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RATE-INDEPENDENT PHASE TRANSITIONS IN ELASTIC MATERIALS: A YOUNG-MEASURE APPROACH

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Abstract. A quasistatic evolution problem for a phase transition model with nonconvex energy density is considered in terms of Young measures. We focus on the particular case of a finite number of phases. The new feature consists in the usage of suitable regularity arguments in order to prove an existence result for a notion of evolution presenting some improvements with respect to the one defined in [13], for infinitely many phases.

1. **Introduction.** In the last years the energetic formulation of rate-independent processes has been widely used to describe mesoscopic models for the isothermal stress-induced transformation in crystalline materials (see, e.g. [2], [15], [19], [21], [25], [26]).

Assuming that the reference configuration of the crystalline material is a bounded region $D \subset \mathbb{R}^d$, the state of the system is determined by two functions: the *deformation* $v\colon D \to \mathbb{R}^N$ and the *internal variable* $z\colon D \to Z \subset \mathbb{R}$, which takes into account the phase transformations of the material.

In our framework, Z is a finite set $\{1, \dots, q\}$, representing the different phases (or phase variants) of the crystal, and z represents the phase distribution of the material. Then the *stored energy* of the system can be written as:

$$\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x))\, dx.$$

From a physical point of view, the energy functional should also depend on the temperature, but we omit this dependence since we are dealing with isothermal transformations. We assume that changes of the phase distribution of the material lead to an energy *dissipation*, which is represented by

$$\int_D H(z_{new}(x), z_{old}(x))\, dx,$$

where H is a metric distance on Z, z_{old} is the old phase distribution and z_{new} the new one. Moreover, we require that the admissible deformations satisfy a prescribed time-dependent boundary condition $\boldsymbol{\varphi}(t)$, which we impose on the whole

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boundary ∂D to avoid some technical difficulties; for the same reason, we neglect any contribution due to external forces.

The natural form for the stored-energy density W is a multiple-well potential form (see [30], [29], [19], [15], [27], [28]), more in general we deal with a density which does not satisfy any convexity assumption with respect to z. As in [13], this lack of convexity gives rise to many technical difficulties, making unsolvable in usual functional spaces the incremental minimum problems used in the construction of approximate solutions (see [23] and references therein); it is also responsible for the formation of microstructures (see, e.g., [19], [22], [31]). To overcome these difficulties, many authors have proposed to introduce suitable regularizing terms in the energy functional (see [2], [15], [20]).

To avoid any artificial regularization, in this paper we follow the same approach of [13], and set the problem in a suitable space of Young measures, where the incremental minimum problems can be solved.

Since we are assuming that the internal variable takes only a finite number of values, we are able to give a more explicit description of the Young measure ν which is going to substitute the pair $(z, \nabla v)$ in our extended setting: ν can be written as

$$\nu = \sum_{\alpha=1}^{q} b_\alpha (\delta_\alpha \otimes \lambda_\alpha),$$

for suitable families $(\lambda_\alpha)_\alpha$ of Young measures on D with values in $\mathbb{R}^{N\times d}$, and $(b_\alpha)_\alpha$ in $L^\infty(D;[0,1])$, with $\sum_\alpha b_\alpha = 1$ a.e. in D. The energy associated to a pair $(b,\lambda) = (b_\alpha, \lambda_\alpha)_\alpha$ of family of coefficients and Young measures will be indicated with $\langle W, (b,\lambda)\rangle$.

In our language, when a Young measure with values in Z is representable by a function z, the corresponding family of coefficients b is defined by

$$b_\alpha = 1_{\{x\in D\,:\,z(x)=\alpha\}} \quad \text{for every } \alpha.$$

In this case the Young measure representation can be interpreted in the following way: the material assumes a *pure phase* distribution, i.e., to every point x is associated a pure phase $\alpha \in Z$. While in the general case we say that the material has a *mixed phase* distribution meaning that at each point x we have a mixture of phases α with volume fractions $b_\alpha(x)$.

Many authors have proposed relaxation of nonconvex problems in terms of Young measures (see [19], [24], [22], [26], [28], [6], [4], [5], [32]). A key point in the analysis of our model is related to the relaxation of the dissipation functional.

In order to express the energy dissipated between two times s and t in terms of Young measures, we need to deal with a measure on $D \times Z^2$, coupling the measures μ_s and μ_t associated to s and t respectively. To this end, some authors consider a measure coupling μ_s and μ_t in an "independent" or "non-correlated" way (i.e., a measure with disintegration $(\mu_s^x \otimes \mu_t^x)_x$), or the Wasserstein distance between μ_s and μ_t (see, e.g., [19], [27], and [22]). In these cases, the dissipation distance is univocally determined by μ_s and μ_t (in our language, by $(b_\alpha^s)_\alpha$ and $(b_\alpha^t)_\alpha$).

We adopt, instead, the approach proposed in [7], and followed in [8] and in [13], based on the notion of compatible systems of Young measures, introduced in [9]. Our discrete setting again allows us to deal with a more explicit expression for these objects: every compatible system of Young measures $\boldsymbol{\mu}$ on D, with time set A and values in Z can be written as

$$\boldsymbol{\mu}_{t_1\ldots t_m} = \sum_{(\alpha_1,\ldots,\alpha_m)} \boldsymbol{b}^{t_1\ldots t_m}_{\alpha_1\ldots\alpha_m} \boldsymbol{\delta}_{(\alpha_1,\ldots,\alpha_m)},$$

for a suitable family $(\boldsymbol{b}^{t^1\ldots t^m}_{\alpha_1\ldots\alpha_m})_{(\alpha_1,\ldots,\alpha_m)}$ in $L^\infty(D;[0,1])$ satisfying

$$\sum_{(\alpha_1,\ldots,\alpha_m)} \boldsymbol{b}^{t^1\ldots t^m}_{\alpha_1\ldots\alpha_m} = 1 \quad \text{a.e. in } D,$$

for every $t_1 < \cdots < t_m$ in A (for the precise definition of compatible systems of Young measures, see Section 3, p. 6). According to our statistical interpretation, if we consider two time instants $s < t$, $\boldsymbol{b}^{st}_{\alpha\beta}(x)$ represents the volume fraction at x undergoing the phase transition from α at time s to β at time t, and the energy dissipated between s and t is given by

$$< H, \boldsymbol{b}^{st} >:= \sum_{\alpha\beta} H(\beta,\alpha) \int_D \boldsymbol{b}^{st}_{\alpha\beta}(x)\, dx.$$

The knowledge of $\boldsymbol{b}^s_\alpha$ and $\boldsymbol{b}^t_\beta$ separately does not keep the complete information about the energy spent in the transition. Indeed, if we consider the case of a homogeneous phase distribution $\boldsymbol{b}^s_\alpha = 1/q$ for every α, and we suppose that the material undergoes a transition from s to t just permuting the phases and leaving the volume fractions unchanged, we have $\boldsymbol{b}^t = \boldsymbol{b}^s$; hence the dissipation computed using only $\boldsymbol{b}^s$ and $\boldsymbol{b}^t$ is 0, while the dissipation energy computed using $\boldsymbol{b}^{st}$ depends on the permutation and it is different from zero. Therefore, our description seems to give a more realistic picture of the dissipation phenomenon, if compared with the one proposed in [19], [27], and [22], which only take into account the contribution of single time instants.

The aim of the paper is to prove an existence result for the quasistatic evolution in a time interval $[0,T]$, defined as a pair $(\boldsymbol{b},\boldsymbol{\lambda})$ formed by a family of coefficients and a family of Young measures, and satisfying an admissibility condition, a suitably reformulated stability condition, and an energy balance.

The admissibility condition requires suitable approximation properties by means of functions which satisfy the boundary condition. Due to the technicalities in this condition, we do not want to enter in the details of these properties here (see Section 5); we just point out that they guarantee the Young measure $\sum_\alpha \boldsymbol{b}^t_\alpha \boldsymbol{\lambda}^t_\alpha$ is a "Gradient Young Measure". This means that it can be generated by a sequence of gradients; in particular it satisfies the conditions proven in the characterization provided by Kinderlehrer and Pedregal in [18]. Unfortunately, in our case these properties are not enough to describe the elements of the admissible set of solutions. Indeed we do not only ask the measure $\sum_\alpha \boldsymbol{b}^t_\alpha \boldsymbol{\lambda}^t_\alpha$ to be a Gradient Young Measure, but we also need a connection between the sequence of gradients approximating $\sum_\alpha \boldsymbol{b}^t_\alpha \boldsymbol{\lambda}^t_\alpha$, and the sequence of functions approximating $\boldsymbol{b}_{t_1\ldots t_m}$ for subsequent times $t_1 < \cdots < t_m$, in order to preserve some good properties of the set of admissible solutions.

The stability condition is a global minimality condition satisfied by the evolution at each time t, but the set of competitors is a proper subset of the admissible pairs; for this reason we call this condition *partial-global stability*.

A natural class of tests for the minimality condition can be obtained by modifying the evolution $(\boldsymbol{b},\boldsymbol{\lambda})$ at time t with permutations of the phase volume fractions $(\boldsymbol{b}^t_\alpha)_\alpha$ and the measures $(\boldsymbol{\lambda}^t_\alpha)_\alpha$. The competitors $(\tilde{b},\tilde{\lambda})$ constructed in this way can be written in the form $(M \cdot \boldsymbol{b}^t, M \cdot \boldsymbol{\lambda}^t)$, where M is a $q \times q$ matrix with entries equal to 0 or 1 and exactly one nonzero entry on each column and row: $M \cdot \boldsymbol{b}^t$, $M \cdot \boldsymbol{\lambda}^t$ represent the product of the matrix M and the vectors $(\boldsymbol{b}^t_\alpha)_\alpha$, $(\boldsymbol{\lambda}^t_\alpha)_\alpha$ respectively. The minimality condition for this class of competitors is as follows

$$\langle W, (\boldsymbol{b}^t, \boldsymbol{\lambda}^t)\rangle \leq \langle W, (\tilde{b}, \tilde{\lambda})\rangle + \mathbb{H}(\tilde{b}, \boldsymbol{b}^t), \tag{1.1}$$

where

$$\mathbb{H}(\tilde{b}, \boldsymbol{b}^t) := \sum_{\alpha\beta} H(\beta, \alpha) \int_D M_{\beta\alpha} \boldsymbol{b}^t_\alpha(x)\; dx.$$

The class of tests can be enlarged by including the pairs obtained with the translation of the measures $(\tilde{\lambda}_\beta)_\beta$ by gradients of functions $\tilde{u} \in H^1_0(D; \mathbb{R}^N)$: hence (1.1) holds true for $(\tilde{b}, \tilde{\lambda}) = (M \cdot \boldsymbol{b}^t, \tilde{\mathcal{T}}_{\nabla \tilde{u}}(M \cdot \boldsymbol{\lambda}^t))$. A further extension of the class of competitors is possible: we are able to prove the minimality not only with respect to permutations (i.e. $M_{\beta\alpha} = 0, 1$) but also with respect to rearrangements of the phase volume fractions $(\boldsymbol{b}^t_\alpha)_\alpha$ and the measures $(\boldsymbol{\lambda}^t_\alpha)_\alpha$: in this case M is a measurable map on D with values in a special set of $q \times q$ real matrices. The elements of this set are the matrices with nonnegative entries such that the sum of the entries of each column is 1; in probabilistic language they are called *stochastic matrices* (see, e.g., [1, Part 2]), and their entries $M_{\beta\alpha}$ represent the probability of a transition from phase α to phase β. In our model, $M_{\beta\alpha}(x)$ is the proportion of the volume fraction at x originally in phase α undergoing a phase transition to β. According to the picture described so far, the quantity

$$H(\beta, \alpha) M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x)$$

can be interpreted as the energy density dissipated at the point x by the phase transition from α to β. Therefore, the following expression

$$\mathbb{H}(\tilde{b}, \boldsymbol{b}^t) = \sum_{\alpha,\beta} H(\beta, \alpha) \int_D M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x)\; dx$$

represents the energy which would be dissipated on the whole domain D, if we performed the microscopic phase transition determined by M.

We observe that any other phase distribution $(\tilde{b}_\beta)_\beta$ can be obtained by the action of a suitable stochastic matrix: indeed, it is enough to choose $M_{\beta\alpha}(x) := \tilde{b}_\beta(x)$ for every α, β.

From the stability property we can deduce a pointwise condition. If we call *active at x* the phases α for which $\boldsymbol{b}^t_\alpha(x) > 0$, then the Euler equation for the internal variable can be written as follows: for a.e. x with active phase α, we have

$$\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\; d(\boldsymbol{\lambda}^t_\alpha)^x(F) \leq \int_{\mathbb{R}^{N\times d}} W(\beta, F)\; d(\boldsymbol{\lambda}^t_\alpha)^x(F) + H(\beta, \alpha),$$

for every β. According to the above physical picture, this condition can be interpreted as an *optimality condition of the active phases*. Clearly, an Euler equation for the deformation can be derived as well: it is the classical *equilibrium condition* on the stress $\boldsymbol{\sigma}$ (see Remark 5.5 for the definition of $\boldsymbol{\sigma}$).

The energy equality expressed in terms of $(\boldsymbol{b}, \boldsymbol{\lambda})$ takes the following form:

$$\langle W, (\boldsymbol{b}^t, \boldsymbol{\lambda}^t)\rangle + \mathrm{Diss}_H(\boldsymbol{b}; 0, t) = \mathcal{W}(z_0, v_0) + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s)\rangle_2\; ds,$$

for every $t \in [0, T]$, where $\boldsymbol{\varphi}$ is the time-dependent boundary condition. The dissipation $\mathrm{Diss}_H(\boldsymbol{b}; 0, t)$ is defined by

$$\mathrm{Diss}_H(\boldsymbol{b}; 0, t) := \sup \sum_{i=1}^{k} \sum_{\alpha\beta} H(\beta, \alpha) \int_D \boldsymbol{b}_{\alpha\beta}^{s_{i-1} s_i}(x)\, dx,$$

where the supremum is taken over all partitions $0 = s_0 < \cdots < s_k = t$ of the interval $[0, t]$.

The proof of the existence theorem (Theorem 6.3) follows the classical scheme of time-discretization, resolution of incremental minimum problems, and passage to the limit in the sequence of approximate solutions.

The main new feature concerns the choice of the solutions to the discretized minimum problems. In the spirit of [11], we use the Ekeland Principle to choose minimizers satisfying special approximability properties. Then the regularity results for quasi-minima of integral functionals (see [16]) are used to prove a uniform bound on the moments of order $2r > 2$ of the selected minimizers, and consequently of the approximate solutions $(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t)$. As a by-product of this selection, we get the continuity of the functional

$$(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t) \mapsto \langle W, (\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t) \rangle.$$

Thanks to this continuity, we are able to obtain in the limit the stability condition and the energy equality written above, which improve the notion of quasistatic evolution proposed in [13]. Under weaker assumptions on W than in [13], we can obtain a better notion of stability, since the minimality property is now satisfied with a quite large set of competitors including all possible rearrangements of the phase distribution. Moreover we can obtain not only an upper energy estimate as in [13], but a complete energy balance.

One technical point in the proof of the stability condition is the approximation of the right hand-side of (1.1) by integrals corresponding to functions satisfying the prescribed boundary condition. This is done by adapting to our problem the classical Riemann-Lebesgue Lemma.

The proof of the lower energy estimate requires a more delicate argument than in the standard case (see e.g. [15, Step 5, p. 7]). Usually the proof of this estimate is based on a suitable minimality property guaranteed by the stability condition. In our case, due to the restriction of the set of competitors in the partial-global stability, we can only prove a weaker version of this minimality property, using the continuity provided by the regularity argument. This fact makes more delicate the last step of the proof, where we need to approximate a Lebesgue integral with Riemann sums.

The outline of this paper is as follows. In Section 2 and 3 we provide some mathematical preliminaries and technical tools. In Section 4 we fix the setting of the problem. In Section 5 we describe the admissible set where we look for the quasistatic evolution, which is defined in Section 6. Section 7 is devoted to the proof of the existence theorem, and finally in Section 8 we derive the Euler equations for the partial-stability condition.

2. **Mathematical preliminaries.** The symbol 1_B indicates the characteristic function of a subset B of $\mathbb{R}^d$. The Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$, is denoted by $\mathcal{L}^d$; we sometimes use the notation $|E|$ for the Lebesgue measure of a measurable subset E of $\mathbb{R}^d$. The Borel σ-algebra on D is denoted by $\mathcal{B}(D)$. For $1 \leq p \leq +\infty$, $\|\cdot\|_p$

is the usual norm on L^p, while $W^{1,p}(D;\mathbb{R}^N)$ denotes the usual Sobolev space of all L^p functions from an open domain $D \subseteq \mathbb{R}^d$ into $\mathbb{R}^N$ with L^p first derivatives. We indicate $W^{1,2}(D;\mathbb{R}^N)$ with $H^1(D;\mathbb{R}^N)$. The symbol $\langle \cdot, \cdot \rangle_2$ denotes the duality product in L^2. Given a function $f \in L^1(D)$ and a measurable subset $Q \subseteq D$, the mean value of f over Q is denoted by $(f)_Q$, i.e.

$$(f)_Q := \frac{1}{|Q|}\int_Q f(x)\, dx.$$

We recall the well-known following lemma.

Lemma 2.1. *Let $f \in L^2(D)$, and consider a finite measurable partition $(D_i)_{i=1}^I$ of D. The projection of f onto the space*

$$\mathcal{K} := \{g \in L^2(D) \ \ : \ g_{|D_i} \text{ is constant for every } i = 1, \dots, I\}$$

is

$$P_{\mathcal{K}}(f) := \sum_{i=1}^{I} (f)_{D_i} 1_{D_i}.$$

The symbol $\mathbf{M}_{St}^{q\times q}$ denotes the set of all stochastic matrices of size $q \times q$, i.e. the set of all matrices $(M_{\beta\alpha})_{\beta,\alpha}$ with

- $0 \le M_{\beta\alpha} \le 1$ for every α, β,
- $\sum_\beta M_{\beta\alpha} = 1$, for every α.

For the notion of quasi-minimum and the related results, we refer to the Appendix.

3. **Young measures and discrete sets of values.** For the mathematical preliminaries about measures and Young measures we refer to [13, Section 2 and 4]. Here we just recall a definition and fix some notation.

Given a sequence $(\mu_k)_k$ of Young measures in $Y(D;\mathbb{R}^n)$, we say that $\mu_k \rightharpoonup \mu$ *p-weakly**, for $1 < p \le \infty$, if

- $\mu_k \rightharpoonup \mu$ in the weak* topology of the space of bounded Radon measures on $D \times \mathbb{R}^n$,
- the p-moments of $(\mu_k)_k$
$$\int_{D\times\mathbb{R}^n} |\xi|^p \; d\mu_k(x,\xi),$$
are equibounded.

Let $(\Omega, \mathcal{F})$ be a measure space, Ξ a finite dimensional Hilbert space, and $\mu \in Y(D;\Xi)$. For every bounded measurable function $g\colon D \to \mathbb{R}$, the product $g\mu$ is defined by

$$\int_{D\times\Xi} \phi(x,\xi)\; d(g\mu)(x,\xi) := \int_{D\times\Xi} g(x)\phi(x,\xi)\; d\mu(x,\xi),$$

for every bounded Borel function $\phi\colon D\times\Xi \to \mathbb{R}$. For every $\mathcal{B}(D\times\Xi)$-$\mathcal{F}$-measurable function $f\colon D\times\Xi \to \Omega$, the image measure, defined by $\mu(f^{-1}(B))$ for every measurable set $B \subseteq \Omega$, will be denoted by $f(\mu)$. In particular, if we define the translation map $\mathcal{T}_G$ associated to a function $G \in L^1(D;\Xi)$ by

$$\mathcal{T}_G(x,\xi) := (x, \xi + G(x)), \quad \text{for a.e. } x \in D \text{ and every } \xi \in \Xi, \tag{3.1}$$

for every measure $\mu \in Y(D;\Xi)$ we can consider the translated measure $\mathcal{T}_G(\mu)$, defined by

$$\int_{D\times\Xi} \phi(x,\xi)\ d\mathcal{T}_G(\mu)(x,\xi) = \int_{D\times\Xi} \phi(x,\xi+G(x))\ d\mu(x,\xi), \tag{3.2}$$

for every bounded Borel function $\phi\colon D\times\Xi \to \mathbb{R}$. If $\boldsymbol{\mu} = (\mu_i)_{i=1}^I$ is a finite family in $Y(D;\Xi)$, we denote by $\tilde{\mathcal{T}}_G(\boldsymbol{\mu})$ the family of translated measures $(\mathcal{T}_G(\mu_i))_{i=1}^I$.

