Homogenization of Penrose tilings

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1 Introduction

In this paper we deal with the problem of the homogenization of integral energies where the spatial dependence follows the geometry of a "Penrose tiling"; that is, we consider functionals of the form

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m) \tag{1}$$

where Ω is an open subset of $\mathbb{R}^2$, and f depends on x also through the shape and the orientation of the cell contaning x in an a-periodic tiling of the space $\mathbb{R}^2$. As an example we may consider mixtures of two (linear) conducting materials with different dielectric constants depending on the type of the tile, and $f(y, Du) = a(y)|Du|^2$, where a takes two values α, β depending whether y is in one type of tile or the other, or the mixture of two elastic materials, etc.. We also include the case when f depends on the orientation of the single tile (so that we have ten different types of tiles), and may be inhomogeneous inside each tile.

We want to show that there exists the Γ-limit of the family $\{F_\varepsilon\}$ as $\varepsilon \to 0$, and that it can be represented as

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du(x))\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m) \tag{2}$$

(for the precise statement, see Theorem 2.1). This will be achieved by using a characterization of Penrose tilings which shows that the corresponding f is Besicovitch almost periodic in y, so that an existing homogenization theorem under very weak almost periodic assumptions can be applied. This method is general and can be applied to other "quasicrystalline" geometries, whenever a characterization in terms of projections from higher-dimensional lattices is available.

2 Statement and proof of the homogenization result

In order to write explicitly the spatial dependence of f and to give the precise statement of the results, we recall some details of the characterization of Penrose tilings through a certain projection of a slice of a five-dimensional cubic lattice onto an "irrational" two-dimensional plane given by de Bruijn [7]. We briefly recall the lines of his construction (see [9] and [10]).

Let Π be the two-dimensional plane in $\mathbb{R}^5$ spanned by the vectors

$$v_1 = \sum_{k=1}^{5} \sin\left(\frac{2(k-1)\pi}{5}\right) e_k \quad \text{and} \quad v_2 = \sum_{k=1}^{5} \cos\left(\frac{2(k-1)\pi}{5}\right) e_k, \tag{3}$$

where e_k is the unit vector on the k-th axis. We note that, considering the matrix M whose action is the permutation of all the coordinate axes in order, then Π is the plane of the vectors v such that the action of M on v is a rotation of $2\pi/5$. Then, we consider the set $\mathcal{Z}$ of the points $z \in \mathbb{Z}^5$ such that $z + (0,1)^5 \cap \Pi \neq \emptyset$, and the function $\phi\colon \mathbb{Z}^5 \to \mathbb{R}^2$ defined as $\phi(z) = \sum_{k=1}^{5} z_k e^{\frac{ik\pi}{5}}$. We set $\phi(\mathcal{Z}) = \mathcal{P}$.

Remark 1 (characterization of Penrose tilings). *The tiling obtained by joining p and p' in $\mathcal{P}$ by an edge if and only if $|p - p'| = 1$ is a Penrose tiling.*

We note that in the original construction of de Bruijn the tiling is obtained, in an equivalent way, by projecting onto Π the points $z \in \mathbb{Z}^5$ such that $z + (0,1)^5 \cap \Pi \neq \emptyset$. Moreover, the construction gives a Penrose tiling for any parallel plane $\gamma + \Pi$ with γ such that $\sum_{k=1}^{5} \gamma_k = 0 \pmod 1$.

We denote by $\mathcal{T}$ the set of the Penrose "cells" of the tiling in $\mathbb{R}^2$; we get two possible shapes of rhombi for the cells $T \in \mathcal{T}$, each one with five possible orientations. Then, we can define a function $a\colon \mathbb{R}^2 \to \{1, \ldots, 10\}$ in $L^\infty(\mathbb{R}^2)$ associating to each x in the inner part of a Penrose cell an index giving the shape and the orientation of the cell. Moreover, in order to fix for each cell one of the vertices, we define $v\colon \mathbb{R}^2 \to \mathcal{P}$ as the function which associates to each $x \in T$ (where T is an open cell) one of the two vertices $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$ corresponding to the angle of $\pi/5$ (or $2\pi/5$) so that $v(x) = p_i$ if $\|y_i\| < \|y_j\|$ or, when $\|y_i\| = \|y_j\|$, if $\|x_i\| < \|x_j\|$.

