NON INTERPENETRATION OF MATTER FOR SBV-DEFORMATIONS OF HYPERELASTIC BRITTLE MATERIALS

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Abstract. We prove that the Ciarlet-Nečas non-interpenetration of matter condition [9] can be extended to the case of SBV-deformations of hyperelastic brittle materials, and can be taken into account for some variational models in fracture mechanics. In order to formulate such a condition, we define the deformed configuration under an SBV-map by means of the approximately differentiable representative, and we prove some connected stability results under weak convergence. We provide an application to the case of brittle Ogden's materials.

Keywords : variational models, energy minimization, free discontinuity problems, brittle fracture, a.e.-injectivity, measure-theoretical image.

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Contents

Introduction

Variational models to describe equilibria of brittle hyperelastic bodies have been largely developed in the recent years. Inspired by Griffith's theory of crack propagation, these models in fracture mechanics are based on the assumption that a pair (u, Γ) is an equilibrium configuration of the body if it minimizes among all admissible configurations a total energy whose basic form is

$$\mathcal{E}(u, \Gamma) = \int_\Omega W(\nabla u)\, dx + k\mathcal{H}^{N-1}(\Gamma). \tag{0.1}$$

Here Γ denotes a crack inside the elastic body $\Omega \subseteq \mathbb{R}^N$ and u is a deformation well defined outside Γ which satisfies suitable boundary conditions. The volume part of $\mathcal{E}(u, \Gamma)$, which depends on the strain ∇u, represents the elastic energy stored in the body, while the surface part, which is proportional to the surface of the crack ($\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure), represents the energy dissipated to produce the crack Γ. More general surface energies may be considered: they could depend, following Barenblatt's theory, on the opening $[u]$ of the lips of the crack, as well as on its orientation.

From a mathematical point of view, the minimization of the total energy (0.1) can be carried out under general assumptions for W if the problem is settled within the theory of SBV-deformations. The functional space SBV of *special functions of bounded variations* has been introduced by De Giorgi and Ambrosio [12] to deal with free discontinuity problems arising in image segmentation, and was proposed by Ambrosio and Braides [4] as a suitable framework for fracture mechanics. A function u belongs to $SBV(\Omega, \mathbb{R}^N)$ (see Section 1.2 for a precise definition) if $u \in L^1(\Omega, \mathbb{R}^N)$ and its derivative in the sense of distributions is a finite Radon measure which is the sum of a part absolutely continuous with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$ with density ∇u (approximate gradient of u) and of a part supported on the complement S_u of the set of Lebsgue points and absolutely continuous with respect to the $(N-1)$-dimensional measure $\mathcal{H}^{N-1}$. Deformations of class SBV are easily interpreted as deformations with cracks inside Ω: the crack is identified with S_u (which is essentially a surface for $N = 3$), and ∇u represents the usual strain in the elastic part of the body outside the crack.

Recently a variational approach to quasi-static crack growth based on time discretization and energy minimization of (0.1) has been proposed by Francfort and Marigo [15], and it has been developed in many subsequent papers in the framework of SBV-functions (we refer to [14], [11], [19] and to the references therein).

The advantage of the SBV-approach to fracture mechanics is that, even if it allows to involve in the minimization process a huge class of cracks, without a priori regularity assumptions, anyway it leads to useful compactness properties (see Ambrossio's Theorem 1.6), so that the minimization can be carried out following the direct method of the Calculus of Variations. The aim of this paper is to introduce and to discuss in this context the constraint of *non-interpenetration of matter*. The introduction of such a constraint would make physically more realistic the equilibria found through the minimization process of the specific considered model.

Non interpenetration of matter for hyperelastic bodies subject to pure traction was first studied by Ball [7] by means of a global inversion theorem for Sobolev maps in $W^{1,p}(\Omega)$ with $p > N$ [7, Theorem 1]: he proved that if u is a.e.-orientation preserving, i.e.,

$$\det \nabla u(x) > 0 \qquad \text{for a.e. } x \in \Omega,$$

and it coincides with a continuous and injective map on $\partial\Omega$, then u is a.e.-injective in Ω, i.e., it is injective outside a negligible subset of Ω. Furthermore [7, Theorem 2], if some suitable energetic assumptions (involving the behavior of $(\nabla u)^{-1}$) are satisfied, u is indeed a homeomorphism between $\overline{\Omega}$ and $u(\overline{\Omega})$. In other words, the non-interpenetration condition can be plugged in the variational theory of nonlinear elasticity introduced by the same author in [6] provided that the strain energy density satisfies suitable growth assumptions.

The problem of non-interpenetration of matter was then considered by Ciarlet-Nečas [9] in the context of more general traction-displacements boundary problems. They consider as admissible deformations Sobolev mappings in $W^{1,p}(\Omega, \mathbb{R}^N)$ with $p > N$ (which are continuous by Sobolev Embedding Theorem) that are a.e.-orientation preserving and which are a.e.-injective in Ω. The key idea in order to take into account this non-interpenetration condition in the minimization process is that the constraint of a.e.-injectivity can be reformulated equivalently (employing the area formula for Sobolev mappings in $W^{1,p}(\Omega,\mathbb{R}^N)$ with $p>N$, see Section 1.1) in the following way

$$\int_\Omega \det \nabla u \, dx \leq \text{Volume}\big(u(\Omega)\big). \tag{0.2}$$

Ciarlet and Nečas proved that this constraint is preserved under weak convergence and so it is suitable to be employed in the minimization of the strain energy. They interpret this minimum problem as a mathematical model of *frictionless self-contact without interpenetration of matter* [9, Theorem 4].

In this paper we will follow the ideas of [9], adapting them to the context of SBV- functions, to prove analogous existence results in the setting of SBV-deformations of elastic bodies with cracks. Given a deformation $u \in SBV(\Omega;\mathbb{R}^N)$, we say that u satisfies the *Ciarlet-Nečas non-interpenetration condition* if u is a.e.-orientation preserving, and u is a.e.-injective. In order to take into account this constraint in a minimization problem, we want to reformulate a.e.-injectivity imposing a constraint on the $\mathcal{L}^N$-volume of the image of the deformation, according to (0.2). To this aim, we have to face the problem of defining what we mean by the *image of* Ω under an SBV-deformation u: in fact u does not admit in general a continuous representative (even outside the crack S_u). The Lebesgue-representative $\tilde{u}$ of u is the natural candidate to define the image of Ω, since it is well defined outside the crack S_u: we prove however that the Lebesgue representative fails to map negligible sets into negligible sets (see Example 2.1), i.e., it does not satisfy what is usually referred to as the N-property, which is the starting point to establish the area formula and recover (0.2). As a consequence, a.e.-injectivity cannot be formulated with the integral constraint (0.2) employing the Lebesgue points. Our example is heavily inspired by that given by Malý and Martio [21] concerning the N-property for the Lebesgue representative of Sobolev functions in $W^{1,N}$: we remark that the N-property fails in SBV even if $\nabla u \in L^p(\Omega,\mathbb{R}^{N^2})$ with $p>N$ (in contrast to Sobolev space case, see Marcus and Mizel [22]).

The "right" notion of image of Ω under u in order to carry out our program is given by the image $\tilde{u}(\Omega_D)$ of the set Ω_D of points of *approximate differentiability* of u (see Section 1.1 for a precise definition) which is only a part of the set Ω_L of Lebesgue points of u. We refer to this image as the *measure theoretical image* of u, and we indicate it as $[u(\Omega)]$. It turns out from general results on the area formula for a.e. approximately differentiable maps (see Section 1.1) that the constraint of a.e.-injectivity for orientation preserving SBV-maps can be formulated through the constraint

$$\int_\Omega \det\nabla u \, dx \leq \text{Volume}([u(\Omega)]),$$

and we prove that this constraint is stable under weak convergence of u in SBV (see Theorem 3.4) provided some control on $\det \nabla u$ is available (which is usually inferred by the energy control in a minimization problem). From a mechanical point of view, we conclude that the set $\Omega_L \setminus \Omega_D$ should be regarded as a set of *damaged points*, even if a mean value of u at those points is well defined, and so they should not be considered to recover the deformed configuration.

