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Nonlinear stability for the surface diffusion flow

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INTRODUCTION

In mathematics, in particular in the field which is in the middle between analysis and geometry called *geometric analysis*, a *geometric flow* is a motion in time of some geometric object or structure, driven by a system of partial differential equations. Such topic got recently an extremely large interest due to its success in solving some famous open problems, notably among them, the Poincaré conjecture by Perelman, via the Ricci flow.

We will consider the evolution in time of smooth subsets E_t of the Euclidean space such that their boundaries ∂E_t, which are smooth hypersurfaces, move, for $t \in [0, T)$, with "outer" normal velocity V_t given by

$$V_t = \Delta_t \mathrm{H}_t \qquad \text{on } \partial E_t, \tag{SDF}$$

where Δ_t and H_t are respectively, the Laplacian and the mean curvature of the surface ∂E_t. The resulting motion is called *surface diffusion flow* and it was first proposed by Mullins in [21] to study thermal grooving in material sciences (see also [6]). We will deal with surfaces in the three–dimensional space, which is a physically relevant case since it describes the evolution of interfaces between solid phases of a system, driven by surface diffusion of atoms under the action of a chemical potential (see for instance [16] and the references therein).

Our main purpose is to prove, following [1], a long time existence result for suitable "initial" sets $E_0 \subseteq \mathbb{R}^3$ in the "periodic" setting, that is, assuming that all the evolving sets E_t (hence, their boundary surfaces) are 1–periodic with respect to the standard integer lattice $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. It is then clear that we can equivalently consider the surface diffusion flow of sets E_t in the ambient space $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, the three–dimensional "flat" torus of unit volume, which is the setting we are going to adopt in all the thesis.

We mention that even if we work in dimension three, all the results and arguments also hold in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, considering in such case moving curves. Moreover, when the dimension of the ambient is larger than three several questions remain open.

The most important property of this flow, which is the basis for the results we are going to discuss, is that it is a *gradient flow* of a functional which clearly gives a natural "energy", decreasing in time during the evolution (in other words the velocity V_t is minus the gradient, that is, the *Euler equation*, of a functional). Precisely, in any dimension $n \in \mathbb{N}$, the surface diffusion flow can be regarded as the H^{-1}–gradient flow of the following *Area* functional, defined for any smooth set E as

$$\mathcal{A}(\partial E) = \mathrm{Area}(\partial E) = \int_{\partial E} d\mu$$

giving the *area* of its $(n-1)$–dimensional smooth boundary, under the constraint that the volume $\mathrm{Vol}(E) = \mathscr{L}^n(E)$ is fixed, choosing a suitable norm on $H^{-1}(\partial E)$. Obviously, μ is the "canonical" measure associated to the Riemannian metric on ∂E induced by the scalar product of $\mathbb{R}^n$, which coincides with the n–dimensional Hausdorff measure $\mathcal{H}^n$. It clearly follows that the volume of the evolving sets $\mathrm{Vol}(E_t)$ is constant in time, while we remark that the convexity of E_t is not necessarily preserved along the flow (see [17]). This is a great difference between our flow and the more famous *mean curvature flow*, which is also a gradient flow of the Area functional (without the constraint on the volume), but with respect to the L^2–norm (see [19]).

Parametrizing the smooth surfaces ∂E_t by some maps (embeddings) $\psi_t : M \to \mathbb{T}^n$ such that $\psi_t(M) = \partial E_t$ (here M is a fixed smooth differentiable surface) and letting ν_t to be the outer unit normal vector to ∂E_t, the evolution law (SDF) can be expressed as

$$\frac{\partial}{\partial t}\psi_t = (\Delta_t \mathrm{H}_t)\nu_t$$

and due to the parabolic nature of this system of PDEs, it is known that for every smooth initial set $E_0 \subseteq \mathbb{T}^n$, the surface diffusion flow E_t with such initial data exists unique and smooth in some positive time interval $[0, T)$. Such short time existence and uniqueness result was proved by Escher, Mayer and Simonett in [6] for the surface diffusion flow in any dimension of a smooth compact hypersurface in the Euclidean space.

Our aim is to present an (expected) "stability" result by Acerbi, Fusco, Julin and Morini in [1], where they prove that if the initial set is sufficiently "close" to a *strictly stable critical* set $F \subseteq \mathbb{T}^3$ for the (volume–constrained) Area functional, then the flow E_t actually exists for all times and asymptotically converges *in some sense* to a "translate" of F. That is, for such special class of initial data we have the existence of a global solution of the evolution problem (SDF). This is clearly related to the fact that the flow is the (volume–constrained) gradient flow of the Area functional in the sense above. For this reason, the analysis of its first and second order behavior (in Chapter 1) is one of the key steps in the proof of such result.

We say that a smooth subset $F \subseteq \mathbb{T}^n$ is *critical* if for any smooth one–parameter family of diffeomorphisms $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$, such that $\mathrm{Vol}(\Phi_t(F)) = \mathrm{Vol}(F)$ and $\Phi_0|_F = \mathrm{Id}$, there holds

$$\frac{d}{dt}\mathcal{A}(\partial\Phi_t(F))\bigg|_{t=0} = 0$$

that is, the first variation of the Area functional $\mathcal{A}$ under the constant volume constraint is zero for F. It follows that F is critical for $\mathcal{A}$ if and only if it satisfies

$$\mathrm{H} = \lambda \in \mathbb{R} \qquad \text{on } \partial F$$

that is, ∂F is a smooth surface with *constant mean curvature.*
The study of the second variation and of the related behavior of the

Area functional around a critical set F, leading to the central notion of *stability*, is more involved. Differently by the original papers, we will compute it with the tools and methods of Riemannian geometry, coherently with the "geometric spirit" of the whole thesis. In particular, we will see that at a critical set F, the second variation of $\mathcal{A}$ only depends on the normal component φ on ∂F of the vector field which is the infinitesimal generator of the family of diffeomorphisms $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$, deforming F keeping its volume constant. This volume constraint on the "admissible" deformations of F implies that the functions φ must have zero integral on ∂F, hence it is natural to define a quadratic form Π_F on such space of functions which is related to the second variation of $\mathcal{A}$ by the following equality

$$\Pi_F(\varphi) = \frac{d^2}{dt^2}\mathcal{A}(\partial\Phi_t(F))\bigg|_{t=0}$$

where $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$ is a one–parameter family of diffeomorphisms satisfying $\mathrm{Vol}(\Phi_t(F)) = \mathrm{Vol}(F)$,

$$\Phi_0\big|_F = \mathrm{Id}_F \qquad \text{and} \qquad \frac{\partial\Phi_t}{\partial t}\bigg|_{t=0} = \varphi\nu_F \quad \text{on } \partial F,$$

where ν_F is the outer unit normal vector of ∂F.
Since the functional $\mathcal{A}$ is clearly *translation invariant*, by choosing for every vector $\eta \in \mathbb{R}^n$ the family of diffeomorphisms Φ_t of the n–torus which simply translate any point by $t\eta$, it is easy to see that the form Π_F vanishes on the finite dimensional vector space given by the functions $\psi = \langle \eta, \nu_F\rangle$. We then say that a smooth critical set $F \subseteq \mathbb{T}^n$ is *strictly stable* if

$$\Pi_F(\varphi) > 0$$

for all non–zero functions $\varphi : \partial F \to \mathbb{R}$, with zero integral and L^2–orthogonal to every function $\psi = \langle \eta, \nu_F\rangle$.
We underline that the presence of such "natural" degenerate space of the quadratic form Π_F (or, equivalently, the translation invariance of $\mathcal{A}$) is the main reason of several technical difficulties in the thesis.

In order to analyze the local behavior of $\mathcal{A}$ around a smooth set $F \subseteq \mathbb{T}^n$, we say that the set E is "$W^{2,p}$–close" to F, if for some $\delta > 0$ "small enough" we have $\mathrm{Vol}(E \triangle F) < \delta$, its boundary ∂E is contained in a suitable *tubular neighborhood* of ∂F and can be described as

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\}$$

for some smooth function $\psi : \partial F \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$. That is, the boundary of E is represented as the "normal graph" on ∂F of the function ψ, which is clearly a very useful way to transform the problem on *sets* into a problem on *functions*.

Our first goal, in the last section of Chapter 1, will be to show the result in [2] that any smooth strictly stable critical set $F \subseteq \mathbb{T}^n$ is a local minimizer of the volume–constrained Area functional ("isolated" up to translations), among all smooth $W^{2,p}$–close sets $E \subseteq \mathbb{T}^n$, if $p > \max\{2, n-1\}$.

Then, it is possible to consider the "dynamic" stability, having heuristically in mind the example of a system whose state is in a "potential well", that is, a region surrounding a local minimum of its potential energy. We will show the following nonlinear stability result, proved in [1] with a line of proof that was new in the literature, based on energy estimates and geometric interpolation inequalities.

Theorem. *Let $F \subseteq \mathbb{T}^3$ be a strictly stable critical set and let N_ε be a tubular neighborhood of ∂F, as in formula (1.36). For every $\alpha \in (0, 1/2)$ there exists $M > 0$ such that, if E_0 is a smooth set satisfying*

- $\mathrm{Vol}(E_0) = \mathrm{Vol}(F)$,
- $\mathrm{Vol}(E_0 \triangle F) \leq M$,
- *the boundary of E_0 is contained in N_ε and can be represented as*

$$\partial E_0 = \{y + \psi_{E_0}(y)\nu_F(y) \,:\, y \in \partial F\},$$

  for some function $\psi_{E_0} : \partial F \to \mathbb{R}$ such that $\|\psi_{E_0}\|_{C^{1,\alpha}(\partial F)} \leq M$,
-
$$\int_{\partial E_0} |\nabla \mathrm{H}_0|^2 \, d\mu_0 \leq M\,,$$

then there exists a unique smooth solution E_t of the surface diffusion flow starting from E_0, which is defined for all $t \geq 0$. Moreover, $E_t \to F + \eta$ exponentially fast in $W^{3,2}$ as $t \to +\infty$, for some $\eta \in \mathbb{R}^3$, with the meaning that the functions $\psi_{\eta,t} : \partial F + \eta \to \mathbb{R}$ representing ∂E_t as "normal graphs" on $\partial F + \eta$, that is,

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{F+\eta}(y) \,:\, y \in \partial F + \eta\},$$

satisfy

$$\|\psi_{\eta,t}\|_{W^{3,2}(\partial F+\eta)} \leq Ce^{-\beta t},$$

for every $t \in [0, +\infty)$, for some positive constants C and β.

The classification of the smooth stable critical sets in $\mathbb{T}^3$ is complete: they are *lamellae*, *balls*, 2–*tori* or *gyroids* (see [26]). The surfaces in the first three classes are actually strictly stable, while the strict stability of gyroids has been established only in some cases (see [14, 15, 27]). Due to the above theorem, such sets are thus "dynamically exponentially stable" for the surface diffusion flow.

The thesis is organized as follows: in Chapter 1 we compute in general the first and second variation of the Area functional and we show the $W^{2,p}$–local minimality of the smooth strictly stable critical sets. In Chapter 2 we introduce the surface diffusion flow and in Chapter 3 we prove the nonlinear stability theorem above. Finally, in Chapter 4 we discuss some results related to our work. In particular, we briefly describe the surface diffusion flow *with elasticity*, introduced recently to study the morphological evolution of strained elastic solids driven by stress and surface mass transport (see [10]) and the modified Mullins–Sekerka flow. Then, we give a classification of the "stable" critical set and we conclude with an overview of possible lines of research.

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1

THE AREA FUNCTIONAL

As we said in the introduction, the surface diffusion flow may be regarded as a suitable gradient flow of the *Area functional*. In this chapter we introduce such functional and we analyze its basic properties. In particular, we compute its first and second variation formulas and we prove a sufficient condition for a set to be local minimizer.

1.1 NOTATION AND GEOMETRIC PRELIMINARIES

We start by recalling that our setting is the n–dimensional (unit) flat torus $\mathbb{T}^n$ (as in [2]), that is the quotient of $\mathbb{R}^n$ with respect to the equivalence relation $x \sim y \iff x - y \in \mathbb{Z}^n$, where $\mathbb{Z}^n$ is the standard integer lattice of $\mathbb{R}^n$. The functional spaces $W^{k,p}(\mathbb{T}^n)$, $k \in \mathbb{N}$, $p \geq 1$, can be identified with the vector subspaces of $W^{k,p}_{loc}(\mathbb{R}^n)$ of functions that are one–periodic with respect to all coordinate directions. Similarly, $C^{k,\alpha}(\mathbb{T}^n)$, for $\alpha \in (0,1)$ may be identified with the space of one–periodic functions in $C^{k,\alpha}(\mathbb{R}^n)$.

A set $E \subseteq \mathbb{T}^n$ will be called smooth (or of class C^k, $W^{k,p}$) if its one–periodic extension to $\mathbb{R}^n$ is smooth (or of class $C^{k,\alpha}$, $W^{k,p}$), with the meaning that its boundary is a smooth hypersurface (or it can be locally described as a graph of a function $C^{k,\alpha}$, $W^{k,p}$, around any of its points).

We now introduce some basic notations and facts about smooth hypersurfaces that we need to compute the first and second variation of the Area functional by "geometric" methods.

We advise the reader that in all our work the convention of summing over the repeated indices will be adopted. Moreover, when it is clear by the contest, we will write div *for both the (Riemannian) divergence operator on a hypersurface (defined by formula* (1.1) *below) and the (standard) divergence in* $\mathbb{T}^n$ *(which is locally* $\mathbb{R}^n$*), but this latter will be instead denoted with* $\operatorname{div}^{\mathbb{T}^n}$ *when it will be computed at a point of a hypersurface, in order to avoid any possibility of misunderstanding. Finally, in all the estimates of the thesis, the constants* C *may vary from a line to another.*

We will consider $(n-1)$–dimensional, compact, smooth hypersurfaces ∂E immersed in $\mathbb{T}^n$ where E is a smooth set, that is, pairs $(\partial E, \psi)$ where $\psi : \partial E \to \mathbb{T}^n$ is a smooth immersion (the rank of the differential $d\psi$ is equal to $n-1$ everywhere on ∂E).

Taking local coordinates around any $x \in \partial E$, we have local bases of the *tangent space* $T_x\partial E$ and of its dual $T^*_x\partial E$, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and 1–forms $\{dx_j\}$. So we denote the vectors on ∂E by

$X = X^i \frac{\partial}{\partial x_i}$ and the 1–forms by $\omega = \omega_j dx_j$, where the indices refer to the local basis.

The manifold ∂E gets in a natural way a metric tensor g turning it into a Riemannian manifold $(\partial E, g)$, where

$$g_{ij} = \left\langle \frac{\partial \psi}{\partial x_i}, \frac{\partial \psi}{\partial x_j} \right\rangle$$

that is the pull–back of the scalar product of $\mathbb{R}^n$ via the immersion map ψ. Then the "canonical" measure induced on ∂E by the metric g is given in a coordinate chart by $\mu = \sqrt{\det(g_{ij})}\,\mathscr{L}^{n-1}$ where $\mathscr{L}^{n-1}$ is the standard Lebesgue measure on $\mathbb{R}^{n-1}$.

The induced covariant derivative on $(\partial E, g)$ of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k , \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji} \omega_k ,$$

where the Christoffel symbols Γ^i_{jk} are expressed by the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right) .$$

Moreover, the gradient ∇f of a function, its Laplacian Δf and the divergence $\mathrm{div} X$ of a tangent vector field at a point $x \in \partial E$ are defined respectively by

$$g(\nabla f(x), v) = df_x(v) \qquad \forall v \in T_x \partial E , \qquad \Delta f = \mathrm{tr} \nabla^2 f ,$$

and

$$\mathrm{div} X = \mathrm{tr} \nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k . \tag{1.1}$$

We recall that by the *divergence theorem* for compact manifolds (without boundary), there holds

$$\int_{\partial E} \mathrm{div} X \, d\mu = 0 \tag{1.2}$$

for every tangent vector field X to ∂E, which in particular implies

$$\int_{\partial E} \Delta f \, d\mu = 0$$

for every smooth function $f : \partial E \to \mathbb{R}$.

Since the hypersurface ∂E is the boundary of a smooth set, we can consider the globally–defined *outer unit normal vector* ν_E at each point of ∂E, then we can define a symmetric 2–form $B = h_{ij}$ called *second fundamental form* as follows,

$$h_{ij} = - \left\langle \frac{\partial^2 \psi}{\partial x_i \partial x_j}, \nu_E \right\rangle$$

whose trace is the *mean curvature* $\mathrm{H} = g^{ij} h_{ij}$ of ∂E.
The symmetry properties of the covariant derivative of B are given by the following Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} . \tag{1.3}$$

Remark 1.1. Let the hypersurface $\partial E \subseteq \mathbb{R}^n$ be locally the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, that is, $\psi(x) = (x, f(x))$, then we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\,, \qquad \nu_E = -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{\mathrm{Hess}f(\nabla f, \nabla f)}{(\sqrt{1+|\nabla f|^2})^3} = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$$

where $\mathrm{Hess}f$ is the Hessian of the function f.

In the sequel, the following *Gauss–Weingarten relations* will be fundamental

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j} = \Gamma^k_{ij}\frac{\partial \psi}{\partial x_k} - h_{ij}\nu_E\,, \qquad \frac{\partial \nu_E}{\partial x_j} = h_{jl} g^{ls}\frac{\partial \psi}{\partial x_s}\,. \tag{1.4}$$

Notice that by these relations it follows

$$\Delta\psi = g^{ij}\nabla^2_{ij}\psi = g^{ij}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\psi}{\partial x_k}\right) = -g^{ij}h_{ij}\nu_E = -\mathrm{H}\nu_E\,. \tag{1.5}$$

Morevover, we have the formula

$$\Delta\nu_E = \nabla\mathrm{H} - |B|^2\nu_E\,, \tag{1.6}$$

indeed, computing in normal coordinates at a point $x \in \partial E$, by the above Gauss–Weingarten relations, we have

$$\begin{aligned}
\Delta\nu_E &= g^{ij}\left(\frac{\partial^2\nu_E}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\nu_E}{\partial x_k}\right)\\
&= g^{ij}\frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial\psi}{\partial x_s}\right)\\
&= g^{ij}\nabla_i h_{jl}g^{ls}\frac{\partial\psi}{\partial x_s} + g^{ij}h_{jl}g^{ls}\frac{\partial^2\psi}{\partial x_i\partial x_s}\\
&= g^{ij}\nabla_l h_{ij}g^{ls}\frac{\partial\psi}{\partial x_s} - g^{ij}h_{jl}g^{ls}h_{is}\nu_E\\
&= \nabla\mathrm{H} - |B|^2\nu_E\,,
\end{aligned}$$

since all Γ^k_{ij} and $\frac{\partial}{\partial x_i}g^{jk}$ are zero at $x \in \partial E$ and we used Codazzi equations (1.3).

Finally, we recall that by straightforward computations the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed by means of the second fundamental form as follows,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g\left(\nabla^2_{ji}\frac{\partial}{\partial x_k} - \nabla^2_{ij}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = h_{ik}h_{jl} - h_{il}h_{jk}\,,\\
\mathrm{Ric}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}\,h_{ij} - h_{il}g^{lk}h_{kj}\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = g^{ij}g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}^2 - |B|^2\,.
\end{aligned} \tag{1.7}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}\nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl} g^{ks} X^l = (h_{ik}h_{jl} - h_{il}h_{jk})\, g^{ks} X^l\,, \\ \nabla_i\nabla_j \omega_k - \nabla_j\nabla_i \omega_k &= \mathrm{R}_{ijkl} g^{ls} \omega_s = (h_{ik}h_{jl} - h_{il}h_{jk})\, g^{ls} \omega_s\,.\end{aligned} \tag{1.8}$$

1.2 FIRST AND SECOND VARIATION OF THE AREA FUNCTIONAL

We define the *Area functional*

$$\mathcal{A}(\partial E) = \int_{\partial E} d\mu$$

on the boundary of any smooth set $E \subseteq \mathbb{T}^n$. Obviously, μ is the canonical measure aforementioned, which coincides with the $(n-1)$–dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on ∂E.

We are interested in computing the first and second variation of the Area functional with respect to volume–preserving variations, that is, the flows Φ as in the following definition.

Definition 1.2. Let $E \subseteq \mathbb{T}^n$ be a smooth set.
We say that a vector field $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ is admissible for E if the associated flow $\Phi : I \times \mathbb{T}^n \to \mathbb{T}^n$, defined by

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,x) = X(\Phi(t,x)) \\ \Phi(0,x) = x \end{cases} \tag{1.9}$$

satisfies

$$\mathrm{Vol}(\Phi(t,E)) = \mathrm{Vol}(E)$$

for all $t \in I$ and $x \in \mathbb{T}^n$.

To do this, we first compute such first and second variations for "general" (not necessarily volume–preserving) variations Φ, generated by (not necessarily admissible) vector fields $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$, then we restrict to X as in this definition. In order to simplify the notation, in the following, we will write often Φ_t in place of $\Phi(t,\cdot)$ and E_t in place of $\Phi_t(E) = \Phi(t,E)$.

As a preliminary computation, we discuss the behavior of the metric tensor g and of the canonical measure μ of ∂E under the effect of the "deformation" of ∂E given by a smooth one parameter family of immersions $\psi_t : \partial E \to \mathbb{T}^n$, with $t \in I$ and $\psi_0 = \psi = \mathrm{Id}$. Defining the field $X = \frac{\partial \psi_t}{\partial t}\Big|_{t=0}$ along ∂E (the *infinitesimal generator* of the variation

ψ_t) and setting $X_\tau = X - \langle X, \nu_E\rangle \nu_E$, to denote the "tangential part" of X, letting ν_E the outer normal unit vector of ∂E, we compute

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} =& \frac{\partial}{\partial t}\left\langle \frac{\partial \psi_t}{\partial x_i}, \frac{\partial \psi_t}{\partial x_j}\right\rangle\Big|_{t=0}\\
=&\left\langle \frac{\partial X}{\partial x_i}, \frac{\partial \psi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j}, \frac{\partial \psi}{\partial x_i}\right\rangle\\
=&\frac{\partial}{\partial x_i}\left\langle X, \frac{\partial \psi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X, \frac{\partial \psi}{\partial x_i}\right\rangle - 2\left\langle X, \frac{\partial^2 \psi}{\partial x_i \partial x_j}\right\rangle\\
=&\frac{\partial}{\partial x_i}\left\langle X_\tau, \frac{\partial \psi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X_\tau, \frac{\partial \psi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X_\tau, \frac{\partial \psi}{\partial x_k}\right\rangle\\
&+ 2h_{ij}\langle X, \nu_E\rangle\,,\end{aligned}$$

where we used the Gauss–Weingarten relations (1.4) in the last step. Letting ω be the 1–form defined by $\omega(Y) = g(X_\tau, Y)$, this formula can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} + 2\Gamma_{ij}^k \omega_k + 2h_{ij}\langle X, \nu_E\rangle\\
&= \nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X, \nu_E\rangle\,, \end{aligned}\tag{1.10}$$

being $\psi : \partial E \to \mathbb{T}^n$ the inclusion (identity) map of ∂E.
We remind that

$$\frac{d}{dt}\det A(t) = \det\left[A^{-1}(t)\frac{d}{dt}A(t)\right]\,,\tag{1.11}$$

for any $n \times n$ squared matrix $A(t)$ dependent on t, then we get

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} &= \frac{1}{2\sqrt{\det g_{ij}}}\frac{\partial}{\partial t}\det g_{ij}\,\Big|_{t=0}\\
&= \frac{\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial}{\partial t}g_{ij}}{2}\,\Big|_{t=0}\\
&= \frac{\sqrt{\det g_{ij}}\, g^{ij}(\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X, \nu_E\rangle}{2}\\
&= \sqrt{\det g_{ij}}\Big(\operatorname{div} X_\tau + \mathrm{H}\langle X, \nu_E\rangle\Big)\end{aligned}\tag{1.12}$$

which can be expressed as

$$\frac{\partial}{\partial t}\mu_t\Big|_{t=0} = \Big(\operatorname{div} X_\tau + \mathrm{H}\langle X, \nu_E\rangle\Big)\mu\,,\tag{1.13}$$

where μ_t is the canonical Riemannian measure of the smooth hypersurface ∂E_t. We are now ready to compute the first variation of the functional $\mathcal{A}$.

Theorem 1.3 (First variation of the Area functional)**.** *Let $E \in \mathbb{T}^n$ a smooth set and $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ the infinitesimal generator of the flow $\Phi : I \times \mathbb{T}^n \to \mathbb{T}^n$. Then,*

$$\frac{d}{dt}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E} \mathrm{H}\langle X, \nu_E\rangle\, d\mu$$

where ν_E and H are respectively, the outer normal and the mean curvature of ∂E (here $E_t = \Phi_t(E)$).

Proof. We let $\psi_t : \partial E \to \mathbb{T}^n$ be given by

$$\psi_t(x) = \Phi(t, x)\,,$$

for $x \in \partial E$ and $t \in I$, then $\psi_t(\partial E) = \partial E_t$ and $\partial_t \psi_t\,|_{t=0} = X$ at every point of ∂E.
Denoting with $g_{ij} = g_{ij}(t)$ the induced metrics (via ψ_t) on the smooth hypersurfaces ∂E_t and setting $\psi_0 = \psi = \mathrm{Id}$, by the above computation, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}(\partial E)\Big|_{t=0} &= \frac{d}{dt}\int_{\partial E_t} d\mu_t\,\Big|_{t=0} \\
&= \frac{d}{dt}\int_{\partial E} \sqrt{\det g_{ij}}\,dx\,\Big|_{t=0} \\
&= \int_{\partial E} \frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} dx \\
&= \int_{\partial E} \sqrt{\det g_{ij}}(\mathrm{div} X_\tau + \mathrm{H}\langle X, \nu_E\rangle)\,dx \\
&= \int_{\partial E} (\mathrm{div} X_\tau + \mathrm{H}\langle X, \nu_E\rangle)\,d\mu \\
&= \int_{\partial E} \mathrm{H}\langle X, \nu_E\rangle\,d\mu
\end{aligned} \tag{1.14}$$

where the last equality follows from the *divergence theorem* (1.2). ☐

It follows that every smooth set E with zero first variation of the Area functional under a volume constraint (for instance, a minimum) must satisfy the condition

$$\int_{\partial E} \mathrm{H}\langle X, \nu_E\rangle\,d\mu = 0 \tag{1.15}$$

for all admissible $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$.