Given $\xi_0 \in \Xi$, the measure $\delta_{\xi_0} \in M_b(\Xi)$ is defined by

$$\int_{\Xi} f(\xi)\ d\delta_{\xi_0}(\xi) = f(\xi_0),$$

for every bounded Borel function $f\colon \Xi \to \mathbb{R}$; fixed a $\mathcal{B}(D)$-$\mathcal{B}(\Xi)$-measurable function $u\colon D \to \Xi$, the Young measure $\boldsymbol{\delta}_u \in Y(D;\Xi)$ is defined by

$$\int_{D\times\Xi} g(x,\xi)\ d\boldsymbol{\delta}_u(x,\xi) = \int_D g(x,u(x))\ dx,$$

for every bounded Borel function $g\colon D\times\Xi \to \mathbb{R}$. In particular $\boldsymbol{\delta}_{\xi_0}$ is the Young measure associated to the constant function $u(x) \equiv \xi_0$, which should not be confused with the measure δ_{ξ_0}.

We recall the statement of a lemma which will be useful in the regularization of the approximate solutions. We will use the statement of Fonseca, Müller, and Pedregal (see [14, Lemma 1.2]).

Lemma 3.1. (Decomposition Lemma) *Let $(v_j)_j$ be a bounded sequence in $H^1(D;\Xi)$. Then there exist a subsequence $(v_{j_k})_k$ of $(v_j)_j$, and a bounded sequence $(w_k)_k$ in $H^1(D;\Xi)$, such that $(|\nabla w_k|^2)_k$ is equiintegrable and*

$$\mathcal{L}^d(\{v_{j_k} \neq w_k \text{ or } \nabla v_{j_k} \neq \nabla w_k\}) \to 0, \tag{3.3}$$

as $k \to \infty$.

In the whole paper D is a bounded connected open subset of $\mathbb{R}^d$ with Lipschitz boundary; Z denotes a nonempty finite subset $\{1,\dots,q\}$ of $\mathbb{R}^m$, and H is a metric on Z; $A \subseteq \mathbb{R}$ denotes a set of indices.

The space of Young measures on D with values in Z is indicated with $Y(D;Z)$ and the space of compatible systems on D with time set A and values in Z is denoted by $SY(A,D;Z)$. We recall that a compatible system of Young measures on D with time set A and values in Z is a family $\boldsymbol{\mu} = (\boldsymbol{\mu}_{t_1\dots t_m})_{t_1\dots t_m}$ of Young measures $\boldsymbol{\mu}_{t_1\dots t_m} \in Y(D;Z^m)$, with $t_1 < \dots < t_m$ varying among all strictly increasing finite sequences of elements of A, satisfying the following projection property:

$$\tilde{\pi}^{t_1\dots t_m}_{s_1\dots s_n}(\boldsymbol{\mu}_{t_1\dots t_m}) = \boldsymbol{\mu}_{s_1\dots s_n},$$

whenever $\{s_1,\dots,s_n\} \subseteq \{t_1,\dots,t_m\}$, where $\tilde{\pi}^{t_1\dots t_m}_{s_1\dots s_n}\colon D\times Z^m \to D\times Z^n$ is defined by $\tilde{\pi}^{t_1\dots t_m}_{s_1\dots s_n}(x,\alpha_{t_1},\dots,\alpha_{t_m}) = (x,\alpha_{s_1},\dots,\alpha_{s_n})$.

It is easy to see that $\mu \in Y(D;Z)$ if and only if it can be written as

$$\mu = \sum_{\alpha=1}^q b_\alpha \boldsymbol{\delta}_\alpha, \tag{3.4}$$

where b_α are functions in $L^\infty(D;[0,1])$ satisfying the condition

$$\sum_{\alpha=1}^q b_\alpha(x) = 1, \quad \text{for a.e. } x \in D. \tag{3.5}$$

In disintegrated form, formula (3.4) can be written as

$$\mu^x = \sum_{\alpha=1}^{q} b_\alpha(x)\delta_\alpha \quad \text{for a.e. } x \in D.$$

Therefore $Y(D;Z)$ can be identified with the set of all families $b = (b_\alpha)_{\alpha=1}^q$ in $L^\infty(D;[0,1])$ satisfying condition (3.5).

Set $\mathscr{A}_q^n := \{1,\dots,q\}^n$. If $\boldsymbol{\mu} \in SY(A,D;Z)$, then for every $t_1 < \cdots < t_n$ in A there exists a finite family $\boldsymbol{b}^{t_1\dots t_n} = (\boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_n^q}$ in $L^\infty(D;[0,1])$, satisfying the property

$$\sum_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n}(x) = 1, \quad \text{for a.e. } x \in D, \tag{3.6}$$

and such that

$$\boldsymbol{\mu}_{t_1\dots t_n} = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n}\boldsymbol{\delta}_{(\alpha_1,\dots,\alpha_n)}, \tag{3.7}$$

for every finite sequence $t_1 < \cdots < t_n$ in A. The projection property of compatible systems can be reformulated in a simpler way using this language: given any finite sequence $t_1 < \cdots < t_n$ in A, we have

$$\boldsymbol{b}^{t_1\dots t_{i-1}t_{i+1}\dots t_n}_{\alpha_1\dots\alpha_{i-1}\alpha_{i+1}\dots\alpha_n} = \sum_{\beta=1}^{q} \boldsymbol{b}^{t_1\dots t_{i-1}t_i t_{i+1}\dots t_n}_{\alpha_1\dots\alpha_{i-1}\beta\alpha_{i+1}\dots\alpha_n}, \tag{3.8}$$

a.e. in D, for every $(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n) \in \mathscr{A}_q^{n-1}$ and every $i = 1,\dots,n$. Therefore we can identify the space $SY(A,D;Z)$ with the set $S(A,D,q)$ of all families $\boldsymbol{b} = (\boldsymbol{b}^{t_1\dots t_n})_{t_1<\cdots<t_n}$, with $t_1 < \cdots < t_n$ varying in A, such that $\boldsymbol{b}^{t_1\dots t_n} = (\boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n}$ satisfy properties (3.6) and (3.8).

If A is a finite set with n elements, we write $\Delta(D,n,q)$ to indicate the set of all families of coefficients $(b_{\alpha_1\dots\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n}$ in $L^\infty(D;[0,1])$ satisfying (3.6). $\Delta(D,n,q)$ can be seen as a generalized version of the Gibbs simplex associated with the pure phases $\hat{e}_1,\dots,\hat{e}_{q^n} \in \mathbb{R}^{q^n}$, where $\hat{e}_j$ is the jth unit vector.

Let $A = [0,T]$. Using the previous identification, we can rewrite the H-variation of a compatible system $\mu \in SY(A,D;Z)$ in the interval $[c,d] \subseteq [0,T]$ (see [13, (4.9)]) in terms of the family $\boldsymbol{b}$ corresponding to $\boldsymbol{\mu}$: $\mathrm{Var}_H(\boldsymbol{\mu};c,d) = \mathrm{Diss}_H(\boldsymbol{b};c,d)$, with

$$\mathrm{Diss}_H(\boldsymbol{b};c,d) := \sup \sum_{i=1}^{k}\sum_{\alpha\beta} H(\beta,\alpha)\int_D b^{t_{i-1}t_i}_{\alpha\beta}(x)\,dx, \tag{3.9}$$

where the supremum is taken over all finite partitions $c = t_0 < \cdots < t_k = d$ of the interval $[c,d]$ (with the convention $\mathrm{Diss}_H(\boldsymbol{b};c,d) = 0$, if $c = d$).

It is easy to see that given a sequence $(\mu^k)_k = (\sum_{\alpha=1}^q b_\alpha^k\boldsymbol{\delta}_\alpha)_k$ in $Y(D;Z)$, $\mu^k \rightharpoonup \mu = \sum_{\alpha=1}^q b_\alpha\boldsymbol{\delta}_\alpha$ weakly* in $Y(D;Z)$ if and only if $b_\alpha^k \rightharpoonup b_\alpha$ L^∞-weakly*, for every $\alpha = 1,\dots,q$. Therefore a compatible system $\boldsymbol{\mu} \in SY([0,T],D;Z)$ is left continuous if and only if the correspondent $\boldsymbol{b} \in S([0,T],D,q)$ satisfies the following property: for every finite sequence $t_1 < \cdots < t_n$ in $[0,T]$

$$\boldsymbol{b}^{s_1\dots s_n}_{\alpha_1\dots\alpha_n} \rightharpoonup \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n} \quad L^\infty\text{-weakly*}, \tag{3.10}$$

as $s_i \to t_i$, with $s_i \in [0,T]$ and $s_i \le t_i$, for every $(\alpha_1,\dots,\alpha_n) \in \mathscr{A}_q^n$. We denote the set of all $\boldsymbol{b} \in S([0,T],D,q)$ satisfying (3.10) by $S_-([0,T],D,q)$.

Definition 3.2. Fixed a sequence $0 = t_1 < \cdots < t_m = T$ in $[0,T]$, and given a family $(b_{\alpha_1\ldots\alpha_m})_{(\alpha_1,\ldots,\alpha_m)\in\mathscr{A}_q^m}$ in $L^\infty(D;[0,1])$ with $\sum_{(\alpha_1,\ldots,\alpha_m)} b_{\alpha_1\ldots\alpha_m} = 1$ a.e. in D, we define $\boldsymbol{b}^{pwc} \in S([0,T],D,q)$ as the family corresponding to the piecewise constant interpolation of the measure $\mu = \sum_{(\alpha_1\ldots\alpha_m)} b_{\alpha_1\ldots\alpha_m}\boldsymbol{\delta}_{(\alpha_1,\ldots,\alpha_m)}$, as defined in [13, Definition 4.12].

More in details, for every finite sequence $\tau_1 < \cdots < \tau_n$ in $[0,T]$ such that for every $j = 1,\ldots,m$ there exists $i = 1,\ldots,n$ with $t_j \le \tau_i < t_{j+1}$, we have

$$(\boldsymbol{b}^{pwc})^{\tau_1\ldots\tau_n}_{\beta_1\ldots\beta_n} := \begin{cases} 0 & \text{if } \beta_i \neq \beta_{i+1} \text{ with } t_j \le \tau_i < \tau_{i+1} < t_{j+1} \\ & \text{for some } i \text{ and } j \\ b_{\alpha_1\ldots\alpha_m} & \text{otherwise,} \end{cases}$$

for every $(\beta_1,\ldots,\beta_n) \in \mathscr{A}_q^n$.

We can reformulate Helly's Theorem for compatible systems of Young measures (see [13, Theorem 4.10]) in the discrete setting as follows.

Theorem 3.3. *Let $(\boldsymbol{b}^k)_k$ be a sequence in $S([0,T],D,q)$ such that $\mathrm{Diss}_H(\boldsymbol{b}^k;0,T) \le C$, for a finite constant $C > 0$ independent of k. Then there exist a subsequence, still denoted by $(\boldsymbol{b}^k)_k$, a set $\mathcal{T} \subseteq [0,T]$ containing 0 and such that $[0,T]\setminus\mathcal{T}$ is at most countable, and $\boldsymbol{b} \in S_-([0,T],D,q)$ with $\mathrm{Diss}_H(\boldsymbol{b};0,T) \le C$, such that*

$$(\boldsymbol{b}^k)^{t_1\ldots t_n}_{\alpha_1\ldots\alpha_n} \rightharpoonup \boldsymbol{b}^{t_1\ldots t_n}_{\alpha_1\ldots\alpha_n} \quad L^\infty\text{-weakly}^*, \tag{3.11}$$

as $k \to \infty$, for every finite sequence $t_1 < \cdots < t_n$ in $\mathcal{T}$, and every $(\alpha_1,\ldots,\alpha_n) \in \mathscr{A}_q^n$.

Now we state a lemma to describe the canonical form of the space $Y^p(D;Z\times\mathbb{R}^{N\times d})$ of all Young measures on D with values in $Z\times\mathbb{R}^{N\times d}$ and finite p-moments, for $1 < p < +\infty$.

Lemma 3.4. *A measure ν is an element of $Y^p(D;Z\times\mathbb{R}^{N\times d})$ if and only if it can be written as*

$$\nu = \sum_{\alpha=1}^{q} b_\alpha(\delta_\alpha \otimes \lambda_\alpha), \tag{3.12}$$

where $(b_\alpha)_{\alpha=1}^q$ is a family in $L^\infty(D;[0,1])$ satisfying (3.5) and, for every $\alpha = 1,\ldots,q$, λ_α is a Young measure on D with values in $\mathbb{R}^{N\times d}$ such that

$$\int_{D\times\mathbb{R}^{N\times d}} b_\alpha(x)|F|^p \, d\lambda_\alpha(x,F) < \infty, \tag{3.13}$$

for every $\alpha = 1,\ldots,q$.

Proof. For every $\alpha = 1,\ldots,q$, let us consider $b_\alpha \in L^\infty(D;[0,1])$ and a Young measure $\lambda_\alpha \in Y(D;\mathbb{R}^{N\times d})$, satisfying (3.5) and (3.13). It is immediate to see that the measure defined by (3.12) is an element of $Y^p(D;Z\times\mathbb{R}^{N\times d})$.

On the other hand, if ν belongs to $\in Y^p(D;Z\times\mathbb{R}^{N\times d})$ and $(\nu^x)_{x\in D}$ is its disintegration, for a.e. $x \in D$ and every $\alpha = 1,\ldots,q$ we define

$$b_\alpha(x) := \nu^x(\{\alpha\}\times\mathbb{R}^{N\times d}); \tag{3.14}$$

let us fix a probability measure ω on $\mathbb{R}^{N\times d}$; for $\alpha = 1,\ldots,q$ and for a.e. $x \in D$ let us define a probability measure λ_α^x on $\mathbb{R}^{N\times d}$ by setting for every $B_\alpha \in \mathcal{B}(\mathbb{R}^{N\times d})$

$$\lambda_\alpha^x(B_\alpha) := \begin{cases} \frac{\nu^x(\{\alpha\}\times B_\alpha)}{b_\alpha(x)} & \text{if } b_\alpha(x) \neq 0 \\ \omega(B_\alpha) & \text{if } b_\alpha(x) = 0 \end{cases} \tag{3.15}$$

By construction b_α is measurable with nonnegative values for every α, $\sum_{\alpha=1}^q b_\alpha(x) = \nu^x(Z \times \mathbb{R}^{N\times d}) = 1$ for a.e. $x \in D$, and $(\lambda_\alpha^x)_x$ is a measurable family of probability measures satisfying (3.13), for every α . It is now immediate to see that the measure $\tilde{\nu}$ whose disintegration is given by

$$\tilde{\nu}^x = \sum_{\alpha=1}^q b_\alpha(x)(\delta_\alpha \otimes \lambda_\alpha^x) \quad \text{for a.e. } x \in D, \tag{3.16}$$

is exactly ν. Indeed every Borel subset B of $Z \times \mathbb{R}^{N\times d}$ can be written as the union of disjoint sets of the form $\{\alpha\} \times B_\alpha$, for suitable $B_\alpha \in \mathcal{B}(\mathbb{R}^{N\times d})$, for $\alpha = 1, \dots, q$; hence we have

$$\begin{aligned} \tilde{\nu}^x(B) &= \sum_{\alpha=1}^q b_\alpha(x)\lambda_\alpha^x(B_\alpha) \\ &= \sum_{\alpha=1}^q b_\alpha(x)\frac{\nu^x(\{\alpha\}\times B_\alpha)}{b_\alpha(x)} = \nu^x(B). \end{aligned}$$

□

Remark 3.5. The functions b_α and the measures $b_\alpha\lambda_\alpha$, $\alpha = 1, \dots, q$, satisfying the properties described in the previous lemma are uniquely determined by ν. In particular if we consider the disintegration of λ_α, $(\lambda_\alpha^x)_{x\in D}$, we obtain that λ_α^x is uniquely determined for a.e. x in $\{x \in D \,:\, b_\alpha(x) > 0\}$.

Remark 3.6. Let $\nu^k = \sum_\alpha b_\alpha^k(\delta_\alpha \otimes \lambda_\alpha^k)$, $\nu = \sum_\alpha b_\alpha(\delta_\alpha \otimes \lambda_\alpha)$ belong to $Y^p(D; Z \times \mathbb{R}^{N\times d})$. A simple computation shows that a sequence $(\nu^k)_k$ in $Y^p(D; Z \times \mathbb{R}^{N\times d})$ p-weakly* converges to $\nu \in Y^p(D; Z \times \mathbb{R}^{N\times d})$ if and only if

$$b_\alpha^k \lambda_\alpha^k \rightharpoonup b_\alpha\lambda_\alpha \quad p\text{-weakly*} \tag{3.17}$$

for every $\alpha = 1, \dots, q$.

Remark 3.7. For every $\alpha = 1, \dots, q$, let $(b_\alpha^h, \lambda_\alpha^h)_h$ be a sequence in $L^\infty(D; [0,1]) \times Y(D; \mathbb{R}^{N\times d})$, satisfying (3.5) for every h, and

$$\sup_h \int_{D\times\mathbb{R}^{N\times d}} b_\alpha^h(x)|F|^p \, d\lambda_\alpha^h(x,F) \le C,$$

for a positive constant C, for every α. Then there exists $(b_\alpha, \lambda_\alpha) \in L^\infty(D; [0,1]) \times Y(D; \mathbb{R}^{N\times d})$, for every α, satisfying (3.5) and (3.13), and such that, up to a subsequence,

$$\begin{aligned} b_\alpha^h &\rightharpoonup b_\alpha \quad L^\infty\text{-weakly*} \\ b_\alpha^h\lambda_\alpha^h &\rightharpoonup b_\alpha\lambda_\alpha \quad p\text{-weakly*}, \end{aligned}$$

as $h \to \infty$.

4. **The mechanical model.** The *reference configuration* is the set D introduced in the previous section.

We indicate the *deformation* with v and the *internal variable* with z.

We denote *the stored energy density* by $W\colon Z \times \mathbb{R}^{N\times d} \to [0, +\infty)$ and the *dissipation rate density* by $H\colon Z^2 \to [0, +\infty)$. For every $\alpha \in Z$ and $F \in \mathbb{R}^{N\times d}$, we make the following assumptions:

(W.1) there exist two positive constants c, C such that

$$c|F|^2 - C \leq W(\alpha, F) \leq C(1 + |F|^2);$$

(W.2) $W(\alpha, \cdot)$ is of class $\mathcal{C}^1$ and

$$\Big|\frac{\partial W}{\partial F}(\alpha, F)\Big| \leq C(1 + |F|);$$

(H.1) H is a metric on Z^2.

Let $\mathcal{W}$ be the functional

$$\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x))\, dx,$$

for every $z \in L^\infty(D; Z)$ and every $v \in H^1(D; \mathbb{R}^N)$, and $\mathcal{H}$ the functional

$$\mathcal{H}(z, \tilde{z}) := \int_D H(z(x), \tilde{z}(x))\, dx,$$

for every $z, \tilde{z} \in L^\infty(D; Z)$.

Given two distinct times $s < t$, the *global dissipation* of a function $\boldsymbol{z}\colon [0, T] \to L^\infty(D; Z)$ in the interval $[s, t]$ will be

$$\mathrm{Var}_H(\boldsymbol{z}; s, t) := \sup \sum_{i=1}^{k} \mathcal{H}(\boldsymbol{z}(\tau_i), \boldsymbol{z}(\tau_{i-1})),$$

where the supremum will be taken among all finite partitions $s = \tau_0 < \tau_1 < \cdots < \tau_k = t$.

The prescribed boundary datum on ∂D at time t is denoted by $\boldsymbol{\varphi}(t)$; we assume $\boldsymbol{\varphi} \in AC([0, T]; W^{1,p}(D; \mathbb{R}^N))$, with $2 < p < +\infty$.

The kinematically admissible values at time t for v are those which make the total energy finite and satisfy the boundary condition, i.e., $v = \boldsymbol{\varphi}(t)$ on ∂D $\mathcal{H}^{d-1}$-a.e. (in the sense of traces).

