CURVES OF MINIMAL ACTION OVER METRIC SPACES

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Abstract. Given a metric space X, we consider a class of action functionals, generalizing those considered in [10] and [3], which measure the cost of joining two given points x_0 and x_1, by means of an absolutely continuous curve. In the case X is given by a space of probability measures, we can think of these action functionals as giving the cost of some congested/concentrated mass transfer problem. We focus on the possibility to split the mass in its *moving part* and its part that (in some sense) has already reached its final destination: we consider new action functionals, taking into account only the contribution of the moving part.

1. Introduction

The recent years have seen a growing interest in the topics of Analysis in metric spaces. This is due to the fact that a number of questions in the Calculus of Variations can be naturally settled in this setting: for instance various PDEs coming from evolution problems can be seen as gradient flows in appropriate metric spaces (see [2]).

In a metric setting, also the problem of geodesics can be treated from a differential viewpoint: in fact, we can look for a curve $\mu : I \to X$, with fixed endpoints, that minimizes the length functional

$$\ell(\mu) = \int_I |\mu'|(t)\ dt,$$

where $|\mu'|$ stands for the *metric derivative* of the curve μ (see Section 2 for the definition).

In the paper [10], Brancolini, Buttazzo and Santambrogio generalize this idea and propose a *geodesic* point of view for congested/concentrated Monge-Kantorovich problems. This has motivated the topic of the present paper: the study of general action functionals defined on the space of absolutely continuous curves in a metric space

$$\mathcal{A}(\mu) = \int_I f(t, \mu(t), |\mu'|(t))\ dt,$$

with applications to the Monge-Kantorovich problem in mind.

The latter, the so called Monge-Kantorovich mass transfer problem, has received a lot of attention in the last years, contributing to the growth of new techniques which can be applied to various fields of mathematics (see [24] and the references therein). Our main interest is in its *dynamical formulation*, which is particularly relevant when we want to study movement of masses, subject to congestion or concentration phenomena: the evolution of urban traffic (see [12], [13], [14]) or the branching of an irrigation network are two examples of such problems (see the papers [5], [6], [8], [22], [25] or the recent monograph [7]).

In all these situations, the cost of moving some mass from a point to another is not only related to the distance between the two points, as in the classical Monge-Kantorovich problem, but also to some physical phenomenon underlying the movement, which can favour the aggregation or the spreading of masses.

Just to clarify the situation, in the case of concentration, you can think of a power supply station which has to give energy to a pair of houses (see Figure 1): one could see this as a transport problem between a Dirac mass (the station) and the sum of two Dirac masses (the houses). Monge-Kantorovich solution imposes to use a wire for every house, but it should be quite clear that in the real world it is preferable to start with a single wire and then split. This means that every model for congested/concentrated dynamics must encode *the cost of the transportation structure* in its formulation.

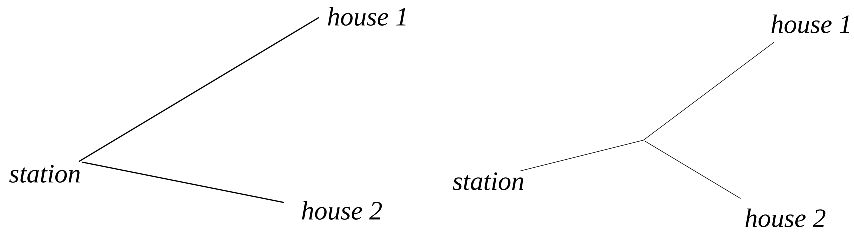

FIGURE 1. The Monge-Kantorovich model (on the left) is not natural in this situation.

The point of view introduced in [10] is the following: the authors propose to study weighted-length functionals of the type

$$\ell_g(\mu) = \int_0^T g(\mu(t))|\mu'|(t)\, dt, \tag{1.1}$$

where $t \mapsto \mu(t)$ is a Lipschitz curve with values in the Wasserstein space $\mathcal{W}_q(\Omega)$. We briefly recall that the latter is a metric space, made up of all Borel probability measures μ over Ω with finite q-momentum, metrized according to the q-Kantorovich-Rubinstein-Wasserstein distance w_q (see Section 6 for the precise definitions).

Minimizers of functionals (1.1), under the constraints $\mu(0) = \mu_0$ and $\mu(T) = \mu_1$, can be seen as geodesics (with respect to a new metric) in the space of probability measures, joining the *points* (which are actually probabilities) μ_0 and μ_1.

With suitable choices of the Riemannian coefficient g, they are able to treat the case of congestion (where the mass spreads over all Ω) as well that of concentration (where on the contrary the mass travels together as much as possible), giving results of existence of a minimizing curve with finite cost and sufficient conditions to ensure that, given an initial distribution of mass μ_0 and a final one μ_1, they can be joined by means of a curve with finite cost (see also [9] for a more detailed analysis of this conditions, in the case of concentration).

Moreover, in the subsequent paper [3], there can be found some necessary optimality condition for a curve to be a minimizer, in the form of an Euler-Lagrange equation for the functional (1.1).

As already observed in [23], for the case of branching transport, in despite of being elegant and relatively simpler, this model has some unnatural behaviours: as a consequence, it happens to give quite different results, with respect to the models proposed by other authors ([5], [22], [25]), which instead turn out to be equivalent with each other (and a proof of these equivalences can be found in [6]). It is however important to underline that the model of Brancolini, Buttazzo and Santambrogio is a *purely dynamical* one: on the contrary, the models of Xia ([25]) and of Bernot, Caselles and Morel ([5], [6]) are static, in the sense that they do not (in a way or another) really depend on time[1].

Let us discuss in some details the unnatural behaviours of the geodesic model, in order to motivate better some of the studies of this paper:

(i) *energetic behaviour*: if one thinks of the curve μ as a quantity of mass which is moving from an initial to a final configuration, one sees that in the Riemannian action (1.1) the term g is a function of the whole μ, which means that if some masses arrive at their destination and then stop, we continue to pay a cost for them until all the process is over.

We try to clarify the situation with an enligthening example: let us take as initial measure $\mu_0 = \delta_{x_0}$, while $\mu_1 = m\delta_{x_1} + (1-m)\delta_{x_2}$ is the final one, with $|x_0 - x_1| = \lambda|x_0 - x_2|$ and $\lambda > 1$ (see Figure 2).

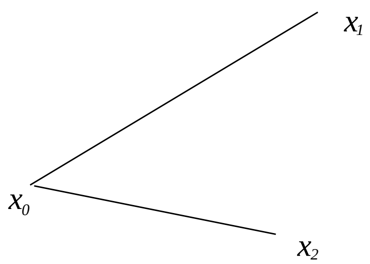

FIGURE 2. The set $\bigcup_{t\in[0,1]} \mathrm{spt}(\mu(t))$ for the curve connecting δ_{x_0} and $m\delta_{x_1} + (1-m)\delta_{x_2}$.

The curve of measures given by

$$\mu(t) = \begin{cases} m\delta_{(1-t)x_0+tx_1} + (1-m)\delta_{(1-\lambda t)x_0+\lambda t x_2}, & t \in [0, 1/\lambda], \\ m\delta_{(1-t)x_0+tx_1} + (1-m)\delta_{x_2}, & t \in [1/\lambda, 1], \end{cases}$$

[1]This is made precise in a recent paper by Bernot and Figalli, where a dynamical extension of the latter model is given: once again, this is equivalent (under appropriate hypothesis on the initial measure μ_0) to the other formulations (we refer to [8] for more details)

is made of two atoms, moving with the same speed $\mathbf{v} = |x_0 - x_1|$: this means that the atom moving towards x_2 and carrying the mass $(1-m)$, arrives before than the other and then it stops. So one sees that for a path like this, the energy given by (1.1) is unnatural, because of the fact that through the coefficient g, we continue to pay the mass $(1-m)$ also after it is stopped.

On the contrary, it would be desiderable to have a model in which g takes into account only the contribution of the moving masses, which for the present example is simply given by the following curve of sub-probabilities

$$\nu(t) = \begin{cases} \mu(t), & t \in [0, 1/\lambda], \\ m\delta_{(1-t)x_0+tx_1}, & t \in [1/\lambda, 1]. \end{cases}$$

This is the reason why, after presenting general action functionals over metric spaces (a topic that we think can be of independent interest), in Section 6 we introduce functionals of the type

$$\tilde{\ell}_g(\nu, \mu) = \int_0^1 g(\nu(t))|\mu'|(t)\, dt, \tag{1.2}$$

where now ν is a curve of sub-probability measures, which represents the mass that is effectively moving, while the pairing (ν, μ) is an *evolution pairing*: roughly speaking, this means that the moving part ν is always less than the total mass μ and that the mass reaching its final destination, given by the difference $\mu - \nu$, has to grow in time (for the precise definition of evolution pairings, see Section 6);

(ii) *scaling behaviour*: it is not clear how to choose the exponent $q \in [1, +\infty]$. This choice influences the energy $\ell_g(\mu)$ through the term $|\mu'|$, which we expect to represent the velocity of the particles: as far as one can see with the following example, it seems that the right choice should be $q = +\infty$. Indeed, let us take μ_0 and μ_1 as before and consider the curve

$$\mu(t) = m\delta_{(1-t)x_0+tx_1} + (1-m)\delta_{(1-t)x_0+tx_2},\ t \in [0,1],$$

then it is easily seen that its metric derivative is given by

$$|\mu'|(t) = w_q(\mu_0, \mu_1) = (m|x_0 - x_1|^q + (1-m)|x_0 - x_2|^q)^{\frac{1}{q}},$$

(in fact μ is actually a constant speed geodesic in the Wasserstein space $\mathcal{W}_q$, see [2] for details).

Anyway the latter quantity, namely the metric derivative, has little to do with the velocities of the single atoms, but it is rather a mass-weigthed sum of these two quantities: the situation changes if we take $q = +\infty$, in fact now the quantity

$$|\mu'|(t) = \max\{|x_0 - x_1|, |x_0 - x_2|\},$$

has the right scaling property. So we are led to study functionals (1.2) also for the case of $\mathcal{W}_\infty$.

The plan of the paper is as follows: in Section 2 we recall some basic facts about spaces of curves in a general metric space; Section 3 is devoted to some preliminary semicontinuity results about *affine* functionals, while Section 4 contains a semicontinuity result for general action functionals over metric spaces. Then (Section 5) we turn to the problem of finding a curve of minimal action joining two given points: we recall the known results and give ours. We specialize (Section 6) the previous results to the case of Wasserstein spaces $\mathcal{W}_q(\Omega)$ with $q \in [1, +\infty]$, introducing the concept of evolution pairing and discussing some of its features. Finally, the last part of the work (Section 7) is devoted to the proof of the existence of minimal evolution pairings, for functionals of the type (1.2).

2. Curves in a metric space

In this paper we will always assume that (X, d) is a Polish space (i.e. a complete and separable metric space), with a given Borel measure $\mathfrak{m}$. Moreover $I = [0, T] \subset \mathbb{R}$ is a compact interval, while by $\mathscr{L}^1$ we mean the 1-dimensional Lebesgue measure.

Let us start recalling some basic facts about spaces of curves in a metric space.

2.1. **Summable curves.** For $p \in [1, +\infty)$, we say that a curve $\mu : I \to X$ belongs to $\mathfrak{L}^p(I; X)$ if μ is Borel measurable and

$$\int_I d(\mu(t), x_0)^p \, dt < +\infty,$$

where x_0 is a point of X (clearly the definition does not depend on the choice of x_0, by means of the triangular inequality).

