we set

$$\mathrm{t}_{\min} := \min_{x\in[0,E]} \mathrm{t}(x) \qquad \mathrm{t}_{\max} := \max_{x\in[0,E]} \mathrm{t}(x),$$

we have that $\gamma(t,\cdot)$ is a regular parametrization for all $t \in [0,\mathrm{t}_{\min}) \cup (\mathrm{t}_{\max}, E/2]$. We can think of $\mathrm{t}_{\min}$ (resp. $\mathrm{t}_{\max}$) as the first (resp. last) singularity time in the periodicity interval $[0,E]$, where by singularity here we mean that the regularity condition of Definition 2.1 fails.

Proposition 5.2. *Let $\gamma \in \mathcal{C}^2([0,\mathrm{t}_{\min}) \times [0,E];\mathbb{R}^2)$ be a solution of (2.14) given by (2.22). Assume that $\gamma(0,\cdot) \in \mathcal{C}^2([0,E])$ is regular and embedded, that $\gamma(0,[0,E])$ encloses a compact centrally symmetric uniformly convex body $K(0)$, and that $\gamma_t(0,\cdot) = 0$. Then γ has a collapsing singularity at time $\mathrm{t}_{\min} = E/4$ with the origin as collapsing point.*

Proof. The assertion follows by observing that $K(0)$ is centrally symmetric, and the function t defined in (5.2) is constant and equals $E/4 = t_{\min}$. □

Remark 5.3. Generically, one can assume that

- the last equality in (5.1) does not hold;
- the set $\{x \in [0,E] : \mathrm{t}(x) = t\}$ is finite for all $t \in [\mathrm{t}_{\min}, \mathrm{t}_{\max}]$, and consists of a single point $x_{\min}$ (resp. $x_{\max}$) for $t = \mathrm{t}_{\min}$ (resp. $t = \mathrm{t}_{\max}$).

From the condition $\mathrm{t}'(x_{\min}) = \mathrm{t}'(x_{\max}) = 0$ we get

$$a''(x_{\min} + \mathrm{t}_{\min}) = -a''(x_{\min} - \mathrm{t}_{\min}) \qquad \text{and} \qquad a''(x_{\max} + \mathrm{t}_{\max}) = -a''(x_{\max} - \mathrm{t}_{\max}),$$

which implies that the images $\gamma(\mathrm{t}_{\min},[0,E])$ and $\gamma(\mathrm{t}_{\max},[0,E])$ are of class $\mathcal{C}^1$. In this generic setting, the formation of singularities has been discussed in [9] (see also [22], [3]), where it is shown that $\mathrm{t_{min}}$ is the first singular time, the singularity has the asymptotic behavior $y \sim x^{\frac{4}{3}}$ in graph coordinates, and two cusps $y \sim x^{\frac{2}{3}}$ appear from the point $x_{\min}$ at time $\mathrm{t_{min}}$, persist for some positive time, and eventually disappear.

We now show that the convexity of the curve is preserved before the onset of singularities, that is on the time interval $[0, \mathrm{t_{min}})$.

Proposition 5.4. *Let $\gamma \in \mathcal{C}^2([0, \mathrm{t_{min}}) \times [0, E]; \mathbb{R}^2)$ be a solution of (2.14) given by (2.22). Assume that $\gamma(0, \cdot) \in \mathcal{C}^2([0, E])$ is embedded and counter-clockwise regularly parametrized, that $\gamma(0, [0, E])$ encloses a compact uniformly convex body $K(0)$, and that $\gamma_t(0, \cdot) = 0$. Then $\gamma(t, \cdot)$ is the regular parametrization of a closed uniformly convex embedded curve of class $\mathcal{C}^2([0, E])$ for all $t \in [0, \mathrm{t_{min}})$. Moreover, letting $K(t)$ the compact convex set enclosed by $\gamma(t, \cdot)$, we have*

$$t_1, t_2 \in [0, \mathrm{t_{min}}),\ t_1 \le t_2 \quad \Rightarrow \quad K(t_1) \subseteq K(t_2). \tag{5.3}$$

