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# EIT and the average conductivity

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**Abstract.** We prove the instability of averages of the conductivity in the inverse boundary value problem of Calderón, also known as the inverse conductivity problem or EIT.

**Key words.** Inverse conductivity problem; stability; G-convergence.

**AMS classification.** 35R30, 35R25.

## 1. Introduction

Given a bounded connected open set $\Omega \in \mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$, and given $K \geq 1$, let $\gamma \in L^\infty(\Omega)$ be a function (which we shall call *conductivity*) such that

$$0 < K^{-1} \leq \gamma \leq K \ , \text{ a. e. in } \Omega \ . \tag{1.1}$$

For any $\varphi \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the unique weak solution to

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in} \quad \Omega \quad , \\ u = \varphi & \text{on} \quad \partial\Omega \quad . \end{cases} \tag{1.2}$$

Let $\Lambda_\gamma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ be the so–called *Dirichlet-to-Neumann map* associated to the Dirichlet problem (1.2), that is the linear mapping defined by

$$\langle \Lambda_\gamma \varphi, v|_{\partial\Omega} \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v, \tag{1.3}$$

for every $\varphi \in H^{1/2}(\partial\Omega)$ and every $v \in H^1(\Omega)$, and with $u$ solution to (1.2). Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, based on the $L^2$ scalar product. Calderón [6] posed the problem of finding $\gamma$ given $\Lambda_\gamma$. This problem has gained popularity in the last two decades also under the name of EIT (electrical impedance tomography), see for instance the review articles [7, 5]. At present, it is known that, when $n > 2$, $\Lambda_\gamma$ uniquely determines $\gamma$ if $\gamma$ is a–priori known to be sufficiently smooth, see the fundamental results by Kohn and Vogelius [13] and Sylvester and Uhlmann [18], see also, for recent developments, [15]. When $n = 2$, Astala and Päivärinta proved that uniqueness holds true with no further assumption, [4].

However it is also well–known that the problem is severely ill–posed, and we refer to [1] for an account on the available examples of instability and results of conditional

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stability. A rather basic question, [3], which until now has remained unanswered is the following

*Does the average conductivity* $\frac{1}{|\Omega|}\int_\Omega \gamma$ *depend continuously upon* $\Lambda_\gamma$?

In this note we prove, by an explicit example, that the answer to this question is *no*.

Before stating the result in detail we need some preparation. For the sake of simplicity, we limit ourselves to the case $n = 2$. For our example, we shall choose $\Omega = B_1(0)$. Let us fix two *distinct* numbers $a,\ b$ such that

$$K^{-1} \le a\ , b \le K\ . \tag{1.4}$$

let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function 1–periodic in $x$ and $y$ separately, such that in the unit square $Q = \{(x,y) \in \mathbb{R}^2 | |x| \le \frac{1}{2}\ , |y| \le \frac{1}{2}\}$ is defined as follows

$$g(x,y) = \begin{cases} a & \text{if} \quad xy \ge 0 \quad , \\ b & \text{if} \quad xy < 0 \quad . \end{cases} \tag{1.5}$$

For any positive integer $h$, we define a conductivity function $\gamma_h : B_1 \to \mathbb{R}$ as follows

$$\gamma_h(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in B_1 \setminus B_{\frac{1}{2}} \quad , \\ g(hx,hy) & \text{if} \quad (x,y) \in B_{\frac{1}{2}} \quad , \end{cases} \tag{1.6}$$

here,and in what follows, we denote by $B_r$ the disk of radius $r$ centered at the origin. Next we introduce a further conductivity function $\gamma : B_1 \to \mathbb{R}$ given by

$$\gamma(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in B_1 \setminus B_{\frac{1}{2}} \quad , \\ \sqrt{ab} & \text{if} \quad (x,y) \in B_{\frac{1}{2}} \quad . \end{cases} \tag{1.7}$$

We can now state our main Theorem.