As in the Euclidean case, we call $L^p(I; X)$ the space of equivalence classes (with respect to the relation *equivalence* $\mathscr{L}^1$*-a.e.*) of functions in $\mathfrak{L}^p(I; X)$: this is clearly a metric space, endowed with the distance

$$d_p(\mu_1, \mu_2) = \left(\int_I d(\mu_1(t), \mu_2(t))^p \, dt \right)^{1/p}.$$

In the case of $p = +\infty$, we define $\mathfrak{L}^\infty(I; X)$ as the space of all curves $\mu : I \to X$ such that

$$\operatorname*{ess\,sup}_{t \in I} d(\mu(t), x_0) < +\infty,$$

for some $x_0 \in X$ and again $L^\infty(I; X)$ is the space of equivalence classes, with the distance

$$d_\infty(\mu_1, \mu_2) = \operatorname*{ess\,sup}_{t \in I} d(\mu_1(t), \mu_2(t)).$$

Remark 1. It is straightforward to see that if X is separable and complete, then for every $p \in [1, +\infty)$ the metric space $L^p(I; X)$ is complete and separable, too. Moreover, as in the Euclidean case, it is possible to show that if $\mu_n \to \mu$ in $L^p(I; X)$, then there exists a subsequence $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ converging to μ $\mathscr{L}^1$-a.e.

2.2. **Continuous curves.** Let $C(I;X)$ be the space of all continuous curves in X, endowed with the topology of the uniform convergence, that is

$$\mu_n \to \mu \text{ in } C(I;X) \iff d_\infty(\mu_n,\mu) = \max_{t\in I} d(\mu_n(t),\mu(t)) \to 0.$$

We recall that by the *metric derivative* of $\mu \in C(I;X)$ at the point $t \in I$, we mean

$$|\mu'|(t) = \lim_{s\to t} \frac{d(\mu(t),\mu(s))}{|s-t|}, \tag{2.3}$$

every time this limit exists.

Remark 2. When $X = \mathbb{R}^N$ with the usual Euclidean distance, if $\mu : I \to X$ is differentiable at the point t_0, then

$$|\mu'|(t_0) = \left\| \frac{d\mu}{dt}(t_0) \right\|,$$

that is $|\mu'|(t_0)$ is nothing but the Euclidean norm of the derivative of μ at the point t_0.

For $p \in [1,+\infty]$, we consider the space $AC^p(I;X) \subset C(I;X)$, defined as follows: we say that $\mu \in AC^p(I;X)$ if there exists some $\psi \in L^p(I;\mathbb{R})$ such that

$$d(\mu(t),\mu(s)) \le \int_s^t \psi(r)\,dr, \quad \text{for every } s,t \in I \text{ such that } s \le t. \tag{2.4}$$

The elements of $AC^p(I;X)$ are called *absolutely continuous curves with finite* p*-energy* (or simply *absolutely continuous curves*, in the case $p = 1$) and they have the nice property of being almost everywhere metric differentiable, as the following Theorem states (see [4] for the proof in the Lipschitz case and [2] for the general case).

Theorem 1. *If $\mu \in AC^p(I;X)$, with $p \ge 1$, then the limit (2.3) exists for $\mathscr{L}^1$-a.e. $t \in I$. The function $t \mapsto |\mu'|(t)$ belongs to $L^p(I;\mathbb{R})$ and*

$$d(\mu(t),\mu(s)) \le \int_s^t |\mu'|(r)\,dr, \quad \textit{for every } s,t \in I \textit{ such that } s \le t.$$

Moreover, we have

$$|\mu'|(t) \le \psi(t), \ \textit{for } \mathscr{L}^1\textit{-a.e. } t \in I,$$

for every $\psi \in L^p(I;\mathbb{R})$ for which (2.4) holds.

Next result is a sort of Poincaré-Wirtinger inequality with a trace term, that holds true for curves in an arbitrary metric space.

Theorem 2 (Poincaré-Wirtinger Inequality). *If $\mu \in AC^p(I;X)$, with $p \in (1,+\infty)$, then for every $x_0 \in X$ we get*

$$\left(\int_0^T d(\mu(t),x_0)^p\,dt\right)^{\frac{1}{p}} \le C(p,T)\left[\left(\int_0^T |\mu'|^p(t)\,dt\right)^{\frac{1}{p}} + \frac{|d(\mu(0),x_0) - d(\mu(T),x_0)|}{T^{\frac{p-1}{p}}}\right] + \xi_p(\mu(0),\mu(T);x_0), \tag{2.5}$$

where the constant $C(p,T)$ *is given by*

$$C(p,T)=\frac{pT}{2\pi(p-1)^{\frac{1}{p}}}\sin\left(\pi\frac{p-1}{p}\right), \tag{2.6}$$

while the function $\xi_p:(X\times X)\times X\to\mathbb{R}$ *is defined by*

$$\xi_p(x,y;z)=\begin{cases}\left(\dfrac{T}{p+1}\dfrac{d(x,z)^{p+1}-d(y,z)^{p+1}}{d(x,z)-d(y,z)}\right)^{\frac{1}{p}}, & d(x,z)\neq d(y,z),\\[2ex] T^{\frac{1}{p}}d(x,z), & d(x,z)=d(y,z).\end{cases}$$

In particular, if $\mu\in AC^p(I;X)$ *happens to be a loop with base point* $x_0\in X$*, that is* $\mu(0)=\mu(T)=x_0$*, then*

$$\left(\int_0^T d(\mu(t),x_0)^p\,dt\right)^{\frac{1}{p}}\le C(p,T)\left(\int_0^T|\mu'|^p(t)\,dt\right)^{\frac{1}{p}}. \tag{2.7}$$

Proof. The proof is the same as in [17], except for the fact that we allow the exponent p to vary in $(1,+\infty)$: we simply use the Poincaré-Wirtinger inequality for real functions of one variable.

Let us set

$$f(t)=d(\mu(t),x_0)-\left(1-\frac{t}{T}\right)d(\mu(0),x_0)-\frac{t}{T}d(\mu(T),x_0),\ t\in[0,T],$$

then it is easily seen that $f\in AC^p(I;\mathbb{R})$, with $f(0)=f(T)=0$, so for it the standard Poincaré-Wirtinger inequality holds true, that is

$$\int_0^T|f(t)|^p\,dt\le C(p,T)\int_0^T|f'(t)|^p\,dt,$$

where the best constant $C(p,T)$ is given by (2.6) (see [18] for example, where the best constant is computed, together with the function that realizes it).

We now observe that

$$|f'(t)|\le|\mu'|(t)+\frac{1}{T}|d(\mu(0),x_0)-d(\mu(T),x_0)|,\ \mathscr{L}^1\text{-a.e. } t\in I,$$

so that Minkowski inequality yields

$$\left(\int_0^T|f(t)|^p\,dt\right)^{\frac{1}{p}}\le C(p,T)\left[\left(\int_0^T|\mu'|^p(t)\,dt\right)^{\frac{1}{p}}+\frac{|d(\mu(0),x_0)-d(\mu(T),x_0)|}{T^{\frac{p-1}{p}}}\right]. \tag{2.8}$$

Moreover, by Minkowski inequality again we get

$$\begin{aligned}\left(\int_0^T d(\mu(t),x_0)^p\,dt\right)^{\frac{1}{p}}&\le\left(\int_0^T|f(t)|^p\,dt\right)^{\frac{1}{p}}\\&+\left(\int_0^T\left(\left(1-\frac{t}{T}\right)d(\mu(0),x_0)+\frac{t}{T}d(\mu(T),x_0)\right)^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

Computing the integral in the right-hand side and using (2.8), we obtain (2.5). □

Another remarkable property of curves in AC^p is that they can be reparametrized by arc length. Precisely, we have the following (see [4] for a proof):

Lemma 1 (Reparametrization Lemma)**.** *For $p \in [1, +\infty]$, suppose $\mu \in AC^p(I;X)$ and let*

$$\ell(\mu) = \int_I |\mu'|(t)\ dt.$$

Then there exists a strictly increasing left-continuous function

$$\mathfrak{t} : [0, \ell(\mu)] \to [0, T],$$

such that:

(1) $\overline{\mu} = \mu \circ \mathfrak{t} \in AC^\infty([0, \ell(\mu)]; X)$;
(2) $\overline{\mu}([0, \ell(\mu)]) = \mu([0, T])$;
(3) $|\overline{\mu}'|(t) = 1$, *for* $\mathscr{L}^1$-*a.e.* $t \in [0, \ell(\mu)]$.

Remark 3. The time rescaling $\mathfrak{t}$ given by the previous Lemma is defined as

$$\mathfrak{t}(s) = \inf \left\{ t \in [0, T]\ :\ s = \int_0^t |\mu'|(r)\ dr \right\}.$$

We remark that in general this is not a continuous function: the important fact is that at its discontinuity points, the jumps of $\mathfrak{t}$ corresponds to time intervals on which μ is constant.

In the sequel, we consider the space $AC^p(I;X)$ endowed with the following notion of convergence: we say that $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X)$ *weakly converges* to some $\mu \in AC^p(I;X)$, and we write $\mu_n \rightharpoonup \mu$, if

(i) $\lim_{n\to\infty} \max_{t\in I} d(\mu_n(t), \mu(t)) = 0$;
(ii) the sequence $\{|\mu_n'|\}_{n\in\mathbb{N}}$ is equi-bounded in $L^p(I;\mathbb{R})$ and equi-integrable;

where we intend that, if $p > 1$, then the equi-integrability condition is redundant.

Finally, we recall a compactness criterion for the space of continuous curves $C(I;X)$.

Theorem 3 (Ascoli-Arzelà)**.** *Given a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset C(I;X)$, this is relatively compact if and only if the following are satisfied:*

(i) *$\{\mu_n\}_{n\in\mathbb{N}}$ is equi-continuous;*
(ii) *for every $t \in I$, the set $\{\mu_n(t)\ :\ n \in \mathbb{N}\}$ is relatively compact in X.*

2.3. **Curves of bounded variation.** Given a curve $\mu : I \to X$, it is possibile to define its *pointwise total variation*

$$\mathrm{Var}(\mu; I) = \sup \left\{ \sum_{i=0}^{k} d(\mu(t_i), \mu(t_{i+1}))\ :\ 0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T \right\}, \tag{2.9}$$

where the supremum is taken over all finite partitions of I and we say that μ is *rectifiable* if $\mathrm{Var}(\mu) < +\infty$. For absolutely continuous curves, we have the following (see [4] for a proof):

Lemma 2. *Let $p \in [1, +\infty]$ and $\mu \in AC^p(I; X)$; then it holds*

$$\mathrm{Var}(\mu; I) = \int_I |\mu'|(t)\ dt. \tag{2.10}$$

In particular, every $\mu \in AC^p(I; X)$ is rectifiable.

We now want to introduce the space of curves of bounded variation: we essentially follow [16].

Let $\mu : I \to X$ be a Borel measurable curve, we say that μ is *approximately continuous* at $t \in I$ if there exists $x \in X$ such that all the sets

$$X_\varepsilon = \{s \in I\ :\ d(\mu(s), x) > \varepsilon\},$$

have 0-density at t, that is

$$\lim_{r \to 0^+} \frac{\mathscr{L}^1((t - r, t + r) \cap X_\varepsilon)}{2r} = 0, \quad \text{for every } \varepsilon > 0.$$

The point x if exists is unique and is called *approximate limit* of μ in t. We indicate with S_μ the set of points where the approximate limit does not exist: we point out that there holds $\mathscr{L}^1(S_\mu) = 0$ (see [16], Theorem 2.9.13).

Remark 4. If $\mu \in L^1(I; X)$, then we have

$$\frac{\mathscr{L}^1((t - r, t + r) \cap X_\varepsilon)}{2r} \leq \frac{1}{\varepsilon} \fint_{t-r}^{t+r} d(\mu(s), x)\ ds,$$

so that every Lebesgue point of μ is in particular a point of approximate continuity.