5. **Admissible set in terms of Young measures.**

Definition 5.1. Given $A \subset \mathbb{R}$ and $\boldsymbol{w}\colon A \to H^1(D; \mathbb{R}^N)$, we define the *admissible set* for the time set A and the boundary datum $\boldsymbol{w}$, $Ad(A, q, \boldsymbol{w})$, as the set of all pairs $(\boldsymbol{b}, \boldsymbol{\lambda}) \in S(A, D, q) \times (Y(D; \mathbb{R}^{N\times d})^q)^A$ such that property (3.13) (for $p = 2$) is satisfied by $\boldsymbol{b}^t_\alpha \boldsymbol{\lambda}^t_\alpha$, for every α and t, and the following condition holds: for every finite sequence $t_1 < \cdots < t_n$ in A, for every $i = 1, \ldots, n$, and every $k \in \mathbb{N}$, there exist a measurable partition $(D^{i,k}_\alpha)^q_{\alpha=1}$ of D and a function $v^k_i \in \boldsymbol{w}(t_i) + H^1_0(D; \mathbb{R}^N)$ such that:

(1) for every $(\alpha_1, \ldots, \alpha_n) \in \mathscr{A}^q_n$

$$1_{D^{1,k}_{\alpha_1}} \cdots 1_{D^{n,k}_{\alpha_n}} \rightharpoonup \boldsymbol{b}^{t_1\ldots t_n}_{\alpha_1\ldots\alpha_n} \quad L^\infty\text{-weakly*, as } k \to \infty;$$

(2) for every $i = 1, \ldots, n$ there exists a subsequence $(k^i_j)_j$, possibly depending on i, such that

$$1_{D^{i,k^i_j}_\alpha} \boldsymbol{\delta}_{\nabla v^i_{k^i_j}} \rightharpoonup \boldsymbol{b}^{t_i}_\alpha \boldsymbol{\lambda}^{t_i}_\alpha \quad 2\text{-weakly*, as } j \to \infty$$

for every $\alpha = 1, \ldots, q$.

The following remark compares the notion of $Ad(A, q, \boldsymbol{w})$ with the notion of admissible set in terms of Young measures $AY(A, Z, \boldsymbol{w})$, as defined in [13, Section 6.2].

Remark 5.2. Given $A \subset \mathbb{R}$ and $\boldsymbol{w}\colon A \to H^1(D;\mathbb{R}^N)$, let us consider $(\boldsymbol{b}, \boldsymbol{\lambda}) \in S(A, D, q) \times (Y(D;\mathbb{R}^{N\times d})^q)^A$, with $(\boldsymbol{b}^t, \boldsymbol{\lambda}^t)$ satisfying (3.13) (for $p = 2$), and $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; Z \times \mathbb{R}^{N\times d})^A \times SY(A, D; Z)$, satisfying

$$\begin{aligned} \boldsymbol{\nu}_{t_i} &= \sum_{\alpha=1}^{q} \boldsymbol{b}_\alpha^{t_i}(\delta_\alpha \otimes \boldsymbol{\lambda}_\alpha^{t_i}), \quad \text{for every } t \in A \\ \boldsymbol{\mu}_{t_1\dots t_n} &= \sum_{(\alpha_1,\dots,\alpha_n)} \boldsymbol{b}_{\alpha_1\dots\alpha_n}^{t_1\dots t_n} \boldsymbol{\delta}_{(\alpha_1,\dots,\alpha_n)} \quad \text{for every } t_1 < \dots < t_n \text{ in } A. \end{aligned}$$

Then $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(A, q, \boldsymbol{w})$ if and only if $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, Z, \boldsymbol{w})$, i.e. for every finite sequence $t_1 < \dots < t_n$ in A there exist sequences $(z_i^k)_k \in L^\infty(D; Z)$, $(v_i^k)_k \subset \boldsymbol{w}(t_i) + H_0^1(D;\mathbb{R}^N)$, for $i = 1, \dots, n$ such that

$(\mathrm{app1})_Z$ we have

$$\boldsymbol{\delta}_{(z_1^k,\dots,z_n^k)} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_n} \quad \text{weakly*}, \tag{5.1}$$

as $k \to \infty$;

$(\mathrm{app2})_Z$ for every $i = 1, \dots, n$, there exists a sequence of integers $(k_j^i)_j$, possibly depending on i, such that

$$\boldsymbol{\delta}_{(z_i^{k_j^i}, \nabla v_i^{k_j^i})} \rightharpoonup \boldsymbol{\nu}_{t_i} \quad 2\text{-weakly*}, \tag{5.2}$$

as $j \to \infty$.

Indeed, given $(D_\alpha^{i,k})_\alpha$satisfying the approximation property for $\boldsymbol{b}_\alpha^{t_i}$ we define z_i^k by $z_i^k(x) = \alpha$ whenever $x \in D_\alpha^{i,k}$, or equivalently, given z_i^k satisfying the approximation property for ν_{t_i}, we consider $D_\alpha^{i,k} := \{x \in D \;:\; z_i^k(x) = \alpha\}$, for $\alpha = 1, \dots, q$.

The closure properties of $Ad(A, q, \boldsymbol{w})$ are described by the following lemma, which is the formulation in our discrete setting of [13, Lemma 6.7].

Lemma 5.3. *Let $(\boldsymbol{w}^j)_j$ be a sequence of functions from A into $H^1(D, \mathbb{R}^m)$, such that $\boldsymbol{w}^j(t) \to \boldsymbol{w}(t)$ strongly in H^1, for every $t \in A$ and let $(\boldsymbol{b}, \boldsymbol{\lambda}) \in S(A, D, q) \times (Y(D;\mathbb{R}^{N\times d})^q)^A$ with $(\boldsymbol{b}^t, \boldsymbol{\lambda}^t)$ satisfying (3.13) for $p = 2$, for every $t \in A$. Assume that for every finite sequence $t_1 < \dots < t_n$ in A there exists a sequence $(\boldsymbol{b}^j, \boldsymbol{\lambda}^j) \in Ad(\{t_1, \dots, t_n\}, q, \boldsymbol{w}^j)$ such that*

$$(\boldsymbol{b}^j)_{\alpha_1\dots\alpha_n}^{t_1\dots t_n} \rightharpoonup \boldsymbol{b}_{\alpha_1\dots\alpha_n}^{t_1\dots t_n} \quad L^\infty\text{-weakly*}, \tag{5.3}$$

as $j \to \infty$ for every $(\alpha_1, \dots, \alpha_n) \in \mathscr{A}_n^q$, and such that for every i there exists a sequence of integers $(j_h^i)_h$, possibly depending on i, satisfying

$$((\boldsymbol{b}^{j_h^i})_\alpha^{t_i}(\boldsymbol{\lambda}^{j_h^i})_\alpha^{t_i}) \rightharpoonup \boldsymbol{b}_\alpha^{t_i}\boldsymbol{\lambda}_\alpha^{t_i}, \quad 2\text{-weakly*}, \tag{5.4}$$

as $h \to \infty$ for every $\alpha = 1, \dots, q$. Then $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(A, q, \boldsymbol{w})$.

The following lemma will be used in order to provide a class of competitors for the discretized minimum problem in Section 7.1.

Lemma 5.4. *Let $0 \le t_1 < \dots < t_m \le T$ be a finite sequence in A. For every $j = 1, \dots, m$, let us consider $v_j \in \boldsymbol{w}(t_j) + H_0^1(D;\mathbb{R}^N)$ and a measurable partition $(D_\alpha^j)_{\alpha=1}^q$ of D. Let $M\colon D \to \mathbf{M}_{St}^{q\times q}$, $x \mapsto (M_{\beta\alpha}(x))_{\beta\alpha}$ be a measurable map, and $\tilde{u}$ an element of $H_0^1(D;\mathbb{R}^N)$.*

Let $(\boldsymbol{\nu},\boldsymbol{\mu}) \in Y^2(D; Z\times\mathbb{R}^{N\times d})^{\{t_1,\dots,t_m\}} \times SY^2(\{t_1,\dots,t_m\},D;Z)$ *be defined by*

$$\begin{aligned}
\boldsymbol{\nu}_{t_m} &:= \sum_{\alpha,\beta} M_{\beta\alpha} 1_{D^m_\alpha} \boldsymbol{\delta}_{(\beta,\nabla v_m+\nabla\tilde u)},\\
\boldsymbol{\nu}_{t_j} &:= \sum_{\alpha=1}^{q} 1_{D^j_\alpha}\boldsymbol{\delta}_{(\alpha,\nabla v_j)} \quad \textit{for every } j<m,\\
\boldsymbol{\mu}_{t_1\dots t_m} &:= \sum_{\alpha_1,\dots,\alpha_{m-1},\alpha,\beta} M_{\beta\alpha} 1_{D^1_{\alpha_1}}\cdot\dots\cdot 1_{D^{m-1}_{\alpha_{m-1}}}\cdot 1_{D^m_\alpha}\boldsymbol{\delta}_{(\alpha_1,\dots,\alpha_{m-1},\beta)}.
\end{aligned}$$

Then $(\boldsymbol{\nu},\boldsymbol{\mu}) \in AY(\{t_1,\dots,t_m\},Z,\boldsymbol{w})$.

Proof. Let us consider first the particular case of $M\colon D\to \mathbf{M}^{q\times q}_{St}$ constant.

Fix $\alpha\in\{1,\dots,q\}$ and define a measure $\boldsymbol{\nu}^\alpha_{t_m}$ on $D\times Z\times\mathbb{R}^{N\times d}$ by

$$\boldsymbol{\nu}^\alpha_{t_m} := \sum_\beta M_{\beta\alpha}1_{D^m_\alpha}\boldsymbol{\delta}_{(\beta,\nabla\tilde v_m)},$$

where $\tilde v_m := v_m+\tilde u$. Consider a measurable partition $(S^\alpha_\beta)_\beta$ of the unitary cube $[0,1]^d$, with $|S^\alpha_\beta| = M_{\beta\alpha}$ for every β (it is possible to find such a partition since $0\le M_{\beta\alpha}\le 1$ and $\sum_\beta M_{\beta\alpha}=1$, by the hypotheses on M). Let us now define a measurable function $\tilde z^\alpha\colon [0,1]^d\to Z$ by setting

$$\tilde z^\alpha(x)=\beta \quad \text{for a.e. } x\in S^\alpha_\beta,$$

for every $\beta=1,\dots,q$, and extend it by periodicity to all $\mathbb{R}^d$. For every $\delta>0$, the function $\tilde z^\alpha_\delta\colon\mathbb{R}^d\to Z$ defined by $\tilde z^\alpha_\delta(x):=\tilde z^\alpha(\frac{x}{\delta})$, for a.e. $x\in\mathbb{R}^d$, is δ-periodic. By the Riemann Lebesgue Theorem, we have

$$1_{\{x\in\mathbb{R}^d\,:\,\tilde z^\alpha_\delta(x)=\beta\}} \rightharpoonup M_{\beta\alpha} \quad L^\infty\text{-weakly*},$$

as $\delta\to 0$. Let now $\psi\in\mathcal{C}_0(D\times Z\times\mathbb{R}^{N\times d}) = \mathcal{C}_0(D\times\mathbb{R}^{N\times d})^Z$; we have

$$1_{D^m_\alpha}(x)\psi(x,\tilde z^\alpha_\delta(x),\nabla\tilde v_m(x)) = 1_{D^m_\alpha}(x)\sum_{\beta=1}^q \psi(x,\beta,\nabla\tilde v_m(x))1_{\{x\in\mathbb{R}^d\,:\tilde z^\alpha_\delta(x)=\beta\}}(x),$$

for a.e. $x\in D$, and the function $x\mapsto 1_{D^m_\alpha}(x)\psi(x,\beta,\nabla\tilde v_m(x))$ is in $L^1(D)$ for every β. Hence we can deduce that

$$\begin{aligned}
&\int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\, d(1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde z^\alpha_\delta,\nabla\tilde v_m)})(x,\gamma,F)\\
&= \int_D 1_{D^m_\alpha}(x)\psi(x,\tilde z^\alpha_\delta(x),\nabla\tilde v_m(x))\,dx\\
&= \sum_\beta\int_D 1_{D^m_\alpha}(x)\psi(x,\beta,\nabla\tilde v_m(x))1_{\{x\in D\,:\tilde z^\alpha_\delta(x)=\beta\}}(x)\,dx\\
&\xrightarrow{\delta\to0}\sum_\beta\int_D M_{\beta\alpha}1_{D^m_\alpha}(x)\psi(x,\beta,\nabla\tilde v_m(x))\,dx\\
&= \int_{D\times Z\times\mathbb{R}^{N\times d}}\psi(x,\gamma,F)\,d(\boldsymbol{\nu}^\alpha_{t_m})(x,\gamma,F).
\end{aligned}$$

Defined $\tilde z_\delta\colon D\to Z$ by $\tilde z_\delta(x):=\tilde z^\alpha_\delta(x)$ if $x\in D^m_\alpha$, we can conclude that $\boldsymbol{\delta}_{(\tilde z_\delta,\tilde v_m)} \rightharpoonup \boldsymbol{\nu}_{t_m}$ 2-weakly*, as $\delta\to0$.

We observe that

$$\begin{aligned}\boldsymbol{\mu}_{t_1\dots t_m} &:= \sum_{\alpha_1,\dots,\alpha_{m-1},\alpha,\beta} M_{\beta\alpha}1_{D^1_{\alpha_1}}\cdots 1_{D^{m-1}_{\alpha_{m-1}}}\cdot 1_{D^m_\alpha}\boldsymbol{\delta}_{(\alpha_1,\dots,\alpha_{m-1},\beta)}\\ &= \sum_{\alpha\beta} M_{\beta\alpha}1_{D^m_\alpha}\boldsymbol{\delta}_{(z^1,\dots,z^{m-1},\beta)},\end{aligned}$$

where $z^j(x) := \alpha$ whenever $x \in D^j_\alpha$, for $j = 1,\dots,m-1$. Therefore, we can apply the same argument used for $\boldsymbol{\nu}_{t_m}$ to $\boldsymbol{\mu}_{t_1\dots t_m}$ and deduce that $\delta_{(z_1,\dots,z_{m-1},\tilde{z}_\delta)} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_m}$ weakly* as $\delta\to 0$. Hence it is enough to consider a sequence $\delta^k \to 0$, and, for every k, $z^k_j = z_j$ for $j < m$, $z^k_m = \tilde{z}_{\delta^k}$, $v^k_j = v_j$ for $j < m$, and $v^k_m = \tilde{v}_m$ to obtain the required approximation properties considered in Remark 5.2.

Consider now the case of $M_{\beta,\alpha}$ in $\mathcal{C}^1(D)$. Fixed a positive parameter ε, consider a finite family $(Q^i_\varepsilon)_{i=1}^{I(\varepsilon)}$ of disjoint cubes in $\mathbb{R}^d$, with diameter ε, covering D, and set

$$(M_{\beta\alpha})^i_\varepsilon := (M_{\beta\alpha})_{Q^i_\varepsilon\cap D} = \frac{1}{|Q^i_\varepsilon\cap D|}\int_{Q^i_\varepsilon\cap D} M_{\beta\alpha}(x)\;dx,$$

for every $i = 1,\dots,I(\varepsilon)$, and every α, β. For a fixed α, we can define a measure $\boldsymbol{\nu}^\alpha_\varepsilon$ on $\mathbb{R}^d\times Z\times\mathbb{R}^{N\times d}$ by setting

$$\boldsymbol{\nu}^\alpha_\varepsilon := \sum_{i=1}^{I(\varepsilon)}\sum_{\beta=1}^{q}(M_{\beta\alpha})^i_\varepsilon 1_{Q^i_\varepsilon}1_{D^m_\alpha}\boldsymbol{\delta}_{(\beta,\nabla\tilde{v}_m)}.$$

Let us fix $i = 1,\dots,I(\varepsilon)$ and reproduce the arguments used in the constant case: consider a measurable partition $((S^i_\varepsilon)^\alpha_\beta)_\beta$ of the unitary cube $[0,1]^d$, with $|(S^i_\varepsilon)^\alpha_\beta| = (M_{\beta\alpha})^i_\varepsilon$, for every β (it is possible to find such a partition since $\sum_\beta (M_{\beta\alpha})^i_\varepsilon = \frac{1}{|Q^i_\varepsilon\cap D|}\int_{Q^i_\varepsilon\cap D}(\sum_\beta M_{\beta\alpha})(x)\;dx = 1$, by the hypotheses on M), and define the map $(\tilde{z}^\alpha)^i_\varepsilon\colon\mathbb{R}^d\to Z$ as the 1-periodic measurable function satisfying

$$(\tilde{z}^\alpha)^i_\varepsilon(x) = \beta \quad \text{for a.e. } x\in(S^i_\varepsilon)^\alpha_\beta,$$

for every $\beta = 1,\dots,q$. For every $\delta > 0$, consider the function $(\tilde{z}^\alpha)^i_{\varepsilon,\delta}\colon\mathbb{R}^d\to Z$ defined by $(\tilde{z}^\alpha)^i_{\varepsilon,\delta}(x) := (\tilde{z}^\alpha)^i_\varepsilon(\frac{x}{\delta})$, for a.e. $x\in\mathbb{R}^d$. Fixed ε, we obtain as before that

$$1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde{z}^\alpha_{\varepsilon,\delta},\tilde{v}_m)} \rightharpoonup \boldsymbol{\nu}^\alpha_\varepsilon \quad \text{2-weakly*}, \tag{5.5}$$

as $\delta\to 0$, where $\tilde{z}^\alpha_{\varepsilon,\delta}\colon\mathbb{R}^d\to Z$ is the function defined by $\tilde{z}^\alpha_{\varepsilon,\delta} := \sum_{i=1}^{I(\varepsilon)}1_{Q^i_\varepsilon}(\tilde{z}^\alpha)^i_{\varepsilon,\delta}$.

Now we want to show that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly* as $\varepsilon\to 0$.

For every $\psi\in\mathcal{C}_0(D\times Z\times\mathbb{R}^{N\times d})$, we have

$$\begin{aligned}&\Big|\int_{D\times Z\times\mathbb{R}^{N\times d}}\psi(x,\gamma,F)\;d\boldsymbol{\nu}^\alpha_{t_m}(x,\gamma,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}}\psi(x,\gamma,F)\;d\boldsymbol{\nu}^\alpha_\varepsilon(x,\gamma,F)\Big|\\ =&\Big|\sum_\beta\int_{D^m_\alpha}\Big[(M_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)}1_{Q^i_\varepsilon}(x)(M_{\beta\alpha})^i_\varepsilon(x)\Big]\psi(x,\beta,\nabla\tilde{v}_m(x))\;dx\Big|.\end{aligned}$$

Since for every $x\in D^m_\alpha$ there exists a unique $i_x = 1,\dots,I(\varepsilon)$ with $x\in Q^{i_x}_\varepsilon$, we have

$$|M_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)}1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon| = |M^n_{\beta\alpha}(x) - (M_{\beta\alpha})^{i_x}_\varepsilon| \le \|\nabla M_{\beta\alpha}\|_\infty\varepsilon,$$

for every $x \in D^m_\alpha$ and every $\beta = 1, \dots, q$. Therefore we have

$$\Big| \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_{t_m}(x,\gamma,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_\varepsilon(x,\gamma,F)\Big|$$
$$\le \sum_\beta \|\nabla M_{\beta\alpha}\|_\infty \|\psi\|_\infty |D|\varepsilon,$$

which tends to 0 as $\varepsilon \to 0$.

Since $Y(D; Z\times\mathbb{R}^{N\times d})$ is contained in a bounded subset of the dual of a separable Banach space, it is metrizable with respect to the weak* topology. Let us denote by d a metric inducing on $Y(D; Z\times\mathbb{R}^{N\times d})$ the weak* topology, so that we have

- $d(\boldsymbol{\nu}^\alpha_\varepsilon, \boldsymbol{\nu}^\alpha_{t_m}) \to 0$ as $\varepsilon \to 0$;
- for every fixed ε, $d(1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde z^\alpha_{\varepsilon,\delta},\nabla\tilde v)}, \boldsymbol{\nu}^\alpha_\varepsilon) \to 0$ as $\delta \to 0$.

Applying as before the same argument to $\boldsymbol{\mu}_{t_1\dots t_m}$, we deduce, using a diagonalization argument, that there exist sequences $\delta_k \to 0$ and $\varepsilon_k \to 0$ such that

(1) for every α, $1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde z^\alpha_{\varepsilon_k,\delta_k},\nabla v^m+\nabla\tilde u)} \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly* as $k\to\infty$;
(2) for every α, we have

$$1_{D^m_\alpha}\boldsymbol{\delta}_{(z^1,\dots,z^m,\tilde z^\alpha_{\varepsilon_k,\delta_k})} \rightharpoonup \sum_{\alpha_1,\dots,\alpha_{m-1},\beta} M_{\beta\alpha}1_{D^1_{\alpha_1}}\cdots 1_{D^{m-1}_{\alpha_{m-1}}}\cdot 1_{D^m_\alpha}\boldsymbol{\delta}_{(\alpha_1,\dots,\alpha_{m-1},\beta)}$$

weakly*, as $k\to\infty$.

Now it is enough to define $\tilde z_{\varepsilon,\delta}\colon D\to Z$, by $\tilde z_{\varepsilon,\delta} := \sum_\alpha 1_{D^m_\alpha}\tilde z^\alpha_{\varepsilon,\delta}$, to prove the thesis.