**Theorem 1.1.** *We have*

$$\lim_{h\to\infty} \Lambda_{\gamma_h} = \Lambda_\gamma\ , \tag{1.8}$$

*in the* $\mathcal{L}(H^{1/2}(\partial B_1), H^{-1/2}(\partial B_1))$*–norm, whereas*

$$\lim_{h\to\infty} \int_{B_1} \gamma_h = \frac{\pi}{4}\left(\frac{a+b}{2}\right) + \frac{3\pi}{4} \ne \frac{\pi}{4}\left(\sqrt{ab}\right) + \frac{3\pi}{4} = \int_{B_1} \gamma\ . \tag{1.9}$$

The underlying theme behind this Theorem is the theory of $G$–convergence initiated by Spagnolo and De Giorgi [16, 9], see Section 2 for a brief account of the basic facts, that we shall need, of this theory. Keeping aside, for the moment, the deep nature of this concept, it suffices to say that the scheme of the proof of Theorem 1.1 will be to show that

(i) $\gamma$ is the $G$–limit of the sequence $\{\gamma_h\}$, see Proposition 2.5,

(ii) due to the special geometry and since $\gamma \equiv \gamma_h \equiv 1$ near $\partial B_1$ we also obtain $\Lambda_{\gamma_h} \to \Lambda_\gamma$ in the *natural* operator norm, see Proposition 2.6.

Step (i) is a minor variant of a well–known fact in the theory of homogenization. Step (ii) seems instead to be new.

It may be worth noticing that (1.9) can be verified directly, but at the same time it can be derived by the more general observation that the sequence $\{\gamma_h\}$ introduced above has indeed a limit in the weak*–topology of $L^\infty(B_1)$ given by

$$\gamma^*(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in B_1 \setminus B_{\frac{1}{2}} \quad , \\ \frac{1}{2}(a+b) & \text{if} \quad (x,y) \in B_{\frac{1}{2}} \quad . \end{cases} \tag{1.10}$$

By the same token, also the sequence $\{\frac{1}{\gamma_h}\}$ has a weak*–limit in $L^\infty(B_1)$, which is given by

$$\frac{1}{\gamma_*}(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in B_1 \setminus B_{\frac{1}{2}} \quad , \\ \frac{1}{2}(\frac{1}{a}+\frac{1}{b}) & \text{if} \quad (x,y) \in B_{\frac{1}{2}} \quad . \end{cases} \tag{1.11}$$

Therefore, under the same assumptions of Theorem1.1, we also obtain.

**Corollary 1.2.** *For every $w \in L^1(B_1)$ such that $\int_{B_{\frac{1}{2}}} w \neq 0$ we have*

$$\lim_{h\to\infty} \int_{B_1} \gamma_h w = \int_{B_1} \gamma^* w \neq \int_{B_1} \gamma w \ , \tag{1.12}$$

*and also*

$$\lim_{h\to\infty} \int_{B_1} \frac{1}{\gamma_h} w = \int_{B_1} \frac{1}{\gamma_*} w \neq \int_{B_1} \frac{1}{\gamma} w \ . \tag{1.13}$$

That is, in particular, for any weight function $w \in L^1(B_1)$ such that $w > 0$ almost everywhere in $B_1$, the weighted average of the conductivity $\gamma$, or of the resistivity $\frac{1}{\gamma}$, does *not* depend continuously upon the Dirichlet to Neumann map.

In another direction, we recall that a well–known special case of EIT is when it is a–priori assumed that the conductivity $\gamma$ has the structure

$$\gamma = 1 + (k-1)\chi_D \ , \tag{1.14}$$

where $D \subset\subset \Omega$ is unknown and $k \neq 1$, $K^{-1} \leq k \leq K$ is also possibly unknown. Uniqueness under a–priori topological and smoothness assumptions has been proved by Isakov [11]. A corresponding stability result has been obtained in [2].

The example provided in Theorem 1.1 shows also that, when the conductivity value $k$ within the inclusion $D$ is considered as an *unknown* and no a–priori regularity assumption is made, the measure (area) of the inclusion $D$ is *not* continuous with respect to $\Lambda_\gamma$. In fact we may fix $a = 1$, $b = k > 1$ and setting

$$D_h = \{(x,y) \in B_{\frac{1}{2}} | \gamma_h(x,y) = k\} \ , \tag{1.15}$$

we can represent the conductivity introduced in (1.6) as

$$\gamma_h = 1 + (k-1)\chi_{D_h} \ . \tag{1.16}$$

Moreover, setting $D = B_{\frac{1}{2}}$ we have that the conductivity given by (1.7) can also be written as

$$\gamma = 1 + (\sqrt{k} - 1)\chi_D \ , \tag{1.17}$$

and therefore

$$\lim_{h\to\infty} |D_h| = \frac{\pi}{8} \neq |D| \ , \tag{1.18}$$

where $|\cdot|$ denotes measure.