Given a Borel measurable curve $\mu : I \to X$, we can also define its left and right approximate limits: for every $t \in I$, we define $x = \mu^+(t)$ if the sets

$$\{s \in I\ :\ t < s,\ d(\mu(s), x) > \varepsilon\}$$

have 0-density at t for every $\varepsilon > 0$. Similarly we set $x = \mu^-(t)$ if

$$\{s \in I\ :\ t > s,\ d(\mu(s), x) > \varepsilon\}$$

have 0-density at t for every $\varepsilon > 0$.

Remark 5. It is easily seen that for every $t \in I \setminus S_\mu$, the limits $\mu^+(t)$ and $\mu^-(t)$ exist and they coincide with the approximate limit of μ in t.

Let $\mu \in L^1(I;X)$ be a summable curve, we define its *essential total variation* as

$$|D\mu|(I) = \sup\left\{\sum_{i=0}^{k} d(\mu(t_i),\mu(t_{i+1}))\ :\ 0 < t_0 < \cdots < t_{k+1} < T\right\}, \tag{2.11}$$

where the supremum is taken over all finite partitions of $I \setminus S_\mu$.

We then say that μ has *bounded variation* if $|D\mu|(I) < +\infty$ and we write $BV(I;X)$ to indicate the space of curves of bounded variation, with values in the metric space X. This is clearly a metric space, too, with distance given by

$$d_{BV}(\mu_1,\mu_2) = d_1(\mu_1,\mu_2) + \big||D\mu_1|(I) - |D\mu_2|(I)\big|,\ \mu_1,\mu_2 \in BV(I;X).$$

Curves of bounded variation posses left and right approximate limits at every point: we give a proof of this fact (see also [16], 2.5.16).

Lemma 3. *If $\mu \in BV(I;X)$, then for every $t \in (0,T)$ there exist $\mu^+(t)$ and $\mu^-(t)$. Furthermore, the same conclusion holds for $\mu^+(0)$ and $\mu^-(T)$.*

Proof. We define the nondecreasing function

$$V(t) = |D\mu|([0,t]),\ t \in I,$$

then for every $t \in I$ we have $V(t^-) \le V(t) \le V(t^+)$, where

$$V(t^-) = \sup\{V(s)\ :\ s < t\} = \lim_{s\to t^-} V(s),$$

$$V(t^+) = \inf\{V(s)\ :\ s > t\} = \lim_{s\to t^+} V(s).$$

We just prove that $\mu^-(t)$ exists for every $t \in (0,T]$: the other part of the statement can be proved in the same way. Indeed, observe that

$$d(\mu(s_1),\mu(s_2)) \le V(t^-) - V(s_1),\ s_1,s_2 \in I \setminus S_\mu \text{ such that } s_1 < s_2 < t,$$

which implies, by means of the completeness of X, the existence of

$$\lim_{s\to t^-} \mu(s) \in X.$$

This has to coincide with the approximate limit $\mu^-(t)$, concluding the proof. □

Remark 6. For every $p \in [1,+\infty]$, if $\mu \in AC^p(I;X)$ we have

$$|D\mu|(I) = \mathrm{Var}(\mu;I).$$

In particular, from Lemma 2 it follows that $AC^p(I;X) \subset BV(I;X)$ and

$$|D\mu|(I) = \int_I |\mu'|(t)\ dt,\ \mu \in AC^p(I;X).$$

We conclude this section with a metric variation of a classical compactness result on BV functions: the proof can be found in [1] (see Theorem 2.4).

Theorem 4. *Let (X,d) be a locally compact, complete and separable metric space. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset BV(I;X)$ be a sequence such that*

$$\sup_{n\in\mathbb{N}} d_{BV}(\mu_n, x_0) < +\infty,$$

for some $x_0 \in X$. Then there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ converging in $L^1(I;X)$ to $\mu \in BV(I;X)$ and

$$|D\mu|(I) \le \liminf_{k\to+\infty} |D\mu_{n_k}|(I).$$

3. Some preliminary semicontinuity results

We start with the following basic result:

Lemma 4. *Let $p \in [1,+\infty]$, for every measurable subset $B \subset I$ such that $\mathscr{L}^1(B) > 0$, the functional*

$$\mu \mapsto \int_B |\mu'|(t)\ dt,\ \ \mu \in AC^p(I;X), \tag{3.1}$$

is sequentially l.s.c. on $AC^p(I;X)$, with respect to the weak topology.

Proof. Let $B \subset I$ be any measurable subset such that $\mathscr{L}^1(B) > 0$ and take $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X)$ a sequence weakly converging to $\mu \in AC^p(I;X)$. We can assume that the sequence $\{|\mu_n'|\}_{n\in\mathbb{N}} \subset L^p(I;\mathbb{R})$ weakly ($*$-weakly if $p = +\infty$) converges to a function $v \in L^p(I;\mathbb{R})$.

Then we have

$$\begin{aligned} d(\mu(s),\mu(t)) &= \lim_{n\to+\infty} d(\mu_n(s),\mu_n(t)) \le \lim_{n\to\infty} \int_s^t |\mu_n'|(r)\ dr \\ &= \int_s^t v(r)\ dr, \quad \text{for every } s,t \in I \ \text{ such that } s \le t, \end{aligned}$$

which clearly shows by Lebesgue Differentiation Theorem that

$$|\mu'|(t) \le v(t), \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in I. \tag{3.2}$$

This in turn implies that

$$\int_B |\mu'|(t)\ dt \le \int_B v(t)\ dt = \liminf_{n\to\infty} \int_B |\mu_n'|(t)\ dt,$$

which gives the lower semicontinuity of (3.1). □

With a little extra work, Lemma 4 can be improved as follows:

Lemma 5. *Let $p \in [1,+\infty]$, for every measurable subset $B \subset I$ such that $\mathscr{L}^1(B) > 0$ and every measurable function $\varphi : B \to \mathbb{R}^+$, the functional*

$$\mu \mapsto \int_B \varphi(t)|\mu'|(t)\ dt,\ \ \mu \in AC^p(I;X), \tag{3.3}$$

is sequentially l.s.c. on $AC^p(I;X)$, with respect to the weak topology.

Proof. Take $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X)$ a sequence weakly converging to $\mu \in AC^p(I;X)$ and call $v \in L^p(I;\mathbb{R})$ the weak ($*$-weak if $p = +\infty$) limit of $\{|\mu_n'|\}_{n\in\mathbb{N}}$. If we assume for the moment that $\varphi \in L^\infty(B;\mathbb{R}^+)$, using (3.2) we get

$$\int_B \varphi(t)|\mu'|(t)\,dt \le \int_B \varphi(t)v(t)\,dt = \lim_{n\to\infty}\int_B \varphi(t)|\mu_n'|(t)\,dt. \tag{3.4}$$

In the general case of φ measurable and positive, it is enough to define the sequence

$$\varphi_k(t) = \min\,\{\varphi(t), k\},\ t \in B,$$

so that $\varphi_k \in L^\infty(B;\mathbb{R}^+)$ and applying (3.4), we get

$$\begin{aligned}\liminf_{n\to\infty}\int_B \varphi(t)|\mu_n'|(t)\,dt &\ge \liminf_{n\to\infty}\int_B \varphi_k(t)|\mu_n'|(t)\,dt\\ &\ge \int_B \varphi_k(t)|\mu'|(t)dt,\ k\in\mathbb{N}.\end{aligned}$$

If we now let $k \to \infty$, we can conclude the proof, by means of the monotone convergence theorem. □

Finally, we get a semicontinuity result for general *affine* functionals. Before this, we need the following definition.

Definition 1. A function $h : I \times X \to \mathbb{R}\cup\{+\infty\}$ is said to be a *Carathéodory integrand* if the following are satisfied:

(i) h is $\mathscr{L}^1 \otimes \mathfrak{m}$-measurable;
(ii) $h(t,\cdot)$ is finite and continuous on X, for $\mathscr{L}^1$-a.e. $t \in I$.

Lemma 6. *Let $a : I \times X \to \mathbb{R}$ and $b : I \times X \to \mathbb{R}^+$ be two Carathéodory integrands. If $p \in [1,+\infty]$, then for every measurable subset $B \subset I$ such that $\mathscr{L}^1(B) > 0$, the functional*

$$\mu \mapsto \int_B \left[a(t,\mu(t)) + b(t,\mu(t))|\mu'|(t)\right]\,dt,\ \mu \in AC^p(I;X), \tag{3.5}$$

is sequentially l.s.c on $AC^p(I;X)$, with respect to the weak topology.

Proof. The sequential semicontinuity of the term

$$\mu \mapsto \int_B a(t,\mu(t))\,dt,\ \mu \in AC^p(I;X),$$

is straightforward: indeed, it is just a consequence of Fatou Lemma. For the term

$$\mu \mapsto \int_B b(t,\mu(t))|\mu'|(t)\,dt,\ \mu \in AC^p(I;X),$$

we observe that, taken a weakly convergent sequence $\mu_n \rightharpoonup \mu$, if we set

$$g_n^k(t) = \min\{k, b(t,\mu_n(t))\},\ t \in B,$$

and

$$g^k(t) = \min\{k, b(t,\mu(t))\},\ t \in B,$$

by means of the assumptions on b, we have that $g_n^k \to g^k$ $\mathscr{L}^1$-a.e. on B. Moreover, $\{g_n^k\}_{n\in\mathbb{N}}$ is equi-bounded in $L^\infty(B;\mathbb{R})$: Lebesgue Dominated Convergence Theorem implies that $g_n^k \to g^k$ strongly, let's say in $L^{\frac{p}{p-1}}(B;\mathbb{R})$, while $|\mu_n'|$ weakly ($*$-weakly if $p=+\infty$) converges in $L^p(B;\mathbb{R})$, so that

$$\liminf_{n\to\infty}\int_B g_n^k(t)|\mu_n'|(t)\ dt = \liminf_{n\to\infty}\int_B g^k(t)|\mu_n'|(t)\ dt.$$

This, Lemma 5 and the positivity of b imply

$$\int_B g^k(t)|\mu'|(t)\ dt \le \liminf_{n\to\infty}\int_B g^k(t)|\mu_n'|(t)\ dt \le \liminf_{n\to\infty}\int_B b(t,\mu_n(t))|\mu_n'|\ dt,$$

which gives the thesis, passing to the limit as $k\to\infty$. □

We recall that a metric space is said to be *proper* if its closed balls are compact: in particular, a proper metric space is locally compact (the converse is not true) . As we will see when dealing with absolutely continuous curves over a space which is not proper (Section 5 and 7), it is of interest also the case of a metric space with different topologies defined on it. First of all, we introduce some definitions.

Definition 2. Let (X,τ) be a topological space and $d: X\times X\to[0,+\infty)$ a metric. We say that d is lower semicontinuous on (X,τ) if the following holds: whenever $x_n \xrightarrow{\tau} x$ and $y_n \xrightarrow{\tau} y$, then

$$d(x,y)\le \liminf_{n\to\infty} d(x_n,y_n).$$

Definition 3. Given a space X with two different metrics d_1 and d_2, we set $X_1=(X,d_1)$ and $X_2=(X,d_2)$. We indicate with $|\mu'|_{d_1}$ and $|\mu'|_{d_2}$ the metric derivative with respect to d_1 and d_2, respectively. Then a sequence $\{\mu_n\}_{n\in\mathbb{N}}\subset AC^p(I;X_1)$ is said to be d_2-weakly convergent if:

(i) $\max_{t\in I} d_2(\mu_n(t),\mu(t))\to 0$;

(ii) the sequence $\{|\mu_n'|_{d_1}\}_{n\in\mathbb{N}}$ is equi-bounded in $L^p(I;\mathbb{R})$ and equi-integrable.

We indicate this convergence by $\mu_n \xrightarrow{d_2} \mu$.

Then we can prove the following slight modification of Lemma 6.