We now note that if $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ is an admissible vector field and Φ is the associated flow, then $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ for all $t \in I$, thus, by the *divergence theorem*, denoting with $J\Phi_t$ the Jacobian of $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$, we have

$$\begin{aligned}
0 &= \frac{d}{dt}\mathrm{Vol}(E_t)\,\Big|_{t=0} \\
&= \frac{d}{dt}\int_{E_t} dx\,\Big|_{t=0} \\
&= \frac{d}{dt}\int_{E} J\Phi(t, z)\,dz\,\Big|_{t=0} \\
&= \int_{E} \frac{\partial}{\partial t} J\Phi_t(z)\,\Big|_{t=0} dz \\
&= \int_{E} \mathrm{div} X(x)\,dx \\
&= \int_{\partial E} \langle X, \nu_E\rangle\,d\mu\,,
\end{aligned}$$

that is, the normal component $\varphi = \langle X, \nu_E \rangle$ of X has zero integral on ∂E.

We remark that we used the fact that

$$\frac{\partial}{\partial t} J\Phi_t \Big|_{t=0} = \operatorname{div} X\,, \tag{1.16}$$

indeed, as $J\Phi_t(z) = \det[d\Phi_t(z)]$, by means of formula (1.11), we obtain

$$\frac{\partial}{\partial t} J\Phi_t(z) = J\Phi_t(z))\,\mathrm{tr}\,[d\Phi_t(z)^{-1} \circ dX(\Phi_t(z)) \circ d\Phi_t(z)],$$

since, by definition of Φ, we have

$$\frac{\partial}{\partial t}[d\Phi_t(z)] = d\Big(\frac{\partial}{\partial t}\Phi_t(z)\Big) = d\Big[X(\Phi_t(z))\Big] = dX(\Phi_t(z)) \circ d\Phi_t(z)\,,$$

then, being the trace of a matrix invariant by conjugation, we conclude

$$\frac{\partial}{\partial t} J\Phi_t(z) = J\Phi_t(z)\,\mathrm{tr}\,[dX(\Phi_t(z))] = J\Phi_t(z))\operatorname{div} X(\Phi_t(z)) \tag{1.17}$$

and considering $t = 0$, we obtain equality (1.16). More in general, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\mathrm{Vol}(E_t) \\ &= \int_E \frac{\partial}{\partial t} J\Phi_t(z)\,dz \\ &= \int_E \operatorname{div} X(\Phi(t,z)) J\Phi(t,z)\,dz \\ &= \int_{E_t} \operatorname{div} X(x)\,dx \\ &= \int_{\partial E_t} \langle X, \nu_{E_t}\rangle\,d\mu_t \end{aligned} \tag{1.18}$$

where ν_{E_t} is the outer unit normal vector of the smooth hypersurface ∂E_t.

Conversely, we have the following lemma whose proof is postponed after Lemma 1.16, since the arguments are very similar.

Lemma 1.4. *Let $\varphi : \partial E \to \mathbb{R}$ a C^∞ function with zero integral. Then, there exists an admissible vector field $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ such that $\varphi = \langle X, \nu_E \rangle$.*

Hence, from equality (1.15) and this lemma, it follows

$$\int_{\partial E} \mathrm{H}\varphi\,d\mu = 0$$

for all $\varphi \in C^\infty(\partial E)$ with zero integral, which is equivalent to say that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H} = \lambda \qquad \text{on } \partial E,$$

That is, ∂E is a smooth hypersurface with *constant mean curvature*.

This motivates the following definition.

Definition 1.5 (Critical sets)**.** We say that a smooth subset $F \subseteq \mathbb{T}^n$ is *critical* for the Area functional $\mathcal{A}$ (under volume constraint), if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H} = \lambda \qquad \text{on } \partial F.$$

Remark 1.6. Clearly, the critical sets for the *unconstrained* Area functional must satisfy

$$\int_{\partial F} \mathrm{H}\langle X, \nu_F\rangle \, d\mu = 0$$

for *every* $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$, which easily implies the *minimal surface* equation $\mathrm{H} = 0$ on ∂F.

Now we turn our attention to the second variation of $\mathcal{A}$.

Theorem 1.7 (Second variation of the Area functional)**.** *Let E and X be as in Theorem 1.3, then*

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} &= \int_{\partial E} \Big(|\nabla\langle X, \nu_E\rangle|^2 - \langle X, \nu_E\rangle^2 |B|^2\Big)\, d\mu \\ &+ \int_{\partial E} \mathrm{H}\Big(\mathrm{H}\langle X, \nu_E\rangle^2 + \langle Z, \nu_E\rangle - 2\langle X_\tau, \nabla\langle X, \nu_E\rangle\rangle + B(X_\tau, X_\tau)\Big)\, d\mu\,,\end{aligned}$$

where B is the second fundamental form on ∂E and $|B|^2$ is its norm, which coincides with the sum of squares of the principal curvatures of ∂E, moreover we set $X_\tau = X - \langle X, \nu_E\rangle \nu_E$ as the tangential *part of X on ∂E and*

$$Z = \frac{\partial^2}{\partial t^2}\Phi(0, \cdot) = \frac{\partial}{\partial t} X(\Phi(0, \cdot)) = dX(X).$$

Proof. We let $\psi_t = \Phi(t, \cdot)|_{\partial E}$ as in Theorem 1.3 where we showed that

$$\frac{d}{dt}\mathcal{A}(\partial E_t) = \frac{d}{dt}\int_{\partial E_t} \sqrt{\det g_{ij}}\, dx = \int_{\partial E_t} \mathrm{H}\langle X, \nu_{E_t}\rangle \, d\mu_t\,.$$

Consequently, we have

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \frac{d}{dt}\int_{\partial E_t} \mathrm{H}\langle X, \nu_{E_t}\rangle \sqrt{\det g_{ij}}\, dx \,\Big|_{t=0}.$$

In order to semplify notations we set $\nu = \nu_E$ and $\varphi = \langle X, \nu_{E_t}\rangle$, moreover we drop the subscript t in ψ_t, that is, we write simply ψ. We need to compute the following derivatives

$$\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0}, \qquad \frac{\partial \varphi}{\partial t}\Big|_{t=0} \qquad \text{and} \qquad \frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0}. \tag{1.19}$$

By formula (1.12), there holds

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} = \Big(\mathrm{div} X_\tau + \mathrm{H}\varphi\Big)\sqrt{\det g_{ij}}$$

hence, the contribution of the third term above to the second variation is given by

$$\int_{\partial E}\Big(\varphi \mathrm{H}\mathrm{div} X_\tau + \varphi^2\mathrm{H}^2\Big)\, d\mu\,.$$

We now compute

$$\frac{\partial\varphi}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\langle X, \nu_{E_t}\rangle\Big|_{t=0} = \Big\langle \frac{\partial X}{\partial t}, \nu\Big\rangle\Big|_{t=0} + \Big\langle X, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \tag{1.20}$$

and, using the fact that $\frac{\partial\nu}{\partial t}$ is tangent to ∂E, we have

$$\frac{\partial\varphi}{\partial t}\Big|_{t=0} = \langle Z, \nu\rangle + \Big\langle X_\tau, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0}.$$

We remember that in coordinates $X_\tau = X_\tau^p \frac{\partial\psi}{\partial x_p}$, then

$$\Big\langle X_\tau, \frac{\partial \nu_{E_t}}{\partial t}\Big|_{t=0}\Big\rangle = X_\tau^p\Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0}.$$

From $\Big\langle \frac{\partial\psi}{\partial x_p}, \nu\Big\rangle = 0$, for every $p \in \{1, \dots, n-1\}$, it follows that

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\Big\langle \frac{\partial\psi}{\partial x_p}, \nu_{E_t}\Big\rangle\Big|_{t=0} \\
&= \Big\langle \frac{\partial X}{\partial x_p}, \nu\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \frac{\partial}{\partial x_p}\langle X, \nu\rangle - \Big\langle X, \frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - \Big\langle X_\tau, \frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q\Big\langle \frac{\partial\psi}{\partial x_q}, \frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0}.
\end{aligned}$$

Now we use the second equality (1.4) to obtain

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_p} &- X_\tau^q\Big\langle \frac{\partial\psi}{\partial x_q}, \frac{\partial \nu}{\partial x_p}\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q\Big\langle \frac{\partial\psi}{\partial x_q}, h_{pl}g^{li}\frac{\partial \nu}{\partial x_i}\Big\rangle + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q h_{pl}g^{li}g_{qi} + \Big\langle \frac{\partial\psi}{\partial x_p}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0},
\end{aligned}$$

that is,

$$\Big\langle \frac{\partial\psi}{\partial x_q}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} = -\frac{\partial\varphi}{\partial x_p} + X_\tau^q h_{pq}\,.$$

Therefore equality (1.20) becomes

$$\begin{aligned}
\frac{\partial\varphi}{\partial t}\Big|_{t=0} &= \langle Z, \nu\rangle + X_\tau^q\Big\langle \frac{\partial\psi}{\partial x_q}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} \\
&= \langle Z, \nu\rangle - X_\tau^q\frac{\partial u}{\partial x_p} + X_\tau^p X_\tau^q h_{pq} \\
&= \langle Z, \nu\rangle - \langle X_\tau, \nabla\langle X, \nu\rangle\rangle + B(X_\tau, X_\tau)\,,
\end{aligned}$$

hence, the contribution of the second term in (1.19) to the second variation formula is

$$\int_{\partial E} \mathrm{H}\Big(\langle Z, \nu\rangle - \langle X_\tau, \nabla\langle X, \nu\rangle\rangle + B(X_\tau, X_\tau)\Big)\, d\mu\,.$$

Finally, we compute $\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0}$, recalling that

$$\mathrm{H} = -\Big\langle \frac{\partial^2\psi}{\partial x_i \partial x_j}, \nu\Big\rangle g^{ij}\,,$$

hence we can write

$$\begin{aligned}\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0} = &-\Big\langle \frac{\partial^2\psi}{\partial x_i \partial x_j}, \nu\Big\rangle \frac{\partial g^{ij}}{\partial t}\Big|_{t=0} - \Big\langle \frac{\partial^2\psi}{\partial x_i \partial x_j}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} g^{ij}\\ &- \Big\langle \frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial x_i \partial x_j}, \nu\Big\rangle\Big|_{t=0} g^{ij}\,. \end{aligned} \tag{1.21}$$

Since we know from formula (1.10) that

$$\frac{\partial g_{ij}}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\Big\langle \frac{\partial\psi}{\partial x_i}, \frac{\partial\psi}{\partial x_j}\Big\rangle\Big|_{t=0} = \nabla_i\omega_j + \nabla_j\omega_i + 2h_{ij}\langle X, \nu\rangle\,,$$

where $\omega(Y) = g(X_\tau, Y)$, and for all indices i, k there holds $g_{ij}g^{jk} = 0$, we get

$$\begin{aligned} 0 = &\frac{\partial g_{ij}}{\partial t} g^{jk}\Big|_{t=0} + g_{ij}\frac{\partial g^{jk}}{\partial t}\Big|_{t=0}\\ = &\,g^{jk}(\nabla_i\omega_j + \nabla_j\omega_i + 2h_{ij}\langle X, \nu\rangle) + g_{ij}\frac{\partial g^{jk}}{\partial t}\Big|_{t=0}\,. \end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial g^{pk}}{\partial t}\Big|_{t=0} &= -g^{jp}g^{ik}(\nabla_i\omega_j + \nabla_j\omega_i + 2h_{ij}\langle X, \nu\rangle)\\ &= -\nabla^p X^k_\tau - \nabla^k X^p_\tau - 2h^{pk}\varphi\,. \end{aligned} \tag{1.22}$$

Furthermore, by the computations above,

$$\begin{aligned}\Big\langle \frac{\partial^2\psi}{\partial x_i \partial x_j}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} g^{ij} &= \Big\langle \Gamma^k_{ij}\frac{\partial\psi}{\partial x_k}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0} g^{ij}\\ &= g^{ij}\Gamma^k_{ij}\Big\langle \frac{\partial\psi}{\partial x_k}, \frac{\partial \nu_{E_t}}{\partial t}\Big\rangle\Big|_{t=0}\\ &= g^{ij}\Gamma^k_{ij}\Big(-\frac{\partial\varphi}{\partial x_k} + X^q_\tau h_{qk}\Big)\,. \end{aligned} \tag{1.23}$$

We now compute the last term in formula (1.21)

$$\begin{aligned}\Big\langle \frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial x_i \partial x_j}, \nu\Big\rangle\Big|_{t=0} g^{ij} &= \Big\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \nu\Big\rangle g^{ij}\\ &= \Big\langle \frac{\partial^2(\varphi\nu)}{\partial x_i \partial x_j}, \nu\Big\rangle g^{ij} + \Big\langle \frac{\partial^2 X_\tau}{\partial x_i \partial x_j}, \nu\Big\rangle g^{ij}\,. \end{aligned}$$

We split this computation, first we consider

$$\begin{aligned}\left\langle\frac{\partial^2(\varphi\nu)}{\partial x_i\partial x_j},\nu\right\rangle g^{ij} &= \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \left\langle\frac{\partial}{\partial x_i}\left(h_{il}g^{lp}\frac{\partial\psi}{\partial x_p}\right),\nu\right\rangle g^{ij}\varphi \\ &= \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \left\langle\frac{\partial}{\partial x_i}\left(h_{jl}g^{lp}\frac{\partial\psi}{\partial x_p}\right),\nu\right\rangle g^{ij}\varphi \\ &= \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \varphi h_{jl}g^{lp}h_{ip}g^{ij} \\ &= \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \varphi|B|^2\,. \end{aligned} \tag{1.24}$$

Then, we compute

$$\begin{aligned}\left\langle\frac{\partial^2 X_\tau}{\partial x_i\partial x_j},\nu\right\rangle g^{ij} &= \frac{\partial}{\partial x_i}\left\langle\frac{\partial X_\tau}{\partial x_j},\nu\right\rangle g^{ij} - \left\langle\frac{\partial X_\tau}{\partial x_j},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= \frac{\partial}{\partial x_i}\left\langle\frac{\partial}{\partial x_j}\left(X^p_\tau\frac{\partial\psi}{\partial x_p}\right),\nu\right\rangle g^{ij} - \left\langle\frac{\partial X_\tau}{\partial x_j},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= \frac{\partial}{\partial x_i}\left[X^p_\tau\left\langle\frac{\partial^2\psi}{\partial x_j\partial x_p},\nu\right\rangle\right]g^{ij} - \left\langle\frac{\partial X_\tau}{\partial x_j},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - \left\langle\frac{\partial X_\tau}{\partial x_j},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - \left\langle\frac{\partial}{\partial x_j}\left(X^p_\tau\frac{\partial\psi}{\partial x_p}\right),\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - X^p_\tau\left\langle\frac{\partial^2\psi}{\partial x_j\partial x_p},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &\quad -\frac{\partial X^p_\tau}{\partial x_i}\left\langle\frac{\partial\psi}{\partial x_p},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - X^p_\tau\Gamma^k_{jp}\left\langle\frac{\partial\psi}{\partial x_k},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &\quad -\frac{\partial X^p_\tau}{\partial x_i}\left\langle\frac{\partial\psi}{\partial x_p},\frac{\partial\nu}{\partial x_i}\right\rangle g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - X^p_\tau\Gamma^k_{jp}h_{il}g^{lq}g_{kq}g^{ij} \\ &\quad -\frac{\partial X^p_\tau}{\partial x_i}h_{il}g^{lq}g_{pq}g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - X^p_\tau\Gamma^k_{jp}h_{ik}g^{ij} - \frac{\partial X^k}{\partial x_i}h_{ik}g^{ij} \\ &= -\frac{\partial}{\partial x_i}\left(X^p_\tau h_{pj}\right)g^{ij} - h_{ij}\nabla^i X^j_\tau \end{aligned} \tag{1.25}$$

where we used formulas (1.4) again.
Using equalities (1.22), (1.23), (1.24) and (1.25) we obtain

$$
\begin{aligned}
\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0} =& h_{ij}(-\nabla^i X^j_\tau - \nabla^j X^i_\tau - 2h^{ij}\varphi)\\
&+ g^{ij}\Gamma^k_{ij}\Big(\frac{\partial\varphi}{\partial x_k} - X^q_\tau h_{qk}\Big) - \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \varphi|B|^2\\
&+ \frac{\partial}{\partial x_i}\Big(X^p_\tau h_{pj}\Big)g^{ij} + h_{ij}\nabla^i X^j_\tau\\
=& -2h_{ij}\nabla^i X^j_\tau - 2\varphi|B|^2 + g^{ij}\Gamma^k_{ij}\Big(\frac{\partial\varphi}{\partial x_k} - X^q_\tau h_{qk}\Big)\\
&- \frac{\partial^2\varphi}{\partial x_i\partial x_j}g^{ij} + \varphi|B|^2 + \frac{\partial}{\partial x_i}\Big(X^p_\tau h_{pj}\Big)g^{ij} + h_{ij}\nabla^i X^j_\tau\\
=& -\varphi|B|^2 - h_{ij}\nabla^i X^j_\tau - \Delta\varphi + g^{ij}\Big[\frac{\partial}{\partial x_i}\Big(X^p_\tau h_{pj}\Big) - \Gamma^k_{ij}X^p_\tau h_{pk}\Big]\\
=& -\varphi|B|^2 - h_{ij}\nabla_i X^j_\tau - \Delta\varphi + g^{ij}\nabla_i\Big(X^p_\tau h_{pj}\Big)\\
=& -\varphi|B|^2 - \Delta\varphi + X^p_\tau \mathrm{div} B_p\\
=& -\varphi|B|^2 - \Delta\varphi + \langle X_\tau, \nabla\mathrm{H}\rangle
\end{aligned}
\tag{1.26}
$$

where in the last equality we used the following consequence of taking the trace in the Codazzi formula (1.3),

$$\mathrm{div} B_i = g^{jk}\nabla_j h_{ki} = \nabla_i \mathrm{H}\,.$$

Hence the contribution of the first term (1.19) is given by

$$\int_{\partial E}\varphi\Big(-\varphi|B|^2 - \Delta\varphi + \langle X_\tau, \nabla\mathrm{H}\rangle\Big)\,d\mu$$

and we have the following second variation of the area functional,

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E}\Big[&-\varphi\Delta\varphi - \varphi^2|B|^2 + \varphi\langle X_\tau, \nabla\mathrm{H}\rangle + \varphi\mathrm{H}\,\mathrm{div} X_\tau + \varphi^2\mathrm{H}^2\\
&+ \mathrm{H}\Big(\langle Z, \nu\rangle - \langle X_\tau, \nabla\varphi\rangle + B(X_\tau, X_\tau)\Big)\Big]\,d\mu\,.
\end{aligned}
$$

Now, integrating by parts

$$\int_{\partial E}\varphi\langle X_\tau, \nabla\mathrm{H}\rangle\,d\mu = -\int_{\partial E}\Big(\mathrm{H}\langle X_\tau, \nabla\varphi\rangle + \varphi\mathrm{H}\,\mathrm{div} X_\tau\Big)\,d\mu$$

and

$$\int_{\partial E} -\varphi\Delta\varphi\,d\mu = \int_{\partial E}|\nabla\varphi|^2\,d\mu\,,$$

we obtain the formula in the statement of the theorem

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E}\Big[&|\nabla\varphi|^2 - \varphi^2|B|^2 + \varphi^2\mathrm{H}^2\\
&+ \mathrm{H}\Big(\langle Z, \nu\rangle - 2\langle X_\tau, \nabla\varphi\rangle + B(X_\tau, X_\tau)\Big)\Big]\,d\mu\,.
\end{aligned}
$$

□

It follows that if we have a critical set E for the *unconstrained* Area functional, hence $\mathrm{H}=0$ on ∂E (see Remark 1.6), the second variation of $\mathcal{A}$ is simply given by

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E}\Big(|\nabla\langle X,\nu_E\rangle|^2 - \langle X,\nu_E\rangle^2|B|^2\Big)\,d\mu$$

which only depends on the normal component of X on ∂E, that is, on $\langle X,\nu_E\rangle$.
We want to see now that the same holds for a critical set of the Area functional under volume constraint. We claim that

$$\begin{aligned}\mathrm{H}\langle X,\nu\rangle^2 + \langle Z,\nu\rangle - 2\langle X_\tau,\nabla\langle X,\nu\rangle\rangle + B(X_\tau,X_\tau)&\\ = \langle X,\nu\rangle\mathrm{div}^{\mathbb{T}^n}X - \mathrm{div}(\langle X,\nu\rangle X_\tau)\,,&\end{aligned} \tag{1.27}$$

where, as in the previous proof, we set $\nu=\nu_E$.
We notice that, being every derivative of ν a tangent vector field,

$$\begin{aligned}\langle X_\tau,\nabla\langle X,\nu\rangle\rangle &= \langle \nu, dX(X_\tau)\rangle + \langle X,\langle X_\tau,\nabla\nu\rangle\rangle\\ &= \langle \nu, dX(X_\tau)\rangle + \langle X_\tau,\langle X_\tau,\nabla\nu\rangle\rangle\\ &= \langle \nu, dX(X_\tau)\rangle + B(X_\tau,X_\tau)\,.\end{aligned}$$

Therefore, recalling that $Z=dX(X)$, we have

$$\begin{aligned}&\mathrm{H}\langle X,\nu\rangle^2 + \langle Z,\nu\rangle - 2\langle X_\tau,\nabla\langle X,\nu\rangle\rangle + B(X_\tau,X_\tau)\\ &\quad= \mathrm{H}\langle X,\nu\rangle^2 + \langle \nu, dX(X)\rangle - \langle X_\tau,\nabla\langle X,\nu\rangle\rangle - \langle \nu, dX(X_\tau)\rangle\\ &\quad= \mathrm{H}\langle X,\nu\rangle^2 + \langle \nu, dX(\langle X,\nu\rangle\nu)\rangle - \langle X_\tau,\nabla\langle X,\nu\rangle\rangle\\ &\quad= \mathrm{H}\langle X,\nu\rangle^2 + \langle X,\nu\rangle\langle \nu, dX(\nu)\rangle + \langle X,\nu\rangle\mathrm{div}X_\tau - \mathrm{div}(\langle X,\nu\rangle X_\tau)\,.\end{aligned}$$

Now we notice that, choosing an orthonormal basis $e_1,\dots,e_{n-1},e_n=\nu$ of $\mathbb{R}^n$ at a point $x\in\partial E$ and letting $X=X^ie_i$, we have

$$\langle e_i,\nabla^\top X^i\rangle = \langle e_i,\nabla X^i - \langle\nabla X^i,\nu\rangle\nu\rangle = \mathrm{div}^{\mathbb{T}^n}X - \langle\nu,dX(\nu)\rangle$$

where the symbol $^\top$ denotes the projection on the tangent space to ∂E. Moreover, if we choose a local parametrizazion of ∂E such that $\frac{\partial\psi}{\partial x_i}=e_i$, for $i\in\{1,\dots,n-1\}$, at $x\in\partial E$ we have $e_i^j=\frac{\partial\psi^j}{\partial x_i}=g^{ij}=\delta_{ij}$ and

$$\begin{aligned}\langle e_i,\nabla^\top X^i\rangle &= \langle e_i,\nabla^\top X^i_\tau\rangle + \langle e_i,\nabla^\top(\langle X,\nu\rangle\nu^i)\rangle\\ &= \langle e_i^\top,\nabla X^i_\tau\rangle + \langle X,\nu\rangle\langle e_i^\top,\nabla\nu^i\rangle\\ &= \langle e_i^\top,\nabla X^i_\tau\rangle + \langle X,\nu\rangle\frac{\partial\psi^j}{\partial x_i}h_{jl}g^{ls}\frac{\partial\psi^i}{\partial x_s}\\ &= \nabla_{e_i}X^i_\tau + \langle X,\nu\rangle h_{ii}\\ &= \mathrm{div}X_\tau + \langle X,\nu\rangle\mathrm{H}\end{aligned}$$

where we used the Gauss–Weingarten relations (1.4) and the fact that the covariant derivative of a vector field along a hypersurface of $\mathbb{R}^n$

can be obtained by differentiating in Euclidean coordinates (a local extension of) the vector field and projecting the result on the tangent space to the hypersurface (see [11], for instance). Hence, we get

$$\langle \nu, dX(\nu)\rangle = \mathrm{div}^{\mathbb{T}^n} X - \langle e_i, \nabla^\top X^i\rangle = \mathrm{div}^{\mathbb{T}^n} X - \mathrm{div} X_\tau - \langle X, \nu\rangle \mathrm{H}$$

and it follows

$$\begin{aligned}\mathrm{H}\langle X, \nu\rangle^2 + \langle Z, \nu\rangle - 2\langle X_\tau, \nabla\langle X, \nu\rangle\rangle + B(X_\tau, X_\tau)\\ = \langle X, \nu\rangle \mathrm{div}^{\mathbb{T}^n} X - \mathrm{div}(\langle X, \nu\rangle X_\tau)\end{aligned}$$

which is equation (1.27).

Then, we can rewrite the second variation of the Area functional as

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} &= \int_{\partial E}\Big(|\nabla\langle X, \nu_E\rangle|^2 - \langle X, \nu_E\rangle^2 |B|^2\Big)\, d\mu\\ &\quad + \int_{\partial E} \mathrm{H}\langle X, \nu_E\rangle \mathrm{div}^{\mathbb{T}^n} X\, d\mu\\ &\quad - \int_{\partial E} \mathrm{H}\,\mathrm{div}(\langle X, \nu_E\rangle X_\tau)\, d\mu\,. \end{aligned} \tag{1.28}$$

Theorem 1.8. *Let $F \subseteq \mathbb{T}^n$ be a critical set for the Area functional, under volume constraint, that is,* H *is constant on ∂F, then for every admissible $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ there holds*

$$\frac{d^2}{dt^2}\mathcal{A}(\partial F_t)\Big|_{t=0} = \int_{\partial F}\Big(|\nabla\langle X, \nu_F\rangle|^2 - \langle X, \nu_F\rangle^2 |B|^2\Big)\, d\mu. \tag{1.29}$$

In particular, the second variation at F only depends on the normal component of X on ∂F, that is, on $\langle X, \nu_F\rangle$.