## 2. Proof of Theorem 1.1

Let us recall here the basic notions and some important properties of the G-convergence. A wide literature is available on this subject, we refer for example to the classical papers [9, 14, 16, 17] and to the books by Jikov, Kozlov and Oleĭnik [12] and by Dal Maso [8]. For any given $K \geq 1$, and a given a bounded connected open set $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, we consider the class of tensors

$$M_K = \{\sigma \in L^\infty(\Omega, M^{n\times n}) \mid K^{-1}|\xi|^2 \leq \sigma(x)\xi\cdot\xi \leq K|\xi|^2 \, , \text{for every } \xi \in \mathbb{R}^n \, , x \in \Omega\} \, , \tag{2.1}$$

here $M^{n\times n}$ denotes the set of $n \times n$ symmetric matrices.

**Definition 2.1.** A sequence $\{\sigma_h\} \subset M_K$ is said to $G$–converge to $\sigma \in M_K$, and we write $\sigma_h \stackrel{G}{\to} \sigma$, if for every $f \in H^{-1}(\Omega)$ the corresponding sequence $\{u_h\} \subset H^1_0(\Omega)$ of solutions to the *inhomogeneous* problems

$$-\operatorname{div}(\sigma_h \nabla u_h) = f \quad \text{in } \Omega \ , \ u_h = 0 \quad \text{on } \partial\Omega \ , \tag{2.2}$$

converges weakly in $H^1_0(\Omega)$ to the solution $u \in H^1_0(\Omega)$ of the problem

$$-\operatorname{div}(\sigma \nabla u) = f \quad \text{in } \Omega \ , \ u = 0 \quad \text{on } \partial\Omega \ . \tag{2.3}$$

It may be worth recalling, first of all, that the $L^1_{\text{loc}}$-strong convergence implies the G-convergence, [16, Proposition 5], [17, Remark 11].

It is well known that $M_K$, with the topology of $G$–convergence, is a *compact* metrizable space, [17, Remark 4].

The property of *convergence of the energies* is also well-known, [17] [8, Theorem 22.9]. More precisely, if $\sigma_h \stackrel{G}{\to} \sigma$, then for every $f \in H^{-1}(\Omega)$ and for every $\varphi \in H^{1/2}(\partial\Omega)$ we have

$$\int_\Omega \sigma_h \nabla u_h \cdot \nabla u_h \to \int_\Omega \sigma \nabla u \cdot \nabla u \ , \tag{2.4}$$

where $u_h, u \in H^1(\Omega)$ are the weak solutions to

$$-\operatorname{div}(\sigma_h \nabla u_h) = f \quad \text{in } \Omega \ , \ u_h = \varphi \quad \text{on } \partial\Omega \ , \tag{2.5}$$

$$-\operatorname{div}(\sigma \nabla u) = f \quad \text{in } \Omega \ , \ u = \varphi \quad \text{on } \partial\Omega \ , \tag{2.6}$$

respectively.

Another important feature of $G$–convergence that we shall use is the property of *localization*, [12, p. 152] [8, Proposition 22.7]. Namely, if $\sigma_h \overset{G}{\to} \sigma$, and $E$ is an open connected subset of $\Omega$ then we also have $\sigma_h|_E \overset{G}{\to} \sigma|_E$.

We now begin with a Lemma which provides a first connection between $G$–convergence and convergence of Dirichlet-to-Neumann maps. In what follows, in the case of isotropic (scalar) conductivities $\gamma_h, \gamma$, we understand $\gamma_h \overset{G}{\to} \gamma$ if we have $\gamma_h \mathrm{Id} \overset{G}{\to} \gamma \mathrm{Id}$ where Id denotes the identity matrix.

**Lemma 2.2.** *Let* $\gamma_h, \gamma \in L^\infty(\Omega)$ , $h = 1, 2, \ldots$ *satisfy* (1.1). *Let* $\Lambda_{\gamma_h}, \Lambda_\gamma$ *be the corresponding Dirichlet-to-Neumann maps as defined in* (1.3). *If* $\gamma_h \overset{G}{\to} \gamma$*, then for every* $\varphi, \psi \in H^{1/2}(\partial\Omega)$

$$\lim_{h\to\infty} \langle \Lambda_{\gamma_h}\varphi, \psi\rangle = \langle \Lambda_\gamma \varphi, \psi\rangle \ , \tag{2.7}$$

*Proof.* By (1.3) and by the above mentioned convergence of energies (2.4), we have

$$\lim_{h\to\infty} \langle \Lambda_{\gamma_h}\varphi, \varphi\rangle = \langle \Lambda_\gamma \varphi, \varphi\rangle \ , \text{ for every } \varphi \in H^{1/2}(\partial\Omega) \ , \tag{2.8}$$

and (2.7) immediately follows by polarization. □

Let us observe that in general, it is not clear whether from the $G$–convergence of the conductivities it does follow the convergence of the Dirichlet-to-Neumann maps in the *strong* operator norm.