The importance of the measure theoretical image $[u(\Omega)]$ (i.e., the image of approximate differentiability points) in the variational approach to perfect finite elasticity has been pointed out by Giaquinta, Modica and Souček [18, Chapter 2] (see also Müller and Spector [23] where a model which allows for cavitation is considered). Also for the case of SBV-maps (i.e., also in the presence of fractures), we prove that the measure theoretical image $[u(\Omega)]$ enjoys interesting variational and stability properties:

(1) it has minimal $\mathcal{L}^N$-measure with respect to any other image $v(\Omega)$, where v is any representative of u (see Proposition 2.5);
(2) it is stable, in a L^1-sense, with respect to weak convergence in $SBV^p(\Omega;\mathbb{R}^N)$ for $p > N$(see Proposition 2.7 and Definition 1.6 for the definition of $SBV^p(\Omega;\mathbb{R}^N)$);
(3) if $u \in SBV(\Omega;\mathbb{R}^N)$ is a.e.-injective, then the function $\mu_{[u]} : E \to \mathcal{L}^N([u(E)])$ is a measure, which says that non overlapping of matter occurs in the deformed configuration (see Proposition (5.3)).

In Section 4 we prove that the Ciarlet-Nečas non-interpenetration condition can be taken into account for hyperelastic brittle materials with an energy W of Ogden's type [24]. In Theorem 4.1 we prove that a minimum energy deformation which does not exhibit interpenetration of matter in the sense of Ciarlet-Nečas can be recovered using the direct method of the Calculus of Variations: this follows easily from the stability property of the measure theoretical images of weakly converging SBV-deformations, and from a lower semicontinuity result in SBV for polyconvex energies of Ogden's type recently proved by Fusco, Leone, March and Verde [16].

In Section 5 we briefly discuss some alternative notions of non-interpenetration of matter which could be taken into account in a minimization problem, pointing out the differences between these notions and the Ciarlet-Nečas non-interpenetration condition through examples. In particular we consider: i) a linearized version of the non-interpenetration condition which involves the behaviour of the deformation near the crack; ii) a notion of non-interpenetration condition in the deformed configuration, based on the assumption that the function $\mu_{[u]} : E \to \mathcal{L}^N([u(E)])$ is a measure; iii) a notion of non-interpenetration during the deformation process.

The paper is organized as follows. In Section 1 we recall some results concerning the area formula for approximately differentiable functions, and we recall some basic facts from the theory of SBV-functions. In Section 2 we prove that SBV^p-functions does not satisfy the N-property even for $p > N$, and we study the properties of the measure theoretical image of SBV-deformations defined through the approximately differentiable representative. Section 3 is devoted to the formulation and the main stability properties of the Ciarlet-Nečas non-interpenetration condition for SBV-maps, while Section 4 contains the application to brittle hyperlastic Ogden's materials. Finally in Section 5 we address the problem of non-interpenetration conditions alternative to that of Ciarlet-Nečas.

1. Preliminaries

In this Section we recall some basic facts which will be employed in the rest of the paper. In what follows, $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, represents an open bounded set. Moreover $\mathcal{L}^N$ stands for the usual N-dimensional outer Lebesgue measure on $\mathbb{R}^N$.

1.1. **Area formula for approximately differentiable maps and a.e.-injectivity.** In this Section we briefly recall the link between a.e.-injectivity and the area formula for a.e.-approximately differentiable maps which is at the basis of Ciarlet-Nečas approach to non-interpenetration of matter for Sobolev deformations (see the Introduction).

Let $u : \Omega \to \mathbb{R}^M$ be a measurable function. Given $x \in \Omega$ we say that u admits an approximate limit l at x, and we write $l = \operatorname{ap}\lim_{y\to x} u(y)$, if for every $\varepsilon > 0$ we have

$$\lim_{r\to 0} r^{-N}\mathcal{L}^N\left(\{y \in B_r(x) \,:\, |u(y) - l| > \varepsilon\}\right) = 0.$$

Here $B_r(x)$ denotes the ball of center x and radius r. We say that u is *approximately continuous* at x if $u(x)$ is the approximate limit of u at x. We say that u is *a.e.-approximately continuous in* Ω if it is approximately continuous at almost every point of Ω.

We say that u is *approximately differentiable* at x if u is approximately continuous at x and there exists an $(M \times N)$-matrix L such that and

$$\operatorname{ap}\lim_{y\to x} \frac{u(y) - u(x) - L(y - x)}{|y - x|} = 0.$$

The matrix L is called the *approximate gradient* of u at x and is usually denoted by $\nabla u(x)$. We say that u is *a.e.-approximately differentiable in* Ω provided that it is approximately differentiable at a.e. $x \in \Omega$.

Let us consider $N = M$, and let us recall the *area formula* for a.e.-approximately differentiable maps. We refer the reader to [17, Chapter 3] for a complete treatment of the subject. For every measurable set $E \subseteq \Omega$, let the number of preimages of a point y in the set E be denoted by

$$m(u, y, E) := \text{cardinality}\{x \in E :\, u(x) = y\}.$$

Let Ω_D be the set of points in Ω at which u is approximately differentiable. The area formula for a.e.-approximately differentiable maps is the following (see e.g. [17, Theorem 1, Section 1.5, Chapter 3]).

Theorem 1.1 (**The area formula).** *Let us assume that $u : \Omega \to \mathbb{R}^N$ is a.e.-approximately differentiable in Ω. Then for every measurable set $E \subseteq \Omega$ the function $\{y \to m(u, y, E \cap \Omega_D)\}$ is measurable, and we have*

$$\int_E |\det \nabla u(x)|\, dx = \int_{\mathbb{R}^N} m(u, y, E \cap \Omega_D)\, dy. \tag{1.1}$$

In order to formulate the area formula without the restriction to the set of approximate differentiability points, we need the notion of N-property.

Definition 1.2 (**N-property).** *We say that $u : \Omega \to \mathbb{R}^N$ has the N-property if for every $\mathcal{L}^N$-negligible set $E \subseteq \Omega$, we have that $u(E)$ is $\mathcal{L}^N$-negligible.*

Notice that if u is measurable and satisfies the N-property, then $u(F)$ is measurable for every measurable set $F \subseteq \Omega$. In view of Theorem 1.1, we get immediately the following area formula.

Theorem 1.3 (**The area formula for a.e.-approximately differentiable maps).** *Let us assume that $u : \Omega \to \mathbb{R}^N$ is a.e.-approximately differentiable in Ω and satisfies the N-property. Then for every measurable set $E \subseteq \Omega$ the function $\{y \to m(u, y, E)\}$ is measurable,* *and we have*

$$\int_E |\det \nabla u(x)|\, dx = \int_{\mathbb{R}^N} m(u, y, E)\, dy. \tag{1.2}$$

Let us come to the link between a.e.-injectivity and the area formula.

Definition 1.4 (A.e.-injective maps). *We say that a measurable map $u : \Omega \to \mathbb{R}^N$ is a.e.-injective if there exists a $\mathcal{L}^N$-negligible set $E \subset \Omega$ such that the restriction of u to $\Omega \setminus E$ is injective.*

The following result is basic to the study of a.e.-injectivity in variational problems (see Section 3).