Lemma 7. *Let $X_1=(X,d_1)$ and $X_2=(X,d_2)$ be two Polish spaces such that d_1 is lower semicontinuous on X_2.*

Fix $p\in[1,+\infty]$. For every pair of Carathéodory integrands $a: I\times X_2\to\mathbb{R}$, $b: I\times X_2\to\mathbb{R}^+$ and every measurable subset $B\subset I$ such that $\mathscr{L}^1(B)>0$, the functional defined on $AC^p(I;X_1)$ by

$$\mu\mapsto\int_B [a(t,\mu(t))+b(t,\mu(t))|\mu'|_{d_1}(t)]\ dt,$$

is sequentially l.s.c. with respect to the d_2-weak convergence.

Proof. First of all, we have to show that $AC^p(I;X_1)$ is closed with respect to the d_2-weak convergence. Indeed if $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X_1)$ is such that $\mu_n \overset{d_2}{\rightharpoonup} \mu$, then by the semicontinuity of d_1 we get

$$d_1(\mu(s),\mu(t)) \leq \liminf_{n\to\infty} d_1(\mu_n(s),\mu_n(t)) \leq \liminf_{n\to\infty} \int_s^t |\mu_n'|_{d_1}(r)\ dr,$$

so that we can still prove property (3.2), that is

$$|\mu'|_{d_1}(t) \leq v(t), \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in I, \tag{3.6}$$

where as above $v \in L^p(I;\mathbb{R})$ is the weak ($*$-weak if $p=+\infty$) limit of $\{|\mu_n'|_{d_1}\}_{n\in\mathbb{N}}$: this precisely means that $\mu \in AC^p(I;X_1)$.

As in the previous case, the key fact is to show that the functional defined on $AC^p(I;X_1)$ by

$$\mu \mapsto \int_B |\mu'|_{d_1}(t)\ dt,$$

is sequentially l.s.c. with respect to the d_2-weak convergence. At this end, it is sufficient to use (3.6): then we can repeat the proof of Lemma 5 and Lemma 6 and get the thesis. □

4. Semicontinuous action functionals over $AC^p(I;X)$

We now want to consider a generic action functional defined on $AC^p(I;X)$ of the form

$$\mathcal{A}(\mu) = \int_I f(t,\mu(t),|\mu'|(t))\ dt,\ \ \mu\in AC^p(I;X), \tag{4.1}$$

for some function $f : I \times X \times \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$, satisfying the following:

$$f \text{ is } \mathscr{L}^1 \otimes \mathfrak{m} \otimes \mathscr{L}^1\text{-measurable}; \tag{4.2}$$

$$f(t,\cdot,\cdot) \text{ is l.s.c. on } X\times\mathbb{R} \text{ for every } t\in I; \tag{4.3}$$

$$f(t,x,\cdot) \text{ is convex and increasing on } \mathbb{R} \text{ for every } t\in I, x\in X. \tag{4.4}$$

We provide some semicontinuity results for such functionals, with respect to the weak convergence in $AC^p(I;X)$.

Remark 7. Let us briefly discuss the monotonicity assumption for the function f: at a first glance, assuming (4.4) could be seem restrictive. Anyway, if you think to the Euclidean case $X=\mathbb{R}^N$, then $g(z)=f(t,x,z)$ would be a function of the modulus $|z|$, that has to be (if we want to ensure the l.s.c. of functional (4.1)) convex in z. Clearly, this is possible if and only if g is convex and increasing (see Figure 3).

As usual, the idea is to seek affine approximations of the function f, satisfying (4.2), (4.3) and (4.4): if this can be done, then semicontinuity of $\mathcal{A}$ will result from the application of Lemma 6.

The following is a crucial result: it is just an adaptation of a classical result, valid in an Euclidean setting (see Lemma 2.2.4 and Remark 2.2.5 of [11]).

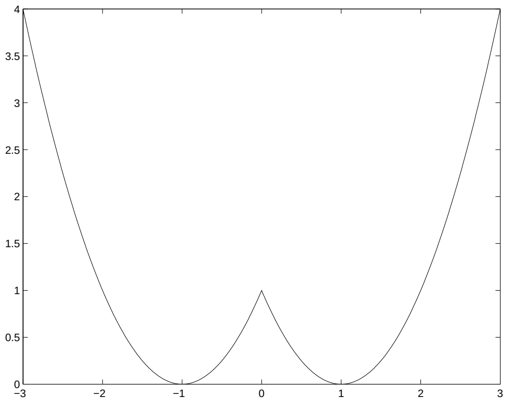

FIGURE 3. A convex function of $|z|$, that is not convex in z

Lemma 8. *Let $f : I\times X\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a function satisfying assumptions (4.2), (4.3) and (4.4). Assume further that for every $t \in I$ the function $f(t,\cdot,\cdot)$ satifies the following condition:*

(4.5)
$$\text{there exists a function } \theta : \mathbb{R} \to \mathbb{R} \text{ such that} \quad \lim_{z\to+\infty} \frac{\theta(z)}{z} = +\infty \quad \text{and} \quad f(t,x,z) \geq \theta(|z|), \quad \text{for every } x \in X,\ z \in \mathbb{R}.$$

Then, there exist two sequences of bounded Carathéodory integrands $a_n : I \times X \to \mathbb{R}$ and $b_n : I \times X \to [0,+\infty)$, such that

$$f(t,x,z) = \sup_{n\in\mathbb{N}} \{a_n(t,x) + b_n(t,x)z\}, \quad \text{for every } t \in I,\ x \in X,\ z \in \mathbb{R}.$$

The next general Lemma will be useful in proving our semicontinuity result: the proof can be found in [11] (Lemma 2.3.2).

Lemma 9. *Let $\Omega \subset \mathbb{R}^N$ be any measurable subset and $g, \{g_n\}_{n\in\mathbb{N}}$ be measurable functions from Ω to $\mathbb{R}\cup\{+\infty\}$, such that $g = \sup\ \{g_n \ :\ n \in \mathbb{N}\}$ and $g_n \geq \varphi$, for a suitable $\varphi \in L^1(\Omega;\mathbb{R})$. Then*

$$\int_\Omega g(x)\ dx = \sup\ \left\{\sum_{i\in I} \int_{B_i} g_i(x)\ dx\right\},$$

where the supremum is taken over all finite partitions of Ω, by pairwise disjoint measurable subsets B_i.

The semicontinuity result now reads as follows:

Theorem 5. *Let $p \in [1,+\infty]$ and let $f : I \times X \times \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a function satisfying (4.2), (4.3) and (4.4). Assume further that there exist two positive constants α, β, a point*

$\overline{x} \in X$ and a function $h \in L^1(I;\mathbb{R})$ such that f satisfies the following estimate

$$f(t,x,z) \geq -\alpha|z| - \beta d(x,\overline{x})^r - h(t), \tag{4.6}$$

for some $r > 0$. Then the functional

$$\mathcal{A}(\mu) = \int_I f(t,\mu(t),|\mu'|(t))\ dt,\ \ \mu \in AC^p(I;X), \tag{4.7}$$

is well-defined, takes its values in $\mathbb{R} \cup \{+\infty\}$ and is sequentially l.s.c. on $AC^p(I;X)$, with respect to the weak topology.

Proof. The fact that the functional $\mathcal{A}$ is well-defined and takes its values in $\mathbb{R} \cup \{+\infty\}$, follows from (4.6).

We now proceed to the proof of the sequential lower semicontinuity: let us first assume that f verifies hypothesis (4.5) of Lemma 8, so that we have

$$f(t,x,z) = \sup\ \{a_n(t,x) + b_n(t,x)z\ :\ n \in \mathbb{N}\},$$

for suitable sequences of bounded Carathéodory integrands $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, with $b_n \geq 0$. By Lemma 9, to conclude the proof we can restrict ourselves to prove that for every $n \in \mathbb{N}$ and $B \subset I$ measurable, the functional

$$\mu \mapsto \int_B [a_n(t,\mu(t)) + b_n(t,\mu(t))|\mu'|(t)]\ dt,\ \ \mu \in AC^p(I;X),$$

is sequentially l.s.c. on $AC^p(I;X)$, with respect to the weak topology: this is just a straightforward consequence of Lemma 6.

We now remove assumption (4.5) on f and assume for the moment that $f \geq 0$. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X)$ be a weakly convergent sequence: $\{|\mu_n'|\}_{n\in\mathbb{N}}$ is equi-integrable, so there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = +\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \int_I \theta(|\mu_n'|(t))\ dt \leq 1.$$

For every $\varepsilon > 0$, we set

$$f_\varepsilon(t,x,z) = f(t,x,z) + \varepsilon\theta(|z|),$$

so that f_ε verifies hypothesis (4.5) of Lemma 8 and we can thus obtain

$$\begin{aligned}
\int_I f(t,\mu(t),|\mu'|(t))\ dt &\leq \int f_\varepsilon(t,\mu(t),|\mu'|(t))\ dt \\
&\leq \liminf_{n\to\infty} \int_I f_\varepsilon(t,\mu_n(t),|\mu_n'|(t))\ dt \\
&= \varepsilon \liminf_{n\to\infty} \int_I \theta(|\mu_n'|)(t)\ dt + \liminf_{n\to\infty} \int_I f(t,\mu_n(t),|\mu_n'|(t))\ dt \\
&\leq \varepsilon + \liminf_{n\to\infty} \int_I f(t,\mu_n(t),|\mu_n'|(t))\ dt,
\end{aligned}$$

proving the semicontinuity of $\mathcal{A}$, by the arbitrariness of ε.

Finally, in the general case of a function f satisfying (4.6), we proceed as follows: for every $k \in \mathbb{N}$, we define

$$f_k(t,x,z) = \max\{f(t,x,z), -k\},$$

and, taken a weakly convergent sequence $\{\mu_n\} \subset AC^p(I;X)$ such that $\mu_n \rightharpoonup \mu$, we set

$$g_n(t) = \alpha|\mu_n'|(t) + \beta d(\mu_n(t), \overline{x})^r + h(t),\ t \in I,$$

$$A_{k,n} = \{t \in I\ :\ f(t, \mu_n(t), |\mu_n'|(t)) < -k\}.$$

We first observe that

$$A_{k,n} \subset \{t \in I\ :\ g_n(t) > k\},$$

and that g_n is equi-integrable, so we obtain

$$\lim_{k\to\infty} \int_{A_{k,n}} g_n(t)\ dt = 0,\ \text{ for every } n \in \mathbb{N}. \tag{4.8}$$

Being f_k bounded from below and using (4.6), we get

$$\begin{aligned}
\int_I f(t,\mu(t),|\mu'|(t))\ dt &\le \int_I f_k(t,\mu(t),|\mu'|(t))\ dt \\
&\le \liminf_{n\to\infty} \int_I f_k(t,\mu_n(t),|\mu_n'|(t))\ dt \\
&= \liminf_{n\to\infty} \left[\int_I f(t,\mu_n(t),|\mu_n'|(t))\ dt - \int_{A_{k,n}} f(t,\mu_n(t),|\mu_n'|(t))\ dt\right] \\
&\le \liminf_{n\to\infty} \int_I f(t,\mu_n(t),|\mu_n'|(t))\ dt + \limsup_{n\to\infty} \int_{A_{k,n}} g_n(t)\ dt,
\end{aligned}$$

and this, taking the limit as $k \to \infty$ and taking into account (4.8), implies the semicontinuity of $\mathcal{A}$. □

Remark 8. In the case $p > 1$, we can weaken assumption (4.6) of Theorem 5, by requiring that there exist two positive constants α, β, a point $\overline{x} \in X$ and a function $h \in L^1(I;\mathbb{R})$ such that

$$f(t,x,z) \ge -\alpha|z|^m - \beta d(x,\overline{x})^r - h(t),$$

for $m < p$ and for $r > 0$. As in the Euclidean case, we cannot expect any semicontinuity result, if the previous is verified with $m = p > 1$ (see [19]).