It remains only to treat the general case of $M_{\beta\alpha}\in L^\infty(D)$. We can reproduce the same construction proposed in the $\mathcal{C}^1$-case; the only difference is that we have to use an approximation argument to show that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly*. Indeed it is enough to consider, for every β, a sequence $(M^n_{\beta\alpha})_n$ in $\mathcal{C}^1(D)$, with $M^n_{\alpha\beta}\to M_{\beta\alpha}$ strongly in $L^1(D)$, as $n\to\infty$, and let $(M^n_{\beta\alpha})^i_\varepsilon := (M^n_{\beta\alpha})_{Q^i_\varepsilon\cap D}$. For every $\psi\in\mathcal{C}_0(D\times Z\times\mathbb{R}^{N\times d})$, we have

$$\Big| \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_{t_m}(x,\gamma,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_\varepsilon(x,\gamma,F)\Big|$$
$$=\Big| \int_{D^m_\alpha}\sum_\beta M_{\beta\alpha}(x)\psi(x,\beta,\nabla\tilde v_m(x))\ dx$$
$$-\int_{D^m_\alpha}\sum_{i=1}^{I(e)} 1_{Q^i_\varepsilon}(x)\sum_\beta (M_{\beta\alpha})^i_\varepsilon\psi(x,\beta,\nabla\tilde v_m(x))\ dx\Big|$$
$$\le\Big|\sum_\beta\int_{D^m_\alpha}\Big[(M_{\beta\alpha}-M^n_{\beta\alpha})(x)-\sum_{i=1}^{I(e)}1_{Q^i_\varepsilon}(x)(M_{\beta\alpha}-M^n_{\beta\alpha})^i_\varepsilon\Big]\psi(x,\beta,\nabla\tilde v_m(x))\ dx\Big|$$
$$+\Big|\sum_\beta\int_{D^m_\alpha}\Big[(M^n_{\beta\alpha}(x)-\sum_{i=1}^{I(\varepsilon)}1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon(x)\Big]\psi(x,\beta,\nabla\tilde v_m(x))\ dx\Big|.$$

We know that

$$|M^n_{\beta\alpha}(x)-\sum_{i=1}^{I(\varepsilon)}1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon| = |M^n_{\beta\alpha}(x)-(M^n_{\beta\alpha})^{i_x}_\varepsilon|\le\|\nabla M^n_{\beta\alpha}\|_\infty\varepsilon,$$

for every $x \in D^m_\alpha$ and every $\beta = 1, \dots, q$. On the other hand, using Lemma 2.1, we can deduce that

$$\begin{aligned}&\int_{D^m_\alpha} |(M_{\beta\alpha} - M^n_{\beta\alpha})(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M_{\beta\alpha} - M^n_{\beta\alpha})^i_\varepsilon)|\ dx \\ &\leq \int_{D^m_\alpha} |M_{\beta\alpha} - M^n_{\beta\alpha}(x)|\ dx.\end{aligned}$$

Let us now fix $\eta > 0$; choosing $\bar n$ such that $\sum_\beta \|M_{\beta\alpha} - M^{\bar n}_{\beta\alpha}\|_1 \|\psi\|_\infty \leq \eta/2$, we have

$$\Big| \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_{t_m}(x,\gamma,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\gamma,F)\ d\boldsymbol{\nu}^\alpha_\varepsilon(x,\gamma,F)\Big| \leq \eta,$$

for every $\varepsilon \leq \varepsilon_\eta := \eta(2\sum_\beta \|\nabla M^{\bar n}_{\beta\alpha}\|_\infty \|\psi\|_\infty |D|)^{-1}$; therefore we obtain that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly* as $\varepsilon \to 0$ and we can prove the thesis as in the previous case. $\square$

Remark 5.5. If $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(A, q, \boldsymbol{w})$, for every $t \in A$ there exists a unique function $\boldsymbol{v}(t) \in \boldsymbol{w}(t) + H^1_0(D;\mathbb{R}^N)$ such that $\nabla \boldsymbol{v}(t) = \sum_\alpha \boldsymbol{b}^t_\alpha \mathrm{bar}(\boldsymbol{\lambda}^t_\alpha)$; moreover, for every $t \in A$, the function $\boldsymbol{\sigma}(t)$ representing the *stress* and defined by

$$\boldsymbol{\sigma}(t,x) := \sum_{\alpha=1}^q \boldsymbol{b}^t_\alpha(x) \int_{\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F) \quad \text{for a.e. } x \in D$$

belongs to $L^2(D;\mathbb{R}^{N\times d})$.

6. **Definition of quasistatic evolution and main result.** First of all we fix some notation and give the definition of quasistatic evolution in the discrete setting.

We will use the following compact notation: given an admissible pair $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(A, q, \boldsymbol{\varphi})$, we will write

$$\begin{aligned}\langle W, (\boldsymbol{b}^t, \boldsymbol{\lambda}^t)\rangle &:= \sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x) W(\alpha, F)\ d\boldsymbol{\lambda}^t_\alpha(x, F), \\ \mathbb{H}(\boldsymbol{b}^t, \boldsymbol{b}^s) = \langle H, \boldsymbol{b}^{st}\rangle &:= \sum_{\alpha,\beta} H(\beta,\alpha) \int_D \boldsymbol{b}^{st}_{\alpha\beta}(x)\ dx,\end{aligned}$$

for every $s < t$ in A. To describe the set of competitors for the stability condition satisfied by the quasistatic evolution, we need to introduce some other notation. Given a measurable map $M\colon D \to \mathbf{M}^{q\times q}_{St}$, we consider the following operators:

$$\begin{aligned}\mathcal{M}_M\colon \Delta(D,1,q) &\longrightarrow \Delta(D,2,q) \\ (b_\alpha)_\alpha &\longmapsto (M_{\beta\alpha} b_\alpha)_{(\alpha,\beta)}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{M}^1_M, \mathcal{M}^2_M)\colon \Delta(D,1,q) \times Y(D;\mathbb{R}^{N\times d})^q &\longrightarrow \Delta(D,1,q) \times Y(D;\mathbb{R}^{N\times d})^q \\ (b,\lambda) &\longmapsto (\mathcal{M}^1_M(b), \mathcal{M}^2_M(b,\lambda)),\end{aligned}$$

where

$$\begin{aligned}(\mathcal{M}^1_M(b))_\beta &:= \sum_\alpha (\mathcal{M}_M(b))_{\alpha\beta} = \sum_\alpha M_{\beta\alpha} b_\alpha \quad \text{a.e. in } D \\ (\mathcal{M}^2_M(b,\lambda))^x_\beta &:= \frac{\sum_\alpha M_{\beta\alpha}(x) b_\alpha(x) \lambda^x_\alpha}{\sum_\alpha M_{\beta\alpha}(x) b_\alpha(x)} \quad \text{if } \sum_\alpha M_{\beta\alpha}(x) b_\alpha(x) > 0,\end{aligned}$$

for a.e. $x \in D$ and every $\beta = 1, \dots, q$.

Given $b \in \Delta(D,1,q)$ and $M \in L^\infty(D;\mathbf{M}^{q\times q}_{St})$, we set

$$\mathbb{H}(\mathcal{M}^1_M(b),b) := \langle H, \mathcal{M}_M(b)\rangle = \sum_{\alpha,\beta} H(\beta,\alpha)\int_D M_{\beta\alpha}(x)b_\alpha(x)\,dx. \tag{6.1}$$

Remark 6.1. If we want to consider a modification of (b,λ) due to a permutation of phase distributions and Young measures, the corresponding M will be independent on x, with entries equals to 0 or 1, and with exactly one nonzero entry on each row. In this case $\mathcal{M}^1_M(b)$ is the product $M\cdot b$ between the matrix M and the vector $(b_\alpha)_\alpha$, while $(\mathcal{M}^2_M(b,\lambda))_\beta = (M\cdot\lambda)_\beta$ for a.e. $x\in D$ such that $(M\cdot b)_\beta(x) > 0$.

Definition 6.2. Given $\boldsymbol{\varphi} \in AC([0,T];W^{1,p}(D;\mathbb{R}^N))$, for $2<p<+\infty$, $T>0$, $z_0\in L^\infty(D;Z)$, and $v_0\in\mathcal{A}(0)$, a *quasistatic evolution of Young measures* with boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0,v_0), in the time interval $[0,T]$, is a pair $(\boldsymbol{b},\boldsymbol{\lambda})\in Ad([0,T],q,\boldsymbol{\varphi})$, with $\boldsymbol{b}\in S_-([0,T],D,q)$, satisfying the following conditions:

(ev0) *initial condition*: with $D^0_\alpha := \{x\in D \ : \ z_0(x)=\alpha\}$, we have $\boldsymbol{b}^0_\alpha = 1_{D^0_\alpha}$ and $(\boldsymbol{\lambda}^0_\alpha)^x = \boldsymbol{\delta}_{\nabla v_0(x)}$ if $x\in D^0_\alpha$, for every α;

(ev1) *partial-global stability*: for every $t\in[0,T]$, we have

$$\langle W, (\boldsymbol{b}^t,\boldsymbol{\lambda}^t)\rangle \le \langle W,(\tilde b,\tilde\lambda)\rangle + \mathbb{H}(\tilde b,\boldsymbol{b}^t),$$

for $(\tilde b,\tilde\lambda)$ varying in the set

$$\{(\mathcal{M}^1_M(\boldsymbol{b}^t), \tilde{\mathcal{T}}_{\nabla\tilde u}(\mathcal{M}^2_M(\boldsymbol{b}^t,\boldsymbol{\lambda}^t))) : \tilde u\in H^1_0(D;\mathbb{R}^N), M\in L^\infty(D;\mathbf{M}^{q\times q}_{St})\},$$

where $\mathbb{H}(\tilde b,\boldsymbol{b}^t) = \mathbb{H}(\mathcal{M}^1_M(\boldsymbol{b}^t),\boldsymbol{b}^t)$ is defined as in (6.1);

(ev2) *energy equality*: if $\boldsymbol{\sigma}$ is the function defined in Remark 5.5, then the map

$$t\mapsto \langle\boldsymbol{\sigma}(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2 \tag{6.2}$$

is integrable on $[0,T]$, and for every $t\in[0,T]$

$$\langle W,(\boldsymbol{b}^t,\boldsymbol{\lambda}^t)\rangle + \mathrm{Diss}_H(\boldsymbol{b};0,t) = \mathcal{W}(z_0,v_0) + \int_0^t \langle\boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\,ds,$$

where $\mathrm{Diss}_H(\boldsymbol{b};0,t)$ is defined by (3.9).

Theorem 6.3. *Let $\boldsymbol{\varphi}\in AC([0,T];H^1(D;\mathbb{R}^N))$ and $T>0$. Assume that the partial-global stability condition is satisfied by $(z_0,v_0)\in L^\infty(D;Z)\times(\boldsymbol{\varphi}(0)+H^1_0(D;\mathbb{R}^N))$. Then there exists a quasistatic evolution of Young measures with boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0,v_0) in the time interval $[0,T]$.*

7. Proof of the main theorem.

The proof is obtained via time-discretization, resolution of incremental minimum problems, and passing to the limit.

7.1. The incremental minimum problem.

The first step of the proof consists in the definition of an approximate solution via an inductive minimization process.

Let us fix a sequence of subdivisions of $[0,T]$, $0=t^0_n<t^1_n<\dots<t^{k(n)}_n=T$, such that $\sup_{i=1,\dots,k(n)}\tau^i_n\to 0$, as $n\to\infty$, where $\tau^i_n := t^i_n - t^{i-1}_n$, for every $i=1,\dots,k(n)$.

For every $i=0,1,\dots,k(n)$ we set $\varphi^i_n := \boldsymbol{\varphi}(t^i_n)$.

We define $(\boldsymbol{b}^i_n,\boldsymbol{\lambda}^i_n)\in Ad(\{t^0_n,\dots,t^i_n\},q,\boldsymbol{\varphi})$ by induction on i: set

$$(\boldsymbol{b}^0_n)_\alpha(\boldsymbol{\lambda}^0_n)_\alpha := 1_{D^0_\alpha}\boldsymbol{\delta}_{\nabla v_0},$$

where $D^0_\alpha := \{x\in D \ : \ z_0(x)=\alpha\}$; for $i>0$ we define $(\boldsymbol{b}^i_n,\boldsymbol{\lambda}^i_n)$ as a pair satisfying the following properties:

(min) $(\boldsymbol{b}^i_n, \boldsymbol{\lambda}^i_n)$ is a minimizer of the functional

$$\begin{aligned}&\langle W, (\boldsymbol{b}^{t^i_n}, \boldsymbol{\lambda}^{t^i_n})\rangle + \langle H, \boldsymbol{b}^{t^{i-1}_n t^i_n}\rangle = \\ =&\sum_\alpha \int_D \boldsymbol{b}^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\lambda}^{t^i_n}_\alpha)^x(F)\Big)\ dx + \\ &+\sum_{\alpha,\beta} H(\beta,\alpha)\int_D \boldsymbol{b}^{t^{i-1}_n t^i_n}_{\alpha\beta}(x)\ dx,\end{aligned} \tag{7.1}$$

over the set $A^i_n(\boldsymbol{b}^{i-1}_n, \boldsymbol{\lambda}^{i-1}_n)$ of all $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(\{t^0_n, \dots, t^i_n\}, q, \boldsymbol{\varphi})$ satisfying

$$\sum_\beta \boldsymbol{b}^{t^0_n\dots t^i_n}_{\alpha_0\dots\alpha_{i-1}\beta} = (\boldsymbol{b}^{i-1}_n)^{t^0_n\dots t^{i-1}_n}_{\alpha_0\dots\alpha_{i-1}} \text{ a.e. in } D, \text{ for every } (\alpha_0,\dots,\alpha_{i-1}) \in \mathscr{A}^q_i \tag{7.2}$$

$$\boldsymbol{\lambda}^{t^j_n}_\alpha = (\boldsymbol{\lambda}^{i-1}_n)^{t^j_n}_\alpha, \quad \text{for every } j < i \text{ and every } \alpha; \tag{7.3}$$

(reg) there exist two constants $r > 1$ and $\gamma > 0$, both independent of i and n, such that

$$\begin{aligned}&\sum_{\alpha=1}^q \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)|F|^{2r}\ d(\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha(x,F) \\ \le&\gamma\Big[1 + \Big(\sum_{\alpha=1}^q \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)|F|^2\ d(\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha(x,F)\Big)^r\Big].\end{aligned} \tag{7.4}$$

The existence of such a pair $(\boldsymbol{b}^i_n, \boldsymbol{\lambda}^i_n)$ is proven in Lemma 7.2 below.

Lemma 7.1. *For every $i > 1$ and every $(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1}) \in Ad(\{t^0_n, \dots, t^{i-1}_n\}, q, \boldsymbol{\varphi})$, the set $A^i_n(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$ is nonempty.*

Proof. Fixed $(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1}) \in Ad(\{t^0_n, \dots, t^{i-1}_n\}, q, \boldsymbol{\varphi})$, let $\boldsymbol{b}$ be the unique element of $S(\{t^0_n, \dots, t^i_n\}, D, q)$ satisfying

$$\boldsymbol{b}^{t^0_n\dots t^{i-1}_n t^i_n}_{\alpha_0\dots\alpha_{i-1}\alpha_i} := \begin{cases}(\boldsymbol{b}^{i-1})^{t^0_n\dots t^{i-1}_n}_{\alpha_0\dots\alpha_{i-1}} & \text{if } \alpha_i = \alpha_{i-1} \\ 0 & \text{otherwise;}\end{cases} \tag{7.5}$$

for every $(\alpha_0, \dots, \alpha_i) \in \mathscr{A}^q_{i+1}$; define $\boldsymbol{\lambda} \in ((Y(D;\mathbb{R}^{N\times d})^q)^{\{t^0_n,\dots,t^i_n\}}$ by

$$\begin{aligned}\boldsymbol{\lambda}^{t^j_n}_\alpha :=&(\boldsymbol{\lambda}^{i-1})^{t^j_n}_\alpha \quad \text{if } j < i, \text{ for every } \alpha, \\ \boldsymbol{\lambda}^{t^i_n}_\alpha :=&\mathcal{T}_{\nabla\varphi^i_n - \nabla\varphi^{i-1}_n}((\boldsymbol{\lambda}^{i-1})^{t^{i-1}_n}_\alpha),\end{aligned} \tag{7.6}$$

where the translated measure $\mathcal{T}_{\nabla\varphi^i_n - \nabla\varphi^{i-1}_n}((\boldsymbol{\lambda}^{i-1})^{t^{i-1}_n}_\alpha)$ is defined as in (3.2). It is immediate to see that $(\boldsymbol{b}, \boldsymbol{\lambda})$ satisfy the properties (7.2) and (7.3). By construction $\boldsymbol{b}^{t^j_n}_\alpha \boldsymbol{\lambda}^{t^j_n}_\alpha \in Y^2(D;\mathbb{R}^{N\times d})$, for every α and every $j = 0, \dots, i$: indeed for $j < i$ it is obvious, while for i we have

$$\begin{aligned}\boldsymbol{b}^{t^i_n}_\alpha &= \sum_{(\alpha_0\dots\alpha_{i-1})} \boldsymbol{b}^{t^0_n\dots t^{i-1}_n t^i_n}_{\alpha_0\dots\alpha_{i-1}\alpha} \\ &= \sum_{(\alpha_0\dots\alpha_{i-2})} (\boldsymbol{b}^{i-1})^{t^0_n\dots t^{i-1}_n}_{\alpha_0\dots\alpha_{i-2}\alpha} = (\boldsymbol{b}^{i-1})^{t^{i-1}_n}_\alpha,\end{aligned}$$

for every α, therefore

$$\begin{aligned}&\int_D \boldsymbol{b}_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}}|F|^2\ d(\boldsymbol{\lambda}_\alpha^{t_n^i})^x(F)\Big)\ dx\\ =&\int_D (\boldsymbol{b}^{i-1})_\alpha^{t_n^{i-1}}(x)\Big(\int_{\mathbb{R}^{N\times d}}|F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x)|^2\ d((\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^{i-1}})^x(F)\Big)\ dx\\ \le&\int_D (\boldsymbol{b}^{i-1})_\alpha^{t_n^{i-1}}(x)\Big(\int_{\mathbb{R}^{N\times d}}|F|^2\ d((\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^{i-1}})^x(F)\Big)\ dx+\|\nabla\varphi_n^i\|_2^2+\|\nabla\varphi_n^{i-1}\|_2^2<+\infty,\end{aligned}$$

for every α. It is now easy to prove the approximations properties (1) and (2) of Definition 5.1 for $(\boldsymbol{b},\boldsymbol{\lambda})$ defined by (7.5) and (7.6). Suppose that for every k and every $j=0,\dots,i-1$, $((D^{i-1})_\alpha^{j,k})_\alpha$ is a measurable partition of D and $(v^{i-1})^{j,k}$ is a function in $\varphi_n^j+H_0^1(D;\mathbb{R}^N)$, which satisfy conditions (1) and (2) for $(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$. Then $(D_\alpha^{j,k})_\alpha$ and $v^{j,k}$, defined by

$$\begin{aligned}D_\alpha^{j,k}&:=(D^{i-1})_\alpha^{j,k}\quad\text{for } j<i\\ D_\alpha^{i,k}&:=(D^{i-1})_\alpha^{i-1,k}\\ v^{j,k}&:=(v^{i-1})^{j,k}\quad\text{for } j<i\\ v^{i,k}&:=(v^{i-1})^{i-1,k}+\varphi_n^i-\varphi_n^{i-1},\end{aligned}$$

for every α and every k, satisfy (1) and (2) for $(\boldsymbol{b},\boldsymbol{\lambda})$. ☐

Lemma 7.2. *There exist constants $\gamma>0$ and $r>1$, such that for every n, every $i=1,\dots,k(n)$, and every $(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})\in Ad(\{t_n^0,\dots,t_n^{i-1},q,\boldsymbol{\varphi})$, the functional (7.1) has a minimizer over $A_n^i(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$, which satisfies (7.4).*

Proof. Let $(\boldsymbol{b}^h,\boldsymbol{\lambda}^h)_h$ be a minimizing sequence. By the bounds on W we have

$$\begin{aligned}&c\sum_\alpha\int_D(\boldsymbol{b}^h)_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}}|F|^2\ d((\boldsymbol{\lambda}^h)_\alpha^{t_n^i})^x(F)\Big)\ dx-C\\ \le&\langle W,((\boldsymbol{b}^h)^{t_n^i}(\boldsymbol{\lambda}^h)^{t_n^i}))\rangle\le C',\end{aligned}\tag{7.7}$$

for every h, for a positive constant C' independent of h. Moreover, we have

$$\sup_h\|(\boldsymbol{b}^h)_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i}\|_\infty\le 1,$$

for every $(\alpha_0,\dots,\alpha_i)\in\mathscr{A}_{i+1}^q$. Therefore, we can deduce that there exists $(b_{(\alpha_0\dots\alpha_i)})_{(\alpha_0\dots\alpha_i)}\in(L^\infty(D;[0,1]))^{q^{i+1}}$ satisfying (3.6) and

$$(\boldsymbol{b}^h)_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i}\rightharpoonup b_{\alpha_0\dots\alpha_i}\quad L^\infty\text{-weakly*},$$

as $h\to\infty$, up to a subsequence. In particular,

$$(\boldsymbol{b}^h)_\alpha^{t_n^i}\rightharpoonup\sum_{(\alpha_0\dots\alpha_{i-1})}b_{\alpha_0\dots\alpha_{i-1}\alpha}\quad L^\infty\text{-weakly*}.\tag{7.8}$$

From (7.7), we can deduce using Remark 3.7 and (7.8) that there exists $\lambda\in Y(D;\mathbb{R}^{N\times d})^q$ such that

$$\int_{D\times\mathbb{R}^{N\times d}}\sum_{(\alpha_0,\dots,\alpha_{i-1})}b_{\alpha_0\dots\alpha_{i-1}\alpha}(x)|F|^2\ d\lambda_\alpha(x,F)<\infty,$$

and

$$(\boldsymbol{b}^h)_\alpha^{t_n^i}(\boldsymbol{\lambda}^h)_\alpha^{t_n^i} \rightharpoonup \sum_{(\alpha_0\ldots\alpha_{i-1})} b_{\alpha_0\ldots\alpha_{i-1}\alpha}\lambda_\alpha \quad \text{2-weakly*},$$

as $h\to\infty$, up to a subsequence. We now define

$$\begin{aligned}
\boldsymbol{\lambda}_\alpha^{t_n^j} &:= (\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^j}, \quad \text{for every } j<i \text{ and every } \alpha,\\
\boldsymbol{\lambda}_\alpha^{t_n^i} &:= \lambda_\alpha \quad \text{for every } \alpha,
\end{aligned}$$

and $\boldsymbol{b}$ as the unique element in $S(\{t_n^0,\ldots,t_n^i\},D,q)$ satisfying

$$\boldsymbol{b}_{\alpha_0\ldots\alpha_i}^{t_n^0\ldots t_n^i} := b_{\alpha_0\ldots\alpha_i} \quad \text{for every } (\alpha_0\ldots\alpha_i)\in\mathscr{A}_{i+1}^q.$$

It is immediate to see that the hypotheses of Lemma 5.3 are satisfied by $(\boldsymbol{b},\boldsymbol{\lambda})$, hence $(\boldsymbol{b},\boldsymbol{\lambda})\in Ad(\{t_n^0,\ldots,t_n^i\},q,\boldsymbol{\varphi})$. Moreover $(\boldsymbol{b},\boldsymbol{\lambda})$ satisfies (7.2) and (7.3), by construction; hence $(\boldsymbol{b},\boldsymbol{\lambda})\in A_n^i(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$.