Proposition 1.5. *Let us assume that $u : \Omega \to \mathbb{R}^N$ is a.e.-approximately differentiable in Ω and satisfies the N-property. If u is a.e.-injective, then*

$$\int_\Omega |\det \nabla u|\, dx \leq \mathcal{L}^N(u(\Omega)). \tag{1.3}$$

Viceversa, if u satisfies (1.3) *and* $\det\nabla u \neq 0$ *a.e. in Ω, then u is a.e.-injective.*

Proof. By the area formula we have that

$$\mathcal{L}^N(u(\Omega)) \leq \int_\Omega |\det \nabla u|\, dx.$$

so that inequality (1.3) is equivalent to

$$\int_\Omega |\det \nabla u|\, dx = \mathcal{L}^N(u(\Omega)). \tag{1.4}$$

From (1.2) we deduce that (1.4) holds if and only if $m(u, y, \Omega) \leq 1$ for a.e. $y \in \mathbb{R}^N$ or equivalently if and only if the set $M := \{y \in \mathbb{R}^N : m(u, y, \Omega) \geq 2\}$ is $\mathcal{L}^N$-negligible.

We can now prove the conclusions of the Proposition. If u is a.e.-injective, then there exists a negligible set E such that the restriction of u to $\Omega \setminus E$ is injective, and hence $M \subseteq u(E)$ is $\mathcal{L}^N$-negligible in view of the N-property of u.

On the other hand let us assume that $\mathcal{L}^N(M) = 0$ and $\det\nabla u \neq 0$ a.e. in Ω. Then by (1.2) we have that also $E := u^{-1}(M)$ is $\mathcal{L}^N$-negligible, so that u is a.e.-injective. □

1.2. **Special functions of bounded variation SBV.** Let us recall some results from the theory of SBV-functions: We refer the reader to [5] for an exhaustive treatment of the subject.

Let Ω be an open subset of $\mathbb{R}^N$, and let $u : \Omega \to \mathbb{R}^M$ be a measurable function. We say that $u \in BV(\Omega; \mathbb{R}^M)$ if $u \in L^1(\Omega; \mathbb{R}^M)$, and its distributional derivative Du is a vector-valued Radon measure on Ω with finite mass.

If $u \in BV(\Omega, \mathbb{R}^M)$, it turns out that u is a.e.-approximately differentiable in Ω. Moreover, denoting with S_u the set of points where the approximate limit of u does not exist, it turns out that S_u is rectifiable, i.e. there exists a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-manifolds such that $S_u \subseteq \bigcup_i M_i$ up to a set of $\mathcal{H}^{N-1}$-measure zero, where $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional measure. In particular S_u admits a normal $\nu_u(x)$ defined for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$. Moreover u admits traces u^+ and u^- on each side of S_u, and for every $A \subseteq \Omega$ we have the representation formula

$$Du(A) = \int_A \nabla u\, dx + \int_{S_u \cap A} (u^+ - u^-) \otimes \nu_u\, d\mathcal{H}^{N-1} + D^c u(A),$$

where $D^c u$ is the Cantor part of Du, which is singular with respect $\mathcal{L}^N$ and $\mathcal{H}^{N-1} \llcorner S_u$.

We say that $u \in SBV(\Omega;\mathbb{R}^M)$ if $u \in BV(\Omega;\mathbb{R}^M)$ and $D^c u = 0$, i.e., the singular part of Du with respect to $\mathcal{L}^N$ is concentrated on S_u. The space $SBV(\Omega;\mathbb{R}^M)$ is called the space of $\mathbb{R}^M$-valued *special functions of bounded variation.*

The space SBV is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to L.Ambrosio (see [1], [2], [3]).

Theorem 1.6. *Let Ω be an open and bounded subset of $\mathbb{R}^N$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $SBV(\Omega;\mathbb{R}^M)$. Assume that there exists $p > 1$ and $C \geq 0$ such that*

$$\int_\Omega |\nabla u_k|^p \, dx + \mathcal{H}^{N-1}(S_{u_k}) + ||u_k||_\infty \leq C$$

for every $k \in \mathbb{N}$. Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and a function $u \in SBV(\Omega;\mathbb{R}^N)$ such that for every open set $A \subseteq \Omega$

$$\begin{aligned} u_{k_h} &\to u \quad \textit{strongly in } L^1(A;\mathbb{R}^M); \\ \nabla u_{k_h} &\rightharpoonup \nabla u \quad \textit{weakly in } L^p(A;\mathbb{R}^{MN}); \\ \mathcal{H}^{N-1}(S_u \cap A) &\leq \liminf_{h\to+\infty} \mathcal{H}^{N-1}(S_{u_{k_h}} \cap A). \end{aligned} \tag{1.5}$$

For applications to fracture mechanics, it is useful to set for $p \geq 1$

$$SBV^p(\Omega;\mathbb{R}^M) := \{u \in SBV(\Omega;\mathbb{R}^M) : \nabla u \in L^p(\Omega,\mathbb{R}^{MN}), \mathcal{H}^{N-1}(S_u) < +\infty\}. \tag{1.6}$$

We will say that u_k converges weakly to u in $SBV^p(\Omega;\mathbb{R}^M)$, and we will write $u_k \rightharpoonup u$ in $SBV^p(\Omega;\mathbb{R}^M)$, if u_k and u satisfy (1.5) for every open subset A of Ω.

2. The measure theoretical image of SBV-maps

In this section we deal with the problem of defining the image for a deformation $u \in SBV(\Omega;\mathbb{R}^N)$ which could be useful for the study of non-interpenetration of matter for cracked hyperelastic bodies. Recall that an SBV-function is formally an equivalence class of maps which coincide almost everywhere in Ω, so that the set $u(\Omega)$ depends on the representative we choose. We look for an image of Ω under u which depends only on the class, and for which an area formula holds, so that a reformulation of a.e.-injectivity in the spirit of Proposition 1.5 is available.

Let us denote by Ω_L and Ω_D the sets of Lebesgue points and of approximate differentiability points of u. From the general theory of BV functions, we have that Ω_L and Ω_D do not depend upon the representative of u, and that they have full measure in Ω. If $\tilde{u}(x)$ is the Lebesgue value of u at $x \in \Omega_L$, two natural candidates for the definition of the image of Ω under u are the representative u_L and u_D defined as

$$u_L(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \Omega_L \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_D(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \Omega_D \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

From Theorem 1.1, we immediately deduce that u_D satisfies the area formula. Concerning the Lebesgue representative u_L, from Section 1.1 we have that the area formula (1.2) holds if and only if u_L satisfies the N-property, i.e., if u_L maps $\mathcal{L}^N$-negligible sets into $\mathcal{L}^N$-negligible sets. The following example shows that this is not the case in general for SBV^p-maps: the construction we employ is inspired by a counterexample given by Malý and Martio [21] concerning the N-property for the Lebesgue representative of Sobolev functions in $W^{1,N}$.

Example 2.1. **(Lebesgue representatives of SBV^p-functions do not satisfy in general the N-property)** Let Ω be the unit ball of $\mathbb{R}^N$. We construct a map $u \in SBV^p(\Omega;\mathbb{R}^N)$ for every $1 \leq p \leq \infty$ which admits a set B of Lebesgue points contained in $J := \{te_1, t \in [0,1]\}$, and such that $u_L(B) = Q$, where $Q = [0,1]^N$ and e_1 is the first vector of the canonical base of $\mathbb{R}^N$.

Let us consider $Y := \{-\frac{1}{2}, \frac{1}{2}\}^N$: for every $y \in Y$ we can find $z_y \in J$ and $r_0 > 0$ such that the balls $B(z_y, r_0)$ are disjoint and contained in Ω, and such that $0 \notin B(z_y, r_0)$. Let us consider the map

$$g_m(x) := \sum_{y \in Y} y 1_{B(z_y, \alpha_m)}$$

where $\alpha_m < r_0$ and $\alpha_m \to 0$ as $m \to +\infty$. Clearly $g_m \in SBV^p(\Omega;\mathbb{R}^N)$ for every $p \in [1, +\infty]$.