Proof. As the vector field X is admissible, by formula (1.17), we have

$$\begin{aligned}0 &= \frac{d^2}{dt^2}\mathrm{Vol}(F_t)\Big|_{t=0}\\ &= \int_F \frac{\partial}{\partial t}\Big[\mathrm{div}X(\Phi_t(x))J\Phi_t(x)\Big]\Big|_{t=0} dx\\ &= \int_F \Big[\langle \nabla \mathrm{div} X, X\rangle + (\mathrm{div} X)^2\Big]\, dx\\ &= \int_F \mathrm{div}[(\mathrm{div}X)X]\, dx\\ &= \int_{\partial F} \langle X, \nu_F\rangle \mathrm{div}^{\mathbb{T}^n} X\, d\mu\,,\end{aligned}$$

hence, being H constant, the first term in the second line of equation (1.28) is zero. The second term is also zero, by the *divergence theorem* (1.2) and again since H is constant, thus we are done. □

By this theorem, the second variation of the Area functional $\mathcal{A}$ at a critical smooth set F is a quadratic form depending only on the normal component of $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ on ∂F, that is, on $\varphi = \langle X, \nu_F\rangle$. This and the fact that the admissible vector fields $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ are in a way "characterized" by having zero integral of such normal component (see the discussion after Theorem 1.3 and Lemma 1.4), suggest the following definitions of the Sobolev space (see [4])

$$\widetilde{H}^1(\partial F) = \left\{ \varphi \in H^1(\partial F) \,:\, \int_{\partial F} \varphi \, d\mu = 0 \right\}, \tag{1.30}$$

and of the quadratic form $\Pi_F : \widetilde{H}^1(\partial F) \to \mathbb{R}$, given by

$$\Pi_F(\varphi) = \int_{\partial F} |\nabla \varphi|^2 \, d\mu - \int_{\partial F} \varphi^2 |B|^2 \, d\mu \,. \tag{1.31}$$

Then, if F is critical, by formula (1.29), we have

$$\frac{d^2}{dt^2} \mathcal{A}(\partial F_t) \Big|_{t=0} = \Pi_F(\langle X, \nu_F \rangle)\,, \tag{1.32}$$

for every smooth vector field X which is admissible for F.

We observe that, by the translation invariance of $\mathcal{A}$, the constant vector field $X = \eta \in \mathbb{R}^n$ is clearly admissible, as the associated flow is given by $\Phi_t(x) = x + t\eta$, then $\mathcal{A}(\partial F_t) = \mathcal{A}(\partial F)$ and

$$0 = \frac{d^2}{dt^2} \mathcal{A}(\partial F_t) \Big|_{t=0} = \Pi_F(\langle \eta, \nu_F \rangle)\,,$$

that is, the form Π_F is zero on the vector subspace

$$T(\partial F) = \left\{ \langle \eta, \nu_F \rangle \,:\, \eta \in \mathbb{R}^n \right\} \subseteq \widetilde{H}^1(\partial F)\,.$$

of dimension less or equal than n. We can then split

$$\widetilde{H}^1(\partial F) = T(\partial F) \oplus T^\perp(\partial F)\,,$$

where $T^\perp(\partial F) \subseteq \widetilde{H}^1(\partial F)$ is the vector subspace L^2–orthogonal to $T(\partial F)$ (with respect to the measure μ on ∂F), that is,

$$\begin{aligned} T^\perp(\partial F) &= \left\{ \varphi \in \widetilde{H}^1(\partial F) \,:\, \int_{\partial F} \varphi \nu_F \, d\mu = 0 \right\} \\ &= \left\{ \varphi \in H^1(\partial F) \,:\, \int_{\partial F} \varphi \, d\mu = 0 \ \text{ and } \ \int_{\partial F} \varphi \nu_F \, d\mu = 0 \right\} \end{aligned}$$

and define the following "stability" conditions.

Definition 1.9 (Stability)**.** We say that a critical set $F \subseteq \mathbb{T}^n$ is *stable* if

$$\Pi_F(\varphi) \geq 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial F)$$

and *strictly stable* if

$$\Pi_F(\varphi) > 0 \qquad \text{for all } \varphi \in T^\perp(\partial F) \setminus \{0\}.$$

Remark 1.10. We observe that there exists an orthonormal frame $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that the functions $\langle \nu_F, e_i \rangle$ are orthogonal in $L^2(\partial F)$, that is

$$\int_{\partial F} \langle \nu_F, e_i \rangle \langle \nu_F, e_j \rangle \, d\mu = 0, \tag{1.33}$$

for all $i \neq j$. Indeed, considering the symmetric $n \times n$–matrix $A = (a_{ij})$ with components $a_{ij} = \int_{\partial F} \nu_F^i \nu_F^j \, d\mu$, where $\nu_F^i = \langle \nu_F, \varepsilon_i \rangle$ for some basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathbb{R}^n$, we have

$$\int_{\partial F} (O\nu_F)_i (O\nu_F)_j \, d\mu = (OAO^{-1})_{ij},$$

for every $O \in SO(n)$. Choosing O such that OAO^{-1} is diagonal and setting $e_i = O^{-1}\varepsilon_i$, relations (1.33) are clearly satisfied.
Hence, the functions $\langle \nu_F, e_i \rangle$ which are not identically zero are an orthogonal basis of $T(\partial F)$. We set

$$\mathrm{I}_F = \Big\{ i \in \{1, \dots, n\} \, : \, \langle \nu_F, e_i \rangle \text{ is not identically zero} \Big\}$$

and

$$\mathrm{O}_F = \mathrm{Span}\{ e_i \, : \, i \in \mathrm{I}_F \}, \tag{1.34}$$

then, given any $\varphi \in \widetilde{H}^1(\partial F)$, its projection on $T^\perp(\partial F)$ is

$$\pi(\varphi) = \varphi - \sum_{i \in \mathrm{I}_F} \frac{\int_{\partial F} \varphi \langle \nu_F, e_i \rangle \, d\mu}{\| \langle \nu_F, e_i \rangle \|^2_{L^2_\mu(\partial F)}} \langle \nu_F, e_i \rangle. \tag{1.35}$$

1.3 $W^{2,p}$–LOCAL MINIMALITY

We will make a large use of Sobolev spaces on smooth hypersurfaces. Most of their properties hold as in $\mathbb{R}^n$, standard references are [3], in the Euclidean space and the book [4] when the ambient is a manifold.

Given a smooth set $F \subseteq \mathbb{T}^n$, for $\varepsilon > 0$ small enough we let (d is the "Euclidean" distance on $\mathbb{T}^n$)

$$N_\varepsilon = \{ x \in \mathbb{T}^n \, : \, d(x, \partial F) < \varepsilon \} \tag{1.36}$$

to be a *tubular neighborhood* of ∂F such that the *orthogonal projection map* $\pi_F : N_\varepsilon \to \partial F$ giving the (unique) closest point on ∂F and the *signed distance function* $d_F : N_\varepsilon \to \mathbb{R}$ from ∂F

$$d_F(x) = \begin{cases} d(x, \partial F) & \text{if } x \notin F, \\ -d(x, \partial F) & \text{if } x \in F \end{cases} \tag{1.37}$$

are well defined and smooth in N_ε. Moreover, for every $x \in N_\varepsilon$, the projection map is given explicitely by

$$\pi_F(x) = x - \nabla d_F^2(x)/2 = x - d_F(x) \nabla d_F(x) \tag{1.38}$$

and the unit vector $\nabla d_F(x)$ is orthogonal to ∂F at the point $\pi_F(x) \in \partial F$, indeed actually $\nabla d_F(x) = \nabla d_F(\pi_F(x)) = \nu_F(\pi_F(x))$, which means that the integral curves of the vector field ∇d_F are straight segments orthogonal to ∂F.
This clearly implies that the map

$$\partial F \times (-\varepsilon, \varepsilon) \ni (y,t) \mapsto L(y,t) = y + t\nabla d_F(y) = y + t\nu_F(y) \in N_\varepsilon \tag{1.39}$$

is a smooth diffeomorphism with inverse

$$N_\varepsilon \ni x \mapsto L^{-1}(x) = (\pi_F(x), d_F(x)) \in \partial F \times (-\varepsilon, \varepsilon),$$

moreover, denoting with JL its (partial and "relative" to the hypersurface ∂F) Jacobian, there holds

$$0 < C_1 \leq JL(y,t) \leq C_2$$

on $\partial F \times (-\varepsilon, \varepsilon)$, for a couple of constants C_1, C_2, depending on F and ε (for a proof of the existence of such tubular neighborhood and of these properties, see [20] for instance).

By means of such tubular neighborhood of a smooth set $F \subseteq \mathbb{T}^n$ and the map L, we can speak of "$W^{k,p}$–closedness" (or "$C^{k,\alpha}$–closedness" to F of another smooth set $E \subseteq \mathbb{T}^n$, asking that for some $\delta > 0$ "small enough", we have $\mathrm{Vol}(F \triangle E) < \delta$ and that ∂E is contained in a tubular neighborhood N_ε of F, as above, described by

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\},$$

for a smooth function $\psi : \partial F \to \mathbb{R}$ with $\|\psi\|_{W^{k,p}(\partial F)} < \delta$ (resp. $\|\psi\|_{C^{k,\alpha}(\partial F)} < \delta$). That is, we are asking that the two sets E and F differ by a set of small measure and that their boundaries are "close" in $W^{k,p}$ (or $C^{k,\alpha}$).

Notice that clearly

$$\psi(y) = \pi_2 \circ L^{-1}\Big(\partial F \cap \{y + \lambda\nu_F(y) \,:\, \lambda \in \mathbb{R}\}\Big),$$

where $\pi_2 : \partial F \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ is the projection on the second factor.
Moreover, given a sequence of smooth sets $E_i \subseteq \mathbb{T}^n$, we will write $E_i \to F$ in $W^{k,p}$ (resp. $C^{k,\alpha}$) if for every $\delta > 0$, there holds $\mathrm{Vol}(E_i \triangle F) < \delta$, the smooth boundary ∂E_i is contained in N_ε and it is described by

$$\partial E_i = \{y + \psi_i(y)\nu_F(y) \,:\, y \in \partial F\},$$

for a smooth function $\psi_i : \partial F \to \mathbb{R}$ with $\|\psi_i\|_{W^{k,p}(\partial F)} < \delta$ (resp. $\|\psi_i\|_{C^{k,\alpha}(\partial F)} < \delta$), for every $i \in \mathbb{N}$ large enough.

From now on, in all the rest of the thesis, we will refer to the volume–constrained Area functional $\mathcal{A}$, sometimes without underlining the presence of such constraint, by simplicity. Morever, with N_ε we will always denote a suitable tubular neighborhood of a smooth set, with the above properties.

Definition 1.11. We say that a smooth set $F \subseteq \mathbb{T}^n$ is a *local minimizer* for the Area functional if there exists $\delta > 0$ such that

$$\mathcal{A}(\partial E) \geq \mathcal{A}(\partial F)$$

for all $E \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(E) = \mathrm{Vol}(F)$ and $\mathrm{Vol}(F \triangle E) < \delta$.
We say that a smooth set $F \subseteq \mathbb{T}^n$ is a $W^{2,p}$*–local minimizer* if there exists $\delta > 0$ such that

$$\mathcal{A}(\partial E) \geq \mathcal{A}(\partial F)$$

for all $E \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(E) = \mathrm{Vol}(F)$, $\mathrm{Vol}(F \triangle E) < \delta$, moreover ∂E is contained in a tubular neighborhood N_ε of F, as above and it is described by

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\},$$

for a smooth function $\psi : \partial F \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$.

We immediately see a *necessary* condition for local minimizers. Notice that a local minimizer is clearly also a $W^{2,p}$–local minimizer.

Proposition 1.12. *Let a smooth set $F \subseteq \mathbb{T}^n$ be a $W^{2,p}$–local minimizer of $\mathcal{A}$, then F is a critical set and*

$$\Pi_F(\varphi) \geq 0 \qquad\qquad \textit{for all } \varphi \in \widetilde{H}^1(\partial F),$$

in particular F is stable.

Proof. If F is a local minimizer of $\mathcal{A}$, for any admissible vector field $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ with associated flow smooth Φ, we have $\mathrm{Vol}(F_t) = \mathrm{Vol}(\Phi_t(F)) = \mathrm{Vol}(F)$ and for every $\delta > 0$, there clearly exists $\overline{\varepsilon} > 0$ such that for $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$ we have $\mathrm{Vol}(F \triangle F_t) < \delta$ and

$$\partial F_t = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon$$

for a smooth function $\psi : \partial F \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$. Hence, the $W^{2,p}$–local minimality of F implies

$$\mathcal{A}(\partial F) \leq \mathcal{A}(\partial F_t),$$

for every $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$. Thus,

$$0 = \frac{d}{dt}\mathcal{A}(\partial F_t)\Big|_{t=0} = \int_{\partial F} \mathrm{H}\langle X, \nu_F\rangle \, d\mu\,,$$

by Theorem 1.3, which implies that F is a critical set, by the subsequent discussion and

$$0 \leq \frac{d^2}{dt^2}\mathcal{A}(\partial F_t)\Big|_{t=0} = \Pi_F(\langle X, \nu_F\rangle),$$

by Theorem 1.7 and equation (1.32).
Since by Lemma 1.4, for every smooth function $\varphi : \partial F \to \mathbb{R}$ with zero integral there exists an admissible vector field $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ such that $\varphi = \langle X, \nu_F\rangle$, we conclude that $\Pi_F(\varphi) \geq 0$ for every $\varphi \in C^\infty(\partial F) \cap \widetilde{H}^1(\partial F)$, then the thesis follows by the density of this space in $\widetilde{H}^1(\partial F)$ (see [4]). ☐

The rest of this section will be devoted to strictly stable sets (see Definition 1.9), in particolar, we will show that the strict stability is a *sufficient* condition for the $W^{2,p}$–local minimality. Precisely, we will prove the following main theorem of this chapter.

Theorem 1.13. *Let $p > \max\{2, n-1\}$ and $F \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set for the Area functional $\mathcal{A}$ (under a volume constraint) as in Definition 1.9, and let N_ε be a tubular neighborhood of ∂F as in formula (1.36). Then, there exist constants $\delta, C > 0$ such that*

$$\mathcal{A}(\partial E) \geq \mathcal{A}(\partial F) + C[\alpha(F,E)]^2$$

for all smooth sets $E \subseteq \mathbb{T}^n$ such that $\mathrm{Vol}(E) = \mathrm{Vol}(F)$, $\mathrm{Vol}(E \triangle F) < \delta$, $\partial E \subseteq N_\varepsilon$ and

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\}$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, where the "distance" $\alpha(F,E)$ is defined as

$$\alpha(F,E) = \min_{\eta \in \mathbb{R}^n} \mathrm{Vol}(F \triangle (E + \eta))\,.$$

As a consequence, F is a $W^{2,p}$–local minimizer of $\mathcal{A}$. Moreover, if E is $W^{2,p}$–close enough to F and $\mathcal{A}(\partial E) = \mathcal{A}(\partial F)$, then E is a translate of F, that is, F is locally the unique $W^{2,p}$–local minimizer, up to translations.

Remark 1.14. We could have introduced the definitions of *strict* local minimizer or *strict* $W^{2,p}$–local minimizer for the Area functional, by asking that the inequalities $\mathcal{A}(\partial E) \leq \mathcal{A}(\partial F)$ in Definition 1.11 are equalities if and only if E is a translate of F. With such notion, the conclusion of this theorem is that F is actually a strict $W^{2,p}$–local minimizer.

Remark 1.15. With some extra effort, it can be proved that in the same hypotheses of Theorem 1.13, the set F is actually a local minimizer (see [2]). Since in the analysis of the surface diffusion flow in the next chapter we do not need such stronger result, we omitted its proof.

We postpone the proof of this result after showing some technical lemmas. We underline that most of the difficulties are due to the presence of the degeneracy subspace $T(\partial F)$ of the form Π_F (that is, where it is zero), related to the translation invariance of the Area functional (recall the discussion before Definition 1.9 of stability).

In the next key lemma we are going to show how to construct admissible smooth vector fields for a smooth set F, "related" to smooth sets which are $W^{2,p}$–close to it. By the same technique we then also prove Lemma 1.4 immediately after, whose proof was postponed from Section 1.2.

Lemma 1.16. *Let $F \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂F as above, in formula (1.36). For all $p > n-1$, there exist constants $\delta, C > 0$ with the following property: if $\psi \in C^\infty(\partial F)$*

and $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, *then there exists a field* $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ *with* $\operatorname{div} X = 0$ *in* N_ε *and with the associated flow* Φ *satisfying*

$$\Phi(1, y) = y + \psi(y)\nu_F(y), \qquad \textit{for all } y \in \partial F. \tag{1.40}$$

Moreover, for every $t \in [0, 1]$, *there holds*

$$\|\Phi(t, \cdot) - \mathrm{Id}\|_{W^{2,p}(\partial F)} \leq C\|\psi\|_{W^{2,p}(\partial F)}\,. \tag{1.41}$$

Finally, if $\mathrm{Vol}(F_1) = \mathrm{Vol}(F)$, *then* $\mathrm{Vol}(F_t) = \mathrm{Vol}(F)$ *for all* $t \in [-1, 1]$, *that is, the vector field* X *is admissible.*

Proof. We start considering the vector field $\widetilde{X} \in C^\infty(N_\varepsilon, \mathbb{R}^n)$ defined as

$$\widetilde{X}(x) = \xi(x)\nabla d_F(x) \qquad \forall x \in N_\varepsilon \tag{1.42}$$

where $d_F : N_\varepsilon \to \mathbb{R}$ is the signed distance and ξ is the function defined as follows: for all $y \in \partial F$ we let

$$f_y : (\varepsilon_0, \varepsilon_0) \to \mathbb{R}$$

to be the unique solution of the ODE

$$\begin{cases} f_y'(t) + f_y(t)\Delta d_F(y + t\nu_F(y)) = 0 \\ f_y(0) = 1 \end{cases}$$

and we set

$$\xi(x) = \xi(y + t\nu_F(y)) = f_y(t) = \exp\Big(-\int_0^t \Delta d_F(y + s\nu_F(y))\, ds\Big),$$

recalling that the map $(y, t) \mapsto x = y + t\nu_F(y)$ is a smooth diffeomorphism between $\partial F \times (-\varepsilon, \varepsilon)$ and N_ε. Notice that the function f is always positive, thus the same holds for ξ and $\xi = 1$, $\nabla d_F = \nu_F$, hence $\widetilde{X} = \nu_F$ on ∂F. Our aim is to prove that the smooth vector field X defined by

$$X(x) = \int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_F(x)))}\, \widetilde{X}(x) \tag{1.43}$$

for every $x \in N_\varepsilon$ and extended smoothly to all $\mathbb{T}^n$, satisfies all the properties of the statement of the lemma.

Step 1. We saw that $\widetilde{X}|_{\partial F} = \nu_F$, now we show that $\operatorname{div}\widetilde{X} = 0$ and analogously $\operatorname{div} X = 0$ in N_ε.

Given any $x = y + t\nu_F(y) \in N_\varepsilon$, with $y \in \partial F$, we have

$$\begin{aligned} \operatorname{div}\widetilde{X}(x) &= \operatorname{div}[\xi(x)\nabla d_F(x)] \\ &= \langle \nabla\xi(x), \nabla d_F(x)\rangle + \xi(x)\Delta d_F(x) \\ &= \frac{\partial}{\partial t}[\xi(y + t\nu_F(y))] + \xi(y + t\nu_F(y))\Delta d_F(y + t\nu_F(y)) \\ &= f_y'(t) + f_y(t)\Delta d_F(y + t\nu_F(y)) \\ &= 0, \end{aligned}$$

where we used the fact that $f_y'(t) = \langle \nabla\xi(y + t\nu_F(y)), \nu_F(y)\rangle$ and that we have $\nabla d_F(y + t\nu_F(y)) = \nu_F(y)$.
Since the function

$$x \mapsto \int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_F(x))} = \theta(x)$$

is constant along the segments $t \mapsto x + t\nabla d_F(x)$, for every $x \in N_\varepsilon$, it follows that

$$0 = \frac{\partial}{\partial t}\Big[\theta(x + t\nabla d_F(x))\Big]\bigg|_{t=0} = \langle \nabla\theta(x), \nabla d_F(x)\rangle,$$

hence,

$$\mathrm{div}X = \langle \nabla\theta, \nabla d_F\rangle\xi + \theta\mathrm{div}\widetilde{X} = 0.$$

Step 2. Recalling that $\psi \in C^\infty(\partial F)$ and $p > n-1$, we have

$$\|\psi\|_{L^\infty(\partial F)} \leq \|\psi\|_{C^1(\partial F)} \leq C_F\|\psi\|_{W^{2,p}(\partial F)},$$

by Sobolev embeddings (see [4]). Then, we can choose $\delta < \varepsilon/C_F$ such that for all $x \in \partial F$ we have that $x \pm \psi(x)\nu_F(x) \in N_\varepsilon$.
To check that equation (1.40) holds, we observe that the integral

$$\int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_E(x)))} = \theta(x)$$

represents the time needed to go from $\pi_F(x)$ to $\pi_F(x) + \psi(\pi_F(x))\nu_F(\pi_F(x))$ along the trajectory of the vector field $\widetilde{X}$, which is the segment connecting $\pi_F(x)$ and $\pi_F(x) + \psi(\pi_F(x))\nu_F(\pi_F(x))$, of length $\psi(\pi_F(x))$, parametrized as

$$s \mapsto \pi_F(x) + s\psi(\pi_F(x))\nu_F(\pi_F(x)),$$

for $s \in [0,1]$ and which is traveled with velocity $\xi(\pi_F(x) + s\nu_F(\pi_F(x))) = f_{\pi_F(x)}(s)$. Therefore, by the above definition of $X = \theta\widetilde{X}$ and the fact that the function θ is constant along such segments, we conclude that

$$\Phi(1,y) - \Phi(0,y) = \psi(y)\nu_F(y)$$

and, equivalently,

$$\Phi(1,y) = y + \psi(y)\nu_F(y)$$

for all $y \in \partial F$.

Step 3. To establish inequality (1.41), we first show that

$$\|X\|_{W^{2,p}(N_\varepsilon)} \leq C\|\psi\|_{W^{2,p}(\partial F)} \tag{1.44}$$

for a constant $C > 0$ depending only on F and ε. This estimate will follow from the definition of X in equation (1.43) and the definition of $W^{2,p}$–norm, that is,

$$\|X\|_{W^{2,p}(N_\varepsilon)} = \|X\|_{L^p(N_\varepsilon)} + \|\nabla X\|_{L^p(N_\varepsilon)} + \|\nabla^2 X\|_{L^p(N_\varepsilon)}\,.$$

As $|\nabla d_F| = 1$ everywhere and the positive function ξ, by its definition at the beginning of the proof, satisfies $0 < C_1 \leq \xi \leq C_2$ in N_ε, for a pair of constants C_1 and C_2, we have

$$\begin{aligned}
\|X\|^p_{L^p(N_\varepsilon)} &= \int_{N_\varepsilon} \left| \int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_F(x)))}\, \xi(x)\nabla d_F(x) \right|^p dx \\
&\leq \|\xi\|^p_{L^\infty(N_\varepsilon)} \int_{N_\varepsilon} \left| \int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_F(x)))} \right|^p dx \\
&\leq \frac{C_1^p}{C_2^p} \int_{N_\varepsilon} |\psi(\pi_F(x))|^p \, dx \\
&= \frac{C_1^p}{C_2^p} \int_{\partial F} \int_{-\varepsilon}^{\varepsilon} |\psi(\pi_F(y + t\nu_F(y)))|^p JL(y,t)\, dt\, d\mu(y) \\
&= \frac{C_1^p}{C_2^p} \int_{\partial F} |\psi(y)|^p \int_{-\varepsilon}^{\varepsilon} JL(y,t)\, dt\, d\mu(y) \\
&\leq C \int_{\partial F} |\psi(y)|^p \, d\mu(y) \\
&= C\|\psi\|^p_{L^p(\partial F)} \,.
\end{aligned}$$

where $L : \partial F \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ the smooth diffeomorphism defined in formula (1.39) and JL its Jacobian. Notice that the constant C depends only on E and ε.

Now we estimate the L^p–norm of ∇X. We compute

$$\begin{aligned}
\nabla X = & \frac{\nabla\psi(\pi_F(x))d\pi_F(x)}{\xi(\pi_F(x) + \psi(\pi_F(x))\nu_F(\pi_F(x)))}\, \xi(x)\nabla d_F(x) \\
& - \left[\int_0^{\psi(\pi_F(x))} \frac{\nabla\xi(\pi_F(x) + s\nu_F(\pi_F(x)))}{\xi^2(\pi_F(x) + s\nu_F(\pi_F(x)))} d\pi_F(x)(\mathrm{Id} + s d\nu_F(\pi_F(x)))\, ds \right] \cdot \\
& \cdot \xi(x)\nabla d_F(x) \\
& + \int_0^{\psi(\pi_F(x))} \frac{ds}{\xi(\pi_F(x) + s\nu_F(\pi_F(x)))} \Big(\nabla\xi(x)\nabla d_F(x) + \xi(x)\nabla^2 d_F(x) \Big)
\end{aligned}$$

and we deal with the integrals in the three terms as before, changing variable by means of the function L. That is, since all the functions $d\pi_F$, $d\nu_F$, $\nabla^2\nu_F$, ξ, $1/\xi$, $\nabla\xi$ are bounded by some constants depending only on E and ε, we easily get (the constant C could vary from line to line)

$$\begin{aligned}
\|\nabla X\|^p_{L^p(N_\varepsilon)} \leq & \, C \int_{N_\varepsilon} |\nabla\psi(\pi_F(x))|^p \, dx + C \int_{N_\varepsilon} |\psi(\pi_F(x))|^p \, dx \\
= & \, C \int_{\partial F} \int_{-\varepsilon}^{\varepsilon} |\nabla\psi(\pi_F(y + t\nu_F(y)))|^p \, JL(y,t)\, dt\, d\mu(y) \\
& + C \int_{\partial F} \int_{-\varepsilon}^{\varepsilon} |\psi(\pi_F(y + t\nu_F(y)))|^p \, JL(y,t)\, dt\, d\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&= C \int_{\partial F} \Big(|\psi(y)|^p + |\nabla\psi(y)|^p\Big) \int_{-\varepsilon}^{\varepsilon} JL(y,t)\, dt\, d\mu(y)\\
&\leq C\|\psi\|^p_{L^p(\partial F)} + C\|\nabla\psi\|^p_{L^p(\partial F)}\\
&\leq C\|\psi\|^p_{W^{1,p}(\partial F)}\,.
\end{aligned}
$$

A very analogous estimate works for $\|\nabla^2 X\|^p_{L^p(N_\varepsilon)}$ and we obtain also

$$
\|\nabla^2 X\|^p_{L^p(N_\varepsilon)} \leq C\|\psi\|^p_{W^{2,p}(\partial F)}\,,
$$

hence, inequality (1.44) follows with $C = C(F,\varepsilon)$.