Note that Theorem 5 can be used to prove lower semicontinuity of *geodesic* functionals, that is functionals of the type

$$\mu \mapsto \int_I g(\mu(t))|\mu'|(t)\ dt,\ \mu \in \mathrm{Lip}(I;X) = AC^\infty(I;X),$$

with $g : X \to [0,+\infty]$ lower semicontinuous, which have been studied in detail in the papers [3] and [10].

We conclude this section, giving some refinements of Theorem 5 which we will need in the sequel.

The first is the following: (X, d_X) and (Y, d_Y) are two Polish spaces and we have an integral functional of the type

$$\mathcal{A}(\nu, \mu) = \int_I f(t, \nu(t), |\mu'|_X(t))\, dt, \ (\nu, \mu) \in L^1(I; Y) \times AC^p(I; X), \tag{4.9}$$

where $|\mu'|_X$ stands for the metric derivative of μ, with respect to the metric d_X. It is straightforward to extend the semicontinuity result of Theorem 5 to this case.

Theorem 6. *Fix $p \in [1, +\infty]$ and let $f : I \times Y \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying hypotheses* (4.2)*,* (4.3) *and* (4.4)*. Suppose moreover that there exist two positive constants α, β, a point $\overline{y} \in Y$ and $h \in L^1(I; \mathbb{R})$ such that f satisfies the following estimate*

$$f(t, y, z) \geq -\alpha|z| - \beta d_Y(y, \overline{y}) - h(t). \tag{4.10}$$

Then the functional $\mathcal{A} : L^1(I; Y) \times AC^p(I; X) \to \mathbb{R} \cup \{+\infty\}$ defined by (4.9) *is well defined and sequentially lower semicontinuous on $L^1(I; Y) \times AC^p(I; X)$, with respect to the strong topology on $L^1(I; Y)$ and the weak topology on $AC^p(I; X)$.*

In the case of a metric space equipped with two different metrics, the following result will be useful: the proof is the same of Theorem 5, with Lemma 7 in place of Lemma 6.

Theorem 7. *Let $X_1 = (X, d_1)$ and $X_2 = (X, d_2)$ be two Polish spaces such that d_1 is lower semicontinuous on X_2.*

Fix $p \in [1, +\infty]$. Let $f : I \times X_2 \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying (4.2)*,* (4.3) *and* (4.4)*. Assume further that there exist two positive constants α, β, a point $\overline{x} \in X$ and a function $h \in L^1(I; \mathbb{R})$ such that f satisfies the following estimate*

$$f(t, x, z) \geq -\alpha|z| - \beta d_2(x, \overline{x})^r - h(t), \tag{4.11}$$

for some $r > 0$. Then the functional

$$\mathcal{A}(\mu) = \int_I f(t, \mu(t), |\mu'|_{d_1}(t))\, dt, \ \mu \in AC^p(I; X_1), \tag{4.12}$$

is well-defined, takes its values in $\mathbb{R} \cup \{+\infty\}$ and is sequentially l.s.c. on $AC^p(I; X_1)$, with respect to the d_2-weak convergence.

Finally, we can easily obtain a variant of Theorem 6 for spaces endowed with two metrics: this is motivated by applications to metric spaces which are not proper.

Theorem 8. *Let $X_1 = (X, d_1)$ and $X_2 = (X, d_2)$ be two Polish spaces such that d_1 is lower semicontinuous on X_2.*

Fix $p \in [1, +\infty]$. Let $f : I \times Y \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying hypotheses (4.2)*,* (4.3) *and* (4.4)*. Suppose moreover that there exist two positive constants α, β, a point $\overline{y} \in Y$ and $h \in L^1(I; \mathbb{R})$ such that f satisfies the following estimate*

$$f(t, y, z) \geq -\alpha|z| - \beta d_Y(y, \overline{y}) - h(t). \tag{4.13}$$

Then the functional

$$\mathcal{A}(\nu,\mu) = \int_I f(t,\nu(t),|\mu'|_{d_1}(t))\;dt,\;\; (\nu,\mu)\in L^1(I;Y)\times AC^p(I;X_1), \tag{4.14}$$

is well-defined, takes its values in $\mathbb{R}\cup\{+\infty\}$ *and is sequentially l.s.c. on* $L^1(I;Y)\times AC^p(I;X_1)$*, with respect to the strong topology on* $L^1(I;Y)$ *and the* d_2*-weak topology on* $AC^p(I;X_1)$*.*

5. Minimizing curves

We now turn to the problem of finding a curve minimizing the general cost functional

$$\mathcal{A}(\mu) = \int_I f(t,\mu(t),|\mu'|(t))\;dt, \tag{5.1}$$

among all curves $\mu\in AC^p(I;X)$ with fixed endpoints. For every $p\in[1,+\infty]$ and $x_0,x_1\in X$, we define

$$\mathcal{C}_p(x_0,x_1) = \{\mu\in AC^p(I;X)\;:\;\mu(0)=x_0,\;\mu(T)=x_1\}. \tag{5.2}$$

Remark 9. In the particular case of $f(t,\mu,|\mu'|)=|\mu'|$, the problem of minimizing the length functional

$$\ell(\mu) = \int_0^T |\mu'|\;dt,$$

in $\mathcal{C}_p(x_0,x_1)$ admits a solution, which is given by every geodesic in X joining x_0 and x_1, provided that $\mathcal{C}_p(x_0,x_1)\neq\emptyset$ and that X is proper (see [4]).

In [10] the authors consider the case with $f(t,\mu,|\mu'|)=g(\mu)|\mu'|$: as already pointed out, this can now be seen as the problem of finding the geodesics in X, with the respect to some sort of Riemannian distance, whose coefficient is given by g. They prove the following:

Theorem 9. *Let* X *be a proper metric space. If* $g:X\to[0,+\infty]$ *is a lower semicontinuos function, bounded from below by a constant* $c>0$*, we define*

$$\ell_g(\mu) = \int_I g(\mu(t))|\mu'|(t)\;dt,\;\;\mu\in AC^\infty(I;X).$$

Then for every pair of points $x_0,x_1\in X$*, the problem of minimizing* ℓ_g *in* $\mathcal{C}_\infty(x_0,x_1)$ *admits a solution, provided that there exists* $\overline{\mu}\in\mathcal{C}_\infty(x_0,x_1)$ *such that* $\ell_g(\overline{\mu})$ *is finite.*

The proof is based on the Reparametrization Lemma, the functional considered being invariant under reparametrization. Observe that this clearly also implies that

$$\inf_{\mathcal{C}_p(x_0,x_1)}\int_I g(\mu(t))|\mu'|(t)\;dt = \inf_{\mathcal{C}_\infty(x_0,x_1)}\int_I g(\mu(t))|\mu'|(t)\;dt.$$

For the case of absolutely continuous curve with finite p-energy, with the general cost functional $\mathcal{A}$ given by (5.1), our existence result reads as follows.

Theorem 10. *Fix $p \in (1, +\infty)$. Let X be a proper metric space and let $f : I \times X \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying* (4.2)*,* (4.3) *and* (4.4)*. Assume further that there exist a point $\overline{x} \in X$ and a function $h \in L^1(I;\mathbb{R})$ such that*

$$f(t, x, z) \geq |z|^p - \beta(t) d(x, \overline{x})^r - h(t), \tag{5.3}$$

where $0 < r < p$ and $\beta \in L^{\frac{p}{p-r}}(I;\mathbb{R}^+)$. Then for every pair of points $x_0, x_1 \in X$, the problem of minimizing $\mathcal{A}$ in $\mathcal{C}_p(x_0, x_1)$ admits a solution, provided that there exists $\overline{\mu} \in \mathcal{C}_p(x_0, x_1)$ with finite $\mathcal{A}$.

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X)$ be some minimizing sequence, we can suppose that, up to a subsequence, there exists M such that

$$\mathcal{A}(\mu_n) \leq M, \quad \text{for every } n \in \mathbb{N}.$$

Thanks to the assumptions on f, we immediately obtain that the sequence $\{|\mu_n'|\}_{n\in\mathbb{N}}$ is equi-bounded in $L^p(I;\mathbb{R})$. Indeed, it is enough to use Poincaré-Wirtinger inequality (2.5)

$$\int_I d(\mu_n(t), \overline{x})^p \, dt \leq C \int_I |\mu_n'|(t)^p \, dt + A,$$

with A depending only on $\overline{x}$ and the enpoints of μ_n, which are fixed. Then we observe that for every $\varepsilon > 0$, applying Young inequality, we get

$$\int_I \beta(t) d(\mu_n(t), \overline{x})^r \, dt \leq \left(1 - \frac{r}{p}\right) \varepsilon^{\frac{r}{r-p}} \int_I \beta(t)^{\frac{p}{p-r}} \, dt + \frac{r}{p}\varepsilon \int_I d(\mu_n(t), \overline{x})^p \, dt,$$

so that, if we now set

$$\tilde{C}(\varepsilon) = \left(1 - \frac{r}{p}\right) \varepsilon^{\frac{r}{r-p}} \int_I \beta(t)^{\frac{p}{p-r}} \, dt + \int_I h(t) \, dt + \frac{r}{p} A\varepsilon,$$

then condition (5.3) implies

$$M \geq \mathcal{A}(\mu_n) \geq \left(1 - \frac{r}{p} C\varepsilon\right) \int_I |\mu_n'|^p(t) \, dt - \tilde{C}(\varepsilon).$$

With a suitable choice of ε, we obtain the boundedness of $\{|\mu_n'|\}_{n\in\mathbb{N}}$.

This in turn implies that the minimizing sequence is equi-Hölder continuous: in fact by the very definition of absolutely continuous curve and Hölder inequality, we get

$$\begin{aligned} d(\mu_n(t), \mu_n(s)) &\leq \int_s^t |\mu_n'|(r) \, dr \leq |t - s|^{\frac{p-1}{p}} \left(\int_I |\mu_n'|^p(t) \, dt\right)^{\frac{1}{p}} \\ &\leq C|t - s|^{\frac{p-1}{p}}, \text{ for every } n \in \mathbb{N}, \ t, s \in I. \end{aligned}$$

Moreover, this sequence is also pointwise relatively compact, because X is proper and there holds

$$d(\mu_n(t), x_0) = d(\mu_n(t), \mu_n(0)) \leq \int_0^t |\mu_n'|(t) \, dt \leq C, \quad \text{for every } n \in \mathbb{N}, \ t \in I.$$

We can thus apply Theorem 3 to obtain that $\mu_n \rightharpoonup \overline{\mu}$, up to subsequences, where $\overline{\mu} \in \mathcal{C}_p(x_0, x_1)$.

Finally, observe that condition (5.3) implies (4.6), so that by means of Theorem 5 the functional $\mathcal{A}$ is l.s.c. on $AC^p(I;X)$, leading us to

$$\mathcal{A}(\overline{\mu}) \le \liminf_{n\to\infty} \mathcal{A}(\mu_n) = \inf_{\mu\in\mathcal{C}_p(x_0,x_1)} \mathcal{A}(\mu),$$

which concludes the proof. □

Remark 10. If condition (5.3) is verified with $r = p$, then Theorem 10 is still valid, provided that the time interval $[0,T]$ is *small enough*, that is we have to guarantee that T is such that

$$C(p,T) < 1,$$

where $C(p,T)$ is the constant given by (2.6) in Poincaré-Wirtinger inequality.

The hypothesis that (X,d) is proper can be a very severe one and it could be relaxed somehow, by substituting it with the request that on X there exists another topology τ, such that:

(τ1) there exists a metric $d_\tau \le d$ which metrizes the topology τ on τ-compact sets;
(τ2) closed balls of (X,d) are τ-compact;
(τ3) d is l.s.c. with respect to τ.

This is a quite standard procedure, which can be also found in [4], for example. A typical case in which this occurs is when X is the dual of a separable Banach space, equipped with the norm topology: in this case, τ is just the $*$-weak topology.

The previous considerations lead us to the following result.