Proof. For any $t \in [0, t_{\min})$ let

$$\nu(t, x) = R\, \frac{\gamma_x(t, x)}{|\gamma_x(t, x)|}, \qquad (t, x) \in [0, \mathrm{t_{min}}) \times [0, E],$$

where $R : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is the counter-clockwise rotation of $\pi/2$. To prove that $\gamma(t, \cdot)$ is a uniformly convex curve, it is enough to show that

$$\langle \gamma_{xx}(t, \cdot), \nu(t, \cdot) \rangle > 0, \qquad t \in [0, \mathrm{t_{min}}) \times [0, E].$$

From $\gamma_x(t, \cdot) \neq 0$ for $t \in [0, \mathrm{t_{min}})$ it follows that

$$a'(x+t) + a'(x-t) \neq 0, \qquad (t, x) \in [0, \mathrm{t_{min}}) \times [0, E]. \tag{5.4}$$

Hence

$$\langle \gamma_{xx}(t, x), \nu(t, x) \rangle \;\; = \;\; \frac{1}{2} \langle a''(x+t) + a''(x-t), \frac{Ra'(x+t) + Ra'(x-t)}{|a'(x+t) + a'(x-t)|} \rangle.$$

Observe now that $|a'| = 1$ implies that $a''(x \pm t) \perp a'(x \pm t)$, so that $a''(x \pm t)$ and $Ra'(x \pm t)$ are parallel. Then (5.4) and the Schwarz inequality imply that

$$\begin{aligned} \langle a''(x+t), Ra'(x+t) + Ra'(x-t) \rangle &\;>\; 0 \\ \langle a''(x-t), Ra'(x+t) + Ra'(x-t) \rangle &\;>\; 0, \end{aligned}$$

which gives

$$\langle \gamma_{xx}(t, x), \nu(t, x) \rangle > 0.$$

It remains to prove (5.3). Equation (2.24) and the uniform convexity $\langle \kappa, \nu \rangle > 0$ imply

$$\langle \mathrm{a}, \nu \rangle = (1 - |v|^2) \langle \kappa, \nu \rangle > 0.$$

Recalling that $\gamma_t(0, \cdot) = 0$ and that $\langle \mathrm{a}, \nu \rangle = \partial_t \langle v, \nu \rangle > 0$, we then get $\langle v, \nu \rangle > 0$ for any $t \in (0, t_{\min})$, and (5.3) follows. $\square$

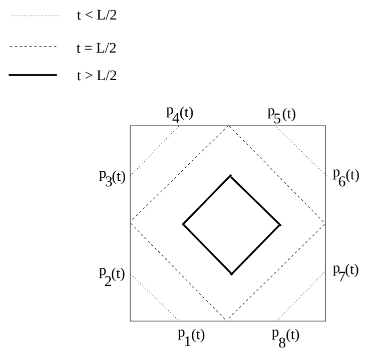

Figure 2: A (weak) evolution of the square with zero initial velocity.

A result analogous to Proposition 5.4 has been obtained in [17] for the equation a $= \kappa$. Differently from our case, for their equation the authors of [17] show that all convex curves shrink to a point in finite time.

Remark 5.5. Assume (as in Proposition 5.2) that the initial uniformly convex set is of class $\mathcal{C}^2$, and that γ has a collapsing singularity at the time $\bar{t} = E/4$, with $p \in \mathbb{R}^2$ as collapsing point. From the representation formula (2.22) with $a = b$, and from Taylor's formula, we get

$$\begin{aligned} \gamma(t,x) &= \frac{1}{2}\left[a(x+\bar{t}) + a(x-\bar{t})\right] + \frac{1}{2}\left[a'(x-\bar{t}) - a'(x+\bar{t})\right]\,(\bar{t}-t) + O(|\bar{t}-t|^2) \\ &= p + a'(x-\bar{t})\,(\bar{t}-t) + O(|\bar{t}-t|^2), \end{aligned}$$

where in the last equality we use $a'(x+\bar{t}) + a'(x-\bar{t}) = 0$ (see (5.2)). It follows that

$$|\gamma(t,x) - p| = |\bar{t}-t| + O(|\bar{t}-t|^2). \tag{5.5}$$

In particular, the asymptotic shape near the collapse is circular, and the blow-up shape of the image of the corresponding map X (see (1.2)) at $(\bar{t}, p)$ is half a light cone.

The conclusion on the asymptotic shape of γ in Remark 5.5 seems not to be true if we drop the $\mathcal{C}^{1,1}$ regularity assumption on the initial convex set, as shown in the following example.