The term of (7.1) containing W is lower semicontinuous with respect to the 2-weak* convergence of Young measures, while the one containing H is L^∞-weakly* continuous; therefore the functional (7.1) is lower semicontinuous with respect to the convergence we are considering, and this implies that $(\boldsymbol{b},\boldsymbol{\lambda})$ is a solution of our minimum problem.

Now we want to construct from $(\boldsymbol{b},\boldsymbol{\lambda})$ a new minimizer $(\boldsymbol{b},\bar{\boldsymbol{\lambda}})$ satisfying property (7.4). Let us set

$$\begin{aligned}
(\boldsymbol{\nu}_n^i)_{t_n^i} &:= \sum_{\alpha=1}^q (\boldsymbol{b}_n^i)_\alpha^{t_n^i}(\delta_\alpha\otimes(\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}),\\
(\boldsymbol{\mu}_n^i)_{t_n^0\ldots t_n^i} &:= \sum_{(\alpha_0,\ldots,\alpha_i)} (\boldsymbol{b}_n^i)_{\alpha_0\ldots\alpha_i}^{t_n^0\ldots t_n^i}\boldsymbol{\delta}_{(\alpha_0,\ldots,\alpha_i)}.
\end{aligned}$$

From the definition of $A_n^i(\boldsymbol{b}^{i-1}\!,\boldsymbol{\lambda}^{i-1})$ it follows that $(\boldsymbol{\nu}_n^i,\boldsymbol{\mu}_n^i)\in AY(\{t_n^0,\ldots,t_n^i\},Z,\boldsymbol{\varphi})$; in particular there exist sequences $(z_{n,k}^{i-1})_k$, $(z_{n,k}^i)_k$ in $L^\infty(D;Z)$, and $(v_{n,k}^i)_k$ in $\mathcal{A}(t_n^i)$ satisfying

$$\begin{aligned}
\boldsymbol{\delta}_{(z_{n,k}^{i-1},z_{n,k}^i)} &\rightharpoonup (\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i} \quad \text{weakly*},\\
\boldsymbol{\delta}_{(z_{n,k}^i,\nabla v_{n,k}^i)} &\rightharpoonup (\boldsymbol{\nu}_n^i)_{t_n^i} \quad \text{2-weakly*},
\end{aligned}$$

as $k\to\infty$. Thanks to Lemma 3.1, we can assume, without loss of generality, that $(|\nabla v_{n,k}^i|^2)_k$ are equiintegrable; hence by the Fundamental Theorem for Young measures (see, e.g., [3]) we may assume that

$$\sup_k \|\nabla v_{n,k}^i\|_2^2 \le \int_{D\times Z\times\mathbb{R}^{N\times d}} |F|^2\, d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\alpha,F)+1, \tag{7.9}$$

$$\int_D W(z_{n,k}^i(x),\nabla v_{n,k}^i(x))\,dx \to \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha,F)\, d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\alpha,F). \tag{7.10}$$

Denote by I_n^i the minimum value of (7.1) over $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$. Thanks to (7.10), we can deduce that

$$\begin{aligned}
&\lim_k \Big[\int_D W(z_{n,k}^i(x), \nabla v_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] \\
&= \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\alpha,F) \\
&\quad + \int_{D\times Z^2} H(\alpha_i, \alpha_{i-1})\ d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x, \alpha_{i-1}, \alpha_i) = I_n^i.
\end{aligned}$$

Now we want to consider the following auxiliary minimum problem, for every k:

$$I_{n,k}^i := \inf_{v\in\varphi_n^i+H_0^1} \int_D W(z_{n,k}^i(x), \nabla v(x))\, dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\, dx. \tag{7.11}$$

For every k, we choose $\hat{v}_{n,k}^i \in \varphi_n^i + H_0^1(D;\mathbb{R}^{N\times d})$ such that

$$\int_D W(z_{n,k}^i(x), \nabla\hat{v}_{n,k}^i(x))\, dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\, dx \le I_{n,k}^i + \tfrac{1}{k}. \tag{7.12}$$

Using $v_{n,k}^i$ as competitor in (7.11), we can easily deduce, from (7.12) and the growth hypothesis on W, that

$$\|\nabla\hat{v}_{n,k}^i\|_2^2 \le \hat{C}(1 + \|\nabla v_{n,k}^i\|_2^2),$$

for a suitable positive constant $\hat{C}$, independent of n. Hence, thanks to (7.9), $\sup_k \|\nabla\hat{v}_{n,k}^i\|_2^2$ is bounded; in particular there exists $\bar{\nu}_n^i \in Y^2(D; Z\times\mathbb{R}^{N\times d})$ such that, up to a subsequence, $\boldsymbol{\delta}_{(z_{n,k}^i, \nabla\hat{v}_{n,k}^i)} \rightharpoonup \bar{\nu}_n^i$ 2-weakly* as $k\to\infty$. Thanks to Lemma 3.1 we can assume, up to a subsequence, that $|\nabla\hat{v}_{n,k}^i|^2$ is equiintegrable in k.

Since $\pi_{D\times Z}(\bar{\nu}_n^i) = \sum_\alpha (\boldsymbol{b}_n^i)_\alpha^{t_n^i} \boldsymbol{\delta}_\alpha$, by Remark 3.5 there exists a family of Young measures $\bar{\lambda}_n^i = ((\bar{\lambda}_n^i)_\alpha)_\alpha$ such that it holds

$$\bar{\nu}_n^i = \sum_{\alpha=1}^q (\boldsymbol{b}_n^i)_\alpha^{t_n^i} (\delta_\alpha \otimes (\bar{\lambda}_n^i)_\alpha); \tag{7.13}$$

since $\bar{\nu}_n^i \in Y^2(D; Z\times\mathbb{R}^{N\times d})$, $(\boldsymbol{b}_n^i)_\alpha^{t_n^i}(\bar{\lambda}_n^i)_\alpha$ satisfies (3.13) for $p = 2$. We have

$$\begin{aligned}
&\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha, F)\ d\bar{\nu}_n^i(x,\alpha,F) + \int_{D\times Z^2} H(\alpha_i, \alpha_{i-1})\ d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x,\alpha,\beta) \\
&\le \liminf_k \Big[\int_D W(z_{n,k}^i(x), \nabla\hat{v}_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] \\
&\le \liminf_k [I_{n,k}^i + 1/k] \\
&\le \liminf_k \Big[\int_D W(z_{n,k}^i(x), \nabla v_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] = I_n^i.
\end{aligned} \tag{7.14}$$

The construction of $\bar{\nu}_n^i$ implies that the pair $(\boldsymbol{b}, \bar{\boldsymbol{\lambda}})$, with

$$\begin{aligned}
\bar{\boldsymbol{\lambda}} &:= (\boldsymbol{\lambda}^{t_n^0}, \ldots, \boldsymbol{\lambda}^{t_n^{i-1}}, \bar{\lambda}_n^i) \\
&= ((\boldsymbol{\lambda}^{i-1})^{t_n^0}, \ldots, (\boldsymbol{\lambda}^{i-1})^{t_n^{i-1}}, \bar{\lambda}_n^i),
\end{aligned}$$

is an element of $Ad(\{t_n^0,\dots,t_n^i\},q,\boldsymbol{\varphi})$; moreover it satisfies the "memory properties" (7.2) and (7.3) required to be in $A_n^i(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$. Hence

$$I_n^i \le \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha,F)\,d\bar\nu_n^i(x,\alpha,F) + \int_{D\times Z^2} H(\alpha_i,\alpha_{i-1})\,d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x,\alpha,\beta); \quad (7.15)$$

we can deduce from (7.14) and (7.15) that $(\boldsymbol{b},\bar{\boldsymbol{\lambda}})$ is a minimizer of (7.1) on the set $A_n^i(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$.

Now we want to apply Ekeland Principle in order to construct a more regular sequence $(\bar v_{n,k}^i)_k$ which, together with $z_{n,k}^i$, generates $\bar\nu_n^i$.

We define $\hat u_{n,k}^i$ as the function $\hat v_{n,k}^i-\varphi_n^i\in H_0^1(D;\mathbb{R}^N)$. Consider the functional $\mathcal{E}$ defined on the Banach space $W_0^{1,1}(D;\mathbb{R}^N)$ by

$$\mathcal{E}(u):=\begin{cases}\int_D W(z_{n,k}^i(x),\nabla\varphi_n^i(x)+\nabla u(x))\,dx & \text{if } u\in H_0^1(D;\mathbb{R}^N);\\ +\infty & \text{otherwise.}\end{cases}$$

This functional is strongly lower semicontinuous with respect to the $W_0^{1,1}$ topology, it is positive and not infinite everywhere: hence we apply Ekeland Principle (see [12, Corollary 6.1, p. 30]) to $W_0^{1,1}(D;\mathbb{R}^N)$ endowed with the norm $\|u\|_{W_0^{1,1}}:=\|\nabla u\|_1$, and we deduce that there exists $\bar u_{n,k}^i\in H_0^1(D;\mathbb{R}^N)$ with the following properties:

$$\begin{aligned}&\int_D W(z_{n,k}^i(x),\nabla\varphi_n^i(x)+\nabla\bar u_{n,k}^i(x))\,dx\\ &\le \inf_{u\in W_0^{1,1}(D;\mathbb{R}^N)}\mathcal{E}(u)+1/k=\\ &=\inf_{u\in H_0^1(D;\mathbb{R}^N)}\int_D W(z_{n,k}^i(x),\nabla\varphi_n^i(x)+\nabla u(x))\,dx+1/k\\ &\le \inf_{v\in\varphi_n^i+H_0^1(D;\mathbb{R}^N)}\int_D\left[W(z_{n,k}^i(x),\nabla v(x))\,dx+H(z_{n,k}^i(x),z_{n,k}^{i-1}(x))\right]dx\\ &\quad+1/k=I_{n,k}^i+1/k;\end{aligned}\quad (7.16)$$

$$\|\nabla\bar u_{n,k}^i-\nabla\hat u_{n,k}^i\|_1\le\frac{1}{\sqrt{k}}; \quad (7.17)$$

$$\begin{aligned}&\int_D W(z_{n,k}^i(x),\nabla\varphi_n^i(x)+\nabla\bar u_{n,k}^i(x))\,dx\\ &\le\int_D\left[W(z_{n,k}^i(x),\nabla\varphi_n^i(x)+\nabla u(x))+\frac{1}{\sqrt{k}}|\nabla u-\nabla\bar u_{n,k}^i|\right]dx,\end{aligned}\quad (7.18)$$

for every $u\in H_0^1(D;\mathbb{R}^N)$.

In particular these properties imply that

$$\sup_k\|\nabla\bar u_{n,k}^i\|_2^2\le\bar C(1+\sup_k\|\nabla\hat u_{n,k}^i\|_2^2), \quad (7.19)$$

for a suitable positive constant $\bar C$ independent of k, n, and i, and

$$\boldsymbol{\delta}_{(z_{n,k}^i,\nabla\varphi_n^i+\nabla\bar u_{n,k}^i)}\rightharpoonup\bar\nu_n^i\quad\text{2-weakly*},$$

as $k\to\infty$.

Using the growth hypotheses on W, it is easy to deduce from (7.18) that, for k sufficiently large, $\bar v_{n,k}^i$ is a Q-quasi-minimum of the functional $\mathcal{F}\colon H^1(D;\mathbb{R}^N)\to\mathbb{R}$

defined by $\mathcal{F}(v) = \int_D (1 + |\nabla v(x)|^2)\ dx$, for a suitable positive constant Q independent of k, n, and i. Indeed, let us consider a cube $Q_R \subset \mathbb{R}^d$ and a function w such that $\bar{v}^i_{n,k} - w \in H^1_0(D \cap Q_R; \mathbb{R}^N)$. We can extend w to $D \setminus Q_R$ by setting $w = \bar{v}^i_{n,k}$ a.e. in $D \setminus Q_R$ and this extension (not relabeled) is in $\varphi^i_n + H^1_0(D; \mathbb{R}^N)$. If we take w as a competitor in (7.18), we obtain

$$\begin{aligned}
&\int_{D\setminus Q_R} W(z^i_{n,k}(x), \nabla \bar{v}^i_{n,k}(x))\ dx + \int_{D\cap Q_R} W(z^i_{n,k}(x), \nabla \bar{v}^i_{n,k}(x))\ dx \leq \\
\leq &\int_{D\setminus Q_R} W(z^i_{n,k}(x), \nabla \bar{v}^i_{n,k}(x))\ dx + \int_{D\cap Q_R} W(z^i_{n,k}(x), \nabla w(x))\ dx + \\
&+ \frac{1}{\sqrt{k}} \int_{D\cap Q_R} |\nabla w(x) - \nabla \bar{v}^i_{n,k}(x)|\ dx;
\end{aligned}$$

using the growth hypotheses on W, the previous inequality implies

$$\begin{aligned}
\int_{D\cap Q_R} (c|\nabla \bar{v}^i_{n,k}(x)|^2 - C)\ dx \quad \leq \quad & \int_{D\cap Q_R} C(|\nabla w|^2 + 1)\ dx \\
& + \frac{1}{\sqrt{k}} \int_{D\cap Q_R} |\nabla w(x) - \nabla \bar{v}^i_{n,k}(x)|\ dx.
\end{aligned}$$

Now we have

$$|\nabla w(x) - \nabla \bar{v}^i_{n,k}(x)| \leq |\nabla w(x) - \nabla \bar{v}^i_{n,k}(x)|^2 + 1 \leq 2(|\nabla w(x)|^2 + |\nabla \bar{v}^i_{n,k}(x)|^2) + 1;$$

hence

$$\begin{aligned}
\int_{D\cap Q_R} c|\nabla \bar{v}^i_{n,k}(x)|^2\ dx \quad \leq \quad & \int_{D\cap Q_R} C(|\nabla w|^2 + 2)\ dx \\
& + \frac{1}{\sqrt{k}} \int_{D\cap Q_R} [2(|\nabla w(x)|^2 + |\nabla \bar{v}^i_{n,k}(x)|^2) + 1]\ dx.
\end{aligned}$$

We can rewrite the previous inequality as follows:

$$\begin{aligned}
&(c - \frac{2}{\sqrt{k}}) \int_{D\cap Q_R} (|\nabla \bar{v}^i_{n,k}(x)|^2 + 1)\ dx \leq \\
\leq &(C + \frac{2}{\sqrt{k}}) \int_{D\cap Q_R} (|\nabla w(x)|^2 + 1)\ dx + \int_{D\cap Q_R} (C + c - \frac{3}{\sqrt{k}})\ dx;
\end{aligned}$$

hence, if k is sufficiently large we can assume that $c/2 \leq c - (3/\sqrt{k}) < c$, so that we obtain

$$\int_{D\cap Q_R} (|\nabla \bar{v}^i_{n,k}(x)|^2 + 1)\ dx \leq \frac{4(C+c)}{c} \int_{D\cap Q_R} (|\nabla w(x)|^2 + 1)\ dx,$$

which proves that $\bar{v}^i_{n,k}$ is a $4(C+c)/c$-quasi-minimum of $\mathcal{F}$.

We can now apply Theorem 8.7, and conclude that there exist two constants $\gamma > 0$ and $r > 1$, both independent of k, n, and i, such that

$$\int_D |\nabla \bar{v}^i_{n,k}(x)|^{2r}\ dx \leq \gamma \Big[1 + \big(\int_D |\nabla \bar{v}^i_{n,k}(x)|^2\ dx\big)^r\Big],$$

for every k. In particular, thanks to (7.19), we have

$$\begin{aligned}
\int_D |\nabla (\bar{v}^i_n)_k(x)|^{2r}\ dx \leq &\gamma \Big[1 + \|\nabla (\bar{v}^i_n)_k\|_2^{2r}\Big] \leq \\
\leq &\tilde{\gamma} \Big[(1 + \|\nabla (\hat{v}^i_n)_k\|_2^{2r}\Big],
\end{aligned} \tag{7.20}$$

for a suitable constant $\tilde{\gamma} > 0$ independent of k, n, and i. Thanks to the equiintegrability of $|\nabla \hat{v}^i_{n,k}|^2$, using the Fundamental Theorem for Young measures we can deduce that

$$\begin{aligned}
&\sum_{\alpha=1}^{q} \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}_\alpha^{t_n^i}(x)|F|^{2r}\ d\bar{\lambda}^i_n(x,F)\\
\le&\liminf_k \int_D |\nabla \bar{v}^i_{n,k}(x)|^{2r}\ dx\\
\le&\tilde{\gamma}[1+(\lim_k \int_D |\nabla \hat{v}^i_{n,k}|^2\ dx)^r]\\
=&\tilde{\gamma}[1+(\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}_\alpha^{t_n^i}(x)|F|^2\ d\bar{\lambda}^i_n(x,F))^r].
\end{aligned}$$

This concludes the proof. □

Using the minimization process described so far, it is possible to construct inductively $(\boldsymbol{b}^i_n, \boldsymbol{\lambda}^i_n)$, for every $i = 1, \dots, k(n)$ and every n.

Set $\tau^n(s) := t^i_n$, whenever $t^i_n \le s < t^{i+1}_n$, where we set $t^{k(n)+1}_n := T + \frac{1}{n}$.

For every i and n we set

$$\sigma^i_n(x) := \sum_{\alpha=1}^{q} \int_{\mathbb{R}^{N\times d}} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x) \frac{\partial W}{\partial F}(\alpha, F)\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F),$$

and define

$$\boldsymbol{\sigma}_n(t,x) := \sigma^i_n(x), \tag{7.21}$$

for a.e. $x \in D$, whenever $t^i_n \le t < t^{i+1}_n$.