Theorem 11. *Let $p \in (1,+\infty)$ and $X_1 = (X,d)$ be a Polish space. Suppose that X can be equipped with another topology τ such that $X_2 = (X,\tau)$ satisfies properties (τ1)-(τ3). Let $f : I \times X_2 \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying* (4.2), (4.3) *and* (4.4). *Assume further that there exist a point $\overline{x} \in X$ and a function $h \in L^1(I;\mathbb{R})$ such that*

$$f(t,x,z) \ge |z|^p - \beta(t) d_\tau(x,\overline{x})^r - h(t), \tag{5.4}$$

where $0 < r < p$ and $\beta \in L^{\frac{p}{p-r}}(I;\mathbb{R}^+)$. Then for every pair of points $x_0, x_1 \in X$, the problem of minimizing

$$\mathcal{A} = \int_I f(t,\mu(t),|\mu'|_d(t))\, dt,$$

in $\mathcal{C}_p(x_0,x_1)$ admits a solution, provided that there exists $\overline{\mu} \in \mathcal{C}_p(x_0,x_1)$ with finite $\mathcal{A}$.

Proof. Taking a minimizing sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset AC^p(I;X_1)$ and arguing as in the proof of Theorem 10 (one has to use (5.4) in combination with $-d_\tau \ge -d$), we can obtain that $\{|\mu_n'|_d\}_{n\in\mathbb{N}}$ is equi-bounded in $L^p(I;\mathbb{R})$, which in turn implies that for every $n \in \mathbb{N}$ and every $t \in I$ we have

$$\mu_n(t) \in \{x \in X \;:\; d(x,x_0) \le R\} = B,$$

for a suitable $R > 0$. We now use the fact that (B, d_τ) is a compact metric space and that, due to the fact that $d_\tau \le d$, we have $\{\mu_n\}_{n\in\mathbb{N}} \in AC^p(I; B) \cap AC^p(I; X_1)$.

Then we apply Ascoli-Arzelà Theorem again: this implies that $\{\mu_n\}_{n\in\mathbb{N}}$ d_τ-weakly converges.

It remains to observe that by Theorem 7 the functional $\mathcal{A}$ is lower semicontinuous with respect to the d_τ-weak convergence, thus concluding the proof. □

Remark 11. We remark that, despite being more general than the case with X proper, Theorem 11 does not cover some interesting and changelling cases: for example, it does not apply to the case of a functional of the type

$$\mathcal{A}(\mu) = \int_I [|\mu'|^p(t) - \beta(t) d(\mu(t), x_0)^r]\ dt, \tag{5.5}$$

because of the fact that, when equipped with the weaker topology τ, the term

$$\mu(t) \mapsto -\int_I \beta(t) d(\mu(t), x_0)^r\ dt,$$

is not τ-l.s.c., due to the lower semicontinuity of d with respect to this topology and to the presence of the $-$ sign.

A remarkable particular case of (5.5) is the following: we choose $(X, d) = (\mathcal{W}_2(\mathbb{R}^N), w_2)$, the 2-Wasserstein metric space (see next Section for more details), and we take the action

$$\mathcal{A}(\mu) = \frac{1}{2} \int_I \left[|\mu'|^2(t) - w_2(\mu(t), \nu_0)^2\right]\ dt,\ \ \mu \in AC^2(I; \mathcal{W}_2(\mathbb{R}^N)),$$

for a given reference probability measure ν_0.

An action like this is considered in the recent paper [17] by Gangbo, Nguyen and Tudorascu: the main interest of such a study is that one can write down explicitely an Euler-Lagrange equation for the action $\mathcal{A}$ and this coincides with the so-called Euler-Monge-Ampère system.

Moreover, when $N = 1$, the minimizers of this action, connecting two prescribed measures μ_0 and μ_1, do exist (provided that $T < \pi$) and they are solutions of the 1-dimensional Euler-Poisson system.

We point out that in the present case, neither Theorem 10 nor Theorem 11 can be applied, in fact $\mathcal{W}_2(\mathbb{R}^N)$ is not locally compact, which implies that it is not proper. Then one can think to equip $\mathcal{W}_2(\mathbb{R}^N)$ with the *narrow topology* given by the duality with $C_b(\mathbb{R}^N)$ (continuous and bounded functions): the fact that w_2 is only l.s.c. with respect to this topology, as already observed, implies that the objective functional is no more l.s.c. with respect to this weaker topology.

We thank the referee for having pointed out to us reference [17].

6. The case of measures: evolution pairings

We now leave the general setting of metric spaces, particularizing the results of the previous sections to the case of action functionals over the space of probability measures.

In particular, we will consider action functionals of the following type (see Theorem 6 and 8):

$$\mathcal{A}(\nu, \mu) = \int_I f(t, \nu(t), |\mu'|_X(t)) \, dt, \ (\nu, \mu) \in L^1(I; Y) \times AC^p(I; X),$$

where X and Y will be suitable spaces of measures which will be made precise in a while.

The main application we have in mind is to provide a dynamical formulation of mass transportation problems, specifically in the context of branching transport (see the Introduction for more details).

We warn the reader that this section contains some technicalities which can be avoided at a first reading: the main fact is the definition of *evolution pairing* (see Definition 4).

Let (Ω, d) be a generic locally compact, complete and separable metric space, not necessarily a subset of $\mathbb{R}^N$. From now on, we make the following choice for the two Polish spaces X and Y:

- given $q \in [1, +\infty]$, X is the q-Wasserstein metric space $\mathcal{W}_q(\Omega)$, that is the space of all Borel probability measures μ over Ω, having finite q-momentum

$$\|d(\cdot, x_0)\|_{L^q(\Omega,\mu)} < +\infty,$$

  (by means of the triangular inequality, property above does not depend on the choice of $x_0 \in \Omega$), equipped with the q-Kantorovich-Rubinstein-Wasserstein distance

$$w_q(\mu_1, \mu_2) = \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \|d(\cdot, \cdot)\|_{L^q(\Omega \times \Omega, \gamma)}, \ \mu_1, \mu_2 \in \mathcal{W}_q(\Omega),$$

  where $\Gamma(\mu_1, \mu_2)$ is the set of *transport plans* between μ_1 and μ_2, that is

$$\Gamma(\mu_1, \mu_2) = \{\gamma \in \mathscr{P}(\Omega \times \Omega) \ : \ (\pi_1)_\sharp \gamma = \mu_1, \ (\pi_2)_\sharp \gamma = \mu_2\},$$

  with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

- Y is the space $\mathcal{M}_1^+(\Omega)$ of positive finite Radon measures over Ω having total variation less than or equal to 1, equipped with the distance

$$\mathbf{d}(\nu_1, \nu_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \int_\Omega \varphi_k \, d(\nu_1 - \nu_2) \right|, \ \nu_1, \nu_2 \in \mathcal{M}_1^+(\Omega), \tag{6.1}$$

  where $\{\varphi_k\}_{k \in \mathbb{N}}$ is a dense subset of $\{\varphi \in C_0(\Omega) \ : \ \varphi \geq 0, \ \|\varphi\|_\infty \leq 1\}$, and as usual $C_0(\Omega)$ is the completion of the space of compactly supported continuous functions over Ω, with respect to the sup-norm $\|\cdot\|_\infty$.

  It is well known that $\mathbf{d}$ metrizes the $*$-weak convergence on the space $\mathcal{M}_1^+(\Omega)$. Moreover, $\mathcal{M}_1^+(\Omega)$ is a compact metric space, so that $\mathbf{d}$ is bounded, which means that

$$L^0(I; \mathcal{M}_1^+(\Omega)) := \{\nu : I \to \mathcal{M}_1^+(\Omega) \ : \ \nu \text{ is Borel measurable}\} = L^\infty(I; \mathcal{M}_1^+(\Omega)). \tag{6.2}$$

Remark 12. We recall that $\mathcal{W}_q(\Omega)$ is a Polish space which is neither compact nor locally compact, unless Ω itself is compact and $q \neq +\infty$. Moreover, for every $q > 1$, as an easy consequence of Hölder inequality we have

$$w_1(\mu_1, \mu_2) \le w_q(\mu_1, \mu_2),\ \ \mu_1, \mu_2 \in \mathcal{W}_q(\Omega),$$

and in the case Ω is bounded, then it is possible to obtain the reverse inequalities

$$w_q(\mu_1, \mu_2) \le \mathrm{diam}(\Omega)^{\frac{q-1}{q}} w_1(\mu_1, \mu_2)^{\frac{1}{q}},\ \ \mu_1, \mu_2 \in \mathcal{W}_q(\Omega),$$

for every $1 < q < \infty$.

For the basic properties of mass transportation problems and Wasserstein spaces, we refer to [2] and [24]: a fairly complete treatment of the *supremal* mass transportation problem, that is the one corresponding to w_∞, can be found in [15]. Finally, we want to point out the reference [21], where an application of the space $\mathcal{W}_\infty(\Omega)$ is given.

It is clear that by means of Stone-Weierstrass Theorem, we can take the functions φ_k to be Lipschitz in the definition (6.1). So, for our purposes it is better to work with the following modified distance

$$\mathfrak{d}(\nu_1, \nu_2) = \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} \left| \int_\Omega \varphi_k\ d(\nu_1 - \nu_2) \right|, \tag{6.3}$$

where $\alpha_k = 1 + \mathrm{Lip}(\varphi_k)$. This distance still metrizes the $*$-weak convergence on $\mathcal{M}_1^+(\Omega)$ and it can be compared with w_q. In fact, we have the following:

Lemma 10. *For every $\mu_1, \mu_2 \in \mathcal{W}_q(\Omega)$, there holds $\mathfrak{d}(\mu_1, \mu_2) \le w_q(\mu_1, \mu_2)$.*

Proof. It is clearly sufficient to prove the thesis in the case $q = 1$. We recall the duality formula of Monge-Kantorovich problem with cost $c(x, y) = d(x, y)$, which reads as (see Theorem 1.14 of [24], for example)

$$\min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\Omega\times\Omega} d(x, y)\ d\gamma(x, y) = \sup_{\varphi \in \mathrm{Lip}_1(\Omega)} \int_\Omega \varphi(x)\ d(\mu_1(x) - \mu_2(x)),$$

where $\mathrm{Lip}_1(\Omega)$ is the space of 1-Lipschitz functions over Ω. Then, for every $\mu_1, \mu_2 \in \mathcal{W}_q(\Omega)$ we have

$$\left| \int_\Omega \varphi_k(x)\ d(\mu_1(x) - \mu_2(x)) \right| \le \alpha_k \sup_{\varphi \in \mathrm{Lip}_1(\Omega)} \int_\Omega \varphi(x)\ d(\mu_1(x) - \mu_2(x)) = \alpha_k\ w_1(\mu_1, \mu_2),$$

being $\mathrm{Lip}(\varphi_k/\alpha_k) \le 1$, so that multiplying by $2^{-k}\alpha_k^{-1}$ and summing up, we get

$$\mathfrak{d}(\mu_1, \mu_2) \le \sum_{k=1}^{\infty} \frac{1}{2^k} w_1(\mu_1, \mu_2) = w_1(\mu_1, \mu_2),$$

proving the assertion. □

Remark 13. By Lemma 10, we obtain for every $p \in [1,+\infty]$ the inclusion $AC^p(I;\mathcal{W}_q(\Omega)) \subset AC^p(I;\mathcal{M}_1^+(\Omega))$, with

$$|\mu'|_{\mathfrak{d}}(t) \leq \ |\mu'|_{w_q}(t), \quad \mathscr{L}^1\text{-a.e. } t \in I,\ \mu \in AC^p(I;\mathcal{W}_q(\Omega)). \tag{6.4}$$

Remark 14. It is worthwile to point out that if $p = q \in [1,+\infty)$, then the elements of $AC^p(I;\mathcal{W}_p(\Omega))$ can be completely characterized in terms of those of $AC^p(I;\Omega)$. Roughly speaking, the idea is that defining the *evaluation map*

$$\begin{array}{rccc} e_t: & C(I;\Omega) & \to & \Omega \\ & \sigma & \mapsto & \sigma(t) \end{array}$$

then for every probability measure $Q \in \mathscr{P}(C(I;\Omega))$ such that $Q(C(I;\Omega) \setminus AC^p(I;\Omega)) = 0$, the push-forward of Q through e_t defines a curve of probability measures, that is

$$\mu(t) := (e_t)_\sharp Q \in AC^p(I;\mathcal{W}_p(\Omega)), \tag{6.5}$$

which additionally satisfies

$$|\mu'|(t) \leq \| |\sigma'|(t) \|_{(L^p(C(I;X));Q)}, \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in I; \tag{6.6}$$

conversely, given $\mu \in AC^p(I;\mathcal{W}_p(\Omega))$, then we can construct $Q \in \mathscr{P}(C(I;\Omega))$ which is concentrated on $AC^p(I;\Omega)$ and such that (6.5) is valid and equality holds in (6.6).