Applying now Lagrange theorem to every component of $\Phi(\cdot, y)$ for every $y \in \partial F$ and $t \in [0,1]$, we have

$$
\Phi_i(t,y) - y_i = \Phi_i(t,y) - \Phi_i(0,y) = tX^i(\Phi(s,y))\,,
$$

for every $i \in \{1,\dots,n\}$, where $s = s(y,t)$ is a suitable value in $(0,1)$. Then, it clearly follows

$$
\|\Phi(t,\cdot) - \mathrm{Id}\|_{L^\infty(\partial F)} \leq C\|X\|_{L^\infty(N_\varepsilon)} \leq C\|X\|_{W^{2,p}(N_\varepsilon)} \leq C\|\psi\|_{W^{2,p}(\partial F)} \tag{1.45}
$$

by estimate (1.44), with $C = C(F,\varepsilon)$ (notice that we used Sobolev embeddings, being $p > n-1$, the dimension of ∂F).
Differentiating the equations relating Φ to X in system (1.9), we have (recall that we use the convention of summing over the repeated indices)

$$
\begin{cases}
\frac{\partial}{\partial t}\nabla^i\Phi_j(t,y) = \nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\\
\nabla^i\Phi_j(0,y) = \delta_{ij}
\end{cases} \tag{1.46}
$$

for every $i,j \in \{1,\dots,n\}$. It follows,

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big|^2 &\leq 2\Big|(\nabla^i\Phi_j(t,y) - \delta_{ij})\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\Big|\\
&\leq 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big|^2\\
&\quad + 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big|
\end{aligned}
$$

hence, for almost every $t \in [0,1]$ where the following derivative exists,

$$
\frac{\partial}{\partial t}\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big| \leq C\|\nabla X\|_{L^\infty(N_\varepsilon)}\Big(\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big| + 1\Big)\,.
$$

Integrating this differential inequality, we get

$$
\Big|\nabla^i\Phi_j(t,y) - \delta_{ij}\Big| \leq e^{tC\|\nabla X\|_{L^\infty(N_\varepsilon)}} - 1 \leq e^{C\|X\|_{W^{2,p}(N_\varepsilon)}} - 1,
$$

as $t \in [0,1]$ and where we used Sobolev embeddings again. Then, by inequality (1.44), we estimate

$$
\sum_{1\leq i,j\leq n} \|\nabla^i\Phi_j(t,\cdot) - \delta_{ij}\|_{L^\infty(\partial F)} \leq C\Big(e^{C\|\psi\|_{W^{2,p}(\partial F)}} - 1\Big) \leq C\|\psi\|_{W^{2,p}(\partial F)}, \tag{1.47}
$$

as $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$, for any $t \in [0,1]$ and $y \in \partial F$, with $C = C(F, \varepsilon, \delta)$.
Differentiating equations (1.46), we obtain

$$\begin{cases} \frac{\partial}{\partial t}\nabla^\ell\nabla^i\Phi_j(t,y) = \nabla^s\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\nabla^\ell\Phi_s(t,y) \\ \qquad\qquad\qquad\qquad + \nabla^k X^j(\Phi(t,y))\nabla^\ell\nabla^i\Phi_k(t,y) \\ \nabla^\ell\nabla^i\Phi(0,y) = 0 \end{cases}$$

(where we sum over s and k), for every $t \in [0,1]$, $y \in \partial F$ and $i, j, \ell \in \{1, \dots, n\}$.
This is a linear *non–homogeneous* system of ODEs such that, if we control $C\|\psi\|_{W^{2,p}(\partial F)}$, the smooth coefficients in the right side multiplying the solutions $\nabla^\ell\nabla^i\Phi_j(\cdot, y)$ are uniformly bounded (as in estimate (1.47), Sobolev embeddings imply that ∇X is bounded in L^∞ by $C\|\psi\|_{W^{2,p}(\partial F)}$).
Then, arguing as before, for almost every $t \in [0,1]$ where the following derivative exists, there holds

$$\begin{aligned} \frac{\partial}{\partial t}\Big|\nabla^2\Phi(t,y)\Big| &\leq C\|\nabla X\|_{L^\infty(N_\varepsilon)}\Big|\nabla^2\Phi(t,y)\Big| + C|\nabla^2 X(\Phi(t,y))| \\ &\leq C\delta\Big|\nabla^2\Phi(t,y)\Big| + C|\nabla^2 X(\Phi(t,y))|\,, \end{aligned}$$

by inequality (1.44) (notice that inequality (1.47) gives an L^∞–bound on $\nabla\Phi$, *not only* in L^p, which is crucial). Thus, by means of Gronwall's lemma (see [25], for instance), we obtain the estimate

$$\Big|\nabla^2\Phi(t,y)\Big| \leq C\int_0^t |\nabla^2 X(\Phi(s,y))| e^{C\delta(t-s)}\,ds \leq C\int_0^t |\nabla^2 X(\Phi(s,y))|\,ds\,,$$

hence,

$$\begin{aligned} \|\nabla^2\Phi(t,\cdot)\|^p_{L^p(\partial F)} &\leq C\int_{\partial F}\Big(\int_0^t |\nabla^2 X(\Phi(s,y))|\,ds\Big)^p\,d\mu(y) \\ &\leq C\int_0^t\int_{\partial F} |\nabla^2 X(\Phi(s,y))|^p\,d\mu(y)ds \\ &= C\int_{N_\varepsilon} |\nabla^2 X(x)|^p JL^{-1}(x)\,dx \\ &\leq C\|\nabla^2 X\|^p_{L^p(N_\varepsilon)} \\ &\leq C\|X\|^p_{W^{2,p}(N_\varepsilon)} \\ &\leq C\|\psi\|^p_{W^{2,p}(\partial F)}\,, \end{aligned} \tag{1.48}$$

by estimate (1.44), for every $t \in [0,1]$, with $C = C(F, \varepsilon, \delta)$.
Clearly, putting together inequalities (1.45), (1.47) and (1.48), we get the estimate (1.41) in the statement of the lemma.

Step 4. Finally, we remind that

$$\frac{d^2}{dt^2}\mathrm{Vol}(F_t) = \int_{\partial F_t} \langle X, \nu_{E_t}\rangle \mathrm{div}^{\mathbb{T}^n} X\,d\mu_t,$$

hence, since by Step 1 we know that $\mathrm{div}X = 0$, we conclude $\frac{d^2}{dt^2}\mathrm{Vol}(F_t) = 0$ for all $t \in [-1,1]$, that is, the function $t \mapsto \mathrm{Vol}(F_t)$ is linear. If $\mathrm{Vol}(F_1) = \mathrm{Vol}(F) = \mathrm{Vol}(F_0)$, it follows that $\mathrm{Vol}(F_t) = \mathrm{Vol}(F)$, for all $t \in [-1,1]$. $\square$

With an argument similar to the one of Step 4 in this proof, we now prove Lemma 1.4.

Proof of Lemma 1.4. Let $\varphi : \partial E \to \mathbb{R}$ a C^∞ function with zero integral, then we define the following smooth vector field in N_ε,

$$X(x) = \varphi(\pi_E(x))\widetilde{X}(x),$$

where $\widetilde{X}$ is the smooth vector field defined by formula (1.42) and we extend it to a smooth vector field $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ on the whole $\mathbb{T}^n$. Clearly, by the properties of $\widetilde{X}$ seen above,

$$\langle X(y), \nu_E(y)\rangle = \varphi(y)\langle \widetilde{X}(y), \nu_E(y)\rangle = \varphi(y)$$

for every $y \in \partial E$.
As the function $x \mapsto \varphi(\pi_E(x))$ is constant along the segments $t \mapsto x + t\nabla d_E(x)$, for every $x \in N_\varepsilon$, it follows, as in Step 1 of the previous proof, that $\operatorname{div} X = 0$ in N_ε. Then, arguing as in Step 4, the function $t \mapsto \operatorname{Vol}(E_t)$ is linear, for t in some interval $(-\delta, \delta)$. Since, by equation (1.18), there holds

$$\frac{d}{dt}\operatorname{Vol}(E_t)\Big|_{t=0} = \int_{\partial E} \langle X, \nu_E\rangle\, d\mu = \int_{\partial E} \varphi\, d\mu = 0,$$

such function $t \mapsto \operatorname{Vol}(E_t)$ must actually be constant.
Hence, $\operatorname{Vol}(E_t) = \operatorname{Vol}(E)$, for all $t \in (-\delta, \delta)$ and X is admissible. □

The next lemma gives a technical estimate needed in the proof of Theorem 1.13.

Lemma 1.17. *Let $p > \max\{2, n-1\}$ and $F \subseteq \mathbb{T}^n$ a smooth strictly stable critical set for the (volume–constrained) functional $\mathcal{A}$. Then, in the hypotheses and notation of Lemma 1.16, there exist constants $\delta, C > 0$ such that if $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$, then $|X| \leq C|\langle X, \nu_{F_t}\rangle|$ on ∂F_t and*

$$\|\nabla X\|_{L^2(\partial F_t)} \leq C\|\langle X, \nu_{F_t}\rangle\|_{H^1(\partial F_t)}, \tag{1.49}$$

hence,

$$\|X\|_{H^1(\partial F_t)} \leq C\|\langle X, \nu_{F_t}\rangle\|_{H^1(\partial F_t)}$$

(here ∇ is the covariant derivative along F_t), for all $t \in [0,1]$, where $X \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ is the smooth vector field defined in formula (1.43).

Proof. Fixed $\overline{\varepsilon} > 0$, from inequality (1.41) it follows that there exists $\delta > 0$ such that if $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$ there holds

$$|\nu_{F_t}(\Phi_t(y)) - \nu_F(y)| \leq \overline{\varepsilon}$$

for every $y \in \partial F$, hence, as $\nabla d_F = \nu_F$ on ∂F, we have

$$|\nabla d_F(\Phi_t^{-1}(y)) - \nu_{F_t}(y)| = |\nu_F(\Phi_t^{-1}(y)) - \nu_{F_t}(y)| \leq \overline{\varepsilon}$$

for every $y \in \partial F_t$. Then, if $\|\psi\|_{W^{2,p}(\partial F)}$ is small enough, Φ_t^{-1} is close to the identity, thus

$$|\nabla d_F(\Phi_t^{-1}(y)) - \nabla d_F(y)| \leq \overline{\varepsilon}$$

on ∂F_t and we conclude

$$\|\nabla d_F - \nu_{F_t}\|_{L^\infty(\partial F_t)} \leq 2\overline{\varepsilon}\,.$$

We estimate $X_{\tau_t} = X - \langle X, \nu_{F_t}\rangle \nu_{F_t}$ (recall that $X = \langle X, \nabla d_F\rangle \nabla d_F$),

$$\begin{aligned}
|X_{\tau_t}| &= |X - \langle X, \nu_{F_t}\rangle \nu_{F_t}| \\
&= |\langle X, \nabla d_F\rangle \nabla d_F - \langle X, \nu_{F_t}\rangle \nu_{F_t}| \\
&= |\langle X, \nabla d_F\rangle \nabla d_F - \langle X, \nu_{F_t}\rangle \nabla d_F + \langle X, \nu_{F_t}\rangle \nabla d_F - \langle X, \nu_{F_t}\rangle \nu_{F_t}| \\
&\leq |\langle X, (\nabla d_F - \nu_{F_t})\rangle \nabla d_F| + |\langle X, \nu_{F_t}\rangle (\nabla d_F - \nu_{F_t})| \\
&\leq 2|X|\,|\nabla d_F - \nu_{F_t}| \\
&\leq 4\overline{\varepsilon}|X|\,,
\end{aligned}$$

then

$$|X_{\tau_t}| \leq 4\overline{\varepsilon}|X_{\tau_t} + \langle X, \nu_{F_t}\rangle \nu_{F_t}| \leq 4\overline{\varepsilon}|X_{\tau_t}| + |\langle X, \nu_{F_t}\rangle|\,,$$

hence,

$$|X_{\tau_t}| \leq C|\langle X, \nu_{F_t}\rangle|\,. \tag{1.50}$$

We now estimate its covariant derivative ∇ along F_t, that is,

$$\begin{aligned}
|\nabla X_{\tau_t}| &= |\nabla X - \nabla(\langle X, \nu_{F_t}\rangle \nu_{F_t})| \\
&= |\nabla(\langle X, \nabla d_F\rangle \nabla d_F) - \nabla(\langle X, \nu_{F_t}\rangle \nu_{F_t})| \\
&= |\nabla(\langle X, \nabla d_F\rangle \nabla d_F) - \nabla(\langle X, \nu_{F_t}\rangle \nabla d_F) \\
&\quad + \nabla(\langle X, \nu_{F_t}\rangle \nabla d_F) - \nabla(\langle X, \nu_{F_t}\rangle \nu_{F_t})| \\
&\leq |\nabla(\langle X, (\nabla d_F - \nu_{F_t})\rangle \nabla d_F)| + |\nabla(\langle X, \nu_{F_t}\rangle (\nabla d_F - \nu_{F_t}))| \\
&\leq C\overline{\varepsilon}\Big[|\nabla X| + |\nabla\langle X, \nu_{F_t}\rangle|\Big] + C|X|\Big[|\nabla(\nabla d_F)| + |\nabla \nu_{F_t}|\Big] \\
&\leq C\overline{\varepsilon}\Big[|\nabla(\langle X, \nu_{F_t}\rangle \nu_{F_t} + X_{\tau_t})| + |\nabla\langle X, \nu_{F_t}\rangle|\Big] \\
&\quad + C\Big(|\langle X, \nu_{F_t}\rangle| + |X_{\tau_t}|\Big)\Big[|\nabla^2 d_F| + |\nabla \nu_{F_t}|\Big]
\end{aligned}$$

hence, using inequality (1.50) and arguing as above, there holds

$$|\nabla X_{\tau_t}| \leq C|\nabla\langle X, \nu_{F_t}\rangle| + C|\langle X, \nu_{F_t}\rangle|\Big[|\nabla^2 d_F| + |\nabla \nu_{F_t}|\Big]\,.$$

Then, we get

$$\begin{aligned}
\|\nabla X_{\tau_t}\|^2_{L^2(\partial F_t)} &\leq C\|\nabla\langle X, \nu_{F_t}\rangle\|^2_{L^2(\partial F_t)} \\
&\quad + C\int_{\partial F_t} |\langle X, \nu_{F_t}\rangle|^2\Big[|\nabla^2 d_F| + |\nabla \nu_{F_t}|\Big]^2\,d\mu \\
&\leq C\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)} \\
&\quad + C\|\langle X, \nu_{F_t}\rangle\|^2_{L^{\frac{2p}{p-2}}(\partial F_t)}\Big\||\nabla^2 d_F| + |\nabla \nu_{F_t}|\Big\|^2_{L^p(\partial F_t)} \\
&\leq C\ \|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}
\end{aligned}$$

where in the last inequality we used as usual Sobolev embeddings, as $p > \max\{2, n-1\}$.
Considering now the covariant derivative of $X = X_{\tau_t} + \langle X, \nu_{F_t}\rangle \nu_{F_t}$, putting together this inequality, the trivial one

$$\|\nabla \langle X, \nu_{F_t}\rangle\|_{L^2(\partial F_t)} \leq \|\langle X, \nu_{F_t}\rangle\|_{H^1(\partial F_t)}$$

and inequality (1.50), we obtain estimate (1.49). □

We now prove that any smooth set E sufficiently $W^{2,p}$–close to another smooth set F, can be "translated" by a vector $\eta \in \mathbb{R}^n$ such that $\partial E - \eta = \{y + \varphi(y)\nu_F(y) \,:\, y \in \partial F\}$, for a function $\varphi \in C^\infty(\partial F)$ having a suitable small "projection" on $T(\partial F)$ (see the definitions and the discussion at the end of the previous section).

Lemma 1.18. *Let $p > n-1$ and $F \subseteq \mathbb{T}^n$ a smooth set with a tubular neighborhood N_ε as above, in formula (1.36). For any $\tau > 0$ there exist constants $\delta, C > 0$ such that if another smooth set $E \subseteq \mathbb{T}^n$ satisfies $\mathrm{Vol}(E \triangle F) < \delta$ and $\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon$ for a function $\psi \in C^\infty(\mathbb{R})$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, then there exist $\eta \in \mathbb{R}^n$ and $\varphi \in C^\infty(\partial F)$ with the following properties:*

$$\partial E - \eta = \{y + \varphi(y)\nu_F(y) \,:\, y \in \partial F\}\,,$$

$$|\eta| \leq C\|\psi\|_{W^{2,p}(\partial F)}, \qquad \|\varphi\|_{W^{2,p}(\partial F)} \leq C\|\psi\|_{W^{2,p}(\partial F)}$$

and

$$\left|\int_{\partial F} \varphi\nu_F \, d\mu\right| \leq \tau\|\varphi\|_{L^2(\partial F)}\,.$$

Proof. We let d_F to be the signed distance function from ∂F. We underline that, throughout all the proof, the various constants will be all independent of $\psi : \partial F \to \mathbb{R}$.

We recall that in Remark 1.10 we saw that there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that the functions $\langle \nu_F, e_i\rangle$ are orthogonal in $L^2(\partial F)$, that is,

$$\int_{\partial F} \langle \nu_F, e_i\rangle\langle \nu_F, e_j\rangle \, d\mu = 0\,, \tag{1.51}$$

for all $i \neq j$ and we let I_F to be the set of the indices $i \in \{1, \dots, n\}$ such that $\|\langle \nu_F, e_i\rangle\|_{L^2(\partial F)} > 0$. Given a smooth function $\psi : \partial F \to \mathbb{R}$, we set $\eta = \sum_{i=1}^n \eta_i e_i$, where

$$\eta_i = \begin{cases} \frac{1}{\|\langle \nu_F, e_i\rangle\|^2_{L^2(\partial F)}} \int_{\partial F} \psi(x)\langle \nu_F(x), e_i\rangle \, d\mu & \text{if } i \in \mathrm{I}_F, \\ \eta_i = 0 & \text{otherwise.} \end{cases} \tag{1.52}$$

Note that, from Hölder inequality, it follows

$$|\eta| \leq C_1\|\psi\|_{L^2(\partial F)}\,. \tag{1.53}$$

Step 1. Let $T_\psi : \partial F \to \partial F$ be the map

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta)\,.$$

It is easily checked that there exists $\varepsilon_0 > 0$ such that if

$$\|\psi\|_{W^{2,p}(\partial F)} + |\eta| \leq \varepsilon_0 \leq 1\,, \tag{1.54}$$

then T_ψ is a smooth diffeomorphism, moreover,

$$\|JT_\psi - 1\|_{L^\infty(\partial F)} \leq C\|\psi\|_{C^1(\partial F)} \tag{1.55}$$

and

$$\|T_\psi - \mathrm{Id}\|_{W^{2,p}(\partial E)} + \|T_\psi^{-1} - \mathrm{Id}\|_{W^{2,p}(\partial E)} \leq C(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|)\,. \tag{1.56}$$

Therefore, setting $\widehat{E} = E - \eta$, we have

$$\partial\widehat{E} = \{z + \varphi(z)\nu_F(z) \,:\, z \in \partial F\}$$

for some function φ, which is linked to ψ by the following relation: for all $y \in \partial F$ we let $z = z(y) \in \partial F$ such that

$$y + \psi(y)\nu_F(y) - \eta = z + \varphi(z)\nu_F(z)\,,$$

then

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta) = \pi_F(z + \varphi(z)\nu_F(z)) = z\,,$$

that is, $y = T_\psi^{-1}(z)$ and

$$\begin{aligned}
\varphi(z) &= \varphi(T_\psi(y)) \\
&= d_F(z + \varphi(z)\nu_F(z)) \\
&= d_F(y + \psi(y)\nu_F(y) - \eta) \\
&= d_F(T_\psi^{-1}(z) + \psi(T_\psi^{-1}(z))\nu_F(T_\psi(y)) - \eta)\,.
\end{aligned}$$

Thus, using inequality (1.56), we have

$$\|\varphi\|_{W^{2,p}(\partial F)} \leq C_2\Big(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|\Big)\,, \tag{1.57}$$

for some constant $C_2 > 1$. We now estimate

$$\begin{aligned}
\int_{\partial F} \varphi(z)\nu_F(z)\,d\mu(z) &= \int_{\partial f} \varphi(T_\psi(y))\nu_F(T_\psi(y))JT_\psi(y)\,d\mu(y) \\
&= \int_{\partial F} \varphi(T_\psi(y))\nu_F(T_\psi(y))\,d\mu(y) + R_1\,,
\end{aligned} \tag{1.58}$$

where

$$\begin{aligned}
|R_1| &= \left|\int_{\partial F} \varphi(T_\psi(y))\nu_F(T_\psi(y))\,[J_{n-1}\nabla T_\psi(y) - 1]\,d\mu(y)\right| \\
&\leq C_3\|\psi\|_{C^1(\partial F)}\|\varphi\|_{L^2(\partial F)}\,,
\end{aligned} \tag{1.59}$$

by inequality (1.55).
On the other hand,

$$
\begin{aligned}
\int_{\partial F} & \varphi(T_\psi(y))\nu_F(T_\psi(y))\,d\mu(y) \\
&= \int_{\partial F} \Big[y + \psi(y)\nu_F(y) - \eta - T_\psi(y)\Big]\,d\mu(y) \\
&= \int_{\partial F} \Big[y + \psi(y)\nu_F(y) - \eta - \pi_F(y + \psi(y)\nu_F(y) - \eta)\Big]\,d\mu(y) \\
&= \int_{\partial F} \Big\{\psi(y)\nu_F(y) - \eta + \Big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\Big]\Big\}\,d\mu(y) \\
&= \int_{\partial F} (\psi(y)\nu_F(y) - \eta)\,d\mu(y) + R_2\,,
\end{aligned}
\tag{1.60}
$$

where

$$
\begin{aligned}
R_2 &= \int_{\partial F} \Big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\Big]\,d\mu(y) \\
&= -\int_{\partial F} d\mu(y) \int_0^1 \nabla\pi_F(y + t(\psi(y)\nu(y) - \eta))(\psi(y)\nu_F(y) - \eta)\,dt \\
&= -\int_{\partial F} \nabla\pi_F(y)(\psi(y)\nu_F(y) - \eta)\,d\mu(y) + R_3\,.
\end{aligned}
\tag{1.61}
$$

In turn, recalling inequality (1.53), we get

$$
\begin{aligned}
|R_3| \leq \int_{\partial F} d\mu(y) \int_0^1 & |\nabla\pi_F(y + t(\psi(y)\nu_F(y) - \eta)) \\
& - \nabla\pi_F(y)|\,|\psi(y)\nu_F(y) - \eta|\,dt \\
\leq C_4 & \|\psi\|^2_{L^2(\partial F)}\,.
\end{aligned}
\tag{1.62}
$$

Since in N_ε, by equation (1.38), we have $\pi_F(x) = x - d_F(x)\nabla d_F(x)$, it follows

$$
\frac{\partial \pi_F^i}{\partial x_j}(x) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(x)\frac{\partial d_F}{\partial x_j}(x) - d_F(x)\frac{\partial^2 d_F}{\partial x_i \partial x_j}(x),
$$

thus, for all $y \in \partial F$

$$
\frac{\partial \pi_F^i}{\partial x_j}(y) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(y)\frac{\partial d_F}{\partial x_j}(y)\,.
$$

From this identity and equalities (1.58), (1.60) and (1.61), we conclude

$$
\begin{aligned}
\int_{\partial F} \varphi(z)\nu_F(z)\,d\mu(z) = \int_{\partial F} & \Big[\psi(x)\nu_F(x) - \langle \eta, \nu_F(x)\rangle\nu_F(x)\Big]\,d\mu(x) \\
& + R_1 + R_3\,.
\end{aligned}
$$

As the integral at the right–hand side vanishes by relations (1.51) and (1.52), estimates (1.59) and (1.62) imply

$$\begin{aligned}\left|\int_{\partial F}\varphi(y)\nu_F(y)\,d\mu(y)\right| \leq& C_3\|\psi\|_{C^1(\partial F)}\|\varphi\|_{L^2(\partial F)}+C_4\|\psi\|^2_{L^2(\partial F)}\\ \leq& C\|\psi\|_{C^1(\partial F)}\Big(\|\varphi\|_{L^2(\partial F)}+\|\psi\|_{L^2(\partial F)}\Big)\\ \leq& C_5\|\psi\|^{1-\vartheta}_{W^{2,p}(\partial F)}\|\psi\|^{\vartheta}_{L^2(\partial F)}\Big(\|\varphi\|_{L^2(\partial F)}\\ &+\|\psi\|_{L^2(\partial F)}\Big)\,,\end{aligned} \tag{1.63}$$

where in the last passage we used a well–known interpolation inequality, with $\vartheta\in(0,1)$ depending only on $p>n-1$ (see [4, Theorem 3.70]).