For every $\alpha = 1, \dots, q$, we define $(\boldsymbol{\lambda}_n)_\alpha \in Y(D; \mathbb{R}^{N\times d})^{[0,T]}$ by

$$(\boldsymbol{\lambda}_n)^s_\alpha := (\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha, \tag{7.22}$$

whenever $t^i_n = \tau^n(s)$, for every $s \in [0,T]$; we define also $\boldsymbol{b}_n \in S([0,T], D; \mathbb{R}^m)$ as the piecewise constant interpolation of $\boldsymbol{b}^{k(n)}_n$ (see Definition 3.2).

Note that $(\boldsymbol{b}_n, \boldsymbol{\lambda}_n) \in Ad([0,T], q, \boldsymbol{\varphi}(\tau^n(\cdot)))$ by construction.

7.2. **A priori estimates.** Set

$$\begin{aligned}
(\boldsymbol{\nu}^i_n)_{t^i_n} &:= \sum_{\alpha=1}^{q} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha (\delta_\alpha \otimes (\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha),\\
(\boldsymbol{\mu}^i_n)_{t^0_n\dots t^i_n} &:= \sum_{(\alpha_0,\dots,\alpha_i)} (\boldsymbol{b}^i_n)^{t^0_n\dots t^i_n}_{\alpha_0\dots\alpha_i} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_i)},
\end{aligned}$$

for every $i = 1, \dots, k(n)$, and

$$\begin{aligned}
(\boldsymbol{\nu}_n)_t &:= \sum_{\alpha=1}^{q} (\boldsymbol{b}_n)^t_\alpha (\delta_\alpha \otimes (\boldsymbol{\lambda}_n)^t_\alpha),\\
(\boldsymbol{\mu}_n)_{t_0\dots t_m} &:= \sum_{(\alpha_0,\dots,\alpha_m)} (\boldsymbol{b}_n)^{t_0\dots t_m}_{\alpha_0\dots\alpha_n} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_m)},
\end{aligned}$$

for every $t \in [0,T]$ and every $t_0 < \dots < t_m$ in $[0,T]$.

As in [13, Section 7.2], we want to deduce a discrete version of the energy inequality for $(\boldsymbol{b}_n, \boldsymbol{\lambda}_n)$. We briefly recall the argument for the reader's convenience.

Using the competitor defined in the proof of Lemma 7.1, we have

$$\begin{aligned}
&\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha,F)\ d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\alpha,F)\\
&+\int_{D\times(Z)^2} H(\alpha_i-\alpha_{i-1})\ d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x,\alpha,\beta)\\
\le&\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha,F)\ d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\alpha,F)\\
&+\int_{D\times Z\times\mathbb{R}^{N\times d}}[W(\alpha,F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x))-W(\alpha,F)]\ d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\alpha,F).
\end{aligned}$$

Let us fix t in $(0,T]$ and suppose that $t_n^i\le t<t_n^{i+1}$, for suitable $i=0,\dots,k(n)+1$; using

$$\begin{aligned}
&\int_{D\times Z\times\mathbb{R}^{N\times d}}[W(\alpha,F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x))-W(\alpha,F)]\ d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\alpha,F)\\
=&\int_{t_n^{i-1}}^{t_n^i}\Big(\int_{D\times Z\times\mathbb{R}^{N\times d}}\frac{\partial W}{\partial F}(\alpha,F+\varepsilon^n(s,x))\nabla\dot{\boldsymbol{\varphi}}(s,x)\ d(\boldsymbol{\nu}_n)_s(x,\alpha,F)\Big)\ ds,
\end{aligned}$$

where $\varepsilon^n(s,x):=\nabla\varphi(s,x)-\nabla\varphi(\tau^n(s),x)$, for every $s\in[0,T]$ and every $x\in D$, and iterating from i to 1, we obtain

$$\begin{aligned}
&\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\alpha,F)\ d(\boldsymbol{\nu}_n)_t(x,\alpha,F)+\mathrm{Var}_H(\boldsymbol{\mu}_n;0,t)\\
\le&\mathcal{W}(z_0,v_0)+\int_0^{\tau^n(t)}\langle\boldsymbol{\sigma}_n(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\ ds\\
&+\int_0^{\tau^n(t)}\Big(\int_{D\times Z\times\mathbb{R}^{N\times d}}\Big[\frac{\partial W}{\partial F}(\alpha,F+\varepsilon^n(s,x))-\frac{\partial W}{\partial F}(\alpha,F)\Big]\\
&\qquad\nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\alpha,F)\Big)\ ds.
\end{aligned}\tag{7.23}$$

From (7.23), we can deduce the following a priori estimates on $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n)$.

Lemma 7.3. *There exists a positive constant C, such that*

$$\sup_n\sup_{t\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|F|^{2r}\ d(\boldsymbol{\nu}_n)_t(x,\alpha,F)\ \le\ C,\tag{7.24}$$

$$\sup_n\mathrm{Var}_H(\boldsymbol{\mu}_n;0,T)\ \le\ C.\tag{7.25}$$

Remark 7.4. Since Z is finite, (7.24) implies that

$$\sup_n\sup_{t\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\alpha,F)|^{2r}\ d(\boldsymbol{\nu}_n)_t(x,\alpha,F)\le C.\tag{7.26}$$

Proof of Lemma 7.3. Using the fact that $\int_0^T\|\dot{\boldsymbol{\varphi}}(t)\|_{H^1}\ dt$ is finite, the hypotheses on W and the inequality

$$\sup_{s\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|F|^2\ d(\boldsymbol{\nu}_n)_s(x,\alpha,F)<\infty,$$

(since $\boldsymbol{\nu}_n$ are piecewise constant interpolations of Young measures with finite second moments), we can deduce from (7.23) that, for n sufficiently large,

$$\begin{aligned}&\int_{D\times Z\times\mathbb{R}^{N\times d}}|F|^2\,d(\boldsymbol{\nu}_n)_t(x,\alpha,F)\\ \le&\tilde{C}+\tilde{C}\sup_{s\in[0,T]}\Big(1+\tilde{c}\int_{D\times Z\times\mathbb{R}^{N\times d}}|F|^2\,d(\boldsymbol{\nu}_n)_s(x,\alpha,F)\Big)^{1/2},\end{aligned}$$

for suitable positive constants $\tilde{C}$ and $\tilde{c}$ independent of t and n.

Since this can be repeated for every $t\in[0,T]$, we deduce

$$\sup_n\sup_{t\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|F|^2\,d(\boldsymbol{\nu}_n)_t(x,\alpha,F)\le C, \tag{7.27}$$

Inequality (7.24) comes now from (7.27) and (7.4), while inequality (7.25) follows from (7.27) and (7.23). ☐

Using Lemma 7.3 and adapting the proof of [13, Lemma 7.5], we can deduce the following discrete version of the energy inequality: for every t in $(0,T]$

$$\begin{aligned}&\int_{D\times Z\times\mathbb{R}^{N\times d}}W(\alpha,F)\,d(\boldsymbol{\nu}_n)_t(x,\alpha,F)+\mathrm{Var}_H(\boldsymbol{\mu}_n;0,t)\\ \le&\mathcal{W}(z_0,v_0)+\int_0^{\tau^n(t)}\langle\boldsymbol{\sigma}_n(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\,ds+\rho_n,\end{aligned}\tag{7.28}$$

where $\rho_n\to0$ as $n\to\infty$.

7.3. **Passage to the limit.** Thanks to (7.25), we can apply Helly's Theorem (Theorem 3.3) to the sequence $(\boldsymbol{b}_n)_n$ and obtain a subsequence, still indicated with $(\boldsymbol{b}_n)_n$, a subset $\mathcal{T}$ of $[0,T]$, containing 0, with $\mathcal{L}^1([0,T]\setminus\mathcal{T})=0$, and $\boldsymbol{b}\in S_-([0,T],D,q)$, such that, for every finite sequence $t_1<\dots<t_l$ in $\mathcal{T}$, we have

$$(\boldsymbol{b}_n)^{t_1\dots t_l}_{\alpha_1\dots\alpha_l}\rightharpoonup\boldsymbol{b}^{t_1\dots t_l}_{\alpha_1\dots\alpha_l}\quad L^\infty\text{-weakly*}, \tag{7.29}$$

as $n\to\infty$, for every $(\alpha_1,\dots,\alpha_l)\in\mathscr{A}^l_q$. Denote by $\boldsymbol{\mu}$ the element in $SY_-([0,T],D;Z)$ corresponding to $\boldsymbol{b}$.

Let $\mathcal{T}'$ be a dense countable subset of $\mathcal{T}$ containing 0. Thanks to (7.26) and Remark 3.7, we can find with a diagonalization process a subsequence of $(\boldsymbol{\lambda}_n)_n$, still indicated by $(\boldsymbol{\lambda}_n)_n$, and $\boldsymbol{\lambda}^t=(\boldsymbol{\lambda}^t_\alpha)_\alpha\in Y(D;\mathbb{R}^{N\times d})^q$ for every $t\in\mathcal{T}'$, such that

$$\int_{D\times\mathbb{R}^{N\times d}}\boldsymbol{b}^t_\alpha(x)|F|^{2r}\,d\boldsymbol{\lambda}^t_\alpha(x,F)\le C, \tag{7.30}$$

and

$$(\boldsymbol{b}_n)^t_\alpha(\boldsymbol{\lambda}_n)^t_\alpha\rightharpoonup\boldsymbol{b}^t_\alpha\boldsymbol{\lambda}^t_\alpha\quad 2r\text{-weakly*, as } n\to\infty, \tag{7.31}$$

for every $t\in\mathcal{T}'$. Note that the family of coefficients $\boldsymbol{b}$ appearing here is the same as in (7.29), because $\pi_{D\times Z}((\boldsymbol{\nu}_n)_t)=(\boldsymbol{\mu}_n)_t$ for every $t\in[0,T]$ and thanks to Remark 3.5; moreover, by construction of $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n)$ we have

$$\boldsymbol{b}^0_\alpha = (\boldsymbol{b}_n)^0_\alpha=1_{D^0_\alpha}, \tag{7.32}$$

$$(\boldsymbol{\lambda}^0_\alpha)^x = ((\boldsymbol{\lambda}_n)^0_\alpha)^x=\delta_{\nabla v_0(x)}\quad\text{for a.e. } x\in D^0_\alpha, \tag{7.33}$$

$$\tag{7.34}$$

where $D^0_\alpha:=\{x\in D\ :\ z_0(x)=\alpha\}$.

For every $t \in \mathcal{T} \setminus \mathcal{T}'$, let us choose an increasing sequence of integers n_k^t, possibly depending on t, such that

$$\limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2 = \lim_k \langle \boldsymbol{\sigma}_{n_k^t}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2 \tag{7.35}$$

(this choice is crucial in order to apply the argument in [10, Section 7]). Again by (7.26) and Remark 3.7, we can extract a further subsequence, still denoted by $(\boldsymbol{\lambda}_{n_k^t})_k$, satisfying (7.35) and such that there exists $\boldsymbol{\lambda}^t \in Y(D; \mathbb{R}^{N\times d})^q$ with

$$\int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}_\alpha^t(x)|F|^{2r}\, d\boldsymbol{\lambda}_\alpha^t(x,F) \quad \leq \quad C, \tag{7.36}$$

$$(\boldsymbol{b}_{n_k^t})_\alpha^t (\boldsymbol{\lambda}_{n_k^t})_\alpha^t \quad \rightharpoonup \quad \boldsymbol{b}_\alpha^t \boldsymbol{\lambda}_\alpha^t \quad 2r\text{-weakly*, as } k\to\infty. \tag{7.37}$$

Note that, thanks to (W.2), we have

$$\limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2 = \langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2, \tag{7.38}$$

where

$$\boldsymbol{\sigma}(t,x) := \sum_\alpha \boldsymbol{b}_\alpha^t(x) \int_{\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\alpha, F)\, d(\boldsymbol{\lambda}_\alpha^t)^x(F),$$

for every $t \in \mathcal{T}$. This implies that the map (6.2) is measurable on $[0,T]$; moreover for every $t \in \mathcal{T}'$ we have

$$\limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2 = \lim_n \langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle_2.$$

The family $\boldsymbol{\nu}$ will denote the element of $Y^{2r}(D; Z\times\mathbb{R}^{N\times d})^{\mathcal{T}}$ corresponding to $(\boldsymbol{b}, \boldsymbol{\lambda})$. Let $t \in [0,T]\setminus\mathcal{T}$, and fix a sequence s_j in $\mathcal{T}$ converging to t with $s_j < t$; by (7.30), and (7.36), we have

$$\sup_j \int_D \boldsymbol{b}_\alpha^{s_j}(x) \Big(\int_{\mathbb{R}^{N\times d}} |F|^{2r}\, d(\boldsymbol{\lambda}_\alpha^{s_j})^x(F) \Big)\, dx \leq C;$$

for every j; again by Remark 3.7, we can find a subsequence, not relabeled, and $\boldsymbol{\lambda}^t \in Y(D; \mathbb{R}^{N\times d})$ such that

$$\int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}_\alpha^t(x)|F|^{2r}\, d\boldsymbol{\lambda}_\alpha^t(x,F) \leq C, \tag{7.39}$$

and

$$\boldsymbol{b}_\alpha^{s_j} \boldsymbol{\lambda}_\alpha^{s_j} \rightharpoonup \boldsymbol{b}_\alpha^t \boldsymbol{\lambda}_\alpha^t \quad 2r\text{-weakly*}, \quad \text{as } j\to\infty. \tag{7.40}$$

Note that, since $\pi_{D\times Z}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$ for every $t \in \mathcal{T}$, the left continuity of $\boldsymbol{b}$ defined in (7.29) ensures that the family of coefficients appearing in (7.40) is the same as in (7.29).

In this way we defined $\boldsymbol{\lambda} \in (Y(D; \mathbb{R}^{N\times d})^q)^{[0,T]}$, and consequently $\boldsymbol{\nu} \in Y^{2r}(D; Z\times \mathbb{R}^{N\times d})^{[0,T]}$. It can be shown that $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad([0,T], q, \boldsymbol{\varphi})$ using Lemma 5.3 and adapting the argument in [13, Section 7.3].

By construction $(\boldsymbol{b}, \boldsymbol{\lambda})$ satisfies (ev0).

7.4. **Stability.** Fix n and $i = 1, \dots, k(n)$. Let

$$\begin{array}{rcl} M\colon D & \to & \mathbf{M}_S^{q\times q} \\ x & \mapsto & (M_{\beta\alpha}(x))_{\beta\alpha}, \end{array}$$

be a measurable map, and let $\tilde{u} \in H_0^1(D;\mathbb{R}^N)$. We define $(\tilde{\boldsymbol{\nu}}_n^i, \tilde{\boldsymbol{\mu}}_n^i) \in Y^2(D; Z \times \mathbb{R}^{N\times d})^{\{t_n^0,\dots,t_n^i\}} \times SY(\{t_n^0,\dots,t_n^i\}, D, Z)$ by

$$\begin{aligned} (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^j} &:= (\boldsymbol{\nu}_n^i)_{t_n^j} \quad \text{if } j < i \\ (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^i} &:= \sum_{\alpha,\beta} M_{\beta\alpha} (\boldsymbol{b}_n^i)_\alpha^{t_n^i} (\delta_\beta \otimes \mathcal{T}_{\nabla\tilde{u}}(\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}), \\ (\tilde{\boldsymbol{\mu}}_n^i)_{t_n^0\dots t_n^i} &:= \sum_{\alpha,\beta} M_{\beta\alpha} (\boldsymbol{b}_n^i)_{\alpha_0\dots\alpha_{i-1}\alpha}^{t_n^0\dots t_n^i} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_{i-1},\beta)}, \end{aligned}$$

where $\mathcal{T}_{\nabla\tilde{u}}$ is defined as in (3.1).

Lemma 7.5. *The pair $(\tilde{\boldsymbol{\nu}}_n^i, \tilde{\boldsymbol{\mu}}_n^i)$ is in $AY(\{t_n^0,\dots,t_n^i\}, Z, \boldsymbol{\varphi})$.*

Proof. Consider $(\boldsymbol{\nu}_n^i, \boldsymbol{\mu}_n^i) \in AY(\{t_n^0,\dots,t_n^i\}, Z, \boldsymbol{\varphi})$: for every $j = 0,\dots,i$, there exist a sequence $(v_k^j)_k$ contained in $\boldsymbol{\varphi}(t_n^j) + H_0^1(D;\mathbb{R}^N)$, and a sequence $((D_\alpha^j)_k)_\alpha$, indexed by k, of measurable partitions of D, such that

(1) we have
$$\sum_{\alpha_0,\dots,\alpha_i} 1_{(D_{\alpha_0}^0)_k} \cdot \dots \cdot 1_{(D_{\alpha_i}^i)_k} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_i)} \rightharpoonup (\boldsymbol{\mu}_n^i)_{t_n^0\dots t_n^i} \quad \text{weakly*},$$
as $k \to \infty$;

(2) for every $j = 0,\dots,i$, there exists a subsequence $(k_l^j)_l$, possibly dependent on j, such that
$$\sum_{\alpha=1}^q 1_{(D_\alpha^j)_{k_l^j}} \boldsymbol{\delta}_{(\alpha,\nabla v_{k_l^j}^j)} \rightharpoonup (\boldsymbol{\nu}_n^i)_{t_n^j} \quad \text{2-weakly*},$$
as $l \to \infty$.

In particular these conditions imply that

$$\begin{aligned} \sum_{\alpha_0,\dots,\alpha_{i-1},\alpha,\beta} M_{\beta\alpha} 1_{(D_{\alpha_0}^0)_k} \cdots 1_{(D_{\alpha_{i-1}}^{i-1})_k} \cdot 1_{(D_\alpha^i)_k} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_{i-1},\beta)} &\overset{k\to\infty}{\rightharpoonup} (\tilde{\boldsymbol{\mu}}_n^i)_{t_n^0\dots t_n^i} \quad \text{weakly*}; \\ \sum_{\alpha,\beta} M_{\beta\alpha} 1_{(D_\alpha^i)_{k_l^i}} \boldsymbol{\delta}_{(\beta,\nabla v_{k_l^i}^i + \nabla\tilde{u})} &\overset{l\to\infty}{\rightharpoonup} (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^i} \quad \text{2-weakly*}; \\ \sum_{\alpha=1}^q 1_{(D_\alpha^j)_{k_l^j}} \boldsymbol{\delta}_{(\alpha,\nabla v_{k_l^j}^j)} &\overset{l\to\infty}{\rightharpoonup} (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^j} \quad \text{2-weakly*}, \end{aligned}$$

for every $j < i$. Thanks to Lemma 5.4, the pair $(\tilde{\boldsymbol{\nu}}^k, \tilde{\boldsymbol{\mu}}^k) \in Y^2(D; Z \times \mathbb{R}^{N\times d})^{\{t_n^0,\dots,t_n^i\}} \times SY^2(\{t_n^0,\dots,t_n^i\}, D; Z)$, defined by

$$\begin{aligned} (\tilde{\boldsymbol{\nu}}^k)_{t_n^i} &:= \sum_{\alpha,\beta} M_{\beta\alpha} 1_{(D_\alpha^i)_k} \boldsymbol{\delta}_{(\beta,\nabla v_k^i + \nabla\tilde{u})}, \\ (\tilde{\boldsymbol{\nu}}^k)_{t_n^j} &:= \sum_{\alpha=1}^q 1_{(D_\alpha^j)_k} \boldsymbol{\delta}_{(\alpha,\nabla v_k^j)} \quad \text{for every } j < i, \\ (\tilde{\boldsymbol{\mu}_n^i})_{t_n^0\dots t_n^i} &:= \sum_{\alpha_0,\dots,\alpha_{i-1},\alpha,\beta} M_{\beta\alpha} 1_{(D_{\alpha_0}^0)_k} \cdot \dots \cdot 1_{(D_{\alpha_{i-1}}^{i-1})_k} \cdot 1_{(D_\alpha^i)_k} \boldsymbol{\delta}_{(\alpha_0,\dots,\alpha_{i-1},\beta)}, \end{aligned}$$

is an element of $AY(\{t_n^0,\dots,t_n^i\},Z,\boldsymbol{\varphi})$, for every k. Therefore the thesis can be deduced using [13, Lemma 6.6]. □