Let us construct a sequence of maps $u_k \in SBV^p(Q, \mathbb{R}^N)$ as follows. Let u_0 be the constant map $(1/2, \dots, 1/2)$. For $k \geq 1$, let us divide the cube Q into cubes

$$Q_i^k := [2^{-k+1}(i_1 - 1), 2^{-k+1} i_1] \times \cdots \times [2^{-k+1}(i_N - 1), 2^{-k+1} i_N]$$

where $i \in \{1, 2, \dots, 2^{k-1}\}^N$: The graph of u_{k-1} enables us to find points $x_i^k \in J$ and a radius r_k such that the mappings u_{k-1} maps the ball $B(x_i^k, r_k)$ to the center of the cube Q_i^k for each $i \in \{1, 2, \dots, 2^{k-1}\}^N$. Let us set

$$h_{m,k} := \begin{cases} 2^{-k} g_m\left(\frac{x - x_i^k}{r_k}\right) & \text{if } |x - x_i^k| \leq r_k \\ 0 & \text{otherwise,} \end{cases}$$

and let us choose m_k in such a way that

$$\mathcal{H}^{N-1}(S_{h_{m_k,k}}) < 2^{-k}. \tag{2.2}$$

Let $B_k := \cup_i B(x_i^k, r_k)$. We set

$$u_k := u_{k-1} + h_{m_k,k}.$$

The sequence u_k converges pointwise to a function $u : \Omega \to \mathbb{R}^N$: in view of (2.2) and of Ambrosio's compactness theorem, we conclude that $u \in SBV^p(\Omega;\mathbb{R}^N)$ for every $p \in [1, +\infty]$. Notice that by construction we have that $B := \cap_{k=1}^{\infty} B_k \subseteq J$ are Lebesgue points for u, and moreover that u_L is continuous on B. As a consequence, we conclude that $u_L(B) = Q$, so that u_L does not satisfy the N-property.

Remark 2.2. In the case of Sobolev functions in $W^{1,p}(\Omega;\mathbb{R}^N)$ with $p > N$, Marcus and Mizel [22] proved that the N-property is satisfied by the continuous representative. Malý and Martio [21] proved that this is no longer the case for functions in $W^{1,N}(\Omega;\mathbb{R}^N)$ (and we used their ideas in the Example above). However, if we add the condition

$$\det \nabla u(x) > 0 \qquad \text{for a.e. } x \in \Omega, \tag{2.3}$$

then the Lebesgue representative of $u \in W^{1,N}(\Omega;\mathbb{R}^N)$ satisfies the N-property (see e.g. [13, Theorem 5.32]). But condition (2.3) does not help to infer further regularity for SBV^p-deformations. In fact, employing the notations of Example 2.1, we can consider the map

$v : \Omega \to \mathbb{R}^N$ defined as follows: we set $v(x) = x$ outside B_1, and if $B_k = \cup_i B(x_i^k, r_k)$ and $B_{k+1} = \cup_j B(x_j^{k+1}, r_{k+1})$, we set

$$v(x) = \lambda_k (x - x_i^k) \qquad \text{for } x \in B(x_i^k, r_k) \setminus B_{k+1}$$

with $\lambda_k \in]0,1[$. Notice that $v \in SBV^p(Q, \mathbb{R}^N)$ for every $p \geq 1$, $\det \nabla v(x) > 0$ for a.e. $x \in Q$, and that $B = \cap_k B_k$ is a set of Lebesgue points for v with $v_L = 0$ on B. As a consequence,

$$w(x) := v(x) + u(x)$$

satisfies $\det \nabla w > 0$ a.e. in Q, and $w_L(B) = Q$, i.e., the Lebesgue representative of w does not satisfy the N-property.

Remark 2.3 (Damaged points). In view of Example 2.1, we deduce that we should consider Lebesgue points of u which are not approximate differentiability points as *damaged points* of the body. Their image under u is not connected to the elastic properties of the deformation.

The previous considerations motivate the choice of u_D instead of u_L as a privileged representative of the map u in order to define the deformed configuration of the body Ω under the action of u. We have the following definition.

Definition 2.4 (Measure theoretical image). *Let $u \in SBV(\Omega; \mathbb{R}^N)$, let u_D be defined in* (2.1)*, and let E be a measurable subset of Ω. We say that $u_D(E)$ is the measure theoretical image of E under the map u, and we denote it by $[u(E)]$.*

Notice that since u_D satisfies the N-property, $[u(E)]$ is indeed a measurable set. The measure theoretical image $[u(\Omega)]$ enjoys the following variational property.

Proposition 2.5. *Let $u \in SBV(\Omega; \mathbb{R}^N)$: then we have*

$$\mathcal{L}^N([u(\Omega)]) = \min \left\{ \mathcal{L}^N(v(\Omega)) : v \text{ is a representative of } u \right\} \tag{2.4}$$

and

$$\mathcal{L}^N\left([u(\Omega)]\right) \leq \int_\Omega |\det \nabla u(x)| \, dx. \tag{2.5}$$

Proof. Inequality (2.5) follows immediately from (1.1) applied to u_D. Let us prove (2.4). Let E be the set where v is different from the representative u_D of u. We have

$$v(\Omega) = v(\Omega \setminus E) \cup v(E) = u_D(\Omega \setminus E) \cup v(E).$$

Since u_D satisfies the N-property and $\mathcal{L}^N(E) = 0$ we deduce

$$\mathcal{L}^N(v(\Omega)) \geq \mathcal{L}^N(u_D(\Omega \setminus E)) = \mathcal{L}^N(u_D(\Omega)),$$

and the proof is concluded. □

In order to prove a stability result for the measure theoretical image of an SBV-map under weak convergence, we need the following lemma.

Lemma 2.6. *Let $(u_h)_{h \in \mathbb{N}}$ be a sequence in $SBV(\Omega; \mathbb{R}^N)$ and let $u \in SBV(\Omega; \mathbb{R}^N)$ be such that*

$$u_h \rightharpoonup u \qquad \text{weakly in } SBV(\Omega; \mathbb{R}^N)$$

according to (1.5)*. Let us assume that $(\det \nabla u_h)_{h \in \mathbb{N}}$ is equintegrable. Then we have*

$$\limsup_{h \to +\infty} \mathcal{L}^N([u_h(\Omega)] \setminus [u(\Omega)]) = 0. \tag{2.6}$$

Proof. Since $u_h \to u$ strongly in $L^1(\Omega;\mathbb{R}^N)$, we can suppose (up to a subsequence) that $u_h \to u$ almost uniformly. As a consequence, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \Omega$ such that $\mathcal{L}^N(\Omega \setminus K_\varepsilon) < \varepsilon$, the restrictions of u_h and u on K_ε are continuous, and $u_h \to u$ uniformly on K_ε. We claim that

$$\limsup_{h\to+\infty} \mathcal{L}^N([u_h(\Omega \setminus K_\varepsilon)]) = c(\varepsilon) \tag{2.7}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that

$$\limsup_{h\to+\infty} \mathcal{L}^N([u_h(K_\varepsilon)] \setminus [u(\Omega)]) = 0. \tag{2.8}$$

Clearly (2.7) and (2.8) imply (2.6). In order to prove claim (2.7), it is sufficient to note that by (2.5) we have

$$\mathcal{L}^N([u_h(\Omega \setminus K_\varepsilon)]) \le \int_{\Omega\setminus K_\varepsilon} |\det\nabla u_h(x)|\, dx.$$

The conclusion follows since $(\det\nabla u_h)_{h\in\mathbb{N}}$ is equintegrable and $\mathcal{L}^N(\Omega \setminus K_\varepsilon) < \varepsilon$. Let us come to (2.8). Since $u_h \to u$ uniformly on K_ε, for every $\eta > 0$ we get that for h large enough

$$u_h(K_\varepsilon) \subseteq A^\eta := \{y \in \mathbb{R}^N : d(y, u(K_\varepsilon)) < \eta\}.$$

We deduce that

$$\limsup_{h\to+\infty} \mathcal{L}^N(u_h(K_\varepsilon) \setminus [u(\Omega)]) \le \mathcal{L}^N(A^\eta \setminus u(K_\varepsilon)).$$

Since $u(K_\varepsilon)$ is compact, we get that $\lim_{\eta\to 0} \mathcal{L}^N(A^\eta \setminus u(K_\varepsilon)) = 0$, so that claim (2.8) is proved. □

The following theorem contains a stability result (in a L^1-sense) for the measure theoretical image of SBV^p-maps with $p > N$ under weak convergence.