For the purposes of Theorem 1.1, where we are just interested in obtaining one example, we shall also assume $n = 2$, $\Omega = B_1$ and that the sequence $\{\gamma_h\}$ is such that $\gamma_h \equiv 1$ in the annulus $B_1 \setminus B_{\frac{1}{2}}$. By the localization property, it is evident that if $\gamma_h \overset{G}{\to} \gamma$, then we also have $\gamma \equiv 1$ in $B_1 \setminus B_{\frac{1}{2}}$. Under such hypotheses we obtain the following bound.

**Lemma 2.3.** *For every* $\varphi \in H^{1/2}(\partial B_1)$*, let* $u$ *be the weak solution to* (1.2). *There exists* $C > 0$*, only depending on* $K$*, such that*

$$\|(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi\|_{H^{-1/2}(\partial B_1)} \le C \int_{B_{\frac{1}{2}}} |\nabla u|^2 \ , \tag{2.9}$$

*Proof.* Let $u_h$ be the weak solution to (1.2) when $\gamma$ is replaced with $\gamma_h$. Setting $v = u_h - u$, we have that $v$ is the weak solution to

$$\begin{cases} \mathrm{div}(\gamma_h \nabla v) = \mathrm{div}((\gamma - \gamma_h)\nabla u) & \text{in} \quad B_1 \quad , \\ v = 0 & \text{on} \quad \partial B_1 \quad . \end{cases} \tag{2.10}$$

We have

$$\|(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi\|^2_{H^{-1/2}(\partial B_1)} = \|\frac{\partial v}{\partial \nu}\|^2_{H^{-1/2}(\partial B_1)} \le c \int_{B_1} |\nabla v|^2 \ , \tag{2.11}$$

where $\nu$ denotes the exterior unit normal to $\partial B_1$ and $c$ is an absolute constant. On the other hand,

$$\int_{B_1} |\nabla v|^2 \leq K \int_{B_1} \gamma_h \nabla v \cdot \nabla v \leq K \|\gamma_h - \gamma\|_{L^\infty(B_1)} \int_{B_{\frac{1}{2}}} |\nabla u||\nabla v| \leq$$

$$\leq (K^2 - 1) \int_{B_{\frac{1}{2}}} |\nabla u||\nabla v| \, , \qquad (2.12)$$

and thus, by Schwarz inequality,

$$\int_{B_1} |\nabla v|^2 \leq (K^2 - 1)^2 \int_{B_{\frac{1}{2}}} |\nabla u|^2 \, . \qquad (2.13)$$

Hence the thesis follows. □

In the following Lemma, we further restrict our attention to the case when, in addition to $\gamma_h \equiv 1$ in $B_1 \setminus B_{\frac{1}{2}}$, we also assume $\gamma = 1 + (k-1)\chi_{B_{\frac{1}{2}}}$ , where $k \in [K^{-1}, K]$ is a given constant. We shall denote, for every $m \in \mathbb{Z}$, $\varphi_m(e^{i\vartheta}) = e^{im\vartheta}$, $\vartheta \in [0, 2\pi]$.

**Lemma 2.4.** *We have*

$$\|(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_m\|_{H^{-1/2}(\partial B_1)} \leq C|m|2^{-2|m|} \, , \textit{ for every } m \in \mathbb{Z} \, , \qquad (2.14)$$

*where $C > 0$ only depends on $K$.*

*Proof.* It suffices to compute, by separation of variables, the solution $u_m$ to (1.2) when $\varphi = \varphi_m$ and then to apply the previous Lemma 2.3. □

The proof of Theorem 1.1 will be an immediate consequence of the following two Propositions.