Now we can define the functional $F_\varepsilon\colon W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ as in (1), where

$$f(y, \xi) = f_{a(y)}(y - v(y), \xi)$$

and, for any $a \in \{1, 2, \ldots, 10\}$, f_a is a positive Borel function, quasiconvex in the second variable, satisfying

$$c_1|\xi|^p - 1 \leq f_a(x, \xi) \leq c_2(1 + |\xi|^p) \tag{4}$$

for some $p > 1$ with $c_1, c_2 > 0$. If the behaviour of f is homogenous inside each cell then simply $f(y, \xi) = f_{a(y)}(\xi)$.

We prove the following homogenization theorem for the sequence of functionals F_ε (for details on Γ-convergence we refer to [3, 4, 6], for the homogenization of multiple integrals by Γ-convergence to [5]).

Theorem 2.1 (homogenization of Penrose tilings)**.** *Let f be as above. Then the sequence of functional F_ε defined in* (1) *Γ-converges on $W^{1,p}(\Omega)$ with respect to the L^p convergence to the functional* (2) *where*

$$f_{\mathrm{hom}}(\xi) = \lim_{S\to+\infty} \inf \left\{ \frac{1}{S^2} \int_{(0,S)^2} f(y, Dv(y) + \xi)\, dy : v \in W_0^{1,p}((0,S)^2; \mathbb{R}^m) \right\}. \qquad (5)$$

The proof is based on the application of a homogenization result for Besicovitch almost-periodic functionals obtained in [2] (see also [5, Th. 17.10]). We recall that a measurable function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a (real) Besicovitch almost-periodic function if it is the limit in the mean of a sequence of trigonometric polynomials (i.e., linear combinations of trigonometric monomials with possibly incommensurable periods) on $\mathbb{R}^n$.

Remark 2 (a criterion for almost periodicity). We recall that a set $A \subset \mathbb{R}^n$ is *relatively dense* if for any there exists an *inclusion length* $L > 0$ such that $A + [0, L)^n = \mathbb{R}^n$.

The function φ is a W^1-*almost periodic function* if it satisfies the following condition (see [1, p. 77], and [8] for the generalization to the n-dimensional case). For any $\eta > 0$, there exists $S_\eta > 0$ large enough that we can find a set A_η relatively dense in $\mathbb{R}^n$ such that for any $\tau \in A_\eta$

$$\sup_{x\in\mathbb{R}^n} \left\{ \frac{1}{S_\eta^n} \int_{x+[0,S_\eta]^n} |\varphi(y+\tau) - \varphi(y)|\, dy \right\} < \eta. \qquad (6)$$

In [1, Ch. 2] some closure theorems are shown for the approximation of almost periodic functions with trigonometric polynomials, and in particular it is proved that if φ is a W^1 almost-periodic function, then it belongs to the closure of the trigonometric polynomials with respect to the mean integral distance (and then, it is a Besicovitch almost-periodic function).

The homogenization theorem of [2] ensures the thesis of Theorem 2.1 if $f(\cdot, \xi)$ is a Besicovitch almost-periodic function for all ξ. By Remark 2 the proof of Theorem 2.1 will then follow from that result if we prove that $f(\cdot, \xi)$ is a W^1 almost-periodic function for any ξ. To that end, we prove the following proposition.

Proposition 2.2 (W^1-almost periodicity of Penrose tilings)**.** *For any $\eta > 0$, there exists $S_\eta \in \mathbb{R}$ large enough that we can find a set A_η relatively dense in $\mathbb{R}^2$ such that the function $f(\cdot, \xi)$ satisfies* (6) *for any $\tau \in A_\eta$.*

Proof. Let us consider the function defined on $\mathbb{R}^2$ by $x \mapsto \mathrm{dist}(\pi(x), \mathbb{Z}^5)$, where π stands for the projection of $\mathbb{R}^2$ onto Π, *i.e.* $\pi(x_1, x_2) = x_1 v_1 + x_2 v_2$, and v_1 and v_2 are defined in (3). Note that, if p stands for the orthogonal projection of $\mathbb{Z}^5$ onto Π, then

$$\phi = \frac{5}{2}\, \pi^{-1} \circ p.$$

Indeed, since $v_1 \cdot v_2 = 0$ and $\|v_1\| = \|v_2\| = \dfrac{5}{2}$, we can write $p(z) = \dfrac{2z\cdot v_1}{5} v_1 + \dfrac{2z \cdot v_2}{5} v_2$, and $\pi(\phi(z)) = (z\cdot v_1)v_1 + (z\cdot v_2)v_2$.