Theorem 2.7. *Let us assume that $p > N$, and let $(u_h)_{h\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega;\mathbb{R}^N)$ weakly converging to $u \in SBV^p(\Omega;\mathbb{R}^N)$ according to* (1.5). *Then we have*

$$1_{[u_h(\Omega)]} \to 1_{[u(\Omega)]} \qquad \textit{strongly in } L^1(\mathbb{R}^N). \tag{2.9}$$

Proof. We have to check that

$$\lim_{h\to+\infty} \mathcal{L}^N([u_h(\Omega)] \setminus [u(\Omega)]) = 0, \tag{2.10}$$

and

$$\lim_{h\to+\infty} \mathcal{L}^N([u(\Omega)] \setminus [u_h(\Omega)]) = 0. \tag{2.11}$$

Equality (2.10) follows immediately from Lemma 2.6 because weak convergence of u_h to u in $SBV^p(\Omega;\mathbb{R}^N)$ with $p > N$ implies weak convergence in $L^1(\Omega)$ of $\det\nabla u_h$ to $\det\nabla u$ (see [5, Corollary 5.31]), so that in particular $(\det\nabla u_h)_{h\in\mathbb{N}}$ is equintegrable.

Let us pass to the proof of (2.11). We claim that given $\varepsilon > 0$, for a.e. $x \in \Omega$ there exists $r_k \to 0$ such that

$$\limsup_{h\to+\infty} \mathcal{L}^N\left([u(B_{r_k}(x))] \setminus [u_h(B_{r_k}(x)]\right) \le \varepsilon r_k^N. \tag{2.12}$$

Then (2.11) follows through a covering argument. In fact, by Besicovitch covering theorem there exists a sequence of points $(x_j)_{j\in\mathbb{N}}$ in Ω and a sequence of radii $(r_j)_{j\in\mathbb{N}}$ such that

$\{B_{r_j}(x_j)\}_{j\in\mathbb{N}}$ is a disjoint covering of Ω up to a set of $\mathcal{L}^N$-measure zero and each $B_{r_j}(x_j)$ satisfies (2.12). We conclude that

$$\limsup_{h\to+\infty} \mathcal{L}^N([u(\Omega)] \setminus [u_h(\Omega)]) \leq \sum_{j=0}^{+\infty} \limsup_{h\to+\infty} \mathcal{L}^N([u(B_{r_j}(x_j))] \setminus [u_h(B_{r_j}(x_j))]) \leq \varepsilon \sum_{j=0}^{+\infty} r_j^N \leq \varepsilon \frac{\mathcal{L}^N(\Omega)}{\omega_N},$$

where ω_N is the volume of the unit ball. Since ε is arbitrary, (2.11) follows.

In order to conclude the proof, we need to establish claim (2.12). Let us consider the measures

$$\mu_h := |\nabla u_h|^p \, dx + \mathcal{H}^{N-1} \llcorner S_{u_h}.$$

By weak convergence of u_h to u, we may assume that up a subsequence

$$\mu_h \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures.}$$

Notice that for a.e. $x \in \Omega$ we have

$$K(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\rho^N} < +\infty \tag{2.13}$$

and

$$H(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\rho^{N-1}} = 0. \tag{2.14}$$

In fact order to prove (2.13), let us assume by contradiction that there exists a Borel set B with positive Lebesgue measure such that $K(x) = +\infty$ on B. Then for every $t > 0$ we have

$$K(x) > t \qquad \text{for every } x \in B,$$

then (see for instance [5, Theorem 2.56]) we deduce that

$$\mu \llcorner B \geq t \mathcal{L}^N \llcorner B,$$

so that $\mu(B) = \infty$. But this is against the fact that μ is finite. In order to prove (2.14), let us assume by contradiction that there exists a Borel set B with positive Lebesgue measure and $t > 0$ such that

$$H(x) \geq t \qquad \text{for every } x \in B.$$

Then (see for instance [5, Theorem 2.56]) we deduce that

$$\mu \llcorner B \geq t \mathcal{H}^{N-1} \llcorner B,$$

so that $\mu(B) = \infty$. But again this is against the fact that μ is finite.

Let Ω_D be the set of approximate differentiability points of u, and let $x \in \Omega_D$ be such that x is a Lebesgue point for $|\nabla u|^p$, x has $(N-1)$-density zero for S_u, and (2.14) and (2.13) hold. Let $r_k \to 0$ and $h_k \to +\infty$ be such that, setting

$$v_k(y) := \frac{u(x + r_k y) - u(x)}{r_k} \tag{2.15}$$

and

$$w_k(y) := \frac{u_{h_k}(x + r_k y) - u(x)}{r_k}, \tag{2.16}$$

and denoting with L the linear map determined by $\nabla u(x)$, we have

$$v_k \to L \qquad \text{strongly in } L^1(B_1;\mathbb{R}^N), \tag{2.17}$$

$$w_k \to L \qquad \text{strongly in } L^1(B_1;\mathbb{R}^N), \tag{2.18}$$

and

$$\limsup_{h\to+\infty} \mathcal{L}^N\left([u(B_{r_k}(x))] \setminus [u_h(B_{r_k}(x)]\right) \le \mathcal{L}^N\left([u(B_{r_k}(x))] \setminus [u_{h_k}(B_{r_k}(x)]\right) + \frac{\varepsilon}{2} r_k^N. \tag{2.19}$$

By assumption on x we have that

$$\|\nabla v_k\|_{L^p(B_1;\mathbb{R}^{N^2})} \le C \quad \text{and} \quad \mathcal{H}^{N-1}(S_{v_k}) \to 0, \tag{2.20}$$

and by (2.13) and (2.14) we have that there exists $C > 0$ such that

$$\|\nabla w_k\|_{L^p(B_1;\mathbb{R}^{N^2})} \le C$$

and

$$\mathcal{H}^{N-1}(S_{w_k}) \to 0.$$

By [20, Lemma 2.1] we get that there exists $z_k \in W^{1,p}(B_1,\mathbb{R}^N)$ such that

$$\mathcal{L}^N\left(\{y \in B_1 : z_k(y) \ne w_k(y) \text{ or } \nabla z_k(y) \ne \nabla w_k(y)\}\right) \to 0$$

and $(|\nabla z_k|^p)_{k\in\mathbb{N}}$ is equintegrable. Since $p > N$, by (2.5) we deduce that

$$\mathcal{L}^N\left([z_k(B_1)]\Delta[w_k(B_1)]\right) \to 0 \tag{2.21}$$

where $A\Delta B$ denotes the symmetric difference of sets. Since

$$\mathcal{L}^N\left([u(B_{r_k}(x))] \setminus [u_{h_k}(B_{r_k}(x)]\right) = r_k^N \mathcal{L}^N\left([v_k(B_1)] \setminus [w_k(B_1)]\right),$$

taking into account (2.19) and (2.21), in order to prove (2.12), it suffices to show that

$$\lim_{k\to+\infty} \mathcal{L}^N\left([v_k(B_1)] \setminus [z_k(B_1)]\right) = 0.$$