Step 2. The previous estimate does not allow to conclude directly, but we have to rely on the following iteration procedure. Fix any number $K>1$ and assume that $\delta\in(0,1)$ is such that (possibly considering a smaller τ)

$$\tau+\delta<\varepsilon_0/2,\qquad C_2\delta(1+2C_1)\leq\tau,\qquad 2C_5\delta^{\vartheta}K\leq\delta\,. \tag{1.64}$$

Given ψ, we set $\varphi_0=\psi$ and we denote by η^1 the vector defined as in (1.52). We set $E_1=E-\eta^1$ and denote by φ_1 the function such that $\partial E_1=\{x+\varphi_1(x)\nu_F(x)\,:\,x\in\partial F\}$. As before, φ_1 satsfies

$$y+\varphi_0(y)\nu_F(y)-\eta^1=z+\varphi_1(z)\nu_F(z)\,.$$

Since $\|\psi\|_{W^{2,p}(\partial F)}\leq\delta$ and $|\eta|\leq C_1\|\psi\|_{L^2(\partial F)}$, by inequalities (1.53), (1.57) and (1.64) we have

$$\|\varphi_1\|_{W^{2,p}(\partial F)}\leq C_2\delta(1+C_1)\leq\tau\,. \tag{1.65}$$

Using again that $\|\psi\|_{W^{2,p}(\partial F)}<\delta<1$, by estimate (1.63) we obtain

$$\left|\int_{\partial F}\varphi_1(y)\nu_F(y)\,d\mu(y))\right|\leq C_5\|\varphi_0\|^{\vartheta}_{L^2(\partial F)}\Big(\|\varphi_1\|_{L^2(\partial F)}+\|\varphi_0\|_{L^2(\partial F)}\Big)\,,$$

where we have $\|\varphi_0\|_{L^2(\partial F)}\leq\delta$.
We now distinguish two cases.
If $\|\varphi_0\|_{L^2(\partial F)}\leq K\|\varphi_1\|_{L^2(\partial F)}$, from the previous inequality and (1.64), we get

$$\begin{aligned}\left|\int_{\partial F}\varphi_1(y)\nu_F(y)\,d\mu(y)\right|&\leq C_5\delta^{\vartheta}\Big(\|\varphi_1\|_{L^2(\partial F)}+\|\varphi_0\|_{L^2(\partial F)}\Big)\\ &\leq 2C_5\delta^{\vartheta}K\|\varphi_1\|_{L^2(\partial F)}\\ &\leq\delta\|\varphi_1\|_{L^2(\partial F)}\,,\end{aligned}$$

thus, the conclusion follows with $\eta=\eta^1$.
In the other case,

$$\|\varphi_1\|_{L^2(\partial F)}\leq\frac{\|\varphi_0\|_{L^2(\partial F)}}{K}\leq\frac{\delta}{K}\leq\delta\,. \tag{1.66}$$

We then repeat the whole procedure: we denote by η^2 the vector defined as in formula (1.52) with ψ replaced by φ_1, we set $E_2 = E_1 - \eta^2 = E - \eta^1 - \eta^2$ and we consider the corresponding φ_2 which satisfies

$$w + \varphi_2(w)\nu_F(w) = z + \varphi_1(z)\nu_F(z) - \eta^2 = y + \varphi_0(y)\nu_F(y) - \eta^1 - \eta^2 \,.$$

Since

$$\begin{aligned}\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \eta^2| &\leq \delta + C_1\delta + C_1\|\varphi_1\|_{L^2(\partial F)} \\ &\leq \delta + C_1\delta\Big(1 + \frac{1}{K}\Big) \leq C_2\delta(1 + 2C_1) \leq \tau \,,\end{aligned}$$

the map $T_{\varphi_0}(y) = \pi_F(y + \varphi_0(y)\nu_F(y) - (\eta^1 + \eta^2))$ is a diffeomorphism thanks to formula (1.54) (having chosen τ and δ small enough).

Thus, by applying inequalities (1.57) (with $\eta = \eta^1 + \eta^2$), (1.53), (1.64) and (1.66), we get

$$\|\varphi_2\|_{W^{2,p}(\partial F)} \leq C_2\Big(\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \eta^2|\Big) \leq C_2\delta\Big(1 + C_1 + \frac{C_1}{K}\Big) \leq \tau \,,$$

as $K > 1$, analogously to conclusion (1.65). On the other hand, by estimates (1.53), (1.65) and (1.66),

$$\|\varphi_1\|_{W^{2,p}(\partial F)} + \eta^2 \leq C_2\delta(1 + C_1) + C_1\frac{\delta}{K} \leq C_2\delta(1 + 2C_1) \leq \tau \,,$$

hence, also the map $T_{\varphi_1}(x) = \pi_F(x + \varphi_1(x)\nu_F(x) - \eta^2)$ is a diffeomorphism satisfying inequalities (1.54) and (1.55). Therefore, arguing as before, we obtain

$$\left|\int_{\partial F} \varphi_2(y)\nu_F(y)\,d\mu(y)\right| \leq C_5\|\varphi_1\|^{\vartheta}_{L^2(\partial F)}\Big(\|\varphi_2\|_{L^2(\partial F)} + \|\varphi_1\|_{L^2(\partial F)}\Big) \,.$$

Since $\|\varphi_1\|_{L^2(\partial F)} \leq \delta$ by inequality (1.66), if $\|\varphi_1\|_{L^2(\partial F)} \leq K\|\varphi_2\|_{L^2(\partial F)}$ the conclusion follows with $\eta = \eta^1 + \eta^2$. Otherwise, we iterate the procedure observing that

$$\|\varphi_2\|_{L^2(\partial F)} \leq \frac{\|\varphi_1\|_{L^2(\partial F)}}{K} \leq \frac{\|\varphi_0\|_{L^2(\partial F)}}{K^2} \leq \frac{\delta}{K^2} \,.$$

This construction leads to three (possibly finite) sequences η^n, E_n and φ_n such that

$$\begin{cases} E_n = E - \eta^1 - \cdots - \eta^n, \qquad |\eta^n| \leq \frac{C_1\delta}{K^{n-1}} \\ \|\varphi_n\|_{W^{2,p}(\partial F)} \leq C_2\Big(\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \cdots + \eta^n|\Big) \leq C_2\delta(1 + 2C_1) \\ \|\varphi_n\|_{L^2(\partial F)} \leq \frac{\delta}{K^n} \\ \partial E_n = \{x + \varphi_n(x)\nu_F(x) \,:\, x \in \partial F\} \end{cases}$$

If for some $n \in \mathbb{N}$ we have $\|\varphi_{n-1}\|_{L^2(\partial F)} \leq K\|\varphi_n\|_{L^2(\partial F)}$, the construction stops, since, arguing as before,

$$\left|\int_{\partial F} \varphi_n(y)\nu_F(y)\,d\mu(y)\right| \leq \delta\|\varphi_n\|_{L^2(\partial F)}$$

and conclusion follows with $\eta = \eta^1 + \cdots + \eta^n$ and $\varphi = \varphi_n$. Otherwise, the iteration continues indefinitely and we reach the conclusion with

$$\eta = \sum_{n=1}^{\infty} \eta^n, \qquad \varphi = 0\,,$$

(notice that the series is converging) which actually means that $E = \eta + F$, hence the thesis is obvious. □

We are now ready to prove the main theorem of this chapter.

Proof of Theorem 1.13.
Step 1. We first want to show that

$$m_0 = \inf\Big\{\Pi_F(\varphi) \,:\, \varphi \in T^{\perp}(\partial F),\, \|\varphi\|_{H^1(\partial F)} = 1\Big\} > 0.$$

We consider a minimizing sequence φ_i for such infimum and we assume that $\varphi_i \rightharpoonup \varphi_0$ weakly in $H^1(\partial F)$, then $\varphi_0 \in T^{\perp}(\partial F)$ (since it is a closed subspace of $H^1(\partial F)$) and if $\varphi_0 \neq 0$, there holds

$$m_0 = \lim_{i\to\infty} \Pi_F(\varphi_i) \geq \Pi_F(\varphi_0) > 0$$

due to the the strict stability of F and the lower semicontinuity of Π_F (recall formula (1.31) and the fact that the weak convergence in $H^1(\partial F)$ implies strong convergence in $L^2(\partial F)$ by Sobolev embeddings). On the other hand, if instead $\varphi_0 = 0$, again by the strong convergence of $\varphi_i \to \varphi_0$ in $L^2(\partial F)$, by looking at formula (1.31), we have

$$m_o = \lim_{i\to\infty} \Pi_F(\varphi_i) = \lim_{i\to\infty} \int_{\partial F} |\nabla \varphi_i|^2 \, d\mu = \lim_{i\to\infty} \|\varphi_i\|^2_{H^1(\partial F)} = 1$$

since $\|\varphi_i\|_{L^2(\partial F)} \to 0$.
Step 2. Now we prove that there exists a constant $\delta_1 > 0$ such that if E is like in the statement and $\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\}$, with $\|\psi\|_{W^{2,p}(\partial F)} < \delta_1$, and $\mathrm{Vol}(E) = \mathrm{Vol}(F)$, then

$$\inf\Big\{\Pi_E(\varphi) \,:\, \varphi \in \widetilde{H}^1(\partial E),\, \|\varphi\|_{H^1(\partial E)} = 1,\, \Big|\int_{\partial E} \varphi \nu_E \, d\mu\Big| < \delta_1\Big\} \geq \frac{m_0}{2}.$$

We argue by contradiction assuming that there exists a sequence of smooth sets E_i with $\partial E_i = \{y + \psi_i(y)\nu_F(y) \,:\, y \in \partial F\}$ with $\|\psi_i\|_{W^{2,p}(\partial F)} \to 0$ and $\mathrm{Vol}(E_i) = \mathrm{Vol}(F)$, and a sequence of smooth functions $\varphi_i \in \widetilde{H}^1(\partial E_i)$ with $\|\varphi_i\|_{H^1(\partial E_i)} = 1$ and $\int_{\partial E_i} \varphi_i \nu_{E_i} \, d\mu_i \to 0$, such that

$$\Pi_{E_i}(\varphi_i) < \frac{m_0}{2}.$$

We then define the following sequence of smooth functions

$$\widetilde{\varphi}_i(y) = \varphi_i(y + \psi_i(y)\nu_F(y)) - \fint_{\partial F} \varphi_i(y + \psi_i(y)\nu_F(y)) \, d\mu(y) \tag{1.67}$$

which clearly belong to $\widetilde{H}^1(\partial F)$. Setting $\theta_i(y) = y + \psi_i(y)\nu_F(y)$, as $p > \max\{2, n-1\}$, by the Sobolev embeddings, $\theta_i \to \mathrm{Id}$ in $C^{1,\alpha}$ and $\nu_{E_i} \circ \theta_i \to \nu_F$ in $C^{0,\alpha}(\partial F)$, hence, the sequence $\widetilde{\varphi}_i$ is bounded in $H^1(\partial F)$ and if $\{e_k\}$ is the special orthonormal basis found in Remark 1.10, we have $\langle \nu_{E_i} \circ \theta_i, e_k\rangle \to \langle \nu_F, e_k\rangle$ uniformly for all $k \in \{1, \dots, n\}$. Thus,

$$\int_{\partial F} \widetilde{\varphi}_i \langle \nu_F, e_k\rangle \, d\mu \to 0,$$

as $i \to \infty$, indeed,

$$\int_{\partial F} \widetilde{\varphi}_i \langle \nu_F, e_k\rangle \, d\mu - \int_{\partial F} \widetilde{\varphi}_i \langle \nu_{E_i} \circ \theta_i, e_k\rangle \, d\mu \to 0$$

and

$$\int_{\partial F} \widetilde{\varphi}_i \langle \nu_{E_i} \circ \theta_i, e_k\rangle \, d\mu = \int_{\partial E_i} \varphi_i \langle \nu_{E_i}, e_k\rangle \, J\theta_i^{-1} d\mu_i \to 0,$$

as the Jacobians (notice that $J\theta_i$ are Jacobians "relative" to the hypersurface ∂E) $J\theta_i^{-1} \to 1$ uniformly and we assumed

$$\int_{\partial E_i} \varphi_i \nu_{E_i} \, d\mu_i \to 0\,.$$

Hence, using expression (1.35) for the projection map π on $T^\perp(\partial F)$, it follows $\|\pi(\widetilde{\varphi}_i) - \widetilde{\varphi}_i\|_{H^1(\partial F)} \to 0$, as $i \to \infty$ and

$$\lim_{i\to\infty} \|\pi(\widetilde{\varphi}_i)\|_{H^1(\partial F)} = \lim_{i\to\infty} \|\widetilde{\varphi}_i\|_{H^1(\partial F)} = \lim_{i\to\infty} \|\varphi_i\|_{H^1(\partial E_i)} = 1, \qquad (1.68)$$

since $\|\psi_i\|_{W^{2,p}(\partial F)} \to 0$, thus $\|\psi_i\|_{C^{1,\alpha}(\partial F)} \to 0$, by looking at the definition of the functions $\widetilde{\varphi}_i$ in formula (1.67).

Note now that the $W^{2,p}$–convergence of E_i to F (computing similarly to Remark 1.1, the second fundamental form $B_{\partial E_i}$ of ∂E_i is "morally" the Hessian of ψ_i) implies

$$B_{\partial E_i} \circ \theta_i \to B_{\partial F} \qquad \text{in } L^p(\partial F)\,,$$

as $i \to \infty$. Moreover, by the Sobolev embeddings again (in particular $H^1(\partial F) \hookrightarrow L^q(\partial F)$ for any $q \in [1, 2^*)$, with $2^* = 2(n-1)/(n-3)$ which is larger than 2) and the $W^{2,p}$–convergence of E_i to F, we get

$$\int_{\partial E_i} |B_{\partial E_i}|^2 \varphi_i^2 \, d\mu_i - \int_{\partial F} |B_{\partial F}|^2 \widetilde{\varphi}_i^2 \, d\mu \to 0\,.$$

Finally, recalling formula (1.31), we conclude

$$\Pi_{E_i}(\varphi_i) - \Pi_F(\widetilde{\varphi}_i) \to 0\,,$$

since we have

$$\|\varphi_i\|_{L^2(\partial E_i)} - \|\widetilde{\varphi}_i\|_{L^2(\partial F)} \to 0\,,$$

which easily follows again by looking at the definition of the functions $\widetilde{\varphi}_i$ in formula (1.67) and taking into account that $\|\psi_i\|_{C^{1,\alpha}(\partial F)} \to 0$, hence limits (1.68) imply

$$\|\nabla\varphi_i\|_{L^2(\partial E_i)} - \|\nabla\widetilde{\varphi}_i\|_{L^2(\partial F)} \to 0\,.$$

By the previous conclusion $\|\pi(\widetilde{\varphi}_i) - \widetilde{\varphi}_i\|_{H^1(\partial F)} \to 0$ and Sobolev embeddings, it this then straightforward, arguing as above, to get also

$$\Pi_F(\widetilde{\varphi}_i) - \Pi_F(\pi(\widetilde{\varphi}_i)) \to 0\,,$$

hence,

$$\Pi_{E_i}(\varphi_i) - \Pi_F(\pi(\widetilde{\varphi}_i)) \to 0\,.$$

Since we assumed that $\Pi_{E_i}(\varphi_i) < m_0/2$, we conclude that for $i \in \mathbb{N}$, large enough there holds

$$\Pi_F(\pi(\widetilde{\varphi}_i)) \leq \frac{m_0}{2} < m_0\,,$$

which is a contradiction to Step 1, as $\pi(\widetilde{\varphi}_i) \in T^{\perp}(\partial F)$.

Step 3. Let us fix E such that $\mathrm{Vol}(E) = \mathrm{Vol}(F)$, $\mathrm{Vol}(E \triangle F) < \delta$ and

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon\,,$$

with $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$, where $\delta > 0$ is smaller than δ_1 given by Step 2. Taking a possibly smaller $\delta > 0$, we consider the smooth vector field X given by Lemma 1.16 and the associated flow Φ. Hence, $\mathrm{div}X = 0$ in N_ε and $\Phi(1, y) = y + \psi(y)\nu_F(y)$, for all $y \in \partial F$, that is, $\Phi(1, \partial F) = \partial E \subseteq N_\varepsilon$ which implies $F_1 = \Phi(1, F) = E$. Then X is an admissible smooth vector field, as $\mathrm{Vol}(F_1) = \mathrm{Vol}(E) = \mathrm{Vol}(F)$, by the last part of such lemma.

By Lemma 1.18, choosing an even smaller $\delta > 0$ if necessary, possibly replacing the set E with a translate $E - \eta$, for some small $\eta \in \mathbb{R}^n$, we can assume that

$$\left|\int_{\partial F} \psi\nu_F \, d\mu\right| \leq \frac{\delta_1}{2}\|\psi\|_{L^2(\partial F)}\,. \tag{1.69}$$

Letting $F_t = \Phi_t(F)$, we now claim that

$$\left|\int_{\partial F_t} \langle X, \nu_{F_t}\rangle \nu_{F_t} \, d\mu_t\right| \leq \delta_1 \|\langle X, \nu_{F_t}\rangle\|_{L^2(\partial F_t)}\,, \tag{1.70}$$

for every $t \in [0, 1]$. To this aim, we write

$$\begin{aligned}
\int_{\partial F_t} \langle X, \nu_{F_t}\rangle \nu_{F_t} \, d\mu &= \int_{\partial F} \langle X \circ \Phi_t, \nu_{F_t} \circ \Phi_t\rangle (\nu_{F_t} \circ \Phi_t)\, J\Phi_t \, d\mu \\
&= \int_{\partial F} \langle X \circ \Phi_t, \nu_F\rangle \nu_F \, d\mu + R_1 \\
&= \int_{\partial F} \langle X, \nu_F\rangle \nu_F \, d\mu + R_1 + R_2 \\
&= \int_{\partial F} \psi\nu_F \, d\mu + R_1 + R_2 + R_3\,.
\end{aligned}$$

By the definition of the vector field X in formula (1.43) (in the proof of Lemma 1.16), the bounds $0 < C_1 \leq \xi \leq C_2$ and $\|J(\pi_F \circ \Phi_t)^{-1}\|_{L^\infty(\partial F)} \leq C_3$ (by inequality (1.41) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we have $\|\Phi(t,\cdot)-\mathrm{Id}\|_{C^{1,\alpha}(\partial F)}\leq C\|\psi\|_{W^{2,p}(\partial F)}\leq C\delta$), the following inequality holds

$$\begin{aligned}
&\int_{\partial F}|X(\Phi_t(y))|\,d\mu(y)\\
&\qquad\qquad=\int_{\partial F}\biggl|\int_0^{\psi(\pi_F(\Phi_t(y)))}\frac{\xi(\Phi_t(y))\nabla d_F(\Phi_t(y))}{\xi(\Phi_t(y)+s\nu_F(\pi_F(\Phi_t(y))))}\,ds\biggr|\,d\mu(y)\\
&\qquad\qquad\leq C\int_{\partial F}|\psi(\pi_F(\Phi_t(y)))|\,d\mu(y)\\
&\qquad\qquad=C\int_{\partial F}|\psi(z)|\,J(\pi_F\circ\Phi_t)^{-1}(z)\,d\mu(z)\\
&\qquad\qquad\leq C\|\psi\|_{L^2(\partial F)},
\end{aligned}\tag{1.71}$$

for every $t\in[0,1]$.
We want now to prove that for every $\overline{\varepsilon}>0$, choosing a suitably small $\delta>0$ we have the estimate

$$|R_1|+|R_2|+|R_3|\leq\overline{\varepsilon}\|\psi\|_{L^2(\partial F)}\,.\tag{1.72}$$

First,

$$\begin{aligned}
R_1=&\int_{\partial F}\langle X\circ\Phi_t,\nu_{F_t}\circ\Phi_t\rangle\nu_{F_t}\circ\Phi_t\,[J\Phi_t-1]\,d\mu\\
&+\int_{\partial F}\langle X\circ\Phi_t,\nu_{F_t}\circ\Phi_t\rangle\nu_{F_t}\circ\Phi_t\,d\mu-\int_{\partial F}\langle X\circ\Phi_t,\nu_F\rangle\nu_F\,d\mu\\
=&\int_{\partial F}\langle X\circ\Phi_t,\nu_{F_t}\circ\Phi_t\rangle\nu_{F_t}\circ\Phi_t\,[J\Phi_t-1]\,d\mu\\
&+\int_{\partial F}\langle X\circ\Phi_t,\nu_{F_t}\circ\Phi_t-\nu_F\rangle\nu_F\,d\mu\\
&+\int_{\partial F}\langle X\circ\Phi_t,\nu_{F_t}\circ\Phi_t\rangle(\nu_{F_t}\circ\Phi_t-\nu_F)\,d\mu\\
\leq&\int_{\partial F}|X\circ\Phi_t|\,\|J\Phi_t-1\|_{L^\infty(\partial F)}\,d\mu\\
&+\int_{\partial F}|X\circ\Phi_t|\,\|\nu_F-\nu_{F_t}\circ\Phi_t\|_{L^\infty(\partial F)}\,d\mu\,,
\end{aligned}$$

then, since by inequality (1.41) it follows that for every $t\in[0,1]$ the two terms

$$\|\nu_F-\nu_{F_t}\circ\Phi_t\|_{L^\infty(\partial F)}\qquad\text{and}\qquad\|J\Phi_t-1\|_{L^\infty(\partial F)}$$

can be made (uniformly in $t\in[0,1]$) small as we want, if $\delta>0$ is small enough, by using inequality (1.71), we obtain

$$|R_1|\leq\overline{\varepsilon}\|\psi\|_{L^2(\partial F)}/3.$$

Then we estimate, by means of inequality (1.41) and where $s = s(t, y) \in [t, 1]$,

$$
\begin{aligned}
|R_2| &\le \int_{\partial F} \Big(|X(\Phi_t(y)) - X(\Phi_1(y))| + |X(\Phi_1(y)) - X(y)| \Big)\, d\mu(y) \\
&\le \int_{\partial F} |X(\Phi_t(y)) - X(\Phi_1(y))|\, d\mu(y) + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial F)} \\
&= \int_{\partial F} (1-t) |\nabla X(\Phi_s(y))| \left| \frac{\partial \Phi_s}{\partial t}(y) \right| d\mu(y) + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial F)} \\
&\le \int_{\partial F} |\nabla X(\Phi_s(y)|\, |X(\Phi_s(y))|\, d\mu(y) + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial F)} \\
&\le C \|\nabla X\|_{L^\infty(N_\varepsilon)} \|\psi\|_{L^2(\partial F)} + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial F)}\,,
\end{aligned}
$$

where in the last inequality we used estimate (1.71). Hence, by inequality (1.44) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we get

$$
|R_2| \le C \|\psi\|_{W^{2,p}(\partial F)} \|\psi\|_{L^2(\partial F)}\,,
$$

then, since $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, we obtain

$$
|R_2| < \overline{\varepsilon} \|\psi\|_{L^2(\partial F)}/3\,,
$$

if $\delta > 0$ is small enough.
Arguing similarly, recalling the definition of X given by formula (1.43), we also obtain $|R_3| \le \overline{\varepsilon}\|\psi\|_{L^2(\partial F)}$, hence estimate (1.72) follows. We can then conclude that, for $\delta > 0$ small enough, we have

$$
\begin{aligned}
\left| \int_{\partial F_t} \langle X, \nu_{F_t} \rangle \nu_{F_t}\, d\mu_t \right| &\le \left| \int_{\partial F} \psi \nu_F\, d\mu \right| + \overline{\varepsilon} \|\psi\|_{L^2(\partial F)} \\
&\le \Big(\frac{\delta_1}{2} + \overline{\varepsilon} \Big) \|\psi\|_{L^2(\partial F)}\,,
\end{aligned}
$$

for any $t \in [0, 1]$, where in the last inequality we used the assumption (1.69), thus choosing $\overline{\varepsilon} = \delta_1/4$ we get

$$
\left| \int_{\partial F_t} \langle X, \nu_{F_t} \rangle \nu_{F_t}\, d\mu_t \right| \le \frac{3\delta_1}{4} \|\psi\|_{L^2(\partial F)}\,.
$$

Along the same line, it is then easy to prove that

$$
\|\langle X, \nu_{F_t} \rangle\|_{L^2(\partial F_t)} \ge (1 - \overline{\varepsilon}) \|\psi\|_{L^2(\partial F)}\,,
$$

for any $t \in [0, 1]$, hence claim (1.70) follows.
As a consequence, since $\langle X, \nu_{F_t} \rangle \in \widetilde{H}(\partial F_t)$, being X admissible for F_t (recalling computation 1.2) and ∂F_t can be described as a graph over ∂F using a function with small norm in $W^{2,p}(\partial F)$ (by estimate (1.41) of Lemma 1.16 and arguing as in Remark 1.1), we can apply Step 2 with $E = F_t$ to the function $\langle X, \nu_{F_t} \rangle / \|\langle X, \nu_{F_t} \rangle\|_{H^1(\partial F_t)}$, concluding

$$
\Pi_{F_t}(\langle X, \nu_{F_t} \rangle) \ge \frac{m_0}{2} \|\langle X, \nu_{F_t} \rangle\|_{H^1(\partial F_t)}\,. \tag{1.73}
$$

By means of Lemma 1.17, for $\delta > 0$ small enough, we now show the following inequality on ∂F_t (here div is the divergence operator and $X_{\tau_t} = X - \langle X, \nu_{F_t}\rangle \nu_{F_t}$ is a tangent vector field on ∂F_t), for any $t \in [0,1]$,

$$\begin{aligned}
\|\mathrm{div}(X_{\tau_t}\langle X, \nu_{F_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial F_t)} &= \|\mathrm{div}X_{\tau_t}\langle X, \nu_{F_t}\rangle + \langle X_{\tau_t}, \nabla\langle X, \nu_{F_t}\rangle\rangle\|_{L^{\frac{p}{p-1}}(\partial F_t)}\\
&\leq C\|\nabla X_{\tau_t}\|_{L^2(\partial F_t)}\|\langle X, \nu_{F_t}\rangle\|_{L^{\frac{2p}{p-2}}(\partial F_t)}\\
&\quad + C\|X_{\tau_t}\|_{L^{\frac{2p}{p-2}}(\partial F_t)}\|\nabla\langle X, \nu_{F_t}\rangle\|_{L^2(\partial F_t)}\\
&\leq C\|X\|_{H^1(\partial F_t)}\|X\|_{L^{\frac{2p}{p-2}}(\partial F_t)}\\
&\leq C\|X\|^2_{H^1(\partial F_t)}\\
&\leq C\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,, \qquad (1.74)
\end{aligned}$$

where we used the Sobolev embedding $H^1(\partial F_t) \hookrightarrow L^{\frac{2p}{p-2}}(\partial F_t)$, as $p > \max\{2, n-1\}$.
Then, we compute (here H_t is the mean curvature of ∂F_t)

$$\begin{aligned}
\mathcal{A}(\partial E) - \mathcal{A}(\partial F) &= \mathcal{A}(\partial F_1) - \mathcal{A}(\partial F)\\
&= \int_0^1 (1-t)\frac{d^2}{dt^2}\mathcal{A}(\partial F_t)\,dt\\
&= \int_0^1 (1-t)\Big(\Pi_{F_t}(\langle X, \nu_{F_t}\rangle)\\
&\quad - \int_{\partial F_t} \mathrm{H}_t\,\mathrm{div}(X_{\tau_t}\langle X, \nu_{F_t}\rangle)\,d\mu_t\Big)\,dt\,,
\end{aligned}$$

by equation (1.28), the definition of Π_{F_t} in formula (1.31) and taking into account that $\mathrm{div}X = 0$ in N_ε.
Hence, by estimate (1.73), we have (recall that $\mathrm{H} = \mathrm{H}_0 = \lambda$, as F is a critical set, hence the mean curvature H of ∂F is constant)

$$\begin{aligned}
\mathcal{A}(\partial E) - \mathcal{A}(\partial F) &\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,dt\\
&\quad - \int_0^1 (1-t)\int_{\partial F_t} \mathrm{H}_t\,\mathrm{div}(X_{\tau_t}\langle X, \nu_{F_t}\rangle)\,d\mu_t\,dt\\
&= \frac{m_0}{2}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,dt\\
&\quad - \int_0^1 (1-t)\int_{\partial F_t} [\mathrm{H}_t - \lambda]\mathrm{div}(X_{\tau_t}\langle X, \nu_{F_t}\rangle)\,d\mu_t\,dt\\
&\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,dt\\
&\quad - \int_0^1 (1-t)\|\mathrm{H}_t - \lambda\|_{L^p(\partial F_t)}\|\mathrm{div}(X_{\tau_t}\langle X, \nu_{F_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial F_t)}\,dt\\
&\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,dt\\
&\quad - C\int_0^1 (1-t)\|\mathrm{H}_t - \lambda\|_{L^p(\partial F_t)}\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\,dt\,,
\end{aligned}$$

by estimate (1.74). If $\delta > 0$ is sufficiently small, as F_t is $W^{2,p}$–close to F (recall again Remark 1.1), we have $\|\mathrm{H}_t - \lambda\|_{L^p(\partial F_t)} < m_0/4C$, hence

$$\mathcal{A}(\partial E) - \mathcal{A}(\partial F) \geq \frac{m_0}{4}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{H^1(\partial F_t)}\, dt\,.$$

Then, we can conclude the proof of the theorem with the following series of inequalities, holding for a suitably small $\delta > 0$ as in the statement,

$$\begin{aligned}
\mathcal{A}(\partial E) &\geq \mathcal{A}(\partial F) + \frac{m_0}{4}\int_0^1 (1-t)\|\langle X, \nu_{F_t}\rangle\|^2_{L^2(\partial F_t)}\, dt \\
&\geq \mathcal{A}(\partial F) + C\|\langle X, \nu_F\rangle\|^2_{L^2(\partial F)} \\
&\geq \mathcal{A}(\partial F) + C\|\psi\|^2_{L^2(\partial F)} \\
&\geq \mathcal{A}(\partial F) + C[\mathrm{Vol}(F \triangle E)]^2 \\
&\geq \mathcal{A}(\partial F) + C[\alpha(F,E)]^2\,,
\end{aligned}$$

where the first inequality is due to the $W^{2,p}$–closedness of F_t to F, the second one by the expression (1.43) of the vector field X on ∂F,

$$|\langle X(y), \nu_F(y)\rangle| = \left|\int_0^{\psi(y)} \frac{ds}{\xi(y + s\nu_F(y))}\right| \leq C|\psi(y)|\,,$$

the third follows by a straighforward computation (involving the map L defined by formula (1.39) and its Jacobian), as ∂F is a "normal graph" over ∂E with ψ as "height function", finally the last one simply by the definition of the "distance" α, recalling that we possibly translated the "original" set E by a vector $\eta \in \mathbb{R}^n$, at the beginning of this step. $\square$

We conclude the chapter by showing two results that deal with strictly stable critical sets, which will be used later. The following lemma says that when a smooth set is sufficiently $W^{2,p}$–close to a strictly stable critical set of the Area functional, then the quadratic form (1.31) remains uniformly positive definite (on the orthogonal complement of its degeneracy subspace, see the discussion at the end of the previous section).