Set

$$\begin{aligned}(\tilde{\boldsymbol{b}}_n^i)^{t_n^0\dots t_n^{i-1}t_n^i}_{\alpha_0\dots\alpha_{i-1}\beta} &:= \sum_\alpha M_{\beta\alpha}(\boldsymbol{b}_n^i)^{t_n^0\dots t_n^{i-1}t_n^i}_{\alpha_0\dots\alpha_{i-1}\alpha},\\ (\tilde{\boldsymbol{\lambda}}_n^i)^{t_n^j}_\beta &:= (\boldsymbol{\lambda}_n^i)^{t_n^j}_\beta \quad \text{for every } j<i,\end{aligned}$$

and

$$((\tilde{\boldsymbol{\lambda}}_n^i)^{t_n^i}_\beta)^x := \frac{\sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)\mathcal{T}_{\nabla\tilde{u}(x)}(((\boldsymbol{\lambda}_n^i)^{t_n^i}_\alpha)^x)}{\sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)} \text{ if } \sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)>0$$

for a.e. $x\in D$, for every β, and every $(\alpha_0,\dots,\alpha_{i-1})\in\mathscr{A}_q^i$, where $\mathcal{T}_{\nabla\tilde{u}(x)}\colon\mathbb{R}^{N\times d}\to\mathbb{R}^{N\times d}$ is the map defined in (3.1); since $(\tilde{\boldsymbol{b}}_n^i,\tilde{\boldsymbol{\lambda}}_n^i)$ is the element corresponding to $(\tilde{\boldsymbol{\nu}}_n^i,\tilde{\boldsymbol{\mu}}_n^i)$, we immediately deduce from Lemma 7.5 that $(\tilde{\boldsymbol{b}}_n^i,\tilde{\boldsymbol{\lambda}}_n^i)$ is in $A_n^i(\boldsymbol{b}_n^{i-1},\boldsymbol{\lambda}_n^{i-1})$. The minimizing property of $(\boldsymbol{b}_n^i,\boldsymbol{\lambda}_n^i)$ implies that

$$\begin{aligned}&\sum_\alpha\int_D(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}}W(\alpha,F)\ d((\boldsymbol{\lambda}_n^i)^{t_n^i}_\alpha)^x(F)\Big)\ dx\\ &\quad+\sum_{\alpha\gamma}H(\alpha,\gamma)\int_D(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx\\ &\le\sum_\alpha\int_D(\tilde{\boldsymbol{b}}_n^i)^{t_n^i}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}}W(\alpha,F)\ d((\tilde{\boldsymbol{\lambda}}_n^i)^{t_n^i}_\alpha)^x(F)\Big)\ dx\\ &\quad+\sum_{\beta\gamma}H(\beta,\gamma)\int_D(\tilde{\boldsymbol{b}}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\beta}(x)\ dx;\end{aligned}$$

in other words

$$\begin{aligned}&\sum_\alpha\int_D(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}}W(\alpha,F)\ d((\boldsymbol{\lambda}_n^i)^{t_n^i}_\alpha)^x(F)\Big)\ dx\\ &\le\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}}W(\beta,F+\nabla\tilde{u})\ d((\boldsymbol{\lambda}_n^i)^{t_n^i}_\alpha)^x(F)\Big)\ dx\\ &+\sum_{\alpha\gamma\beta}H(\beta,\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx-\sum_{\alpha\gamma}H(\alpha,\gamma)\int_D(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx.\end{aligned}\tag{7.41}$$

Since $\sum_\beta M_{\beta\alpha}(x)=1$ for a.e. $x\in D$ and every α, we can deduce, using the triangle inequality, that

$$\begin{aligned}&\sum_{\alpha\beta\gamma}H(\beta,\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx-\sum_{\alpha\gamma}H(\alpha,\gamma)\int_D(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx\\ &=\sum_{\alpha\beta\gamma}H(\beta,\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx-\sum_{\alpha\beta\gamma}H(\alpha,\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx\\ &\le\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)\sum_\gamma(\boldsymbol{b}_n^i)^{t_n^{i-1}t_n^i}_{\gamma\alpha}(x)\ dx=\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)^{t_n^i}_\alpha(x)\ dx.\end{aligned}$$

Hence we deduce from (7.41) that

$$\begin{aligned}&\sum_{\alpha}\int_D (\boldsymbol{b}_n^i)_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha,F)\ d((\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i})^x(F)\Big)\ dx\\ \leq&\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\beta,F+\nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i})^x(F)\Big)\ dx\\ &+\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n^i)_\alpha^{t_n^i}(x)\ dx,\end{aligned}$$

for every n and $i = 1,\dots,k(n)$; we can rewrite the previous inequality in the following form

$$\begin{aligned}&\sum_{\alpha}\int_D (\boldsymbol{b}_n)_\alpha^{t}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha,F)\ d((\boldsymbol{\lambda}_n)_\alpha^{t})^x(F)\Big)\ dx\\ \leq&\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)_\alpha^{t}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\beta,F+\nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}_n)_\alpha^{t})^x(F)\ dx\\ &+\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)_\alpha^{t}(x)\ dx,\end{aligned}\tag{7.42}$$

for every $t \in \mathcal{T}\setminus\{0\}$ and every n. From (7.29) we can deduce that

$$\begin{aligned}&\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)_\alpha^{t}(x)\ dx\\ \longrightarrow&\sum_{\alpha\beta}H(\beta,\alpha)\int_D M_{\beta\alpha}(x)\boldsymbol{b}_\alpha^{t}(x)\ dx,\end{aligned}\tag{7.43}$$

as $n\to\infty$, for every $t\in\mathcal{T}\setminus\{0\}$.

Consider $(\bar{\boldsymbol{\nu}}_n)_t := \sum_{\alpha\beta} M_{\beta\alpha}(\boldsymbol{b}_n)_\alpha^t(\delta_\beta\otimes(\boldsymbol{\lambda}_n)_\alpha^t)$, for every $t\in(0,T]$; we have

$$\begin{aligned}&\sup_n\sup_{t\in(0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\alpha,F)|^{2r}\ d(\bar{\boldsymbol{\nu}}_n)_t(x,\alpha,F)\\ =&\sup_n\sup_{t\in(0,T]}\int_D\sum_{\alpha\beta}M_{\beta\alpha}(x)(\boldsymbol{b}_n)_\alpha^t(x)\Big(\int_{\mathbb{R}^{N\times d}}|(\beta,F)|^{2r}\ d((\boldsymbol{\lambda}_n)_\alpha^t)^x(F)\Big)\ dx\\ \leq& q\sup_n\sup_{t\in(0,T]}\int_D\sum_{\alpha}(\boldsymbol{b}_n)_\alpha^t(x)\Big(\int_{\mathbb{R}^{N\times d}}|(\alpha,F)|^{2r}\ d((\boldsymbol{\lambda}_n)_\alpha^t)^x(F)\Big)\ dx\\ &+\sup_\alpha|\alpha|^{2r}\\ =& q\sup_n\sup_{t\in(0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\alpha,F)|^{2r}\ d(\boldsymbol{\nu}_n)_t(x,\alpha,F)+K,\end{aligned}$$

for $K := \sup_\alpha|\alpha|^{2r}$; therefore we can deduce from (7.26) that

$$\sup_n\sup_{t\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\alpha,F)|^{2r}\ d(\bar{\boldsymbol{\nu}}_n)_t(x,\alpha,F)\leq K+qC.$$

In particular for every $t\in\mathcal{T}\setminus\{0\}$, we deduce from (7.31) and (7.37) that

$$(\bar{\boldsymbol{\nu}}_{n_k^t})_t\rightharpoonup\bar{\boldsymbol{\nu}}_t:=\sum_{\alpha\beta}M_{\beta\alpha}\boldsymbol{b}_\alpha^t(\delta_\beta\otimes\boldsymbol{\lambda}_\alpha^t)\quad 2r\text{-weakly}^*,\tag{7.44}$$

as $k \to \infty$, where $(\boldsymbol{b}, \boldsymbol{\lambda})$ is the pair defined by (7.29), (7.31), and (7.37). Since $|W(\alpha, F + \nabla\tilde{u}(x))| \le C(1 + |\nabla\tilde{u}(x)|^2) + C|F|^2$, we can use a suitable version of [13, Remark 4.3] in the case $2r$ instead of 2, to deduce from (7.44) that

$$\begin{aligned}&\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_{n_k^t})^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\beta, F + \nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}_{n_k^t})^t_\alpha)^x(F)\ dx\\ \longrightarrow &\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)\boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\beta, F + \nabla\tilde{u}(x))\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\ dx,\end{aligned} \tag{7.45}$$

as $k \to \infty$. Analogously we deduce that

$$\begin{aligned}&\sum_{\alpha}\int_D (\boldsymbol{b}_{n_k^t})^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d((\boldsymbol{\lambda}_{n_k^t})^t_\alpha)^x(F)\Big)\ dx\\ \longrightarrow &\sum_{\alpha}\int_D \boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx,\end{aligned} \tag{7.46}$$

as $k \to \infty$; therefore using (7.42), (7.46), (7.45), and (7.43), we can deduce immediately (ev1), for every $t \in \mathcal{T} \setminus \{0\}$, while for $t = 0$ it is an obvious consequence of (ev0) and the hypothesis on the initial datum. For $t \in [0, T] \setminus \mathcal{T}$, (ev1) can be easily proved using (7.40) and (ev1) for $t \in \mathcal{T}$, as in [13, Section 7.3].

7.5. **Upper energy estimate.** Let us fix $t \in \mathcal{T}$. We have

$$\begin{aligned}&\sum_{\alpha}\int_D \boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; 0, t)\\ \le &\liminf_k \Big[\sum_{\alpha}\int_D (\boldsymbol{b}_{n_k^t})^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d((\boldsymbol{\lambda}_{n_k^t})^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}_{n_k^t}; 0, t)\Big];\end{aligned}$$

using (7.28), we can deduce that

$$\begin{aligned}&\sum_{\alpha}\int_D \boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; 0, t)\\ \le &\mathcal{W}(z_0, v_0) + \limsup_n \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\ ds;\end{aligned}$$

since $\sup_t \sup_n \|\boldsymbol{\sigma}_n(t)\|_2$ is finite, we have by Fatou Lemma

$$\begin{aligned}&\limsup_n \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\ ds\\ \le &\int_0^T \limsup_n [1_{[0,\tau^n(t)]}(s)\langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2]\ ds\\ = &\int_0^t \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\ ds.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{\alpha}\int_D \boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; 0, t)\\ \le &\mathcal{W}(z_0, v_0) + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle_2\ ds\end{aligned} \tag{7.47}$$

Using (7.40), (7.40), and the left continuity of $\boldsymbol{b}$, the same argument as in [13, Section 7.3] proves (ev2) for $t \in [0, T] \setminus \mathcal{T}$.

7.6. **Lower energy estimate.** The standard procedure to prove the lower energy estimate uses a special minimality condition satisfied by the limit of the approximate solutions thanks to the stability property (see [15, Step 5, p. 7]). In our case, the partial-global stability is not powerful enough to guarantee this minimality property, because of the restriction on the set of competitors. Nevertheless, the desired minimality can be partially recovered using the properties of the approximate solutions.

Let us consider first $s < t$ with $s \in \mathcal{T}'$ and $t \in \mathcal{T}$.

Thanks to the minimality property satisfied by the approximate solutions defined in Subsection 7.1, and to the triangle inequality for H, we get

$$\langle W, (\boldsymbol{b}_n^s, \boldsymbol{\lambda}_n^s)\rangle \leq \langle W, (\boldsymbol{b}_n^t, \tilde{\mathcal{T}}_{\nabla\varphi(\tau^n(s))-\nabla\varphi(\tau^n(t))}(\boldsymbol{\lambda}_n^t))\rangle + \langle H, \boldsymbol{b}_n^{st}\rangle. \tag{7.48}$$

In other words, we have

$$\begin{aligned}
&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)_\alpha^s W(\alpha, F)\ d(\boldsymbol{\lambda}_n)_\alpha^s(x,F) \\
\leq &\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)_\alpha^t W(\alpha, F - \nabla\varphi(\tau^n(t)) + \nabla\varphi(\tau^n(s)))\ d(\boldsymbol{\lambda}_n)_\alpha^t(x,F) \\
&+ \sum_{\alpha\beta} H(\beta,\alpha) \int_D (\boldsymbol{b}_n)_{\alpha\beta}^{st}(x)\ dx.
\end{aligned}$$

We can rewrite the previous inequality as follows

$$\begin{aligned}
&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)_\alpha^s W(\alpha, F)\ d(\boldsymbol{\lambda}_n)_\alpha^s(x,F) \\
\leq &\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)_\alpha^t W(\alpha, F - \nabla\varphi(t) + \nabla\varphi(s))\ d(\boldsymbol{\lambda}_n)_\alpha^t(x,F) \\
&+ \sum_{\alpha\beta} H(\beta,\alpha) \int_D (\boldsymbol{b}_n)_{\alpha\beta}^{st}(x)\ dx + \rho_n',
\end{aligned} \tag{7.49}$$

with

$$\begin{aligned}
\rho_n' \quad := \quad &\Big| \sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)_\alpha^t \big[W(\alpha, F - \nabla\varphi(\tau^n(t)) + \nabla\varphi(\tau^n(s))) + \\
&\qquad\qquad -W(\alpha, F - \nabla\varphi(t) + \nabla\varphi(s))\big]\ d(\boldsymbol{\lambda}_n)_\alpha^t(x,F)\Big|.
\end{aligned}$$

Since $s \mapsto \boldsymbol{\varphi}(s)$ is continuous from $[0,T]$ into $W^{1,p}(D;\mathbb{R}^N)$, using (W.2), Hölder inequality, and (7.24), we can easily deduce that $\rho_n' \to 0$, as $n \to \infty$.

Since $s \in \mathcal{T}'$ and $t \in \mathcal{T}$, we will have

$$\begin{aligned}
(\boldsymbol{b}_n)_{\alpha\beta}^{st} &\rightharpoonup \boldsymbol{b}_{\alpha\beta}^{st} \quad L^\infty\text{-weakly* as } n \to \infty \\
(\boldsymbol{b}_n)_\alpha^s(\boldsymbol{\lambda}_n)_\alpha^s &\rightharpoonup \boldsymbol{b}_\alpha^s\boldsymbol{\lambda}_\alpha^s \quad 2r\text{-weakly*, as } n \to \infty, \\
(\boldsymbol{b}_{n_k^t})_\alpha^t(\boldsymbol{\lambda}_{n_k^t})_\alpha^t &\rightharpoonup \boldsymbol{b}_\alpha^t\boldsymbol{\lambda}_\alpha^t \quad 2r\text{-weakly*, as } k \to \infty,
\end{aligned}$$

for every α, β, where n_k^t is the sequence of integers chosen in (7.35), if $t \notin \mathcal{T}'$. Hence passing to the limit in (7.49) we get

$$\begin{aligned}
&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^s_\alpha(x) W(\alpha, F)\ d\boldsymbol{\lambda}^s_\alpha(x,F) \\
=&\lim_n \sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_n)^s_\alpha(x) W(\alpha, F)\ d(\boldsymbol{\lambda}_n)^s_\alpha(x,F) \\
=&\lim_k \sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_{n_k^t})^s_\alpha(x) W(\alpha, F)\ d(\boldsymbol{\lambda}_{n_k^t})^s_\alpha(x,F) \\
\le&\lim_k \Big[\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} (\boldsymbol{b}_{n_k^t})^t_\alpha(x) W(\alpha, F-\nabla\varphi(t)+\nabla\varphi(s))\ d(\boldsymbol{\lambda}_{n_k^t})^t_\alpha(x,F) \\
&\qquad + \sum_{\alpha\beta} H(\beta,\alpha)\int_D (\boldsymbol{b}_{n_k^t})^{st}_{\alpha\beta}(x)\ dx + \rho'_{n_k^t}\Big] \\
=&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x) W(\alpha, F-\nabla\varphi(t)+\nabla\varphi(s))\ d\boldsymbol{\lambda}^t_\alpha(x,F) \\
&+\sum_{\alpha\beta} H(\beta,\alpha)\int_D \boldsymbol{b}^{st}_{\alpha\beta}(x)\ dx.
\end{aligned} \tag{7.50}$$

Hence we have obtained

$$\begin{aligned}
&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^s_\alpha(x) W(\alpha, F)\ d\boldsymbol{\lambda}^s_\alpha(x,F) \\
\le&\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x) W(\alpha, F-\nabla\varphi(t)+\nabla\varphi(s))\ d\boldsymbol{\lambda}^t_\alpha(x,F) \\
&+\sum_{\alpha\beta} H(\beta,\alpha)\int_D \boldsymbol{b}^{st}_{\alpha\beta}(x)\ dx.
\end{aligned} \tag{7.51}$$

If $\tau \in [0,T]\setminus\mathcal{T}$, thanks to the left continuity of $\boldsymbol{b}$ and to (7.40) there exists a sequence s_j in $\mathcal{T}$ with $s_j \le \tau$ and converging to τ such that

$$\boldsymbol{b}^{s_j}_\alpha \boldsymbol{\lambda}^{s_j}_\alpha \rightharpoonup \boldsymbol{b}^\tau_\alpha \boldsymbol{\lambda}^\tau_\alpha \quad 2r\text{-weakly}^*, \quad \text{as } j\to\infty. \tag{7.52}$$

For every j (7.51) holds true for $t = s_j$, hence we use (7.52), (W.2), Hölder inequality, and the continuity of the map $t \mapsto \nabla\varphi(t)$ to pass to the limit as $j\to\infty$ and to obtain (7.51) for $s \in \mathcal{T}'$ and $\tau \in [0,T]\setminus\mathcal{T}$.

By changing the choice of the subsequence in (7.50), we can obtain (7.51) in the case of $s \in \mathcal{T}$ and $t \in \mathcal{T}'$, and consequently for $s \in [0,T]$ and $t \in \mathcal{T}'$.

In particular, we observe that, unlike the classical case in which it comes from the stability condition, we are not able to obtain (7.51) if both s and t do not belong to $\mathcal{T}'$.

We can rewrite (7.51) as follows

$$\begin{aligned}
\langle W, (\boldsymbol{b}^s, \boldsymbol{\lambda}^s)\rangle \le& \langle W, (\boldsymbol{b}^t, \boldsymbol{\lambda}^t)\rangle + \langle H, \boldsymbol{b}^{st}\rangle \\
&- \int_s^t \langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\ d\tau + \rho(s,t),
\end{aligned} \tag{7.53}$$

where

$$\rho(s,t) := \int_s^t \Big\{ \sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x) \Big[-\frac{\partial W}{\partial F}(\alpha, F + \nabla\boldsymbol{\varphi}(\tau) - \nabla\boldsymbol{\varphi}(t)) \\ + \frac{\partial W}{\partial F}(\alpha, F)\Big] \nabla\dot{\boldsymbol{\varphi}}(\tau)\, d\boldsymbol{\lambda}^t_\alpha(x,F) \Big\}\, d\tau.$$

Let us now fix $t \in [0,T]$. We consider a sequence of partitions $0 = s_j^0 < s_j^1 \cdots < s_j^{i_j} = t$ with $s_j^1 \le 1/j, t - s_j^{i_j-1} \le 1/j$ and $s_j^i - s_j^{i-1} = 1/j$ for $i = 2, \dots, i_j - 1$; we would like to choose these partitions in such a way that

$$\text{for every } i = 1,\dots,i_j, \quad s_j^i \in \mathcal{T}' \quad \text{or} \quad s_j^{i-1} \in \mathcal{T}'; \tag{7.54}$$

in this way (7.53) would hold true for s_j^{i-1}, s_j^i, for every i, and we could iterate it between 0 and t and pass to the limit as $j \to \infty$. Unfortunately, to recover the lower energy estimate in the limit, we need to approximate a Lebesgue integral by Riemann sums (see [15, 4, Step 5]): this can be done for a careful choice of the sequence of partitions $0 = s_j^0 < s_j^1 \cdots < s_j^{i_j} = t$ (see e.g. [10, Lemma 4.12]), but nothing guarantees that the appropriate sequence of partitions satisfies (7.54). We recall the statement of the measure theoretic result for the reader's convenience.