Notice that in view of (2.17) and (2.20), $v_k \to L$ weakly in $SBV^p(B_1;\mathbb{R}^N)$, and since $p > N$, $\det\nabla v_k \rightharpoonup \det L$ weakly in $L^1(B_1)$. From Lemma 2.6 we get

$$\lim_{k\to+\infty} \mathcal{L}^N([v_k(B_1)] \setminus L(B_1)) = 0.$$

Then in order to conclude, it suffices to show that

$$\lim_{k\to+\infty} \mathcal{L}^N(L(B_1) \setminus [z_k(B_1)]) = 0. \tag{2.22}$$

Notice that $z_k \to L$ weakly in $W^{1,p}(B_1,\mathbb{R}^N)$, and since $p > N$, the convergence is uniform. If $\det L = 0$, then there is nothing to prove; otherwise (2.22) is a consequence of the stability of the degree for continuous maps under uniform convergence (see [13]). The proof is thus concluded. □

Remark 2.8. Notice that Theorem 2.7 does not hold in the case $p \le N$ even in the case of Sobolev spaces, because cavitation effects may occur (see Ball-Murat [8]). Convergence (2.9) still holds if non-interpenetration condition for u_h and suitable estimates on $\det\nabla u_h$ are assumed (see Theorem 3.4).

3. Ciarlet-Nečas non-interpenetration condition for SBV-deformations

The aim of this section is to show that a non-interpenetration condition for SBV-maps can be taken into account in some problems arising in the variational approach to fracture mechanics.

Following the ideas of Ciarlet-Nečas, we will consider a.e.-injective deformations as admissible deformations which do not present interpenetration of matter.

Definition 3.1 (A.e.-injective SBV-maps). *We say that $u \in SBV(\Omega;\mathbb{R}^N)$ is a.e.-injective if for every representative v of u there exists a $\mathcal{L}^N$-negligible set $E \subset \Omega$ such that the restriction of v on $\Omega \setminus E$ is injective.*

By Proposition 1.5 applied to the approximately differentiable representative, we get immediately that a.e.-injectivity for SBV-maps can be reformulated in the following way.

Proposition 3.2. *If $u \in SBV(\Omega;\mathbb{R}^N)$ is a.e.-injective, then*

$$\int_\Omega |\det\nabla u|\, dx \le \mathcal{L}^N([u(\Omega)]), \tag{3.1}$$

where $[u(\Omega)]$ denotes the image of Ω under u according to Definition 2.4. Viceversa, if u satisfies (3.1) *and* $\det\nabla u \neq 0$ *a.e. in Ω, then u is a.e.-injective.*

We can now give the definition of the Ciarlet-Nečas non-interpenetration condition for SBV-maps.

Definition 3.3 (Ciarlet-Nečas non-interpenetration condition for SBV-maps). *We say that $u \in SBV(\Omega;\mathbb{R}^N)$ satisfies the Ciarlet-Nečas non-interpenetration condition if* $\det\nabla u(x) > 0$ *for a.e. $x \in \Omega$ and if u is a.e.-injective or, equivalently, if it satisfies*

$$\int_\Omega \det\nabla u\, dx \le \mathcal{L}^N([u(\Omega)]),$$

where $[u(\Omega)]$ denotes the image of Ω under u according to Definition 2.4.

As mentioned in the Introduction, the condition $\det\nabla u(x) > 0$ for a.e. $x \in \Omega$ means that, in a weak sense, u is orientation preserving, while the a.e.-injectivity prevents overlapping of matter.

Maps which satisfy the Ciarlet-Nečas non-interpenetration condition are essentially closed under weak convergence with stability for their measure theoretical images. The precise statement is the following.

Theorem 3.4. *Let $(u_h)_{h\in\mathbb{N}}$ be a sequence of maps in $SBV(\Omega;\mathbb{R}^N)$ satisfying inequality* (3.1)*, and let $u \in SBV(\Omega;\mathbb{R}^N)$ be such that $u_h \rightharpoonup u$ weakly in $SBV(\Omega;\mathbb{R}^N)$. Let us assume that* $\det\nabla u_h \rightharpoonup \det\nabla u$ *weakly in $L^1(\Omega)$. Then we have*

$$1_{[u_h(\Omega)]} \to 1_{[u(\Omega)]} \qquad \text{strongly in } L^1(\mathbb{R}^N), \tag{3.2}$$

and u satisfies inequality (3.1)*. If in addition* $\det\nabla u(x) > 0$ *for a.e. $x \in \Omega$, then u is a.e.-injective, and hence satisfies the Ciarlet-Nečas non-interpenetration condition.*

Proof. To prove (3.2) we have to check that

$$\lim_{h\to+\infty} \mathcal{L}^N([u_h(\Omega)] \setminus [u(\Omega)]) = 0, \tag{3.3}$$

and

$$\lim_{h\to+\infty} \mathcal{L}^N([u(\Omega)] \setminus [u_h(\Omega)]) = 0. \tag{3.4}$$

Inequality (3.3) follows immediately from Lemma 2.6. Let us prove (3.4). By assumption and by (2.5) we have

$$\mathcal{L}^N([u_h(\Omega)]) = \int_\Omega |\mathrm{det}\nabla u_h|\, dx.$$

By (2.5) and since $\mathrm{det}\nabla u_h \rightharpoonup \mathrm{det}\nabla u$ weakly in $L^1(\Omega)$, we obtain

$$\mathcal{L}^N([u(\Omega)]) \le \int_\Omega |\mathrm{det}\nabla u|\, dx \le \liminf_{h\to+\infty} \int_\Omega |\mathrm{det}\nabla u_h|\, dx = \liminf_{h\to+\infty} \mathcal{L}^N([u_h(\Omega)]).$$

This relation together with (3.3) implies (3.4), and the proof of (3.2) is concluded. Since

$$\int_\Omega |\mathrm{det}\nabla u|\, dx \le \liminf_{h\to+\infty} \int_\Omega |\mathrm{det}\nabla u_h|\, dx = \liminf_{h\to+\infty} \mathcal{L}^N([u_h(\Omega)]) = \mathcal{L}^N([u(\Omega)]),$$

we get that u satisfies (3.1). Finally, the last statement follows by Proposition 3.2. □

4. An application to brittle Ogden's materials

In this section we show how the Ciarlet-Nečas non-interpenetration condition given in Definition 3.3 can be taken into account in the analysis of brittle materials of Ogden's type [24]. Let us consider $\Omega \subseteq \mathbb{R}^N$ open, bounded and with Lipschitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology. Let $\mathbb{M}$ denote the set on $N \times N$ matrices, and let $\mathbb{M}_+$ be the subset of $\mathbb{M}$ given by those with positive determinants. Let $W : \mathbb{M}_+ \to \mathbb{R}$ be a stored energy density such that the following assumptions hold.

(a) *Polyconvexity of* W: there is a convex function $\mathbb{W} : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$ such that

$$W(F) = \mathbb{W}(\mathcal{M}(F)) \qquad \text{for all } F \in \mathbb{M},$$

where $\mathcal{M}(F)$ denotes the vector whose components are all the minors of the matrix F, and τ is the dimension of $\mathcal{M}(F)$.

(b) *Behavior as* $\mathrm{det}F \to 0^+$:

$$W(F) \to +\infty \qquad \text{as } \mathrm{det}F \to 0^+. \tag{4.1}$$

(c) *Coerciveness*: we have the growth estimate

$$W(F) \ge \beta_1|F|^p + \sum_{k=2}^{N-1} \beta_k|\mathrm{adj}_k F|^{p_k} + \beta_N|\mathrm{det}F|^{p_N} \qquad \text{for all } F \in \mathbb{M}_+, \tag{4.2}$$

where $\beta_k > 0$ for every k, and

$$p \ge 2, \qquad p_k \ge \frac{p}{p-1} \text{ if } k = 2, \dots, N-1, \qquad p_N > 1,$$

and where $\mathrm{adj}_k F$ denotes the vector whose components are the minors of the matrix F of order k.