Lemma 1.19. *Let $p > \max\{2, n-1\}$ and $F \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε a tubular neighborhood of ∂F as in formula (1.36). Then, for every $\theta \in (0,1]$ there exist $\sigma_\theta, \delta > 0$ such that if a smooth set $E \subseteq \mathbb{T}^n$ is $W^{2,p}$–close to F, that is, $\mathrm{Vol}(E \triangle F) < \delta$ and $\partial E \subseteq N_\varepsilon$ with*

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\}\,,$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, there holds

$$\Pi_E(\varphi) \geq \sigma_\theta\|\varphi\|^2_{H^1(\partial E)}$$

for all $\varphi \in \widetilde{H}^1(\partial E)$ satisfying

$$\min_{\eta \in \mathrm{O}_F} \|\varphi - \langle \eta, \nu_E\rangle\|_{L^2(\partial E)} \geq \theta\|\varphi\|_{L^2(\partial E)}\,,$$

where O_F is defined by formula (1.34).

Proof.
Step 1. We first claim that the strict stability of F implies

$$\Pi_F(\varphi) > 0 \qquad \text{for all } \varphi \in \widetilde{H}(\partial F) \setminus T(\partial F). \tag{1.75}$$

By means of formula (1.6)

$$\Delta\nu_F = \nabla \mathrm{H} - |B|^2\nu_F\,,$$

since F (being critical) satisfies $\mathrm{H} = \lambda$ for some constant $\lambda \in \mathbb{R}$, we have $\Delta\nu_F = -|B|^2\nu_F$ on ∂F. This equation can be written as $L(\nu_i) = 0$, for every $i \in \{1, \dots, n\}$, where L is the self–adjoint, linear operator defined as

$$L(\varphi) = -\Delta\varphi - |B|^2\varphi\,,$$

then, if we "decompose" a smooth function $\varphi \in \widetilde{H}(\partial F) \setminus T(\partial F)$ as $\varphi = \psi + \langle \eta, \nu_F\rangle$, for some $\eta \in \mathbb{R}^n$ and $\psi \in T^\perp(\partial F) \setminus \{0\}$, we have (recalling formula (1.31))

$$\begin{aligned}
\Pi_F(\varphi) &= \int_{\partial F} \langle L(\varphi), \varphi\rangle\, d\mu \\
&= \int_{\partial F} \Big(\langle L(\psi), \psi\rangle + 2\langle L(\langle \eta, \nu_F\rangle), \psi\rangle + \langle L(\langle \eta, \nu_F\rangle), \langle \eta, \nu_F\rangle\rangle \Big)\, d\mu \\
&= \Pi_F(\psi)\,.
\end{aligned}$$

By approximation with smooth functions, we conclude that this equality holds for every function in $\widetilde{H}(\partial F) \setminus T(\partial F)$, hence $\Pi_F(\varphi) = \Pi_F(\psi) > 0$ for every $\varphi \in \widetilde{H}(\partial F) \setminus T(\partial F)$, by the strict stability assumption on F.

We now show that for every $\theta \in (0, 1]$ there holds

$$\begin{aligned}
m_\theta = \inf\Big\{ \Pi_F(\varphi) \,:\, \varphi \in \widetilde{H}^1(\partial F)\,, \|\varphi\|_{H^1(\partial F)} = 1 \quad \text{and} \\
\min_{\eta \in \mathrm{O}_F} \|\varphi - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)} \geq \theta\|\varphi\|_{L^2(\partial F)} \Big\} > 0\,.
\end{aligned} \tag{1.76}$$

Indeed, let φ_i be a minimizing sequence for this infimum and assume that $\varphi_i \rightharpoonup \varphi_0 \in \widetilde{H}^1(\partial F)$ weakly in $H^1(\partial F)$.
If $\varphi_0 \neq 0$, as the weak convergence in $H^1(\partial F)$ implies strong convergence in $L^2(\partial F)$ by Sobolev embeddings, for every $\eta \in \mathrm{O}_F$ we have

$$\begin{aligned}
\|\varphi_0 - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)} &= \lim_{i\to\infty} \|\varphi_i - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)} \\
&\geq \lim_{i\to\infty} \theta\|\varphi_i\|_{L^2(\partial F)} \\
&= \theta\|\varphi_0\|_{L^2(\partial F)},
\end{aligned}$$

hence,

$$\min_{\eta \in \mathrm{O}_F} \|\varphi_0 - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)} \geq \theta\|\varphi_0\|_{L^2(\partial F)} > 0,$$

thus, we conclude $\varphi_0 \in \widetilde{H}^1(\partial F) \setminus T(\partial F)$ and

$$m_\theta = \lim_{i\to\infty} \Pi_E(\varphi_i) \geq \Pi_E(\varphi_0) > 0\,,$$

where the last inequality follows from estimate (1.75).
If $\varphi_0 = 0$, then again by the strong convergence of $\varphi_i \to \varphi_0$ in $L^2(\partial F)$, by looking at formula (1.31), we have

$$m_\theta = \lim_{i\to\infty} \Pi_F(\varphi_i) = \lim_{i\to\infty} \int_{\partial F} |\nabla \varphi_i|^2 \, d\mu = \lim_{i\to\infty} \|\varphi_i\|^2_{H^1(\partial F)} = 1$$

since $\|\varphi_i\|_{L^2(\partial F)} \to 0$.

Step 2. In order to conclude the proof it is enough to show the existence of some $\delta > 0$ such that if $\mathrm{Vol}(E \triangle F) < \delta$ and $\partial E = \{y + \psi(y)\nu_F(y) : y \in \partial F\}$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, then

$$\begin{aligned}\inf\Big\{ \Pi_E(\varphi) \,:\, \varphi \in \widetilde{H}^1(\partial F)\,, \|\varphi\|_{H^1(\partial E)} = 1 \ \text{ and}\\ \min_{\eta \in \mathrm{O}_F} \|\varphi - \langle \eta, \nu_E \rangle\|_{L^2(\partial E)} \geq \theta \|\varphi\|_{L^2(\partial E)} \Big\} \geq \sigma_\theta = \frac{1}{2} \min\{m_{\theta/2}, 1\}\,,\end{aligned} \tag{1.77}$$

where $m_{\theta/2}$ is defined by formula (1.76), with $\theta/2$ in place of θ.
Assume by contradiction that there exist a sequence of smooth sets $E_i \subseteq \mathbb{T}^n$, with $\partial E_i = \{y + \psi_i(y)\nu_F(y) : y \in \partial F\}$ and $\|\psi_i\|_{W^{2,p}(\partial F)} \to 0$, and a sequence $\varphi_i \in \widetilde{H}^1(\partial E_i)$, with $\|\varphi_i\|_{H^1(\partial E_i)} = 1$ and

$$\min_{\eta \in \mathrm{O}_F} \|\varphi_i - \langle \eta, \nu_{E_i} \rangle\|_{L^2(\partial E_i)} \geq \theta \|\varphi_i\|_{L^2(\partial E_i)}\,,$$

such that

$$\Pi_{E_i}(\varphi_i) < \sigma_\theta \leq m_{\theta/2}/2\,. \tag{1.78}$$

Let us suppose first that $\lim_{i\to\infty} \|\varphi_i\|_{L^2(\partial E_i)} = 0$ and observe that by Sobolev embeddings $\|\varphi_i\|_{L^q(\partial E_i)} \to 0$ for some $q > 2$, thus, since the functions ψ_i are uniformly bounded in $W^{2,p}(\partial F)$ for $p > \max\{2, n-1\}$, recalling formula (1.31), it is easy to see that

$$\lim_{i\to\infty} \Pi_{E_i}(\varphi_i) = \lim_{i\to\infty} \int_{\partial E_i} |\nabla \varphi_i|^2 \, d\mu_i = \lim_{i\to\infty} \|\varphi_i\|^2_{H^1(\partial E_i)} = 1\,,$$

which is a contradiction with assumption (1.78).
Hence, we may assume that

$$\lim_{i\to\infty} \|\varphi_i\|_{L^2(\partial E_i)} > 0. \tag{1.79}$$

The idea now is to write every φ_i as a function on ∂F. We define the functions $\widetilde{\varphi}_i(\partial F) \to \mathbb{R}$, given by

$$\widetilde{\varphi}_i(y) = \varphi_i\Big(y + \psi_i(y)\nu_F(y)\Big) - \fint_{\partial F} \varphi_i(y + \psi_i(y)\nu_F(y))\, d\mu(y)\,,$$

for every $y \in \partial F$.
As $\psi_i \to 0$ in $W^{2,p}(\partial F)$, we have in particular that

$$\widetilde{\varphi}_i \in \widetilde{H}^1(\partial F)\,, \qquad \|\widetilde{\varphi}_i\|_{H^1(\partial F)} \to 1 \qquad \text{and} \qquad \frac{\|\widetilde{\varphi}_i\|_{L^2(\partial F)}}{\|\varphi_i\|_{L^2(\partial E_i)}} \to 1\,.$$

Moreover, note also that $\nu_{E_i}(\cdot + \psi_i(\cdot)\nu_F(\cdot)) \to \nu_F$ in $W^{1,p}(\partial F)$ and thus in $C^{0,\alpha}(\partial F)$ for some $\alpha \in (0,1)$ (depending on p) by Sobolev embeddings. Using this fact and taking into account the third limit above and inequality (1.79), one can easily show that

$$\varliminf_{i\to\infty} \frac{\min_{\eta\in \mathrm{O}_F} \|\widetilde{\varphi}_i - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)}}{\|\widetilde{\varphi}_i\|_{L^2(\partial F)}} \geq \varliminf_{i\to\infty} \frac{\min_{\eta\in \mathrm{O}_F} \|\varphi_i - \langle \eta, \nu_{E_i}\rangle\|_{L^2(\partial E_i)}}{\|\varphi_i\|_{L^2(\partial E_i)}}$$

which is larger or equal than θ.
Hence, for $i \in \mathbb{N}$ large enough, we have

$$\|\widetilde{\varphi}_i\|_{H^1(\partial F)} \geq 3/4 \qquad \text{and} \qquad \min_{\eta\in \mathrm{O}_F} \|\widetilde{\varphi}_i - \langle \eta, \nu_F\rangle\|_{L^2(\partial F)} \geq \frac{\theta}{2}\|\widetilde{\varphi}_i\|_{L^2(\partial F)}\,,$$

then, in turn, by Step 1, we infer

$$\Pi_F(\widetilde{\varphi}_i) \geq \frac{9}{16} m_{\theta/2}\,. \tag{1.80}$$

Moreover, the $W^{2,p}$–convergence of E_i to E imply (see the proof of Theorem 1.13 for more details) the convergence of the second fundamental forms

$$B_{\partial E_i}\Big(\cdot + \psi_i(\cdot)\nu_F(\cdot)\Big) \to B_{\partial F} \;\text{ in } L^p(\partial F),$$

for $i \to \infty$ and, since $p > 2$, the Sobolev embeddings and the $W^{2,p}$–convergence of E_i to F imply

$$\int_{\partial E_i} |B_{\partial E_i}|^2 \varphi_i^2 \, d\mu - \int_{\partial F} |B_{\partial F}|^2 \widetilde{\varphi}_i^2 \, d\mu \to 0\,.$$

Combining all these convergences, we conclude that all the terms of $\Pi_{E_i}(\varphi_i)$ are asymptotically close to the corresponding terms of $\Pi_F(\widetilde{\varphi}_i)$, thus

$$\Pi_{E_i}(\varphi_i) - \Pi_E(\widetilde{\varphi}_i) \to 0\,.$$

which is a contradiction, by inequalities (1.78) and (1.80). This establishes inequality (1.77) and concludes the proof. □

The next proposition states the fact that in the neighborhood of a strictly stable critical set there are no other critical sets, up to translations.

Proposition 1.20. *Let $p > \max\{2, n-1\}$ and $F \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε a tubular neighborhood of ∂F as in formula (1.36). Then, there exists $\delta > 0$ such that if $F' \subseteq \mathbb{T}^n$ is a smooth critical set such that $\mathrm{Vol}(F') = \mathrm{Vol}(F)$, $\mathrm{Vol}(F \triangle F') < \delta$, $\partial F' \subseteq N_\varepsilon$ and*

$$\partial F' = \{y + \psi(y)\nu_F(y) : y \in \partial F\},$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, then F' is a traslate of F.

Proof. We have seen in Step 3 of the proof of Theorem 1.13 that in these hypotheses of F and F', if $\delta > 0$ is small enough, we may find a small vector $\eta \in \mathbb{R}^n$ and an admissible smooth vector field X for F, with the associated flow Φ satisfying $\Phi_0(F) = F$, $\Phi_1(F) = F' - \eta$ and

$$\frac{d^2}{dt^2}\mathcal{A}(\partial\Phi_t(F)) \geq C[\mathrm{Vol}(F \triangle (F' - \eta))]^2 \,,$$

for all $t \in [0,1]$, where C is a positive constant independent of F'.
Assume that F' is a smooth critical set as in the statement, which is not a translate of F, then

$$\frac{d}{dt}\mathcal{A}(\partial\Phi_t(F))\Big|_{t=0} = 0 \,,$$

but from the above formula it follows

$$\frac{d}{dt}\mathcal{A}(\partial\Phi_t(F))\Big|_{t=1} > 0 \,,$$

which implies that $F' - \eta$ cannot be critical, hence neither F', which is a contradiction. Indeed, $-X$ is an admissible vector field for $F' - \eta$ with an associate flow Ψ satisfying $\Psi_s(F' - \eta) = \Phi_{1-s}(F)$, for every $s \in [0,1]$, hence

$$\frac{d}{ds}\mathcal{A}(\partial\Psi_s(F' - \eta))\bigg|_{s=0} = \frac{d}{ds}\mathcal{A}(\partial\Phi_{1-s}(F))\bigg|_{s=0} = -\frac{d}{dt}\mathcal{A}(\partial\Phi_t(F))\bigg|_{t=1} < 0 \,,$$

showing that $F' - \eta$ is not critical. $\square$

2

THE SURFACE DIFFUSION FLOW

In this chapter we introduce the *surface diffusion flow*, we discuss its basic properties and we prove some technical lemmas in order to show a global existence result in the three–dimensional case, in the next chapter.

2.1 DEFINITION AND BASIC PROPERTIES

Definition 2.1 (Smooth flow of sets)**.** Let $E_t \subseteq \mathbb{T}^n$ for $t \in [0, T)$ be a one–parameter family of sets, we say that it is a *smooth flow* if there exists a smooth *reference set* $F \subseteq \mathbb{T}^n$ and a map $\Psi \in C^\infty([0, T) \times \mathbb{T}^n, \mathbb{T}^n)$ such that $\Psi_t = \Psi(t, \cdot)$ is a smooth diffeomorphism from $\mathbb{T}^n$ to $\mathbb{T}^n$ and $E_t = \Psi_t(F)$, for all $t \in [0, T)$.

The *velocity* of the motion of any point $x = \Psi_t(y)$ of the set E_t, with $y \in F$, is then given by

$$X_t(x) = \frac{\partial \Psi_t}{\partial t}(y)\,,$$

hence,

$$\frac{\partial \Psi_t}{\partial t}(y) = X_t(\Psi_t(y))$$

for every $y \in F$. Notice that, in general, the smooth vector field X_t is not independent of t, so *it is not* the infinitesimal generator of the flow Ψ, but we will see that in the computations in the sequel, it will behave similarly.

When $x \in \partial E_t$, we define the *outer normal velocity* of the flow of the boundaries, which are smooth hypersurface of $\mathbb{T}^n$,

$$V_t(x) = \langle X_t(x), \nu_{E_t}(x)\rangle\,, \tag{2.1}$$

for every $t \in [0, T)$, where ν_{E_t} is the outer unit normal vector to E_t.

Definition 2.2 (Surface diffusion flow)**.** Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that a smooth flow E_t for $t \in [0, T)$, with $E_0 = E$, is a *surface diffusion flow* starting from E if the outer normal velocity V_t of the moving boundaries ∂E_t is given by

$$V_t = \Delta_t \mathrm{H}_t \quad \text{for all } t \in [0, T), \tag{2.2}$$

where Δ_t and H_t are respectively the Laplacian and the mean curvature of ∂E_t.

Parametrizing the smooth hypersurfaces $M_t = \partial E_t$ of $\mathbb{T}^n$ by some smooth embeddings $\psi_t : M \to \mathbb{T}^n$ such that $\psi_t(M) = \partial E_t$ (here M is a fixed smooth differentiable $(n-1)$–dimensional manifold and the map $(t,p) \mapsto \psi(t,p) = \psi_t(p)$ is smooth), the geometric evolution law (2.2) can be expressed equivalently as

$$\left\langle \frac{\partial \psi_t}{\partial t}, \nu_t \right\rangle = \Delta_t \mathrm{H}_t \,, \tag{2.3}$$

where we denoted with ν_t the outer unit normal to $M_t = \partial E_t$.
Moreover, as the moving hypersurfaces $M_t = \partial E_t$ are compact, it is always possible to smoothly reparametrize them with maps (that we still call) ψ_t such that

$$\frac{\partial \psi_t}{\partial t} = (\Delta_t \mathrm{H}_t)\nu_t \,, \tag{2.4}$$

describing such surface diffusion flow. This follows by the *invariance by tangential perturbations of the velocity*, shared by the flow due to its geometric nature and can be proved following the line in Section 1.3 of [19], where the analogous property is shown in full detail for the (more famous) mean curvature flow. Roughly speaking, the tangential component of the velocity of the points of the moving hypersurfaces, does not affect the global "shape" during the motion.

Formula (2.4) is actually the more standard way to define the surface diffusion flow, in the more general situation of smooth and possibly *immersed–only* hypersurfaces (usually in $\mathbb{R}^n$), without being the boundary of any set.
By means of equation (1.5), this system can be rewritten as

$$\frac{\partial \psi_t}{\partial t} = -\Delta_t \Delta_t \psi_t + \text{ lower order terms} \tag{2.5}$$

and it can be seen that it is a fourth order, *quasilinear* and *degenerate*, parabolic system of PDEs. Indeed, it is quasilinear, as the coefficients (as second order partial differential operator) of the Laplacian associated to the induced metrics g_t on the evolving hypersurfaces, that is,

$$\Delta_t \psi_t(p) = \Delta_{g_t(p)} \psi_t(p) = g_t^{ij}(p) \nabla_i^{g_t(p)} \nabla_j^{g_t(p)} \psi_t(p)$$

depend on the first order derivatives of ψ_t, as g_t (and the coefficient of $\Delta_t\Delta_t$ on the third order derivatives). Moreover, the operator at the right hand side of system (2.4) is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms.

Like the Area functional, the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{T}^n$ (or $\mathbb{R}^n$). Moreover, if $\psi : [0,T) \times M \to \mathbb{T}^n$ is a surface diffusion flow and $\Phi : [0,T) \times M \to M$ is a time–dependent family of smooth diffeomorphisms of M, then it is easy to check that the reparametrization $\widetilde{\psi} : [0,T) \times M \to \mathbb{T}^n$ defined as $\widetilde{\psi}(t,p) = \psi(t, \Phi(t,p))$ is still a surface diffusion flow (in the sense of equation (2.3)). This property can be reread as "the flow is invariant under reparametrization", suggesting that the really relevant objects of the flow are actually the subsets $M_t = \psi_t(M)$ of $\mathbb{T}^n$.

We now state a short time existence and uniqueness result (and also of dependence on the initial data) of the surface diffusion flow starting from a smooth hypersurface, proved by Escher, Mayer and Simonett in [6], which is expected due to the parabolic nature of the system (2.4), made evident by formula (2.5). It deals with the evolution in the whole space $\mathbb{R}^n$ of a generic hypersurface, only immersed, hence possibly with self–intersections. It is then straightforward to adapt the same arguments to our case, when the ambient is the flat torus $\mathbb{T}^n$ and we are looking for the surface diffusion flow of the boundaries of the sets E_t as in Definition 2.2, getting a (unique) surface diffusion flow in a positive time interval $[0, T)$, for every initial smooth set $E_0 \subseteq \mathbb{T}^n$.

Theorem 2.3 (Short time existence and uniqueness)**.** *Let $\psi_0 : M \to \mathbb{R}^n$ be a smooth and compact, immersed hypersurface. Then, there exists a unique smooth surface diffusion flow $\psi : [0, T) \times M \to \mathbb{R}^n$, starting from $M_0 = \psi_0(M)$ and solving system (2.4), for some maximal time of existence $T > 0$.*
Moreover, the maximal time of existence depend continuosly on the $C^{2,\alpha}$ norm of the inital hypersurface.

As an easy consequence, we have the following proposition, better suited for our situation.

Proposition 2.4. *Let $F \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂F, as in formula (1.36), Then, for every $\alpha \in (0, 1)$ and $M \in (0, \varepsilon/2)$ small enough, there exists $T = T(F, M, \alpha) > 0$ such that if $E_0 \in \mathfrak{C}^{2,\alpha}_M(F)$ there exists a unique smooth surface diffusion flow, starting from E_0, in the time interval $[0, T)$.*

It is well–known that the surface diffusion flow of boundaries of sets is volume–preserving, that is, the volume of the moving sets E_t is constant, while neither convexity (see [17]) is maintained (nor the embeddedness, in the "stand–alone" formulation of motion of hypersurfaces, as in formula (2.4), see [13]), nor there holds the so–called "comparison property" asserting that if two initial sets are one contained in the other, they stay so during the two respective flows, due to the lack of *maximum principle* for parabolic equations or systems of order larger than two (these two properties holds instead for the mean curvature flow, see [19, Chapter 2] for instance).

The volume–preserving property follows immediately arguing as in computation (1.18), indeed, if $E_t = \Psi_t(F)$ is a surface diffusion flow, described by $\Psi \in C^\infty([0, T) \times \mathbb{T}^n, \mathbb{T}^n)$, with associated smooth vector field X_t satisfying

$$\frac{\partial \Psi_t}{\partial t}(y) = X_t(\Psi_t(y))$$

as we said above, we have

$$
\begin{aligned}
0 &= \frac{d}{dt}\mathrm{Vol}(E_t)\\
&= \int_F \frac{\partial}{\partial t} J\Psi_t(y)\,dy\\
&= \int_F \mathrm{div} X_t(\Psi(t,y))J\Psi(t,y)\,dy\\
&= \int_{E_t} \mathrm{div} X_t(x)\,dx\\
&= \int_{\partial E_t} \langle X, \nu_{E_t}\rangle\,d\mu_t\\
&= \int_{\partial E_t} V_t\,d\mu_t\\
&= \int_{\partial E_t} \Delta_t H_t\,d\mu_t\\
&= 0\,,
\end{aligned}
$$

where μ_t is in the canonical measure induced on ∂E_t by the flat metric of $\mathbb{T}^n$ and in the last equality we applied the divergence theorem (1.2).