We point out that no compactness properties of Ω are needed for these results to hold: see [20], Theorem 4 and 5, for more details.

We now introduce the key concept of *evolution pairing*, which formalizes the idea of associating to every curve of probability measures, a curve which describes the mass that is effectively moving.

Definition 4. Let $(\nu,\mu) \in L^0(I;\mathcal{M}_1^+(\Omega)) \times AC^p(I;\mathcal{W}_q(\Omega))$ be two curves of measures, such that the following are satisfied:

(E1) $\nu(t) \leq \mu(t)$ in the sense of measures, for $\mathscr{L}^1$-a.e. $t \in I$;

(E2) $\mu(t) - \nu(t)$ is monotone nondecreasing, that is: there exists an $\mathscr{L}^1$-negligible subset $M \subset I$ such that

$$\mu(s) - \nu(s) \leq \mu(t) - \nu(t), \quad \text{for every } s,t \in I \setminus M, \text{ with } s < t;$$

Then we say that (ν,μ) is an evolution pairing and we write $\nu \preceq \mu$.

Remark 15. We can think of ν as the *moving mass*, while μ is the *total mass*: in this sense, condition (E2) means that the mass that has actually reached its final destination must increase, while (E1) simply states that the moving mass is always less than or equal to the total mass.

Observe also that the increasing monotonicity of the *arrived mass* $\mu - \nu$, implies the monotonicity of the quantity

$$t \mapsto |\nu(t)|(\Omega),$$

while it does not imply that ν has a monotone decreasing (in the sense of measures) behaviour. As an easy counterexample, let us take

$$\sigma_1(t) = (1-t)x_0 + tx_1,\ t \in [0,1],$$

and

$$\sigma_2(t) = \begin{cases} (1-2t)x_0 + 2tx_2, & t \in [0,1/2], \\ x_2, & t \in [1/2,1], \end{cases}$$

and consider the curve of probability measures

$$\mu(t) = m\delta_{\sigma_1(t)} + (1-m)\delta_{\sigma_2(t)},\ t \in [0,1],$$

joining $\mu_0 = \delta_{x_0}$ and $\mu_1 = m\delta_{x_1} + (1-m)\delta_{x_2}$, with $x_0, x_1, x_2 \in \mathbb{R}^N$ pairwise distinct and $m \in (0,1)$. If we take

$$\nu(t) = \begin{cases} \mu(t), & t \in [0,1/2], \\ m\delta_{\sigma_1(t)}, & t \in [1/2,1] \end{cases}$$

then it is easy to verify that $\nu \preceq \mu$, but

$$\nu(t+h) \not\leq \nu(t),\ t \in I,\ h > 0.$$

This example should also clarify that in general the curve ν has no continuity properties.

We exploit the more relevant consequence of evolution pairings in the next Lemma.

Lemma 11. *Let $(\nu,\mu) \in L^0(I;\mathcal{M}_1^+(\Omega)) \times AC^p(I;\mathcal{W}_q(\Omega))$ be an evolution pairing. Then $\nu \in BV(I;\mathcal{M}_1^+(\Omega))$ and in particular we get*

$$|D\nu|_{\mathfrak{d}}(I) \leq |D\Phi|(I) + \int_I |\mu'|_{w_q}(t)\ dt, \tag{6.7}$$

where $\Phi : I \to \mathbb{R}^+$ is the monotone nondecreasing function defined by

$$\Phi(t) = \mathfrak{d}(\nu(t),\mu(t)).$$

Proof. Before proving the main assertion, we first collect some easy consequences of the definition of evolution pairing:

if the pair (ν,μ) satisfies (E1), this means that

$$\int_\Omega \varphi_k\ d\nu(t) \leq \int_\Omega \varphi_k\ d\mu(t),\quad \text{for } \mathscr{L}^1\text{-a.e.} t \in I,\ \text{for every } k \in \mathbb{N}.$$

If we now take $\tilde{\nu} \in \mathcal{M}_1^+(\Omega)$ such that $\tilde{\nu} \leq \nu(t)$ for $\mathscr{L}^1$-a.e. $t \in I$, we obtain

$$0 \leq \int_\Omega \varphi_k\ d(\nu(t) - \tilde{\nu}) \leq \int_\Omega \varphi_k\ d(\mu(t) - \tilde{\nu}),$$

and so, multiplying by $2^{-k}\alpha_k^{-1}$ and summing up, we get

$$\mathfrak{d}(\nu(t),\tilde{\nu}) \leq \mathfrak{d}(\mu(t),\tilde{\nu}),\quad \text{for } \mathscr{L}^1\text{-a.e. } t \in I. \tag{6.8}$$

By hypothesis (E2), we also get

$$0 \leq \int_\Omega \varphi_k\ d(\mu(s) - \nu(s)) \leq \int_\Omega \varphi_k\ d(\mu(t) - \nu(t)),\ \text{for every } s < t \in I \setminus M,$$

that is $\Phi(t) = \mathfrak{d}(\nu(t),\mu(t))$ is a real monotone nondecreasing function of a real variable.

To obtain that $\nu \in BV(I;\mathcal{M}_1^+(\Omega))$, it is sufficient to write

$$\nu(t)-\nu(s) = [\nu(t)-\mu(t)] - [\nu(s)-\mu(s)] + [\mu(t)-\mu(s)],\ \ s,t \in I\setminus M,$$

and then we use again (E2), so that for every φ_k

$$\left|\int_\Omega \varphi_k\ d(\nu(t)-\nu(s))\right| \le \Phi(t)-\Phi(s) + \left|\int_\Omega \varphi_k\ d(\mu(t)-\mu(s))\right|,\ \text{for every } s<t\in I\setminus M.$$

Again, multiplying by $2^{-k}\alpha_k^{-1}$ and summing up, we get

$$\mathfrak{d}(\nu(t),\nu(s)) \le \Phi(t)-\Phi(s) + \mathfrak{d}(\mu(t),\mu(s)),\ \ s<t\in I\setminus M. \tag{6.9}$$

Finally, we observe that

$$|D\Phi|(I) = \lim_{t\to T-}\Phi(t) - \lim_{t\to 0^+}\Phi(t) = \Phi^-(T)-\Phi^+(0),$$

then it follows from (6.9) and the definition of essential total variation that

$$\begin{aligned}\sum_{i=0}^k \mathfrak{d}(\nu(t_i),\nu(t_{i+1})) &\le \sum_{i=0}^k[\Phi(t_{i+1})-\Phi(t_i)] + \sum_{i=0}^k \mathfrak{d}(\mu(t_i),\mu(t_{i+1}))\\ &\le \Phi^-(T)-\Phi^+(0) + \int_I |\mu'|_{w_q}(t)\ dt,\end{aligned}$$

for every finite partitions $0<t_0<\cdots<t_{k+1}<1$ of $I\setminus(M\cup S_\nu)$, proving (6.7). □

Remark 16. We observe that $\nu^+(0)$ and $\nu^-(T)$ are well defined, thanks to Lemma 3. Moreover, by the very definition of evolution pairings, we have that if $\nu \preceq \mu$, then $\nu^+(0)\le \mu(0)$ and $\nu^-(T)\le\mu(T)$, in the sense of measures. Indeed, let us prove the first: suppose that there exist $\varphi\in C_0(\Omega;\mathbb{R}^+)$ and $\varepsilon>0$ such that

$$\int_\Omega \varphi\ d\nu^+(0) = \int_\Omega \varphi\ d\mu(0) + 4\varepsilon.$$

We can clearly assume that $\|\varphi\|_\infty\le 1$ and we observe that $t\mapsto \int_\Omega \varphi\ d\mu(t)$ is a uniformly continuous real function of one variable: then there exists $r_0<T$ such that

$$\int_\Omega \varphi\ d\mu(t) < \int_\Omega \varphi\ d\mu(0)+\varepsilon,\ \ t\in(0,r_0),$$

which implies

$$\int_\Omega \varphi\ d\nu(t) \le \int_\Omega \varphi\ d\mu(t) < \int_\Omega \varphi\ d\nu^+(0) - 3\varepsilon,\quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,r_0).$$

Then

$$3\varepsilon < \int_\Omega \varphi\ d\nu^+(0) - \int_\Omega \varphi\ d\nu(t) = \int_\Omega \varphi\ d(\nu^+(0)-\nu(t)),\quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,r_0),$$

and if φ_m is such that $\|\varphi-\varphi_m\|_\infty<\varepsilon$, the previous yields

$$\varepsilon(3-|\nu^+(0)-\nu(t)|(\Omega)) \le \int_\Omega \varphi_m\ d(\nu^+(0)-\nu(t)),\quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,r_0).$$

It is enough to observe that $\nu^+(0) - \nu(t)$ is a signed Radon measure, with total variation less than or equal to 2, so that we simply obtain

$$\varepsilon \leq \int_\Omega \varphi_m \, d(\nu^+(0) - \nu(t)), \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, r_0),$$

and multiplying the terms on both sides by $c = 2^{-m}\alpha_m^{-1}$, we have

$$c\,\varepsilon \leq \mathfrak{d}(\nu(t), \nu^+(0)), \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, r_0).$$

So, if we denote

$$X_\varepsilon^+(0) = \{t > 0 \ : \ \mathfrak{d}(\nu(t), \nu^+(0)) > c\,\varepsilon\},$$

we have proved (up to an $\mathscr{L}^1$-negligible set) the inclusion $(0, r_0) \subset X_\varepsilon^+(0)$: this in turn contradicts the fact that, by definition of $\nu^+(0)$, the set $X_\varepsilon^+(0)$ must have 0-density.

The fact that $\nu^-(T) \leq \mu(T)$ can be proved in the same way.

As already observed, the space $\mathcal{W}_q(\Omega)$ is not locally compact, which in particular means that it is not proper. However, this is not a great trouble, as far as we can endow it with the weaker topology given by $\mathfrak{d}$ and conditions (τ_1)-(τ_3) of Section 5 are satisfied. This is the content of the next Lemma.