**Proposition 2.5.** *Let $\gamma_h$, $\gamma$ be given by* (1.6), (1.7)*, respectively, then*

$$\gamma_h \stackrel{G}{\to} \gamma \, . \qquad (2.15)$$

*Proof.* The starting point is a well–known result in homogenization of checkerboard composites, due to Dykhne [10], which says that, given the periodic function $g$ introduced in (1.5), the sequence $\{g_h\}$, defined by $g_h(x,y) = g(hx, hy)$, satisfies

$$g_h \stackrel{G}{\to} \sqrt{ab} \, . \qquad (2.16)$$

Consequently, by the localization property, and by the definitions (1.6), (1.7), we deduce that

$$\gamma_h|_{B_{\frac{1}{2}}} \stackrel{G}{\to} \gamma|_{B_{\frac{1}{2}}} \, , \qquad (2.17)$$

on the other hand, we obviously have

$$\gamma_h|_{B_1\setminus B_{\frac{1}{2}}} \stackrel{G}{\to} \gamma|_{B_1\setminus B_{\frac{1}{2}}} \ . \tag{2.18}$$

It is also a well–known consequence of the localization property and of compactness, [12, p. 165], that the separate $G$–convergence on the two disjoint domains $B_{\frac{1}{2}}$ and $B_1 \setminus B_{\frac{1}{2}}$ implies the global $G$–convergence on their union, and (2.15) follows. □

**Proposition 2.6.** *If we assume* $\gamma_h \equiv 1$ *in* $B_1 \setminus B_{\frac{1}{2}}$, $\gamma = 1 + (k-1)\chi_{B_{\frac{1}{2}}}$ , *with* $k \in [K^{-1}, K]$ *constant and also* $\gamma_h \stackrel{G}{\to} \gamma$, *then we have*

$$\lim_{h\to\infty} \|\Lambda_{\gamma_h} - \Lambda_\gamma\|_{\mathcal{L}(H^{1/2}(\partial B_1),H^{-1/2}(\partial B_1))} = 0 \ . \tag{2.19}$$

*Proof.* Being $\Lambda_{\gamma_h}$ , $\Lambda_\gamma$ selfadjoint operators between the dual Hilbert spaces $H^{1/2}(\partial B_1)$ , $H^{-1/2}(\partial B_1)$ , it suffices to prove that

$$\sup\{\langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi, \varphi\rangle \,|\varphi \in H^{1/2}(\partial B_1) \ , \ \|\varphi\|_{H^{1/2}(\partial B_1)} = 1\} \to 0 \ , \tag{2.20}$$

as $h \to \infty$. Let us fix $\varphi \in H^{1/2}(\partial B_1)$ such that $\|\varphi\|_{H^{1/2}(\partial B_1)} = 1$ . Let

$$\varphi(e^{i\vartheta}) = \sum_{l\in\mathbb{Z}} a_l e^{il\vartheta} \tag{2.21}$$

be its Fourier series representation. With no loss of generality, we may assume $a_0 = 0$ and we have

$$\sum_{l\in\mathbb{Z}} |l||a_l|^2 = \|\varphi\|^2_{H^{1/2}(\partial B_1)} = 1 \ . \tag{2.22}$$

On the other hand, using again the notation $\varphi_m(e^{i\vartheta}) = e^{im\vartheta}$ we have

$$\langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi, \varphi\rangle = \sum_{l,m\in\mathbb{Z}} a_l\overline{a_m} \langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m\rangle \ . \tag{2.23}$$

For any fixed $N \in \mathbb{N}$, we split the summation on the right hand side above as follows

$$\langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi, \varphi\rangle = S_1 + S_2 + S_3 \ , \tag{2.24}$$

where

$$S_1 = \sum_{|l|,|m|\le N} a_l\overline{a_m} \langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m\rangle \ , \tag{2.25}$$

$$S_2 = \sum_{|m|\le|l|,|l|>N} a_l\overline{a_m} \langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m\rangle \ , \tag{2.26}$$

$$S_3 = \sum_{|l|<|m|,|m|>N} a_l\overline{a_m} \langle(\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m\rangle \ , \tag{2.27}$$

and we trivially have

$$|S_2 + S_3| \leq 2 \sum_{|m| \leq |l|, |l| > N} |a_l||a_m| \, |\langle (\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m \rangle| \, , \tag{2.28}$$

now, recalling Lemma 2.4, we obtain

$$\begin{aligned} |S_2 + S_3| &\leq C \sum_{|m| \leq |l|, |l| > N} |a_l||a_m| \sqrt{|l|} 2^{-|l|} \sqrt{|m|} \leq \\ &\leq C \sum_{|l| > N} |a_l| \sqrt{|l|} 2^{-|l|} \sum_{|m| \leq |l|} |a_m| \sqrt{|m|} \leq \\ \leq C \sum_{|l| > N} |a_l| \sqrt{|l|} 2^{-|l|} &\left( \sum_{m \in \mathbb{Z}} |m||a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{0 < |m| \leq |l|} 1 \right)^{\frac{1}{2}} \leq \\ &\leq C \sum_{|l| > N} |a_l| \sqrt{|l|} 2^{-|l|} \|\varphi\|_{H^{1/2}(\partial B_1)} \, (2|l|)^{\frac{1}{2}} \leq \\ &\leq C \, N 2^{-N} \, , \end{aligned} \tag{2.29}$$