Moreover, the surface diffusion flow can be regarded as the $\widetilde{H}^{-1}$–gradient flow of the volume–constrained Area functional, in the following sense (see [12], for instance). For a smooth set $E \subseteq \mathbb{T}^n$, we let the space $\widetilde{H}^{-1}(\partial E) \subseteq L^2(\partial E)$ to be the dual of $\widetilde{H}^1(\partial E)$ with the norm $\|u\|_{\widetilde{H}^1(\partial E)} = \int_{\partial E} |\nabla u|^2\,d\mu$ (the functions in $H^1(\partial E)$ with zero integral) and the pairing between $\widetilde{H}^1(\partial E)$ and $\widetilde{H}^{-1}(\partial E)$ simply being the integral of the product of the functions on ∂E.

Then, it follows easily that the norm of a smooth function $v \in \widetilde{H}^{-1}(\partial E)$ is given by

$$\|v\|^2_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} v(-\Delta)^{-1} v\,d\mu = \int_{\partial E} \langle \nabla(-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\rangle\,d\mu$$

and, by polarization, we have the $\widetilde{H}^{-1}(\partial E)$–scalar product between a pair of smooth functions $u, v : \partial E \to \mathbb{R}$ with zero integral,

$$\langle u, v\rangle_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} \langle \nabla(-\Delta)^{-1}u, \nabla(-\Delta)^{-1}v\rangle\,d\mu = \int_{\partial E} u(-\Delta)^{-1}v\,d\mu\,,$$

integrating by parts.

This scalar product, extended to the whole space $\widetilde{H}^{-1}(\partial E)$, make it a Hilbert space, hence, by *Riesz representation theorem*, there exists a function $\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} \in \widetilde{H}^{-1}(\partial E)$ such that, for every smooth function $v \in \widetilde{H}^{-1}(\partial E)$, there holds

$$\int_{\partial E} v\mathrm{H}\,d\mu = \delta\mathcal{A}_{\partial E}(v) = \langle v, \nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A}\rangle_{\widetilde{H}^{-1}(\partial E)} = \int_{\partial E} v(-\Delta)^{-1}\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A}\,d\mu\,,$$

by Theorem 1.3.

Then, by the *fundamental lemma of calculus of variations*, we conclude

$$(-\Delta)^{-1}\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} = \mathrm{H} + c\,,$$

for a constant $c \in \mathbb{R}$, that is,

$$\nabla^{\widetilde{H}^{-1}}_{\partial E}\mathcal{A} = -\Delta \mathrm{H}\,.$$

It clearly follows that the outer normal velocity of the moving boundaries of a surface diffusion flow $V_t = \Delta_t \mathrm{H}_t$ is minus the $\widetilde{H}^{-1}$–gradient of the volume–constrained functional $\mathcal{A}$.

Remark 2.5. Arguing analogously, we can see easily that the mean curvature flow, where $V_t = -\mathrm{H}_t$ is the L^2–gradient flow of the Area functional (without constraints).

2.2 ENERGY IDENTITIES AND TECHNICAL LEMMAS

In this section we prove some auxiliary results that we need in the sequel. From now on we drop the t–subscript on H_t, B_t, Δ_t, μ_t and we simply write H, B, Δ, μ for the mean curvature, second fundamental form, Laplacian and canonical measure, respectively, when it is clear that they refer to the set E_t and its boundary.

Lemma 2.6 (Energy identities). *Let $E_t \subseteq \mathbb{T}^n$ be a surface diffusion flow. Then, the following identities hold:*

$$\frac{d}{dt}\mathcal{A}(\partial E_t) = -\int_{\partial E_t} |\nabla \mathrm{H}|^2\, d\mu\,, \tag{2.6}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\partial E_t} |\nabla \mathrm{H}|^2\, d\mu = &-\Pi_{E_t}(\Delta \mathrm{H}) - \int_{\partial E_t} B(\nabla \mathrm{H}, \nabla \mathrm{H})\Delta \mathrm{H}\, d\mu \\ &+ \frac{1}{2}\int_{\partial E_t} \mathrm{H}|\nabla \mathrm{H}|^2 \Delta \mathrm{H}\, d\mu\,, \end{aligned}\tag{2.7}$$

where Π_{E_t} is the quadratic form defined in formula (1.31).

Proof. Let ψ_t the smooth family of maps describing the flow as in formula (2.4). By formula (1.13), where X is the smooth (velocity) vector field $X_t = \frac{\partial \psi_t}{\partial t} = (\Delta \mathrm{H})\nu_{E_t}$ along ∂E_t, hence $X_\tau = X_t - \langle X_t, \nu_{E_t}\rangle \nu_{E_t} = 0$ (as usual ν_{E_t} is the outer normal unit vector of ∂E_t), following computation (1.14), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{A}(\partial E_t) &= \frac{d}{dt}\int_{\partial E_t} d\mu \\ &= \int_{\partial E_t} (\mathrm{div} X_\tau + \mathrm{H}\langle X, \nu_{E_t}\rangle)\, d\mu \\ &= \int_{\partial E_t} \mathrm{H}\Delta \mathrm{H}\, d\mu \\ &= -\int_{\partial E_t} |\nabla \mathrm{H}|^2\, d\mu\,,\end{aligned}$$

where the last equality follows integrating by parts. This establishes relation (2.6).

In order to get relation (2.7) we also need the time derivatives of the evolving metric and of the mean curvature of ∂E_t, that we already computed in formulas (1.10), (1.22) and (1.26) (where the function φ in this case is equal to $\Delta\mathrm{H}$ and $X_\tau = 0$), that is,

$$\frac{\partial g_{ij}}{\partial t} = 2h_{ij}\Delta\mathrm{H} \qquad \text{and} \qquad \frac{\partial g^{ij}}{\partial t} = -2h^{ij}\Delta\mathrm{H}\,,$$

$$\frac{\partial\mathrm{H}}{\partial t} = -|B|^2\Delta\mathrm{H} - \Delta\Delta\mathrm{H}$$

Then, we compute

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\partial E_t}|\nabla\mathrm{H}|^2\,d\mu &= \frac{1}{2}\int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu - \int_{\partial E_t}h^{ij}\nabla_i\mathrm{H}\nabla_j\mathrm{H}\,\Delta\mathrm{H}\,d\mu\\
&\quad - \int_{\partial E_t}g^{ij}\nabla_i\mathrm{H}\nabla_j\Big(|B|^2\Delta\mathrm{H} + \Delta\Delta\mathrm{H}\Big)\,d\mu\\
&= \frac{1}{2}\int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu - \int_{\partial E_t}B(\nabla\mathrm{H},\nabla\mathrm{H})\,\Delta\mathrm{H}\,d\mu\\
&\quad + \int_{\partial E_t}|B|^2(\Delta\mathrm{H})^2\,d\mu + \int_{\partial E_t}\Delta\mathrm{H}\,\Delta\Delta\mathrm{H}\,d\mu\\
&= \frac{1}{2}\int_{\partial E_t}\mathrm{H}|\nabla\mathrm{H}|^2\,\Delta\mathrm{H}\,d\mu - \int_{\partial E_t}B(\nabla\mathrm{H},\nabla\mathrm{H})\,\Delta\mathrm{H}\,d\mu\\
&\quad + \int_{\partial E_t}|B|^2(\Delta\mathrm{H})^2\,d\mu - \int_{\partial E_t}|\nabla\Delta\mathrm{H}|^2\,d\mu\,,
\end{aligned}$$

which is formula (2.7), recalling the definition of Π_{E_t} in formula (1.31). $\square$

Given a smooth set $F \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂F, as in formula (1.36), for any $M \in (0,\varepsilon/2)$ (recall the discussion at the beginning of Section 1.3 about our notion of "closedness" of sets), we denote by $\mathfrak{C}^1_M(F)$, the class of all smooth sets $E \subseteq F \cup N_\varepsilon$ such that $\mathrm{Vol}(E \triangle F) \leq M$ and

$$\partial E = \{x + \psi_E(x)\nu_F(x) : x \in \partial F\}\,, \tag{2.8}$$

for some $\psi_E \in C^\infty(\partial F)$, with $\|\psi_E\|_{C^1(\partial F)} \leq M$ (hence, $\partial E \subseteq N_\varepsilon$). For every $k \in \mathbb{N}$ and $\alpha \in (0,1)$, we also denote by $\mathfrak{C}^{k,\alpha}_M(F)$ the collection of sets $E \in \mathfrak{C}^1_M(F)$ such that $\|\psi_E\|_{C^{k,\alpha}(\partial F)} \leq M$.

From now on, we restrict ourselves to the three–dimensional case, that is, we will consider smooth subsets of $\mathbb{T}^3$ with boundaries which then are smooth embedded (2–dimensional) surfaces.

In the estimates in the following series of lemmas, we will be interested in having uniform constants for the families $\mathfrak{C}^{1,\alpha}_M(F)$, given a smooth set $F \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂F as in formula (1.36), for any $M \in (0,\varepsilon/2)$ and $\alpha \in (0,1)$. This is guaranteed if the constants in the Sobolev, Gagliardo–Nirenberg interpolation and Calderón–Zygmund *inequalities, relative to all the smooth hypersurfaces ∂E boundaries of the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$, are uniform, as it is proved in detail in [5].*

We remind that in all the inequalities, the constants C may vary from one line to another.

The following lemma is an easy consequence of Theorem 3.70 in [4], with $j = 0$, $m = 1$, $n = 2$ and $r = q = 2$, taking into account the previous discussion.

Lemma 2.7 (Interpolation on boundaries)**.** *Let $F \subseteq \mathbb{T}^3$ be a smooth set. In the previous notations, for every $p \in [2, +\infty)$ there exists a constant $C = C(F, M, \alpha, p) > 0$ such that for every set $E \in \mathfrak{C}^{1,\alpha}_M(F)$ and $g \in H^1(\partial E)$, we have*

$$\|g\|_{L^p(\partial E)} \leq C\big(\|\nabla g\|^{\theta}_{L^2(\partial E)}\|g\|^{1-\theta}_{L^2(\partial E)} + \|g\|_{L^2(\partial E)}\big)\,,$$

with $\theta = 1 - 2/p$.
Moreover, the following Poincaré inequality holds

$$\|g - \overline{g}\|_{L^p(\partial E)} \leq C\|\nabla g\|_{L^2(\partial E)}\,,$$

where $\overline{g}(x) = \fint_\Gamma g\, d\mu$, if x belongs to a connected component Γ of ∂E.

Then, we have the following mixed "analytic–geometric" estimate.

Lemma 2.8 (H^2–estimates on boundaries)**.** *Let $F \subseteq \mathbb{T}^3$ be a smooth set. Then there exists a constant $C = C(F, M, \alpha, p) > 0$ such that if $E \in \mathfrak{C}^{1,\alpha}_M(F)$ and $f \in H^1(\partial E)$ with $\Delta f \in L^2(\partial E)$, then $f \in H^2(\partial E)$ and*

$$\|\nabla^2 f\|_{L^2(\partial E)} \leq C\|\Delta f\|_{L^2(\partial E)}(1 + \|\mathrm{H}\|^2_{L^4(\partial E)})\,.$$

Proof. We first claim that the following inequality holds,

$$\int_{\partial E} |\nabla^2 f|^2\, d\mu \leq \int_{\partial E} |\Delta f|^2\, d\mu + C\int_{\partial E} |B|^2|\nabla f|^2\, d\mu\,. \tag{2.9}$$

Indeed, if we integrate by parts the left–hand side, we obtain (the Hessian of a function is symmetric)

$$\int_{\partial E} g^{ik}g^{jl}\nabla^2_{ij}f\nabla^2_{kl}f\, d\mu = -\int_{\partial E} g^{ik}g^{jl}\nabla_k\nabla_j\nabla_i f\nabla_l f\, d\mu\,.$$

Hence, using relations (1.7) and (1.8), interchanging the covariant derivatives and integrating by parts, we get

$$\begin{aligned}-\int_{\partial E} g^{ik}g^{jl}\nabla_k\nabla_j\nabla_i f\nabla_l f\, d\mu = &-\int_{\partial E} g^{ik}g^{jl}\nabla_j\nabla_k\nabla_i f\nabla_l f\, d\mu\\ &-\int_{\partial E} g^{ik}g^{jl}R_{kjip}g^{ps}\nabla_s f\nabla_l f\, d\mu\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\partial E} g^{jl}\nabla_j \Delta f \nabla_l f \, d\mu \\
&\quad - \int_{\partial E} \mathrm{Ric}(\nabla f, \nabla f)\, d\mu \\
&= \int_{\partial E} |\Delta f|^2 \, d\mu \\
&\quad + \int_{\partial E} \Big[|B|^2|\nabla f|^2 - \mathrm{H}B(\nabla f, \nabla f)\Big]\, d\mu \\
&\le \int_{\partial E} |\Delta f|^2 \, d\mu + C\int_{\partial E} |B|^2|\nabla f|^2 \, d\mu \,,
\end{aligned}
$$

thus, inequality (2.9) holds (in the last passage we applied Cauchy–Schwarz inequality and the well known relation $|\mathrm{H}| \le \sqrt{2}|B|$, then $C = 1 + \sqrt{2}$).

We now estimate the last term in formula (2.9) by means of Lemma 2.7 (which is easily extended to vector valued functions $g : \partial E \to \mathbb{R}^m$) with $g = \nabla f$ and $p = 4$:

$$
\begin{aligned}
\int_{\partial E} |B|^2|\nabla f|^2 \, d\mu &\le \|B\|^2_{L^4(\partial E)}\|\nabla f\|^2_{L^4(\partial E)} \\
&\le C\|B\|^2_{L^4(\partial E)}\Big(\|\nabla^2 f\|^{1/2}_{L^2(\partial E)}\|\nabla f\|^{1/2}_{L^2(\partial E)} + \|\nabla f\|_{L^2(\partial E)}\Big)^2 \\
&\le C\|B\|^2_{L^4(\partial E)}\Big(\|\nabla^2 f\|_{L^2(\partial E)}\|\nabla f\|_{L^2(\partial E)} + \|\nabla f\|^2_{L^2(\partial E)}\Big)\,.
\end{aligned}
$$

Hence, expanding the product on the last line, using Peter–Paul (Young) inequality on the first term of such expansion and "adsorbing" in the left hand side of inequality (2.9) the small fraction of the term $\|\nabla^2 f\|^2_{L^2(\partial E)}$ that then appears, we obtain

$$
\begin{aligned}
\|\nabla^2 f\|^2_{L^2(\partial E)} &\le C(\|\Delta f\|^2_{L^2(\partial E)} + \|\nabla f\|^2_{L^2(\partial E)}(\|B\|^2_{L^4(\partial E)} + \|B\|^4_{L^4(\partial E)})) \\
&\le C(\|\Delta f\|^2_{L^2(\partial E)} + \|\nabla f\|^2_{L^2(\partial E)}(1 + \|B\|^4_{L^4(\partial E)}))\,. \qquad (2.10)
\end{aligned}
$$

By the fact that Δf has zero average on each connected component of ∂E, there holds

$$
\begin{aligned}
\|\nabla f\|^2_{L^2(\partial E)} &= -\int_{\partial E} f\Delta f \, d\mu \\
&= -\int_{\partial E} (f - \overline{f})\Delta f \, d\mu \\
&\le \|f - \overline{f}\|_{L^2(\partial E)}\|\Delta f\|_{L^2(\partial E)} \\
&\le C\|\nabla f\|_{L^2(\partial E)}\|\Delta f\|_{L^2(\partial E)}\,,
\end{aligned}
$$

where we used Lemma 2.7 again, hence,

$$
\|\nabla f\|_{L^2(\partial E)} \le C\|\Delta f\|_{L^2(\partial E)}\,. \qquad (2.11)
$$

Thus, from inequality (2.10), we deduce

$$
\|\nabla^2 f\|^2_{L^2(\partial E)} \le C\|\Delta f\|^2_{L^2(\partial E)}(1 + \|B\|^4_{L^4(\partial E)})\,. \qquad (2.12)
$$

Now, by means of Calderón–Zygmund estimates, it is possible to show (see [5]) that there exists a constant $C > 0$ depending only on F, M, α and $q > 1$ such that for every $E \in \mathfrak{C}^{1,\alpha}_M(F)$, there holds

$$\|B\|_{L^q(\partial E)} \leq C(1 + \|\mathrm{H}\|_{L^q(\partial E)})\,. \tag{2.13}$$

Then, since it is easy to check that also all the other constant in the previous inequalities (and the ones coming from Lemma 2.7 also) depend only on F, M, α and p, if $E \in \mathfrak{C}^{1,\alpha}_M(F)$, substituting this estimate, with $q = 4$, in formula (2.12), the thesis of the lemma follows. □

The following lemma provides a crucial "geometric interpolation" that will be needed in the proof of the main theorem.

Lemma 2.9 (Geometric interpolation)**.** *Let $F \subseteq \mathbb{T}^3$ be a smooth set. Then there exists a constant $C = C(F, M, \alpha) > 0$ such that the following estimates holds*

$$\int_{\partial E} |B||\nabla \mathrm{H}|^2 |\Delta \mathrm{H}|\, d\mu \leq C\|\nabla\Delta \mathrm{H}\|^2_{L^2(\partial E)}\, \|\nabla \mathrm{H}\|_{L^2(\partial E)}\, (1 + \|\mathrm{H}\|^3_{L^6(\partial E)})\,,$$

for every $E \in \mathfrak{C}^{1,\alpha}_M(F)$.

Proof. First, by a standard application of Hölder inequality, we have

$$\int_{\partial E} |B||\nabla \mathrm{H}|^2 |\Delta \mathrm{H}|\, d\mu \leq \|\Delta \mathrm{H}\|_{L^3(\partial E)} \left(\int_{\partial E} |B|^{\frac{3}{2}} |\nabla \mathrm{H}|^3\, d\mu\right)^{2/3}.$$

Then, using the Poincaré inequality stated in Lemma 2.7 and the fact that $\Delta\mathrm{H}$ has zero average on each connected component of ∂E, we get

$$\|\Delta \mathrm{H}\|_{L^3(\partial E)} \leq C\|\nabla\Delta \mathrm{H}\|_{L^2(\partial E)}.$$

Now, we use Hölder inequality again

$$\left(\int_{\partial E} |B|^{\frac{3}{2}} |\nabla \mathrm{H}|^3\, d\mu\right)^{2/3} \leq \left(\int_{\partial E} |\nabla \mathrm{H}|^4\, d\mu\right)^{1/2} \left(\int_{\partial E} |B|^6\, d\mu\right)^{1/6},$$

and we apply Lemma 2.7 with $p = 4$,

$$\left(\int_{\partial E} |\nabla \mathrm{H}|^4\, d\mu\right)^{1/2} \leq C(\|\nabla^2 \mathrm{H}\|_{L^2(\partial E)}\|\nabla \mathrm{H}\|_{L^2(\partial E)} + \|\nabla \mathrm{H}\|^2_{L^2(\partial E)})\,.$$

Combining all these inequalities, we conclude

$$\begin{aligned}\int_{\partial E} |B||\nabla \mathrm{H}|^2 |\Delta \mathrm{H}|\, d\mu \leq C\, \|\nabla\Delta \mathrm{H}\|_{L^2(\partial E)}\, \|B\|_{L^6(\partial E)}\, \|\nabla \mathrm{H}\|_{L^2(\partial E)} \cdot \\ \cdot (\|\nabla^2 \mathrm{H}\|_{L^2(\partial E)} + \|\nabla \mathrm{H}\|_{L^2(\partial E)})\,.\end{aligned}$$

By Lemma 2.8 and estimate (2.11), with H in place of f, the right–hand side of the previous inequality can be bounded from above by

$$C\|\nabla\Delta \mathrm{H}\|_{L^2(\partial E)}\, \|B\|_{L^6(\partial E)}\, \|\Delta \mathrm{H}\|_{L^2(\partial E)}\, \|\nabla \mathrm{H}\|_{L^2(\partial E)}\, (1 + \|H\|^2_{L^4(\partial E)}).$$

Hence, using again Poincaré inequality and estimate (2.13) with $q = 6$, we have

$$\|\Delta \mathrm{H}\|_{L^2(\partial E)} \leq C\|\nabla \Delta \mathrm{H}\|_{L^2(\partial E)}$$

and

$$\|B\|_{L^6(\partial E)} \leq C(1 + \|\mathrm{H}\|_{L^6(\partial E)}) \,.$$

Finally, using this relations and Hölder inequality, we obtain the thesis

$$\int_{\partial E} |B||\nabla \mathrm{H}|^2 |\Delta \mathrm{H}| \, d\mu \leq C\|\nabla \Delta \mathrm{H}\|^2_{L^2(\partial E)} \|\nabla \mathrm{H}\|_{L^2(\partial E)} \,(1 + \|\mathrm{H}\|^3_{L^6(\partial E)}) \,.$$

□

We now remind that since ∂E can be disconnected (as in the case of lamellae), the Poincaré inequality could fail for ∂E. However, if E is sufficiently close to a stable critical set then it is true for the mean curvature of ∂E.

Lemma 2.10 (Geometric Poincaré inequality). *Fixed $p > 2$ and a smooth strictly stable critical set $F \subseteq \mathbb{T}^3$, let $\delta > 0$ be the constant provided by Lemma 1.19, with $\theta = 1$. Then, for M small enough, there exists a constant $C = C(F, M, \alpha, p) > 0$ such that*

$$\int_{\partial E} |\mathrm{H} - \overline{\mathrm{H}}|^2 \, d\mu \leq C \int_{\partial E} |\nabla \mathrm{H}|^2 \, d\mu \,, \tag{2.14}$$

for every set $E \in \mathfrak{C}^{1,\alpha}_M(F)$ such that $\partial E \subseteq N_\varepsilon$ with

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \,,$$

for a smooth function ψ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$.

Proof. Since

$$\int_{\partial E} (\mathrm{H} - \overline{\mathrm{H}})\nu_E \, d\mu = 0 \,,$$

there holds

$$\begin{aligned}\int_{\partial E} |\mathrm{H} - \overline{\mathrm{H}} - \langle \eta, \nu_E\rangle|^2 \, d\mu &= \|\mathrm{H} - \overline{\mathrm{H}}\|^2_{L^2(\partial E)} + \int_{\partial E} \langle \eta, \nu_E\rangle^2 \, d\mu \\ &\geq \|\mathrm{H} - \overline{\mathrm{H}}\|^2_{L^2(\partial E)}\end{aligned}$$

for all $\eta \in \mathbb{R}^n$. Choosing $M < \delta$, we may then apply Lemma 1.19 with $\theta = 1$ and $\varphi = \mathrm{H} - \overline{\mathrm{H}}$, obtaining

$$\sigma_1 \int_{\partial E} |\mathrm{H} - \overline{\mathrm{H}}|^2 \, d\mu \leq \int_{\partial E} |\nabla \mathrm{H}|^2 \, d\mu - \int_{\partial E} |B|^2 |\mathrm{H} - \overline{\mathrm{H}}|^2 \, d\mu \leq \int_{\partial E} |\nabla \mathrm{H}|^2 \, d\mu \,.$$

□

The following lemma is straightforward.

Lemma 2.11. *Let $E \subseteq \mathbb{T}^3$ be a smooth set. If $f \in H^1(\partial E)$ and $g \in W^{1,4}(\partial E)$, then*

$$\|\nabla(fg)\|_{L^2(\partial E)} \leq C\|\nabla f\|_{L^2(\partial E)}\|g\|_{L^\infty(\partial E)} + C\|f\|_{L^4(\partial E)}\|\nabla g\|_{L^4(\partial E)}\,,$$

for a constant C independent of E.

Proof. We estimate with Cauchy–Schwarz inequality,

$$\begin{aligned}\|\nabla(fg)\|^2_{L^2(\partial E)} &\leq 2\|\nabla f\|^2_{L^2(\partial E)}\|g\|^2_{L^\infty(\partial E)} + 2\int_{\partial E}|f|^2|\nabla g|^2\,d\mu\\ &\leq 2\|\nabla f\|^2_{L^2(\partial E)}\|g\|^2_{L^\infty(\partial E)} + 2\|f\|^2_{L^4(\partial E)}\|\nabla g\|^2_{L^4(\partial E)}\,,\end{aligned}$$

hence the thesis follows. □

As a corollary, we prove the following result.

