A direct uniqueness proof for equations involving the p-Laplace operator

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Abstract: *We provide a simple convexity argument for some known uniqueness theorems. Previous proofs were more technical and had to pay attention to the behaviour of solutions near the boundary.*

Let $p \in (1, \infty)$, and suppose that $\Omega \subset I\!R^n$ is a bounded simply connected domain. A well-known result in nonlinear partial differential equations states that positive (weak) solutions of

$$\begin{aligned} \Delta_p u + \lambda |u|^{p-2} u = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned} \tag{1}$$

are unique modulo scaling, in other words the corresponding eigenvalue is simple. These functions are also called first eigenfunctions or nonlinear ground states of the p-Laplace operator, and $\Delta_p v := \operatorname{div}(|\nabla v|^{p-2} \nabla v)$. Various proofs were given in [1–3, 7–12].

To prove the uniqueness result for problem (1), recall that modulo scaling u is characterized as a critical point, and in fact a minimizer, of the functional

$$J_p(v) = \int_\Omega |\nabla v|^p \ dx \quad \text{on} \quad K := \{ \ v \in W_0^{1,p}(\Omega) \mid ||v||_{L^p(\Omega)} = 1 \ \}, \tag{2}$$

Our proof is based on the observation, made in [8, Prop. 4], that for nonnegative functions u the functional $J_p(v)$ is convex in v^p. We attribute this idea to R.Benguria, see [4] and the remarks therein. If there are two positive solutions u and U of (2), then the function $w = \eta^{1/p}$ with $\eta := (u^p + U^p)/2$ is admissible in (2), because $\int_\Omega w^p \ dx = (\int_\Omega u^p \ dx + \int_\Omega U^p \ dx)/2 = 1$.

Now we calculate $\nabla w = \eta^{-1+1/p}[u^{p-1}\nabla u + U^{p-1}\nabla U]/2$, so that

$$\begin{aligned} |\nabla w|^p &= \eta^{1-p} \left| \frac{1}{2} \left(u^{p-1} \nabla u + U^{p-1} \nabla U \right) \right|^p \\ &= \eta \left| \frac{1}{2} \left(\frac{u^p}{\eta} \frac{\nabla u}{u} + \frac{U^p}{\eta} \frac{\nabla U}{U} \right) \right|^p \\ &= \eta \left| s(x) \frac{\nabla u}{u} + (1 - s(x)) \frac{\nabla U}{U} \right|^p \text{ with } s(x) = \frac{u^p}{u^p + U^p} \in (0,1) \\ &\leq \eta \left[s(x) \left| \frac{\nabla u}{u} \right|^p + (1 - s(x)) \left| \frac{\nabla U}{U} \right|^p \right] \\ &= \ \frac{1}{2} \left(u^p \left| \frac{\nabla u}{u} \right|^p + \ U^p \left| \frac{\nabla U}{U} \right|^p \right) = \ \frac{1}{2} \left(|\nabla u|^p + |\nabla U|^p \right) \end{aligned}$$

and

$$\int_\Omega |\nabla w|^p \, dx \le \frac{1}{2}\left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |\nabla U|^p \, dx\right). \tag{3}$$

Because u and U are both solutions of (2), equality must hold in (3), i.e.

$$\frac{\nabla u}{u} = \frac{\nabla U}{U} \qquad \text{a.e. in } \Omega. \tag{4}$$

But (4) implies that $\nabla(u/U) = 0$ a.e. in Ω, so that $u = const.\ U$. This completes the proof for problem (1).

The proof can be easily generalized to recover a related result (from [5,6]), which states that positive (weak) solutions of

$$\begin{aligned} \Delta_p u + f(x,u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5}$$

are unique, provided $f : \Omega \times [0,\infty)$ satisfies the hypotheses
(H2) for a.e. $x \in \Omega$ the map $r^{1-p} f(x,r)$ is strictly decreasing in $r \in [0,\infty)$.
(H3) There ex. $c > 0$ with $f(x,r) \le c(r^{p-1} + 1)$ for a.e. $x \in \Omega$ and $r \in [0,\infty)$.

To give the proof for problem (5), one has to observe that solutions of (5) are critical points of a functional

$$H_p(v) := \int_\Omega \left[\frac{1}{p}|\nabla v|^p - F(x,v)\right] dx \tag{6}$$

with $F(x,v) := \int_0^v f(x,|s|)\, ds$. Because of (H3), the functional H_p is well defined on $W_0^{1,p}(\Omega)$. By definition it is even in v and its first part convex in v^p. The second part $-\int F(x,v)\, dx$ is even strictly convex in v^p due to (H2). Hence H_p can have at most one positive critical point. We should remark that the strict convexity of H_p in v^p was mentioned in [5,6], but not exploited in such a direct way. Moreover these papers start with nonnegative solutions and first show their positivity under an additional assumption (H1) on f.

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