where it is understood that the constant $C > 0$ may vary from line to line. Therefore, for every $N \in \mathbb{N}$, we have

$$|\langle (\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi, \varphi \rangle| \leq \Big| \sum_{|l|, |m| \leq N} a_l \overline{a_m} \, \langle (\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m \rangle \Big| + C \, N 2^{-N} \, . \tag{2.30}$$

Given $\varepsilon > 0$, let $N$ be such that

$$C \, N 2^{-N} < \frac{\varepsilon}{2} \, , \tag{2.31}$$

next, by Lemma 2.2, there exists $H > 0$ such that, for every $h > H$, we have

$$|\langle (\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi_l, \varphi_m \rangle| \leq \frac{\varepsilon}{4N} \, , \text{ for every } |l|, |m| \leq N \, . \tag{2.32}$$

Consequently, for every $\varepsilon > 0$, we have found $H > 0$ such that for every $h > H$, we have

$$\begin{aligned} |\langle (\Lambda_{\gamma_h} - \Lambda_\gamma)\varphi, \varphi \rangle| &\leq \frac{\varepsilon}{4N} \sum_{|l| \leq N} |a_l| \sum_{|m| \leq N} |a_m| + \frac{\varepsilon}{2} \leq \\ &\leq \frac{\varepsilon}{2} \left[ \frac{1}{2N} \left( \sum_{|l| \leq N} \frac{1}{|l|} \right) \left( \sum_{l \in \mathbb{Z}} |l||a_l|^2 \right) + \right] \leq \varepsilon \, , \end{aligned} \tag{2.33}$$

and (2.19) follows. □

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## References

1. G. Alessandrini, Open issues of stability for the inverse conductivity problem. *J. Inverse Ill-Posed Probl.* **15** (2007), 451–460.
2. G. Alessandrini and M. Di Cristo, Stable determination of an inclusion by boundary measurements. *SIAM J. Math. Anal.* **37** (2005), 200–217.
3. G. Alessandrini and S. Vessella, Lipschitz stability for the inverse conductivity problem. *Adv. in Appl. Math.* **35** (2005), 207–241.
4. K. Astala and L. Päivärinta, Calderón's inverse conductivity problem in the plane. *Ann. of Math. (2)* **163** (2006), 265–299.
5. L. Borcea, Electrical impedance tomography. *Inverse Problems* **18** (2002), R99–R136.
6. A.-P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980, 65–73.
7. M. Cheney, D. Isaacson and J. C. Newell, Electrical impedance tomography. *SIAM Rev.* **41** (1999), 85–101).
8. G. Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.
9. E. De Giorgi and S. Spagnolo, Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital. (4)* **8** (1973), 391–411.
10. A. M. Dykhne, Conductivity of a two-dimensional two-phase system. *Zhurnal eksperimental'noi i teoreticheskoi fiziki/Akademiia Nauk SSSR* **59** (1970). English translation in *Sov. Phys. JETP* 32, 63–65 (1971).
11. V. Isakov, On uniqueness of recovery of a discontinuous conductivity coefficient. *Comm. Pure Appl. Math.* **41** (1988), 865–877.
12. V. V. Jikov, S. M. Kozlov and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994, Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].
13. R. Kohn and M. Vogelius, Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.* **37** (1984), 289–298.
14. A. Marino and S. Spagnolo, Un tipo di approssimazione dell'operatore $\sum_1^n ijD_i(a_{ij}(x)D_j)$ con operatori $\sum_1^n jD_j(\beta(x)D_j)$. *Ann. Scuola Norm. Sup. Pisa (3)* **23** (1969), 657–673.
15. L. Päivärinta, A. Panchenko and G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities. *Rev. Mat. Iberoamericana* **19** (2003), 57–72.
16. S. Spagnolo, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3)* **22** (1968), 673.
17. ———, *Convergence in energy for elliptic operators*, Numerical solution of partial differential equations, III (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975), Academic Press, New York, 1976, 469–498.

18. J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)* **125** (1987), 153–169.

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