Lemma 7.6. *Let X be a Banach space, and let $F\colon [0,t] \to X$ be a Bochner integrable function. Then there exists a sequence of partitions $\mathcal{S} := \{s_j^i,\ 0 \le i \le i_j, j \in \mathbb{N}\}$ of the interval $[0,t]$, with*

$$0 = s_k^0 < \cdots < s_j^{i_j-1} < s_j^{i_j} = t,$$

$$s_j^1 \le 1/j, \quad t - s_j^{i_j-1} \le 1/j, \tag{7.55}$$

$$s_j^i - s_j^{i-1} = 1/j \quad \textit{for } i = 2,\dots,i_j - 1, \tag{7.56}$$

such that

$$\lim_j \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \|F(s_j^i) - F(\tau)\|\, d\tau = 0. \tag{7.57}$$

We apply this Lemma to the functional defined by

$$\begin{array}{rcl} F\colon [0,t] & \to & L^2(D;\mathbb{R}^N) \times \mathbb{R} \\ \tau & \mapsto & (\nabla\dot{\boldsymbol{\varphi}}(\tau), \langle \boldsymbol{\sigma}(\tau), \nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2), \end{array} \tag{7.58}$$

in order to find a sequence of partitions $\mathcal{S} := \{s_j^i,\ 0 \le i \le i_j, j \in \mathbb{N}\}$ of $[0,t]$ satisfying (7.55) and (7.56), and such that

$$\lim_j \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} \|\nabla\dot{\boldsymbol{\varphi}}(s_j^i) - \nabla\dot{\boldsymbol{\varphi}}(\tau)\|_2\, d\tau = 0, \tag{7.59}$$

$$\lim_j \sum_{i=1}^{i_j} \int_{s_j^{i-1}}^{s_j^i} |\langle \boldsymbol{\sigma}(s_j^i), \nabla\dot{\boldsymbol{\varphi}}(s_j^i)\rangle_2 - \langle \boldsymbol{\sigma}(\tau), \nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2|\, d\tau = 0. \tag{7.60}$$

Whenever both s_j^{i-1} and s_j^i belong to $[0,T]\setminus\mathcal{T}'$, we consider $t_j^{i-1}\in(s_j^{i-1},s_j^{i-1}+1/j^2)\cap\mathcal{T}'$, so that (7.53) holds true for s_j^{i-1},t_j^{i-1} and t_j^{i-1},s_j^i. Hence we get

$$\begin{aligned}\langle W,(\boldsymbol{b}^{s_j^{i-1}},\boldsymbol{\lambda}^{s_j^{i-1}})\rangle \le&\langle W,(\boldsymbol{b}^{s_j^i},\boldsymbol{\lambda}^{s_j^i})\rangle+\langle H,\boldsymbol{b}^{s_j^{i-1}t_j^{i-1}}\rangle+\langle H,\boldsymbol{b}^{t_j^{i-1}s_j^i}\rangle\\&-\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\\&-\int_{s_j^{i-1}}^{t_j^{i-1}}\langle\boldsymbol{\sigma}(t_j^{i-1})-\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\\&+\rho(s_j^{i-1},t_j^{i-1})+\rho(t_j^{i-1},s_j^i),\end{aligned}$$

where

$$\begin{aligned}\rho(s_j^{i-1},t_j^{i-1})=\int_{s_j^{i-1}}^{t_j^{i-1}}\Big\{\sum_\alpha\int_{D\times\mathbb{R}^{N\times d}}\boldsymbol{b}_\alpha^{t_j^{i-1}}(x)\Big[&-\frac{\partial W}{\partial F}(\alpha,F+\nabla\boldsymbol{\varphi}(\tau)-\nabla\boldsymbol{\varphi}(t_j^{i-1}))\\&+\frac{\partial W}{\partial F}(\alpha,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(\tau)\;d\boldsymbol{\lambda}_\alpha^{t_j^{i-1}}(x,F)\Big\}\,d\tau,\\ \rho(t_j^{i-1},s_j^i)=\int_{t_j^{i-1}}^{s_j^i}\Big\{\sum_\alpha\int_{D\times\mathbb{R}^{N\times d}}\boldsymbol{b}_\alpha^{s_j^i}(x)\Big[&-\frac{\partial W}{\partial F}(\alpha,F+\nabla\boldsymbol{\varphi}(\tau)-\nabla\boldsymbol{\varphi}(s_j^i))\\&+\frac{\partial W}{\partial F}(\alpha,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(\tau)\;d\boldsymbol{\lambda}_\alpha^{s_j^i}(x,F)\Big\}\,d\tau.\end{aligned}$$

We iterate now from 0 to t; using (ev0) and

$$\sum_{i=1}^{i_j}\big(\langle H,\boldsymbol{b}^{s_j^{i-1}t_j^{i-1}}\rangle+\langle H,\boldsymbol{b}^{t_j^{i-1}s_j^i}\rangle\big)\le \mathrm{Diss}_H(\boldsymbol{b};0,t),$$

we get

$$\begin{aligned}&\mathcal{W}(z_0,v_0)-\langle W,(\boldsymbol{b}^t,\boldsymbol{\lambda}^t)\rangle-\mathrm{Diss}_H(\boldsymbol{b};0,t)\\&\le-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{t_j^{i-1}}\langle\boldsymbol{\sigma}(t_j^{i-1})-\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\\&\quad+\sum_{i=1}^{i_j}\big[\rho(s_j^{i-1},t_j^{i-1})+\rho(t_j^{i-1},s_j^i)\big].\end{aligned}$$

Reasoning as in [13, Lemma 7.5], we can deduce that

$$\sum_{i=1}^{i_j}\big[\rho(s_j^{i-1},t_j^{i-1})+\rho(t_j^{i-1},s_j^i)\big]\to0\quad\text{as } j\to\infty.$$

Moreover, using Hölder inequality and the fact that $\sup_t \|\boldsymbol{\sigma}(t)\|_2$ is bounded, we can deduce that

$$\begin{aligned}
&\Big|\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{t_j^{i-1}}\langle \boldsymbol{\sigma}(t_j^{i-1})-\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\Big|\\
\leq&2\sup_t\|\boldsymbol{\sigma}(t)\|_2\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{t_j^{i-1}}\|\nabla\dot{\boldsymbol{\varphi}}(\tau)\|_2\,d\tau\\
=&2\sup_t\|\boldsymbol{\sigma}(t)\|_2\int_{\bigcup_{i=1}^{i_j}[s_j^{i-1},t_j^{i-1}]}\|\nabla\dot{\boldsymbol{\varphi}}(\tau)\|_2\,d\tau.
\end{aligned}$$

Since $\mathcal{L}^1(\bigcup_i[s_j^{i-1},t_j^{i-1}])\leq (tj+2)/j^2\to 0$ as $j\to\infty$, and $\nabla\dot{\boldsymbol{\varphi}}\in L^1([0,T];L^2(D;\mathbb{R}^{N\times d}))$, we get $\int_{\bigcup_i[s_j^{i-1},t_j^{i-1}]}\|\nabla\dot{\boldsymbol{\varphi}}(\tau)\|\,d\tau\to 0$ as $j\to\infty$.

Hence it remains only to prove that

$$\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau-\int_0^t\langle\boldsymbol{\sigma}(\tau)\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\longrightarrow 0,\tag{7.61}$$

as $j\to\infty$.

Let us first prove that

$$\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(s_j^i)\rangle_2\,d\tau\longrightarrow 0,\tag{7.62}$$

as $j\to\infty$.

Thanks to (7.59), using Hölder inequality and the fact that $\sup_{t\in[0,T]}\|\boldsymbol{\sigma}(t)\|_2$ is finite we deduce that

$$\begin{aligned}
&\Big|\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau-\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(s_j^i)\rangle_2\,d\tau\Big|\\
\leq&\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\|\boldsymbol{\sigma}(s_j^i)\|_2\|\nabla\dot{\boldsymbol{\varphi}}(\tau)-\nabla\dot{\boldsymbol{\varphi}}(s_j^i)\|_2\,d\tau\leq\\
\leq&\sup_{t\in[0,T]}\|\boldsymbol{\sigma}(t)\|_2\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\|\nabla\dot{\boldsymbol{\varphi}}(\tau)-\nabla\dot{\boldsymbol{\varphi}}(s_j^i)\|_2\,d\tau\to 0,
\end{aligned}$$

as $j\to\infty$.

Finally

$$\Big|\sum_{i=1}^{i_j}\int_{s_j^{i-1}}^{s_j^i}\langle\boldsymbol{\sigma}(s_j^i),\nabla\dot{\boldsymbol{\varphi}}(s_j^i)\rangle_2\,d\tau-\int_0^t\langle\boldsymbol{\sigma}(\tau)\nabla\dot{\boldsymbol{\varphi}}(\tau)\rangle_2\,d\tau\Big|\longrightarrow 0,\tag{7.63}$$

as $j\to\infty$, thanks to (7.60).

8. **Euler conditions.** In this section we derive the Euler equations for the partial-global stability condition.

Theorem 8.1. *Let $(b, \lambda) \in L^\infty(D; [0,1])^q \times Y(D; \mathbb{R}^{N\times d})^q$ satisfy (3.5) and (3.13) with $p = 2$. Assume that (b, λ) satisfies*

$$\begin{aligned}
&\sum_{\alpha=1}^{q} \int_D b_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\alpha, F)\, d\lambda_\alpha^x(F)\Big)\, dx \\
&\leq \sum_{\alpha,\beta=1}^{q} \int_D M_{\beta\alpha}(x) b_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\beta, F + \nabla\tilde{u}(x))\, d\lambda_\alpha^x(F)\Big)\, dx \\
&\quad + \sum_{\alpha,\beta=1}^{q} H(\beta,\alpha) \int_D M_{\beta\alpha}(x) b_\alpha(x)\, dx,
\end{aligned} \tag{8.1}$$

for every $\tilde{u} \in H^1_0(D; \mathbb{R}^N)$ and every measurable map $M\colon D \to \mathbf{M}^{q\times q}_{St}$, and denote by σ the stress, i.e.,

$$\sigma(x) := \sum_{\alpha=1}^{q} b_\alpha(x) \int_{\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\alpha, F)\, d\lambda_\alpha^x(F), \quad \text{for a.e. } x \in D.$$

Then the following conditions are satisfied:

(ec)$_1$ equilibrium condition*: $\mathrm{div}\sigma(t) = 0$;*

(ec)$_2$ optimality of active phases*: for every $\alpha, \beta = 1, \ldots, q$ and every $t \in [0, T]$, we have*

$$\int_{\mathbb{R}^{N\times d}} [W(\alpha, F) - W(\beta, F)]\, d\lambda_\alpha^x(F) \leq H(\beta, \alpha),$$

for a.e. $x \in D$ with $b_\alpha(x) > 0$.

Remark 8.2. We say that a phase α is *active* at x if $b_\alpha(x) > 0$. Hence the condition (ec)$_2$ can be rephrased as follows: if α is active at x, then α is a minimizer over Z of the functional

$$\beta \mapsto \int_{\mathbb{R}^{N\times d}} W(\beta, F)\, d\lambda_\alpha^x(F) + H(\beta, \alpha)$$

This is the reason why we call (ec)$_2$ optimality of active phases.

Remark 8.3. Note that from (ec)$_2$ it descends immediately that

$$\sum_{\alpha,\beta} M_{\beta\alpha} b_\alpha(x) \int_{\mathbb{R}^{N\times d}} [W(\alpha, F) - W(\beta, F) - H(\beta, \alpha)]\, d\lambda_\alpha^x(F) \leq 0,$$

for a.e. $x \in D$ and every stochastic matrix $M \in \mathbf{M}^{q\times q}_{St}$.

Proof of Theorem 8.1. Let (b, λ) satisfy the prescribed hypotheses. Choosing in (8.1) the map M associating to every $x \in D$ the identity matrix $\mathbb{I}$, we obtain

$$\sum_{\alpha} \int_D b_\alpha(x)\Big[\int_{\mathbb{R}^{N\times d}} \big[W(\alpha, F + \nabla\tilde{u}(x)) - W(\alpha, F)\big]\, d\lambda_\alpha)^x(F)\Big]\, dx \geq 0,$$

for every $\tilde{u} \in H^1_0(D; \mathbb{R}^N)$, which implies immediately (ec)$_1$.

Let us denote by $(e_\gamma)_{\gamma=1}^q$ the canonical basis of the vector space $\mathbb{R}^q$. Fixed α, β in $\{1, \ldots, q\}$, define $\bar{M} \in \mathbf{M}^{q\times q}_{St}$ by

$$\begin{aligned}
\bar{M} e_\gamma &= e_\gamma \quad \text{for every } \gamma \neq \alpha \\
\bar{M} e_\alpha &= e_\beta.
\end{aligned}$$

Let us choose now in (8.1) $\tilde{u} = 0$ and $M := \mathbb{I}(1 - 1_A) + \bar{M}1_A$, for any measurable subset A of D: we obtain

$$\begin{aligned}&\int_A b_\alpha(x)\Big[\int_{\mathbb{R}^{N\times d}} \big[W(\alpha, F) - W(\beta, F)\big]\; d\lambda^x_\alpha(F)\Big]\; dx\\ &\le \int_A H(\beta, \alpha) b_\alpha(x)\; dx;\end{aligned} \tag{8.2}$$

By the free choice of A among all measurable subsets of D, from (8.2) we deduce immediately $(\mathrm{ec})_2$. □

Appendix. In this Appendix we briefly recall the notion of cubic quasi-minimum, introduced by Giaquinta and Giusti in [16], and the related results.

Given $\varphi \in H^1(D;\mathbb{R}^N)$, let $\mathcal{G}$ be the functional defined by

$$\mathcal{G}(v) = \mathcal{G}(v, D) := \int_D G(x, \nabla v(x))\; dx$$

for every $v \in \varphi + H^1_0(D;\mathbb{R}^N)$, where $G\colon D \times \mathbb{R}^{N\times d} \to \mathbb{R}$ is a function satisfying

$$|G(x, F)| \le L(|F|^2 + 1)$$

for a suitable positive constant L, for every $(x, F) \in D \times \mathbb{R}^{N\times d}$.

Definition 8.4. Let $Q > 0$. A function $v \in H^1(D;\mathbb{R}^{N\times d})$ is said to be a *cubic Q-quasi-minimum for the functional $\mathcal{G}$ if for every cube of side R, $Q_R \subset\subset D$, and for every function $w \in H^1(D;\mathbb{R}^{N\times d})$, with* $\operatorname{supp}(v - w) \subseteq Q_R$*, we have*

$$\mathcal{G}(v, Q_R) \le Q\mathcal{G}(w, Q_R).$$

We restrict our analysis to the particular case of $G(F) = 1 + |F|^2$, since this is the integrand we will consider; for the reader's convenience, we recall the statement and the proof of the Caccioppoli inequality for quasi-minima of the corresponding integral functional: for our purposes, we need a slightly different statement of the result contained in [17, Theorem 6.5]; our statement does not involve the L^{2^*}-norm of the quasi-minimum but it is valid for every cube Q_R. The precise result we will use is the following.

Theorem 8.5. *Let $v \in H^1(D;\mathbb{R}^N)$ be a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w|^2)\; dx.$$

Then there exist a positive constant $C > 0$, depending only on Q, such that

$$\fint_{Q_{R/2}} |\nabla v|^2\; dx \le C\Big\{\Big(\fint_{Q_R} |\nabla v|^{2m}\; dx\Big)^{\frac{1}{m}} + 1\Big\}, \tag{8.3}$$

for every cube $Q_R \subset\subset D$, where $m = \dfrac{d}{2+d}$.

Proof. Let $R/2 < t < s \le R$. We consider a cut-off function $\eta \in \mathcal{C}^\infty_0(Q_s)$, with $0 \le \eta \le 1$, $\eta \equiv 1$ on Q_t, and $|\nabla\eta| \le \frac{2}{s-t}$. Let $\phi := \eta(v - v_s)$, where v_s denotes the mean value $(v)_{Q_s}$ of v over Q_s; define a function w by $w := v - \phi$, so that $w = v_s + (1 - \eta)(v - v_s)$. We have

$$\int_{Q_s} |\nabla\phi|^2\; dx \le \int_{Q_s} (|\nabla v|^2 + 1)\; dx + \int_{Q_s} \big||\nabla\phi|^2 - |\nabla v|^2\big|\; dx; \tag{8.4}$$

Since by construction $\nabla\phi = \nabla v$ on Q_t, we have

$$\begin{aligned}\int_{Q_s} \big||\nabla\phi|^2 - |\nabla v|^2\big|\,dx &= \int_{Q_s\setminus Q_t} \big||\nabla\phi|^2 - |\nabla v|^2\big|\,dx \\ &\le 2\Big[\int_{Q_s\setminus Q_t} |\nabla v|^2\,dx + \int_{Q_s} |\nabla w|^2\,dx\Big].\end{aligned} \tag{8.5}$$

Moreover, by the quasi-minimum property of v, we have

$$\int_{Q_s} (|\nabla v|^2 + 1)\,dx \le Q \int_{Q_s} (|\nabla w|^2 + 1)\,dx; \tag{8.6}$$

therefore (8.4), (8.5), and (8.6) imply

$$\begin{aligned}\int_{Q_t} |\nabla v|^2\,dx = \int_{Q_t} |\nabla\phi|^2\,dx &\le \int_{Q_s} |\nabla\phi|^2\,dx \\ &\le (Q+2)\int_{Q_s} (|\nabla w|^2+1)\,dx + 2\int_{Q_s\setminus Q_t} |\nabla v|^2\,dx.\end{aligned} \tag{8.7}$$

Using the relation

$$|\nabla w|^2 = |(1-\eta)\nabla v + (v - v_s)\nabla\eta|^2 \le c[(1-\eta)^2|\nabla v|^2 + (s-t)^{-2}|v - v_s|^2],$$

we obtain from (8.7)

$$\begin{aligned}\int_{Q_t} |\nabla v|^2\,dx \le &(c+1)(Q+2)\Big\{\int_{Q_s\setminus Q_t} |\nabla v|^2\,dx \\ &+ \frac{1}{(s-t)^2}\int_{Q_s} |v - v_s|^2\,dx + |Q_s|\Big\}.\end{aligned} \tag{8.8}$$

Since we have

$$\int_{Q_s} |v - v_s|^2\,dx \le c\int_{Q_R} |v - v_R|^2\,dx,$$

(8.8) implies

$$\begin{aligned}\int_{Q_t} |\nabla v|^2\,dx \le &(c+1)(Q+2)\Big\{\int_{Q_s\setminus Q_t} |\nabla v|^2\,dx \\ &+ \frac{1}{(s-t)^2}\int_{Q_R} |v - v_R|^2\,dx + |Q_s|\Big\}.\end{aligned} \tag{8.9}$$

Now we use the "hole filling" method: we add to both terms of (8.9) the quantity

$$(c+1)(Q+2)\int_{Q_t} |\nabla v|^2\,dx,$$

to get

$$\int_{Q_t} |\nabla v|^2\,dx \le \alpha\int_{Q_s} |\nabla v|^2\,dx + \frac{1}{(s-t)^2}\int_{Q_R} |v - v_R|^2\,dx + |Q_R|, \tag{8.10}$$

with $1 > \alpha := \frac{(c+1)(Q+2)}{(c+1)(Q+2)+1}$. Therefore, we are in the position to apply the same technical Lemma as in [17] (see [17, Lemma 6.1]), obtaining

$$\int_{Q_{R/2}} |\nabla v|^2\,dx \le c\Big\{\frac{1}{R^2}\int_{Q_R} |v - v_R|^2\,dx + |Q_R|\Big\}. \tag{8.11}$$

Set now $2_* := \frac{2d}{2+d}$, we have $2_* < d$ and

$$(2_*)^* = \frac{2_* d}{d - 2_*} = 2;$$

hence, by the Sobolev-Poincaré inequality (see [17, formula (3.32)]), we have

$$\int_{Q_R} |v - v_R|^2 \, dx \le c\Big(\int_{Q_R} |\nabla v|^{2_*} \, dx\Big)^{2/2_*} = c\Big(\int_{Q_R} |\nabla v|^{2m} \, dx\Big)^{1/m},$$

which together with (8.11) gives (8.3). □

If we deal with quasi-minima satisfying a prescribed boundary condition, the following result can be proved with similar arguments (see [17, Section 6.5]).

Theorem 8.6. *Let $V \in W^{1,p}(D;\mathbb{R}^N)$, for $2 < p$, and let $v \in V + H^1_0(D;\mathbb{R}^N)$ be a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w(x)|^2) \, dx,$$

i.e., for every cube $Q_R \subset \mathbb{R}^d$, and every function w such that $v - w \in H^1_0(D \cap Q_R)$ we have

$$\int_{(Q_R \cap D)} (1 + |\nabla v|^2) \, dx \le Q \int_{Q_R \cap D} (1 + |\nabla w|^2) \, dx.$$

Then there exist a positive constant $C > 0$, depending only on Q, such that

$$\fint_{Q_{R/2}} |\nabla(v - V)|^2 \, dx \le C\Big\{\Big(\fint_{Q_R} |\nabla(v - V)|^{2m} \, dx\Big)^{\frac{1}{m}} + 1\Big\}, \tag{8.12}$$

for every cube $Q_R \subset \mathbb{R}^d$, where $m = \dfrac{d}{2+d}$ and $v - V$ is extended to 0 in $Q_R \setminus D$.

Using Theorem 8.5 and Theorem 8.6, we can obtain as in [17, Theorem 6.8] the following result

Theorem 8.7. *Let $V \in W^{1,p}(D;\mathbb{R}^N)$, for $2 < p$, and let $v \in V + H^1_0(D;\mathbb{R}^N)$ a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w(x)|^2) \, dx.$$

Then there exist constants $\gamma > 0$ and $r > 1$, depending only on Q and V, such that

$$\int_D |\nabla v|^{2r} \, dx \le \gamma\Big\{\big(\int_D |\nabla v|^2 \, dx\big)^r + 1\Big\}. \tag{8.13}$$

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