Lemma 12. *The distance w_q is $\mathfrak{d}$-lower semicontinuous. Moreover, all bounded sets in $\mathcal{W}_q(\Omega)$ are $\mathfrak{d}$-relatively compact.*

Proof. The proof is the same as in [10] (Lemma 4.2 and Lemma 4.3), the only difference being the fact that $\mathfrak{d}$ metrizes the $*$-weak convergence, instead of the narrow convergence, which is the one induced by the duality with the space $C_b(\Omega)$ of continuous and bounded functions over Ω. Anyway, having assumed that Ω is locally compact, we have that at the level of probability measures, $*$-weak and narrow convergence are actually equivalent:

let us take $\{\mu_n^1\}_{n\in\mathbb{N}}, \{\mu_n^2\}_{n\in\mathbb{N}} \subset \mathcal{W}_q(\Omega)$ such that

$$\mu_n^i \overset{*}{\rightharpoonup} \mu^i \in \mathcal{W}_q(\Omega), \ i = 1, 2.$$

The two sequences are equi-tight (here we use the equivalence between $*$-weak and narrow convergence), so that if for every $n \in \mathbb{N}$ we take $\gamma_n \in \Gamma(\mu_n^1, \mu_n^2)$ to be an optimal transport plan, that is

$$w_q(\mu_n^1, \mu_n^2) = \|d(\cdot,\cdot)\|_{(L^q(\Omega\times\Omega);\gamma_n)},$$

then the equi-tightness of the marginals, implies that of $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(\Omega \times \Omega)$. Thus by Prokhorov Theorem we have that, up to a subsequence, γ_n narrowly converges to γ and clearly $\gamma \in \Gamma(\mu_1, \mu_2)$. This yields

$$\begin{aligned} w_q(\mu^1, \mu^2) \leq \|d(\cdot,\cdot)\|_{(L^q(\Omega\times\Omega);\gamma)} &\leq \liminf_{n\to\infty} \|d(\cdot,\cdot)\|_{(L^q(\Omega\times\Omega);\gamma_n)} \\ &= \liminf_{n\to\infty} w_q(\mu_n^1, \mu_n^2), \end{aligned}$$

proving the first statement.

For the second statement, let us take $x_0 \in \Omega$: we observe that setting

$$B = \{\mu \in \mathcal{W}_q(\Omega) \ : \ w_q(\mu, \delta_{x_0}) < R\},$$

then every $\{\mu_n\}_{n\in\mathbb{N}} \subset B$ is equi-tight, by means of Markov Inequality: using again Prokhorov Theorem, we get that $\mu_n \overset{*}{\rightharpoonup} \mu$ (up to subsequences). It remains to observe that μ has finite q-momentum: this is just a consequence of the lower semicontinuity of the functional

$$\mu \mapsto w_q(\mu, \delta_{x_0}),$$

thus concluding the proof. $\square$

7. Minimizing evolution pairings

Let $p \in [1, +\infty]$, for every pair $\mu_0, \mu_1 \in \mathcal{W}_q(\Omega)$, we define the following subset of $L^0(I; \mathcal{M}_1^+(\Omega)) \times AC^p(I; \mathcal{W}_q(\Omega))$:

$$\mathcal{D}_{p,q}(\mu_0, \mu_1) = \{\nu \preceq \mu \ : \ \mu(0) = \mu_0, \ \mu(T) = \mu_1\}.$$

We are interested in the existence of an evolution pairing (ν, μ) minimizing

$$\overline{\mathcal{A}}(\nu, \mu) = \int_I f(t, \nu(t), |\mu'|_{w_q}(t)) \, dt, \tag{7.1}$$

over the set $\mathcal{D}_{p,q}(\mu_0, \mu_1)$: at this end, we have to prove that the latter set is closed, with respect to some reasonable topology.

Lemma 13. *Let $\{(\nu_n, \mu_n)\} \subset \mathcal{D}_{p,q}(\mu_0, \mu_1)$ be such that $\nu_n \to \nu$ $\mathscr{L}^1$-a.e. and $\mu_n \overset{\mathfrak{d}}{\rightharpoonup} \mu$, then*

$$(\nu, \mu) \in \mathcal{D}_{p,q}(\mu_0, \mu_1).$$

Proof. We first show that (ν, μ) is an evolution pairing: for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_\Omega \varphi_k \, d\nu(t) &= \lim_{n\to\infty} \int_\Omega \varphi_k \, d\nu_n(t) \le \lim_{n\to\infty} \int_\Omega \varphi_k \, d\mu_n(t) \\ &= \int_\Omega \varphi_k \, d\mu(t), \text{ for } \mathscr{L}^1\text{-a.e. } t \in I, \end{aligned}$$

so (ν, μ) verifies (E1).

Then let $M_n \subset I$ be the $\mathscr{L}^1$-negligible set corresponding to ν_n in (E2) and define $M = \bigcup_{n\in\mathbb{N}} M_n$: this is still an $\mathscr{L}^1$-negligible subset of I, on which we have

$$\begin{aligned} \int_\Omega \varphi_k \, d(\mu(s) - \nu(s)) &= \lim_{n\to\infty} \int_\Omega \varphi_k \, d(\mu_n(s) - \nu_n(s)) \\ &\le \lim_{n\to\infty} \int_\Omega \varphi_k \, d(\mu_n(t) - \nu_n(t)) \\ &= \int_\Omega \varphi_k \, d(\mu(t) - \nu(t)), \text{ for every } s, t \in I \setminus M, \text{ such that } s < t, \end{aligned}$$

proving property (E2).

It remains to show that $\mu \in AC^p(I;\mathcal{W}_q(\Omega))$ and that it still verifies the conditions on the endpoints: the first is just a consequence of the fact that w_q is $\mathfrak{d}$-l.s.c., while the second straightforwardly follows from the uniform convergence, together with the fact that $\mu_n(0)=\mu_0$ and $\mu_n(T)=\mu_1$, for every $n\in\mathbb{N}$. □

We are in position to obtain the existence of a minimal evolution pairing, under the usual appropriate growth conditions on the integrand f.

Theorem 12. *Fix $p\in(1,+\infty)$. Let $f: I\times\mathcal{M}^1_+(\Omega)\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ be a function satisfying hypotheses* (4.2), (4.3) *and* (4.4). *Assume further that there exist a measure $\overline{\nu}\in\mathcal{M}^+_1(\Omega)$ and a summable function h such that*

$$f(t,\nu,z)\geq |z|^p-\beta(t)\mathfrak{d}(\nu,\overline{\nu})^r-h(t), \tag{7.2}$$

where $0<r<p$ and $\beta\in L^{\frac{p}{p-r}}(I;\mathbb{R}^+)$. Then for every pair $\mu_0,\mu_1\in\mathcal{W}_q(\Omega)$, the minimization problem

$$\inf_{(\nu,\mu)\in\mathcal{D}_{p,q}(\mu_0,\mu_1)}\overline{\mathcal{A}}(\nu,\mu),$$

admits a solution, provided there exists $(\overline{\nu},\overline{\mu})\in\mathcal{D}_{p,q}(\mu_0,\mu_1)$ with finite $\overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is defined by (7.1).

Proof. Take a minimizing sequence $\{(\nu_n,\mu_n)\}_{n\in\mathbb{N}}\subset\mathcal{D}_{p,q}(\mu_0,\mu_1)$ and suppose that

$$\overline{\mathcal{A}}(\nu_n,\mu_n)\leq L,\quad \text{for every } n\in\mathbb{N}.$$

We consider on $\mathcal{W}_q(\Omega)$ the weaker topology given by $\mathfrak{d}$: then we can repeat the same arguments of Theorem 11, in combination with Lemma 12, to get the $\mathfrak{d}$-weak convergence in $AC^p(I;\mathcal{W}_q(\Omega))$ (up to a subsequence) of $\{\mu_n\}_{n\in\mathbb{N}}$ to $\widehat{\mu}\in AC^p(I;\mathcal{W}_q(\Omega))$.

In order to get the convergence of $\{\nu_n\}_{n\in\mathbb{N}}$, we want to use Theorem 4: indeed, it is trivially true that

$$\sup_{n\in\mathbb{N}}\int_I \mathfrak{d}(\nu_n(t),0)\ dt<+\infty.$$

If we want to obtain a bound on the total variations, we can simply use the fact that every (ν_n,μ_n) is an evolution pairing: if we indicate

$$\Phi_n(t)=\mathfrak{d}(\nu_n(t),\mu_n(t)),\ t\in I,\ n\in\mathbb{N},$$

we have already seen that these are monotone increasing functions. Moreover, they are equi-bounded, because of the boundedness of $\mathfrak{d}$.

This, together with Lemma 11, implies that $\{\nu_n\}_{n\in\mathbb{N}}\in BV(I;\mathcal{M}^+_1(\Omega))$, with a uniform bound on the total variations. Indeed we have

$$\sup_{n\in\mathbb{N}}|D\nu_n|_{\mathfrak{d}}(I)\leq\sup_{n\in\mathbb{N}}(\Phi_n^-(T)-\Phi_n^+(0))+\sup_{n\in\mathbb{N}}\int_I|\mu_n'|_{w_q}(t)\ dt<+\infty,$$

where we have used that

$$\sup\int_I|\mu_n'|_{w_q}(t)\ dt<+\infty,$$

by the first part of the proof.

So we can apply Theorem 4, obtaining the convergence of $\{\nu_n\}_{n\in\mathbb{N}}$ in $L^1(I;\mathcal{M}_1^+(\Omega))$ (up to a subsequence) to a curve $\widehat{\nu}\in BV(I;\mathcal{M}_1^+(\Omega))$.

It only remains to observe that, by Lemma 13 we have $(\widehat{\nu},\widehat{\mu})\in\mathcal{D}_{p,q}(\mu_0,\mu_1)$, while by Theorem 8 the functional $\overline{\mathcal{A}}$ is lower semicontinuous, so that

$$\overline{\mathcal{A}}(\widehat{\nu},\widehat{\mu})\leq\liminf_{n\to+\infty}\overline{\mathcal{A}}(\nu_n,\mu_n)=\min_{(\nu,\mu)\in\mathcal{D}_{p,q}(\mu_0,\mu_1)}\overline{\mathcal{A}}(\nu,\mu),$$

concluding the proof. □

Finally, as far as the case $p=+\infty$ is concerned, we can prove an analogue of Theorem 9: namely, we have the existence of an evolution pairing minimizing a *geodesic* functional

$$\tilde{\ell}_g(\nu,\mu)=\int_I g(\nu(t))|\mu'|_{w_q}(t)\ dt. \tag{7.3}$$

Theorem 13. *Suppose that $g:\mathcal{M}_1^+(\Omega)\to[0,+\infty]$ is lower semicontinuous and bounded from below by a positive constant $c>0$. Then for every $\mu_0,\mu_1\in\mathcal{W}_q(\Omega)$, the problem*

$$\inf_{(\nu,\mu)\in\mathcal{D}_{\infty,q}(\mu_0,\mu_1)}\tilde{\ell}_g(\nu,\mu),$$

admits a solution, provided that there exists an evolution pairing in $\mathcal{D}_{\infty,q}(\mu_0,\mu_1)$, with finite energy.

Proof. Indeed, taking a minimizing sequence $\{(\nu_n,\mu_n)\}_{n\in\mathbb{N}}$, it should be clear that it is sufficient to obtain the convergence of $\{\mu_n\}_{n\in\mathbb{N}}$: then one can argue as in Theorem 12.

We suppose

$$\tilde{\ell}_g(\nu_n,\mu_n)\leq C,$$

otherwise the result is trivial.

The functional under consideration is invariant by reparametrization and moreover, we observe that if $\nu\preceq\mu$ and $\tilde{\mu}=\mu\circ\mathfrak{t}$ is a reparametrization of μ, then $\nu\circ\mathfrak{t}=\tilde{\nu}\preceq\tilde{\mu}$. So up to reparametrization, we can suppose that

$$|\mu_n'|_{w_q}(t)\equiv L_n,$$

then

$$cL_n=c\int_I|\mu_n'|_{w_q}(t)\ dt\leq\tilde{\ell}_g(\nu_n,\mu_n)\leq C,$$

giving that $\{\mu_n\}_{n\in\mathbb{N}}$ is equi-Lipschitz (with respect to w_q) and

$$\mu_n(t)\in\{\mu\ :\ w_q(\mu,\mu_0)\leq R\},\ n\in\mathbb{N},\ t\in I,$$

for a suitable $R>0$. This implies the $\mathfrak{d}$-weak convergence of the sequence, with the same line of reasoning of Theorem 11.

Then one can conclude by applying the semicontinuity result of Theorem 8. □

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