The stored energy density W models a large class of hyperelastic materials known as Ogden's materials [24].

Let K be a given compact set in $\mathbb{R}^N$. Let us consider as family of admissible deformations the set

$$\mathcal{A}(K) := \left\{u \in SBV^p(\Omega, \mathbb{R}^N) : u \text{ satisfies Definition 3.3 and } [u(\Omega)] \subseteq K\right\}.$$

As explained in the previous Section, the Ciarlet-Nečas non-interpenetration condition requires that u is an a.e.-injective and orientation preserving (in a weak sense) map. The relation $[u(\Omega)] \subseteq K$ can be interpreted as a *confinement condition.*

The problem we are going to consider is the following. Let $g \in \mathcal{A}(K) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ be such that $\int_\Omega W(\nabla g)\,dx < +\infty$. We consider the total energy on $\mathcal{A}(K)$ defined as

$$F(u) := \int_\Omega W(\nabla u)\,dx + \mathcal{H}^{N-1}(S_u^g) \tag{4.3}$$

where

$$S_u^g := S_u \cup \{x \in \partial_D\Omega : g(x) \neq u(x)\}\,,$$

and the inequality is intented for the traces of g and u on $\partial\Omega$. The set S_u^g takes into account the crack formed inside Ω, and the part of the $\partial_D\Omega$ where u does not agree with the imposed deformation g (which is thus considered as a part of the crack which has reached the boundary). As mentioned in the Introduction, the minimization of (4.3) can be interpreted as a mathematical model for equilibrium configurations of Ogden's materials with cracks. The minimization on $\mathcal{A}(K)$ leads to *non-interpenetrating* equilibrium configurations.

The main result of the Section is the following.

Theorem 4.1. *The minimum problem*

$$\min\{F(u)\,:\,u \in \mathcal{A}(K)\} \tag{4.4}$$

has a solution.

Proof. Let $(u_h)_{h\in\mathbb{N}}$ be a minimizing sequence for F. Since $F(u_h) \leq F(g) = \int_\Omega W(\nabla g)\,dx < +\infty$, we get by (4.2)

$$\sup_h \left(\|\nabla u_h\|_{L^p} + \sum_{k=2}^{N-1} \|\mathrm{adj}_k \nabla u_h\|_{L^{p_k}} + \|\det\nabla u_h\|_{L^{p_N}} + \mathcal{H}^{N-1}(S_{u_h}^g) \right) < +\infty.$$

Since $[u_h(\Omega)] \subseteq K$, and K is compact, we obtain that u_h is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^N)$. By Ambrosio's theorem 1.6 we get that, up to a subsequence

$$u_h \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega, \mathbb{R}^N).$$

By [16, Theorem 3.4], we obtain that, up to a subsequence, for every $k = 2, \ldots, N-1$

$$\mathrm{adj}_k \nabla u_h \rightharpoonup \mathrm{adj}_k \nabla u \qquad \text{weakly in } L^{p_k}(\Omega, \mathbb{R}^{\tau_k})$$

(τ_k is the number of minors of order k) and

$$\det\nabla u_h \rightharpoonup \det\nabla u \qquad \text{weakly in } L^{p_N}(\Omega).$$

By Theorem 3.4 and the fact that $[u_h(\Omega)] \subseteq K$, we get that $[u(\Omega)] \subseteq K$. Moreover, since $\det\nabla u_h > 0$ a.e. in Ω, we obtain $\det\nabla u \geq 0$ a.e. in Ω. By polyconvexity of W we deduce that

$$\int_\Omega W(\nabla u)\,dx \leq \liminf_{h\to+\infty} \int_\Omega W(\nabla u_h)\,dx$$

and by Ambrosio's theorem (applied to the extension of u_h and u to $\mathbb{R}^N$ by setting $u_h = u = g$ outside Ω) we get

$$\mathcal{H}^{N-1}(S_u^g) \leq \liminf_{h\to+\infty} \mathcal{H}^{N-1}(S_{u_h}^g).$$

We thus finally obtain

$$F(u) \leq \liminf_{h\to+\infty} F(u_h) = \min_{\mathcal{A}(K)} F.$$

Since $F(u) < +\infty$, by (4.1) we get that $\det\nabla u > 0$ a.e. in Ω. By Theorem 3.4 we deduce that $u \in \mathcal{A}(K)$, and the proof is concluded. □

5. Further discussions and remarks

The non-interpenetration of matter for SBV-deformation which we have studied in the previous sections following the ideas of Ciarlet-Nečas relies on the notion of a.e.- injectivity, and it is based on the area formula for a.e.-approximately differentiable maps. We have seen that the constraint of non-interpenetration is closed with respect to weak convergence in SBV under mild additional energetic assumptions (see Theorem 3.4).

Different notions of non-interpenetration can be considered. The aim of this section is to discuss briefly some of them, pointing out the differences through examples.

5.1. **Linearized self-contact condition.** A local non-interpenetration condition based on the self-contact of the crack's surface can be introduced for linearized elasticity as follows. We say that a displacement $u : \Omega \to \mathbb{R}^N$ satisfies the *linearized self-contact condition* if for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ we have

$$(u^+(x) - u^-(x)) \cdot \nu(x) \geq 0. \tag{5.1}$$

This condition is *local* because it takes into account the behaviour near each point of the crack, prescribing that the opening does not generate interpenetration of matter. Clearly this condition has not the global character carried by a.e.-injectivity. It can be proved that the linearized self-contact condition is closed with respect to weak convergence in SBV [10].

It is clear that even if a displacement function u satisfies the linearized self-contact condition (5.1), the associated deformation function $v(x) := x+u(x)$ is not in general a.e.-injective. For instance it is very easy to find continuous deformations (trivially satisfying (5.1)), which are not a.e.-injective. Also the viceversa is false: a.e.-injective functions do not satisfy in general the linearized non-interpenetration condition. An easy example is given as follows.

Example 5.1. Let $\Omega := B_1$, let w be a fixed vector, and let u be the displacement function defined by

$$u(x) := \begin{cases} 0 & \text{if } |x| \geq \frac{1}{2}, \\ w & \text{if } |x| < \frac{1}{2}. \end{cases} \tag{5.2}$$

If $|w|$ is big enough, we clearly have that the deformation function $v(x) := x + u(x)$ is a.e.-injective, while u does not satisfy the linearized self-contact condition.

However, for small displacements, a.e-injectivity implies the linearized condition. A rigorous statement is given in the following Proposition.

Proposition 5.2. *Let $u \in SBV^p(\Omega;\mathbb{R}^N)$ with $p > N$. Let $t_n \searrow 0$, and assume that for every $n \in \mathbb{N}$ the function $v_n(x) := x + t_n u(x)$ satisfies inequality* (3.1)*. Then u satisfies the linearized self-contact condition* (5.1)*.*

Proof. Assume by contradiction that there exists a set $E \subseteq S_u$ with $\mathcal{H}^{N-1}(E) > 0$ and such that for $x_0 \in E$

$$(u^+(x_0) - u^-(x_0)) \cdot \nu(x_0) < 0. \tag{5.3}$$

Let us consider the function $z_\infty : B_1 \to \mathbb{R}^N$ defined by

$$z_\infty(y) := \begin{cases} y + \lambda u^+(x_0) & \text{if } y \cdot \nu(x_0) \geq 0, \\ y + \lambda u^-(x_0) & \text{if } y \cdot \nu(x_0) < 0, \end{cases}$$

where $\lambda > 0$ is a positive constant. In view of (5.3) we can choose λ (small enough) such that the function z_∞ is not a.e.-injective.