The function defined by $x \mapsto \mathrm{dist}(\pi(x), \mathbb{Z}^5)$ is a quasi-periodic function (that is, it is a diagonal function of a periodic function), and it is continuous; hence it is uniformly almost periodic. Then, by the characterization of uniformly almost periodic functions [1], the set

$$\widetilde{A}_\eta = \left\{x \in \mathbb{R}^2 \ : \ \mathrm{dist}(\pi(x), \mathbb{Z}^5) < \eta\right\}$$

is relatively dense in $\mathbb{R}^2$, and the set $A_\eta = \widetilde{A}_\eta \cap \mathcal{P}$ is relatively dense too, since the points in this set are the projections of the points in $\mathbb{Z}^5$ with distance less than η from Π. Now, we have to show that there exists S_η large enough such that for every τ in A_η equation (6) holds.

We set $R_\eta = \left\{y \in \mathbb{R}^5 \ : \ \mathrm{dist}(y, \mathbb{Z}^5) < \eta\right\}$, $G(y) = \chi_{R_\eta}(y)$ and g the function defined in $\mathbb{R}^2$ by $g(x) = G(\pi(x))$. Since the function g is quasi-periodic, then Birkhoff's ergodic theorem can be applied (see *e.g.* [5, Th. A.13]). It follows that

$$\lim_{S\to+\infty} \frac{1}{S^2} \int_{x_0+[0,S]^2} g(x)\, dx = \frac{1}{|K|} \int_K G(y)\, dy$$

where the limit exists uniformly in $x_0 \in \mathbb{R}^2$ and $K = (0,1)^5$ is the periodicity torus in $\mathbb{R}^5$. Then, we get that

$$\lim_{S\to+\infty} \frac{1}{S^2} \int_{x_0+[0,S]^2} g(x)\, dx = c\eta^5$$

uniformly in $x_0 \in \mathbb{R}^2$. The same holds for the function $\overline{G}$ constructed with $R_{2\eta}$, and for the corresponding $\overline{g}$, so that we get

$$\lim_{S\to+\infty} \frac{1}{S^2} \int_{x_0+[0,S]^2} \overline{g}(x)\, dx = c'\eta^5.$$

Let $B_\rho(z)$ be the open ball with centre z and radius ρ. We note that if $B_\eta(z) \cap \Pi \neq \emptyset$, then $B_{2\eta}(z) \cap \Pi \neq \emptyset$, and the measure of the latter intersection is greater than $\tilde{c}\eta^2$ for some positive constant $\tilde{c}$. It follows that for any fixed η there exists S_η such that if $S > S_\eta$ then

$$\#\left\{z \in \mathbb{Z}^5 \ : \ \sum_{k=1}^{5} z_k e^{\frac{ik\pi}{5}} \in x_0 + [0,S]^2,\ B_\eta(z) \cap \Pi \neq \emptyset\right\} \leq CS^2\eta^3. \tag{7}$$

By construction, if $y \in \mathbb{R}^2$ belongs to a cell T such that all the vertices in $V(T)$ correspond to points $z \in \mathbb{Z}^5$ such that $\mathrm{dist}(z, \Pi) \geq \eta$, then, for a given $\tau \in A_\eta$, we get that $y+\tau$ belongs to a translate cell $T' = \tau + T$ of the same kind, hence $f(y+\tau, \xi) = f(y, \xi)$. This implies that for any $\tau \in A_\eta$

$$\frac{1}{S^2} \int_{x+[0,S]^2} |f(y+\tau,\xi) - f(y,\xi)|\, dy = \frac{1}{S^2} \sum_{T \in \mathcal{T}_S^\eta} \int_T |f(y+\tau,\xi) - f(y,\xi)|\, dy,$$

where

$$\mathcal{T}_S^\eta = \left\{T \in \mathcal{T} \ : \ T \subset a + [0,S]^2, \text{and } \exists\ v \in V(T) \text{ such that } v = \phi(z) \text{ with } \mathrm{dist}(z, \Pi) < \eta\right\}.$$

Now, estimate (7) and the growth hypothesis on f give, for $S > S_\eta$

$$\begin{aligned}\frac{1}{S^2}\sum_{T\in\mathcal{T}_S^\eta}\int_T |f(y+\tau,\xi)-f(y,\xi)|\,dy &\leq \frac{1}{S^2}\sum_{T\in\mathcal{T}_S^\eta}\int_T 2c_2(1+|\xi|^p)\,dy \\ &\leq \frac{1}{S^2}\#\mathcal{T}_S^\eta \sup_{T\in\mathcal{T}_S^\eta}|T|\; 2c_2(1+|\xi|^p) \leq \tilde{C}\eta^3\end{aligned}$$

concluding the proof. □

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