Example 5.6. Assume $n = 2$, let $L > 0$ and let $a = b : \mathbb{R} \to \mathbb{R}^2$ be $4L$-periodic, and such that $a : [0, 4L] \to \mathbb{R}^2$ be the counterclock-wise arc-length parametrization of the boundary of the square $Q_0 = [-L/2, L/2]^2$ (sending for instance $\{0\}$ into the point $x^1 = -L/2$, $x^2 = -L/2$). Obviously $a \in \mathcal{C}^1([0, 4L] \setminus \{0, L, 2L, 3L\}; \mathbb{R}^2)$, and a is Lipschitz continuous in $[0, 4L]$.

Then, letting $\gamma(t,x) := \frac{1}{2}\left[a(x+t)+a(x-t)\right]$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}$, we have that $\gamma(t,\cdot)$ is a Lipschitz parametrization of $\partial Q(t)$, where $Q(t)$ is defined as

$$Q(t) := Q_0 \cap \left\{(x^1,x^2)\in\mathbb{R}^2 : |x^1|+|x^2| \le L-t\right\}, \qquad t\in[0,L].$$

For times larger than L the solution is continued periodically, hence γ is Lipschitz in $\mathbb{R}\times[0,E]$, and therefore it is almost everywhere differentiable. Observe that

(i) the map $X(t,x) := (t,\gamma(t,x))$ is Lipschitz, and at those points of $X(\mathbb{R}\times[0,4L])$ where there exists the tangent plane such a plane is time-like.

(ii) For $t\in[0,L/2)$ the set $Q(t)$ is a shrinking octagon. Its vertices, denoted by $p_1(t),\dots,p_8(t)$, are depicted in Fig. 2. For this interval of times the conservation law (2.17) is satisfied, since

$$\int_{\gamma(t,[0,4L])} \frac{1}{\sqrt{1-|\gamma_t^\perp|^2}}\, d\mathcal{H}^1 = 4\left[|p_8(t)-p_1(t)| + \sqrt{2}|p_8(t)-p_7(t)|\right] = 4L.$$

Moreover, for $t\in[0,L/2)$ the map γ is strictly admissible, in the sense that $|\gamma^\perp|^2<1$ almost everywhere.

(iii) For $t\in[L/2,L)$ the set $Q(t)$ is a shrinking rotated square of side $\sqrt{2}(L-t)$ (depicted in bold in Fig. 2). It shrinks to the point $(0,0)$ at $t=L$ (collapsing singularity). Its normal velocity is constantly equal to $\frac{1}{\sqrt{2}}$. Therefore (2.17) *cannot* be satisfied, since the time derivative of the length of $\gamma(t,\cdot)$ is nonzero. However, the function

$$t\in[L/2,L) \to \int_{\gamma(t,[0,4L])} \frac{1}{\sqrt{1-|\gamma_t^\perp(t,\cdot)|^2}}\, d\mathcal{H}^1$$

is nonincreasing.

(iv) Given $t\in(L/2,L)$, we have $\gamma_x(t,x)=0$ when x belongs to the union $I(t)$ of four intervals of length $2t-L$, and centered at the centers of the four sides of ∂Q_0. Indeed, $\gamma_x(t,x)=0$ when $a'(x+t)=-a'(x-t)$, hence, for instance assuming x to be the center of $[-L/2,L/2]\times\{-L/2\}$, when $x+t$ and $x-t$ belong to opposite vertical sides of ∂Q_0. Therefore, for $t\in(L/2,L)$ and $x\in I(t)$, we have that $\gamma(t,\cdot)$ is not regular,

$$\mathcal{L}(\gamma_t(t,x),\gamma_x(t,x)) = 0, \qquad |\gamma_t(t,x)|^2 = 1,$$

and (2.3) is not satisfied.

Note that the blow-up of X at $(L,0)$ is not half a light cone as in Remark 5.5, but is the half-cone $\{(t,x_1,x_2) : |t-\bar{t}|+|x_1|+|x_2|=1\}$ with square section, inscribed in half the light-cone.

This example proposes an example of nonsmooth evolution, however the conclusion that such a solution is the correct weak solution starting from ∂Q_0 with zero initial velocity is questionable, since the map γ is not regular anymore and, perhaps more importantly, the conservation law (2.16) fails for $t\in(L/2,L)$.

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