Let $z_n : B_1 \to \mathbb{R}^N$ be defined as

$$z_n(y) := y + \lambda u \left(x_0 + \frac{t_n}{\lambda} y \right).$$

Note that by assumption the functions z_n satisfy inequality (3.1). Moreover (see [5, Theorem 3.78]) we can assume that $x_0 \in E$ is chosen in such a way that $z_n \to z_\infty$ strongly in $L^1(B_1)$. Since $\nabla u \in L^p(\Omega, \mathbb{R}^{N^2})$ with $p > N$, we deduce that

$$\det \nabla z_n \to 1 = \det \nabla z_\infty \qquad \text{strongly in } L^1(B_1).$$

By Theorem 3.4 we deduce that also z_∞ satisfies inequality (3.1) and that it is a.e. injective, which clearly provides a contradiction. □

5.2. **Non-interpenetration in the deformed configuration.** The Ciarlet-Nečas non-interpenetration condition requires that a map u satisfies

$$\int_\Omega |\det \nabla u| \, dx \leq \mathcal{L}^N([u(\Omega)]), \tag{5.4}$$

and that u preserves orientation, i.e. $\det \nabla u(x) > 0$ for a.e. $x \in \Omega$. If we let $\det \nabla u(x) \geq 0$ for a.e. $x \in \Omega$, we obtain a weaker notion of non-interpenetration in the deformed configuration $[u(\Omega)]$ as shown in the following Proposition.

Proposition 5.3. *Let $u \in SBV(\Omega; \mathbb{R}^N)$. Then u satisfies inequality* (5.4) *if and only if the set function $\mu_{[u]}$ defined by $\mu_{[u]}(E) = \mathcal{L}^N([u(E)])$ is a measure. In particular $\mu_{[u]}$ is a measure whenever u is a.e.-injective.*

Proof. Let us assume that (5.4) holds. By the area formula (1.2) applied to the approximately differentiable representative u_D of u we have that

$$m(u_D, y, \Omega) = 1 \qquad \text{for a.e. } y \in [u(\Omega)]. \tag{5.5}$$

In order to prove that $\mu_{[u]}$ is a measure, it suffices to show the additivity of $\mu_{[u]}$ on disjoint sets. Let E_1, E_2 be two measurable disjoint subset of Ω. By the fact that u_D satisfies the N-property we have that $u_D(E_1)$ and $u_D(E_2)$ are measurable subsets of $\mathbb{R}^N$. Moreover, by (5.5) we deduce that their intersection is negligible, so that $\mu_{[u]}(E_1 \cup E_2) = \mu_{[u]}(E_1) + \mu_{[u]}(E_2)$.

Let us assume now that $\mu_{[u]}$ is a measure. In view of the area formula (1.1), the proof reduces to showing that the multiplicity function $m(u_D, y, \Omega) = 1$ for a.e. $y \in \mathbb{R}^N$. To this aim let

$$E_n := \{ y \in \mathbb{R}^N : \text{ there exist } x_1, x_2 \in \Omega \text{ with } |x_1 - x_2| \geq 1/n, \, u_D(x_1) = u_D(x_2) = y \}.$$

The union of these sets E_n gives exactly the set of points $y \in \mathbb{R}^N$ with $m(u_D, y, \Omega) \neq 1$. Therefore we have to prove that each E_n has measure zero. To this aim let us fix $n \in \mathbb{N}$, and let us cover Ω by means of cubes Q_i of size $m(n)$, where $m(n)$ is chosen so small that if $|x_1 - x_2| \geq 1/n$, then x_1 and x_2 belong to two disjoint cubes. Let Q_1 and Q_2 be two disjoint cubes. By the fact that $\mu_{[u]}$ is a measure, we obtain that $[u(Q_1 \cap \Omega)] \cap [u(Q_2 \cap \Omega)]$ has measure zero. Since E_n by construction is contained in a finite union of such intersections, we deduce that E_n has measure zero.

Finally, if u is a.e.-injective, the conclusion follows by Proposition 3.2. □

In view of Proposition 5.3, we conclude that no overlapping of matter in the deformed configuration occurs on a set of positive measure. On the other hand, inequality (3.1) does not prevent that a set of positive measure in the reference configuration is mapped on a single point. These considerations lead to the following definition.

Definition 5.4. *We say that* $u \in SBV(\Omega;\mathbb{R}^N)$ *satisfies the non-interpenetration condition in the deformed configuration if* $\det\nabla u \geq 0$ *for a.e.-*$x \in \Omega$*, and*

$$\int_\Omega \det\nabla u \, dx \leq \mathcal{L}^N([u(\Omega)]).$$

Theorem 3.4 ensures that the non-interpenetration condition in the deformed configuration is preserved along any sequence $u_n \rightharpoonup u$ in $SBV(\Omega;\mathbb{R}^N)$ such that $\det\nabla u_n \rightharpoonup \det\nabla u$ weakly in $L^1(\Omega)$. Therefore this condition can be involved in minimization problems in alternative to the Ciarlet-Nečas non-interpenetration condition. The convenience of this notion is that it does not require the condition $\det\nabla u > 0$ for a.e. $x \in \Omega$, which in some cases could be difficult to check. An application of the non-interpenetration in the deformed configuration which explains this point is given in the following paragraph, where we take into account the deformation process.

5.3. **Non-interpenetration during the deformation process.** Example 5.1 shows that there are very unphysical deformations which satisfy the Ciarlet-Nečas non-interpenetration condition. The point is that it seems difficult to imagine a deformation process whose result is the deformation function $v(x) := x + u(x)$ with $u(x)$ defined as in (5.2). It looks then natural to consider a notion of non-interpenetration which takes into account the deformation process. More precisely, given a deformation $v \in SBV(\Omega;\mathbb{R}^N)$, we could consider v admissible if there exists a time dependent deformation process satisfying at each time a non-interpenetration condition, which starts from the identity map and whose final result is the given deformation v. To make this notion rigorous, we have also to specify which are the admissible deformation processes. We will consider here the simplest deformation process, which is progressive and linear in time. More precisely given $v \in SBV(\Omega;\mathbb{R}^N)$, let $V : [0,1] \times \Omega \to \mathbb{R}^N$ be defined by

$$V(t,x) := x + t(v(x) - x).$$

The function V represents the deformation process, while the function $t(v(x) - x)$ represents the displacement function at time t, which is assumed to be linear with respect to time. Note that $V(0,\cdot)$ is the identity map, while $V(1,\cdot) \equiv v$.

We say that a deformation v satisfies the *progressive non-interpenetration condition* if for every $t \in [0,1]$ the map $V(t,\cdot)$ satisfies the non-interpenetration condition in the deformed configuration (see Definition 5.4). Clearly every deformation v which satisfies the progressive non-interpenetration condition and with $\det\nabla v > 0$ a.e. in Ω is in particular a.e.-injective (since $v \equiv V(1,\cdot)$ is a.e.-injective by Proposition 3.2) and hence it satisfies the Ciarlet-Nečas non-interpenetration condition. The converse is not true in general, as we saw in Example 5.1.

The progressive non-interpenetration condition can be clearly taken as a constraint in the minimization problem (4.4) relative to brittle Odgen materials provided that the boundary datum g satisfies the same condition. Indeed, by Theorem 3.4 we easily deduce that the progressive non-interpenetration condition is closed along sequences of SBV-deformations whose minors weakly converge in L^1. We deduce that the minimum problem (4.4) has a solution in the class of Ciarlet-Nečas admissible deformations which satisfy also the progressive non-interpenetration condition.

Finally, another interesting feature of the progressive non-interpenetration condition is that, in view of Proposition 5.2, it implies the linearized self-contact condition.

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