Lemma 2.12. *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E \in \mathfrak{C}^{1,\alpha}_M(F)$. Then, for M small enough, there holds*

$$\|\psi_E\|_{W^{3,2}(\partial F)} \leq C(F, M, \alpha)(1 + \|\mathrm{H}\|^2_{H^1(\partial E)})\,,$$

where H *is the mean curvature of ∂E (the function ψ_E is defined by formula (2.8)).*

Proof. By a standard localization/partition of unity/straightening argument, we may reduce ourselves to the case where the function ψ_E is defined in a disk $D \subseteq \mathbb{R}^2$ and $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$. Fixed a smooth cut–off function φ with compact support in D and equal to one on a smaller disk $D' \subseteq D$, we have (see Remark 1.1)

$$\Delta(\varphi\psi_E) - \frac{\nabla^2(\varphi\psi_E)\nabla\psi_E\nabla\psi_E}{1 + |\nabla\psi_E|^2} = \varphi\mathrm{H}\sqrt{1 + |\nabla\psi_E|^2} + R(x, \psi_E, \nabla\psi_E)\,,$$

where the remainder term $R(x, \psi_E, \nabla\psi_E)$ is a smooth Lipschitz function. Then, using Lemma 2.11 and recalling that $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$, we estimate

$$\begin{aligned}\|\nabla\Delta(\varphi\psi_E)\|_{L^2(D)} \leq C(F, M, \alpha)\Big(&M^2\|\nabla^3(\varphi\psi_E)\|_{L^2(D)}\\ &+ \|\nabla\mathrm{H}\|_{L^2(\partial E)}(1 + \|\nabla\psi_E\|_{L^\infty(D)})\\ &+ \|\mathrm{H}\|_{L^4(\partial E)}(1 + \|\psi_E\|_{W^{2,4}(D)})\\ &+ 1 + \|\psi_E\|_{W^{2,4}(D)}\Big)\,.\end{aligned}$$

We now use the fact that, by a simple integration by part argument, if u is a smooth function with compact support in $\mathbb{R}^2$, there holds

$$\|\nabla\Delta u\|_{L^2(\mathbb{R}^2)} = \|\nabla^3 u\|_{L^2(\mathbb{R}^2)}\,,$$

hence,

$$\begin{aligned}\|\nabla^3(\varphi\psi_E)\|_{L^2(D)} &= \|\nabla\Delta(\varphi\psi_E)\|_{L^2(D)}\\ &\leq C(F,M,\alpha)\Big(M^2\|\nabla^3(\varphi\psi_E)\|_{L^2(D)}\\ &\qquad\qquad + \|\nabla \mathrm{H}\|_{L^2(\partial E)}(1+\|\nabla\psi_E\|_{L^\infty(D)})\\ &\qquad\qquad + \|\mathrm{H}\|_{L^4(\partial E)}(1+\|\psi_E\|_{W^{2,4}(D)})\\ &\qquad\qquad + 1+\|\psi_E\|_{W^{2,4}(D)}\Big)\,,\end{aligned}$$

then, if M is small enough, we have

$$\|\nabla^3(\varphi\psi_E)\|_{L^2(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^1(\partial E)})(1+\|\mathrm{Hess}\,\psi_E\|_{L^4(D)})\,, \tag{2.15}$$

as

$$\|\mathrm{H}\|_{L^4(\partial E)} \leq C(F,M,\alpha)\|\mathrm{H}\|_{H^1(\partial E)}\,, \tag{2.16}$$

by Theorem 3.70 in [4].
By the Calderón–Zygmund estimates (holding uniformly for every hypersurface ∂E, with $E \in \mathfrak{C}^{1,\alpha}_M(F)$, see [5]),

$$\|\mathrm{Hess}\,\psi_E\|_{L^4(D)} \leq C\|\Delta\psi_E\|_{L^4(D)} + C\|\psi_E\|_{L^4(D)} \tag{2.17}$$

and the expression of the mean curvature (Remark (1.1))

$$\mathrm{H} = \frac{\Delta\psi_E}{\sqrt{1+|\nabla\psi_E|^2}} - \frac{\mathrm{Hess}\,\psi_E(\nabla\psi_E\nabla\psi_E)}{(\sqrt{1+|\nabla\psi_E|^2})^3}\,,$$

we obtain

$$\begin{aligned}\|\Delta\psi_E\|_{L^4(D)} &\leq 2M\|\mathrm{H}\|_{L^4(\partial E)} + M^2\|\mathrm{Hess}\,\psi_E\|_{L^4(D)}\\ &\leq 2M\|\mathrm{H}\|_{L^4(\partial E)} + CM^2(\|\psi_E\|_{L^4(D)} + \|\Delta\psi_E\|_{L^4(D)})\,,\end{aligned}$$

hence, possibly choosing a smaller M, we conclude

$$\|\Delta\psi_E\|_{L^4(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{L^4(\partial E)}) \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^1(\partial E)})\,,$$

again by inequality (2.16).
Thus, by estimate (2.17), we get

$$\|\mathrm{Hess}\,\psi_E\|_{L^4(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^1(\partial E)})\,, \tag{2.18}$$

and using this inequality in estimate (2.15),

$$\|\nabla^3(\varphi\psi_E)\|_{L^2(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^1(\partial E)})^2\,,$$

hence,

$$\|\nabla^3\psi_E\|_{L^2(D')} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^1(\partial E)})^2 \leq C(F,M,\alpha)(1+\|\mathrm{H}\|^2_{H^1(\partial E)})\,.$$

The inequality in the statement of the lemma then easily follows by this inequality, estimate (2.18) and $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$, with a standard covering argument. □

Lemma 2.13 (Compactness)**.** *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E_n \subseteq \mathfrak{C}^{1,\alpha}_M(F)$ a sequence of smooth sets such that*

$$\sup_{n\in\mathbb{N}} \int_{\partial E_n} |\nabla \mathrm{H}_n|^2 \, d\mu_n < +\infty \, .$$

Then, if $\alpha \in (0, 1/2)$ and M is small enough, there exists a smooth set $F' \in \mathfrak{C}^1_M(F)$ such that, up to a (non relabeled) subsequence, $E_n \to F'$ in $W^{2,p}$ for all $1 \le p < +\infty$ (recall the definition of convergence of sets at the beginning of Section 1.3).
Moreover, if inequality (2.14) holds for every set E_n with a constant C independent of n and

$$\int_{\partial E_n} |\nabla \mathrm{H}_n|^2 \, d\mu_n \to 0 \, ,$$

then F' is critical for the volume–constrained Area functional $\mathcal{A}$ and the convergence $E_n \to F'$ is in $W^{3,2}$.

Proof. We first claim that

$$\sup_{n\in\mathbb{N}} \|\mathrm{H}_n\|_{H^1(\partial E_n)} < +\infty. \tag{2.19}$$

We set $\widetilde{\mathrm{H}}_n = \fint_{\partial E_n} \mathrm{H}_n \, d\mu_n$, then, by the "geometric" Poincaré inequality of Lemma 2.10, which holds with a "uniform" constant $C = C(F, M, \alpha)$, for all the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$ (see [5]), if M is small enough, we have

$$\|\mathrm{H}_n - \widetilde{\mathrm{H}}_n\|^2_{H^1(\partial E_n)} \le \sup_{n\in\mathbb{N}} \int_{\partial E_n} |\nabla \mathrm{H}_n|^2 \, d\mu_n < C < +\infty$$

with a constant C independent of $n \in \mathbb{N}$.
Then, we note that by the uniform $C^{1,\alpha}$–bounds on ∂E_n, we may find a solid cylinder of the form $C = D \times (-L, L)$, with $D \subseteq \mathbb{R}^2$ a ball centered at the origin and functions f_n, with

$$\sup_{n\in\mathbb{N}} \|f_n\|_{C^{1,\alpha}(\overline{D})} < +\infty \, , \tag{2.20}$$

such that $\partial E_n \cap C = \{(x', x_n) \in D \times (-L, L) : x_n = f_n(x')\}$ with respect to a suitable coordinate frame (depending on $n \in \mathbb{N}$).

$$\begin{aligned} \int_D (\mathrm{H}_n - \widetilde{\mathrm{H}}_n) \, dx' + \widetilde{\mathrm{H}}_n \mathrm{Area}(D) &= \int_D \mathrm{div}\Big(\frac{\nabla_{x'} f_n}{\sqrt{1 + |\nabla_{x'} f_n|^2}} \Big) \, dx' \\ &= \int_{\partial D} \frac{\nabla_{x'} f_n}{\sqrt{1 + |\nabla_{x'} f_n|^2}} \cdot \frac{x'}{|x'|} \, d\sigma \, . \end{aligned}$$

where σ is the canonical (standard) measure on the circle ∂D.
Hence, recalling the uniform bound (2.20) and the fact that $\|\mathrm{H}_n - \widetilde{\mathrm{H}}_n\|_{H^1(\partial E_n)}$ are equibounded, we get that $\widetilde{\mathrm{H}}_n$ are also equibounded (by a standard "localization" argument, "uniformly" applied to all the hypersurfaces ∂E_n). Therefore, the claim (2.19) follows.

By applying the Sobolev embedding theorem on each connected component of ∂F, we have that

$$\|\mathrm{H}_n\|_{L^p(\partial E_n)} \le C\|\mathrm{H}_n\|_{H^1(\partial E_n)} < C < +\infty \qquad \text{for all } p \in [1,+\infty).$$

for a constant C independent of $n \in \mathbb{N}$.
Now, by means of inequality (2.13), we obtain

$$\|B\|_{L^p(\partial E)} \le C(1 + \|\mathrm{H}\|_{L^p(\partial E)}) .$$

for every $E \in \mathfrak{C}^{1,\alpha}_M(F)$ with a uniform constant C. Then, if we write

$$\partial E_n = \{y + \psi_n(y)\nu_F(y) : y \in \partial F\} ,$$

we have $\sup_{n\in\mathbb{N}} \|\psi_n\|_{W^{2,p}(\partial F)} < +\infty$, for all $p \in [1,+\infty)$ (taking into account Remark 1.1).
Thus, by the Sobolev compact embedding $W^{2,p}(\partial F) \hookrightarrow C^{1,\alpha}(\partial F)$, up to a subsequence (not relabeled), there exists a set $F' \in \mathfrak{C}^{1,\alpha}_M(F)$ such that

$$\psi_n \to \psi_{F'} \text{ in } C^{1,\alpha}(\partial F),$$

for all $\alpha \in (0,1/2)$ and $\beta \in (0,1)$.
From estimate (2.19) and Lemma 2.12 (possibly choosing a smaller M), we have then that the functions ψ_n are bounded in $W^{3,2}(\partial F)$. Hence, possibly passing to another subsequence (again not relabeled), we conclude that $E_n \to F'$ in $W^{2,p}$ for every $p \in [1,+\infty)$, by the Sobolev compact embeddings.

For the second part of the lemma, we first observe that if

$$\int_{\partial E_n} |\nabla \mathrm{H}_n|^2 \, d\mu_n \to 0 ,$$

then there exists $\lambda \in \mathbb{R}$ and a subsequence E_n (not relabeled) such that

$$\mathrm{H}_n\Big(\cdot + \psi_n(\cdot)\nu_F(\cdot)\Big) \to \lambda = \mathrm{H}\Big(\cdot + \psi_{F'}(\cdot)\nu_F(\cdot)\Big)$$

in $H^1(\partial F)$, where H is the mean curvature of F'. Hence F' is critical.
To conclude the proof we only need to show that ψ_n converge to $\psi = \psi_{F'}$ in $W^{3,2}(\partial F)$.
Fixed $\delta > 0$, arguing as in the proof of Lemma 2.12, we reduce ourselves to the case where the functions ψ_n are defined on a disk $D \subseteq \mathbb{R}^2$, are bounded in $W^{3,2}(D)$, converge in $W^{2,p}(D)$ for all $p \in [1,+\infty)$ to $\psi \in W^{3,2}(D)$ and $\|\nabla\psi\|_{L^\infty(D)} \le \delta$. Then, fixed a smooth cut–off function φ with compact support in D and equal to one on a smaller disk $D' \subseteq D$, we have

$$\begin{aligned}
\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|\nabla\psi_n|^2}} - \frac{\Delta(\varphi\psi)}{\sqrt{1+|\nabla\psi|^2}} &= (\nabla^2(\varphi\psi_n) - \nabla^2(\varphi\psi))\frac{\nabla\psi\nabla\psi}{(1+|\nabla\psi|^2)^{3/2}} \\
&\quad + \nabla^2(\varphi\psi_n)\Big(\frac{\nabla\psi_n\nabla\psi_n}{(1+|\nabla\psi_n|^2)^{3/2}} - \frac{\nabla\psi\nabla\psi}{(1+|\nabla\psi|^2)^{3/2}}\Big) \\
&\quad + \varphi(\mathrm{H}_n - \mathrm{H}) + R(x,\psi_n,\nabla\psi_n) - R(x,\psi,\nabla\psi) ,
\end{aligned}$$

where R is a smooth Lipschitz function.
Then, using Lemma 2.11, an argument similar to the one of the proof of Lemma 2.12 shows that

$$\begin{aligned}
\left\|\nabla\Big(\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|\nabla\psi_n|^2}}-\frac{\Delta(\varphi\psi)}{\sqrt{1+|\nabla\psi|^2}}\Big)\right\|_{L^2(D)}&\\
\leq C(M)\Big(\delta^2\|\nabla^3(\varphi\psi_n)-\nabla^3(\varphi\psi)\|_{L^2(D)}&\\
+\|\nabla^2(\varphi\psi_n)-\nabla^2(\varphi\psi)\|_{L^4(D)}\|\nabla^2\psi\|_{L^4(D)}&\\
+\|\nabla^3(\varphi\psi_n)\|_{L^2(D)}\|\nabla\psi_n-\nabla\psi\|_{L^\infty(D)}&\\
+\|\nabla^2(\varphi\psi_n)\|_{L^4(D)}(\|\nabla^2\psi_n\|_{L^4}+\|\nabla^2\psi\|_{L^4(D)})&\\
+\|\nabla \mathrm{H}_n-\nabla \mathrm{H}\|_{L^2(D)}+\|\psi_n-\psi\|_{W^{2,4}(D)}\Big)\,.&
\end{aligned}$$

Using Lemma 2.11 again and arguing again as in the proof of Lemma 2.12, we finally get

$$\begin{aligned}
\|\nabla^3(\varphi\psi_n)-\nabla^3(\varphi\psi)\|_{L^2(D)}\leq C(M)\Big(\,&\|\psi_n-\psi\|_{W^{2,4}(D)}\\
&+\|\nabla\psi_n-\nabla\psi\|_{L^\infty(D)}\\
&+\|\nabla \mathrm{H}_n-\nabla \mathrm{H}\|_{L^2(D)}\Big)\,,
\end{aligned}$$

hence,

$$\begin{aligned}
\|\nabla^3\psi_n-\nabla^3\psi\|_{L^2(D')}\leq C(M)\Big(\,&\|\psi_n-\psi\|_{W^{2,4}(D)}\\
&+\|\nabla\psi_n-\nabla\psi\|_{L^\infty(D)}\\
&+\|\nabla \mathrm{H}_n-\nabla \mathrm{H}\|_{L^2(D)}\Big)\,,
\end{aligned}$$

from which the conclusion follows, by the first part of the lemma and a standard covering argument.

□

3 GLOBAL EXISTENCE AND ASYMPTOTIC STABILITY

We finally prove the main result of this thesis, following the line in [1]. By means of the lemmas in the previous chapter, we will show that if the initial set E_0 is "sufficiently close" to a strictly stable critical set $F \subseteq \mathbb{T}^3$, then the *surface diffusion flow* starting from E_0 exists for all time and converges asymptotically to a translate of F.

Theorem 3.1. *Let $F \subseteq \mathbb{T}^3$ be a strictly stable critical set and let N_ε be a tubular neighborhood of ∂F, as in formula (1.36). For every $\alpha \in (0, 1/2)$ there exists $M > 0$ such that, if E_0 is a smooth set in $C^{1,\alpha}_M(F)$ satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(F)$ and*

$$\int_{\partial E_0} |\nabla \mathrm{H}_0|^2 \, d\mu_0 \leq M,$$

then the unique smooth solution E_t of the surface diffusion flow starting from E_0, given by Proposition 2.4, is defined for all $t \geq 0$. Moreover, $E_t \to F + \eta$ exponentially fast in $W^{3,2}$ as $t \to +\infty$ (recall the definition of convergence of sets at the beginning of Section 1.3), for some $\eta \in \mathbb{R}^3$, with the meaning that the functions $\psi_{\eta,t} : \partial F + \eta \to \mathbb{R}$ representing ∂E_t as "normal graphs" on $\partial F + \eta$, that is,

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{F+\eta}(y) \,:\, y \in \partial F + \eta\},$$

satisfy

$$\|\psi_{\eta,t}\|_{W^{3,2}(\partial F+\eta)} \leq Ce^{-\beta t},$$

for every $t \in [0, +\infty)$, for some positive constants C and β.

Remark 3.2. We already said that the property of a set E_0 to belong to $\mathfrak{C}^{1,\alpha}_M(F)$ is a "closedness" in L^1 of E_0 and F, and in $C^{1,\alpha}$ of their boundaries. The extra condition in the theorem on the L^2–smallness of the gradient of H_0 (see the second part of Lemma 2.13 and its proof) implies that the mean curvature of ∂E_0 (that from now on we renamed as H_0) is "close" to be constant, as it is for the set F (or actually for any critical set). Notice that this is a second order condition for the boundary of E_0, in addition to the first order one $E_0 \in \mathfrak{C}^{1,\alpha}_M(F)$.

Proof of Theorem 3.1. Throughout the whole proof C will denote a constant depending only on F, M and α, whose value may vary from line to line.

Assume that the surface diffusion flow E_t is defined for t in the maximal time interval $[0, T(E_0))$, where $T(E_0) \in (0, +\infty]$ and let the moving boundaries ∂E_t be represented as "normal graphs" on ∂F as

$$\partial E_t = \{y + \psi_t(y)\nu_F(y) \,:\, y \in \partial F\},$$

for some smooth functions $\psi_t : \partial F \to \mathbb{R}$. As before we set $\nu_t = \nu_{E_t}$.

We recall that, by Proposition 2.4, for every $E \in \mathfrak{C}^{2,\alpha}_M(F)$, the flow is defined in the time interval $[0,T)$, with $T = T(F,M,\alpha) > 0$.

We show the theorem for the smooth sets $E_0 \subseteq \mathbb{T}^3$ satisfying

$$\mathrm{Vol}(E_0 \Delta F) \le M_1, \quad \|\psi_0\|_{C^{1,\alpha}(\partial F)} \le M_2 \quad \text{and} \quad \int_{\partial E_0} |\nabla \mathrm{H}_0|^2 \, d\mu_0 \le M_3\,, \tag{3.1}$$

for some positive constants M_1, M_2, M_3, then we get the thesis by setting $M = \min\{M_1, M_2, M_3\}$.

For any set $E \in \mathfrak{C}^{1,\alpha}_M(F)$ we introduce the following quantity

$$D(E) = \int_{E \Delta F} d(x, \partial F)\, dx = \int_E d_F\, dx - \int_F d_F\, dx, \tag{3.2}$$

where d_F is the signed distance function defined in formula (1.37). We observe that

$$\mathrm{Vol}(E\Delta F) \le C\|\psi_E\|_{L^1(\partial F)} \le C\|\psi_E\|_{L^2(\partial F)}$$

for a constant C depending only on F and, as $E \subseteq N_\varepsilon$,

$$D(E) \le \int_{E\Delta F} \varepsilon\, dx \le \varepsilon \mathrm{Vol}(E\Delta F).$$

Moreover,

$$\begin{aligned}
\|\psi_E\|^2_{L^2(\partial F)} &= 2\int_{\partial F}\int_0^{|\psi_E(y)|} t\, dt\, d\mu(y)\\
&= 2\int_{\partial F}\int_0^{|\psi_E(y)|} d(L(y,t), \partial F)\, dt\, d\mu(y)\\
&= 2\int_{E\Delta F} d(x, \partial F)\, JL^{-1}(x)\, dx\\
&\le CD(E)\,.
\end{aligned}$$

where $L : \partial F \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ the smooth diffeomorphism defined in formula (1.39) and JL its Jacobian. As we already said, the constant C depends only on F and ε. This clearly implies

$$\mathrm{Vol}(E\Delta F) \le C\|\psi_E\|_{L^1(\partial F)} \le C\|\psi_E\|_{L^2(\partial F)} \le C\sqrt{D(E)}\,. \tag{3.3}$$

Hence, by this discussion, the initial smooth set $E_0 \in \mathfrak{C}^{1,\alpha}_M(F)$ satisfies $D(E_0) \le M \le M_1$ (having chosen $\varepsilon < 1$).

By rereading the proof of Lemma 2.13, it follows that for M_2, M_3 small enough, if $\|\psi_E\|_{C^{1,\alpha}(\partial F)} \le M_2$ and

$$\int_{\partial E} |\nabla \mathrm{H}|^2\, d\mu \le M_3\,,$$

then

$$\|\psi_E\|_{W^{2,6}(\partial F)} \le \omega(\max\{M_2, M_3\})\,, \tag{3.4}$$

where $s \mapsto \omega(s)$ is a positive nondecreasing function (defined on $\mathbb{R}$) such that $\omega(s) \to 0$ as $s \to 0^+$. This clearly implies (recalling Remark 1.1)

$$\|\nu_E\|_{W^{1,6}(\partial E)} \leq \omega'(\max\{M_2, M_3\}) ,$$

for a function ω' with the same properties of ω. Both ω and ω' only depend on F and α, for M small enough. Moreover, thanks to Lemma 2.10, there exists $C > 0$ such that, choosing M_2, M_3 small enough, in order that $\omega(\max\{M_2, M_3\})$ is small enough, we have

$$\int_{\partial E} |\mathrm{H} - \overline{\mathrm{H}}|^2 \, d\mu \leq C \int_{\partial E} |\nabla \mathrm{H}|^2 \, d\mu , \tag{3.5}$$

where, as usual, $\overline{\mathrm{H}}$ is the average of H over ∂E.

We split the proof of the theorem into steps.

Step 1. (*Stopping–time*) Let $\overline{T} \leq T(E_0)$ be the maximal time such that

$$\mathrm{Vol}(E_t \Delta F) \leq 2M_1, \quad \|\psi_t\|_{C^{1,\alpha}(\partial F)} \leq 2M_2 \quad \text{and} \quad \int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq 2M_3 , \tag{3.6}$$

for all $t \in [0, \overline{T})$. Hence,

$$\|\psi_t\|_{W^{2,6}(\partial F)} \leq \omega(2\max\{M_2, M_3\}) \tag{3.7}$$

for all $t \in [0, \overline{T}')$, as in formula (3.4). Note that such a maximal time is clearly positive, by the hypotheses on E_0.
We claim that by taking M_1, M_2, M_3 small enough, we have $\overline{T} = T(E_0)$.

Step 2. (*Estimate of the translational component of the flow*) We want to show that there exists a small constant $\theta > 0$ such that

$$\min_{\eta \in \mathrm{O}_F} \|\Delta \mathrm{H}_t - \langle \eta, \nu_t \rangle\|_{L^2(\partial E_t)} \geq \theta \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \qquad \text{for all } t \in [0, \overline{T}) , \tag{3.8}$$

where O_F is defined by formula (1.34).
If M is small enough, clearly there exists a constant $C_0 = C_0(F, M, \alpha) > 0$ such that, for every $i \in \mathrm{I}_F$, we have $\|\langle e_i, \nu_t \rangle\|_{L^2(\partial E_t)} \geq C_0 > 0$, holding $\|\langle e_i, \nu_F \rangle\|_{L^2(\partial F)} > 0$. It is then easy to show that the vector $\eta_t \in \mathrm{O}_F$ realizing such minimum is unique and satisfies

$$\Delta \mathrm{H}_t = \langle \eta_t, \nu_t \rangle + g, \tag{3.9}$$

where $g \in L^2(\partial E_t)$ is a function L^2–orthogonal (with respect to the measure μ_t on ∂E_t) to the vector subspace of $L^2(\partial E_t)$ spanned by $\langle e_i, \nu_t \rangle$, with $i \in \mathrm{I}_F$, where $\{e_1, \dots, e_3\}$ is the orthonormal basis of $\mathbb{R}^3$ given by Remark 1.10). Moreover, the inequality

$$|\eta_t| \leq C \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \tag{3.10}$$

holds, with a constant C depending only on F, M and α.
We now argue by contradiction, assuming $\|g\|_{L^2(\partial E_t)} < \theta \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)}$.

First we recall that $\Delta \mathrm{H}_t$ has zero average. Then, setting $\overline{\mathrm{H}} = \fint_{\partial E_t} \mathrm{H}\, d\mu_t$, and recalling relation (3.5), we get

$$\begin{aligned}
\|\mathrm{H}_t - \overline{\mathrm{H}}_t\|^2_{L^2(\partial E_t)} &\leq C \int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \\
&= -C \int_{\partial E_t} \mathrm{H}_t \Delta \mathrm{H}_t \, d\mu_t \\
&= -C \int_{\partial E_t} \Delta \mathrm{H}_t (\mathrm{H}_t - \overline{\mathrm{H}}_t) \, d\mu_t \\
&\leq C \|\mathrm{H}_t - \overline{\mathrm{H}}_t\|_{L^2(\partial E_t)} \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \,, \qquad (3.11)
\end{aligned}$$

Hence, we conclude

$$\|\mathrm{H}_t - \overline{\mathrm{H}}_t\|_{L^2(\partial E_t)} \leq C \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \,. \qquad (3.12)$$

Since, there holds

$$\int_{\partial E_t} \mathrm{H}_t \, \nu_t \, d\mu_t = \int_{\partial E_t} \nu_t \, d\mu_t = 0 \,,$$

by multiplying relation (3.9) by $\mathrm{H}_t - \overline{\mathrm{H}}_t$, integrating over ∂E_t, and using inequality (3.12), we get

$$\begin{aligned}
\left| \int_{\partial E_t} (\mathrm{H}_t - \overline{\mathrm{H}}_t) \Delta \mathrm{H}_t \, d\mu_t \right| &= \left| \int_{\partial E_t} (\mathrm{H}_t - \overline{\mathrm{H}}_t) g \, d\mu_t \right| \\
&< \theta \|\mathrm{H}_t - \overline{\mathrm{H}}_t\|_{L^2(\partial E_t)} \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \\
&\leq C\theta \|\Delta \mathrm{H}_t\|^2_{L^2(\partial E_t)} \,.
\end{aligned}$$

Recalling now estimate (3.10), as g is orthogonal to $\langle \eta_t, \nu_t \rangle$, computing as in the first three lines of formula (3.11), we have

$$\begin{aligned}
\|\langle \eta_t, \nu_t \rangle\|^2_{L^2(\partial E_t)} &= \int_{\partial E_t} \Delta \mathrm{H}_t \langle \eta_t, \nu_t \rangle \, d\mu_t \\
&= - \int_{\partial E_t} \langle \nabla \mathrm{H}_t, \nabla \langle \eta_t, \nu_t \rangle \rangle \, d\mu_t \\
&\leq |\eta_t| \|\nabla \nu_t\|_{L^2(\partial E_t)} \|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)} \\
&\leq C \|\nabla \nu_t\|_{L^2(\partial E_t)} \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \left| \int_{\partial E_t} (\mathrm{H}_t - \overline{\mathrm{H}}_t) \Delta \mathrm{H}_t \, d\mu_t \right|^{1/2} \\
&\leq C\sqrt{\theta} \|\nabla \nu_t\|_{L^2(\partial E_t)} \|\Delta \mathrm{H}_t\|^2_{L^2(\partial E_t)} \\
&\leq C\sqrt{\theta} \|\Delta \mathrm{H}_t\|^2_{L^2(\partial E_t)} \,,
\end{aligned}$$

where in the last inequality we estimated $\|\nabla \nu_t\|_{L^2(\partial E_t)}$ with $C\|\psi_t\|_{W^{2,6}(\partial E_t)}$ (keeping into account Remark 1.1) and we used inequality (3.7).
If then $\theta > 0$ is chosen so small that $C\sqrt{\theta} + \theta^2 < 1$ in the last inequality, then we have a contradiction with equality (3.9) and the fact that $\|g\|_{L^2(\partial E_t)} < \theta \|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)}$, as they imply (by L^2–orthogonality) that

$$\|\langle \eta_t, \nu_t \rangle\|^2_{L^2(\partial E_t)} > (1 - \theta^2) \|\Delta \mathrm{H}_t\|^2_{L^2(\partial E_t)} \,.$$

All this argument shows that for such a choice of θ condition (3.8) holds. By Propositions 1.19 and 1.20, there exist positive constants σ_θ and δ with the following properties: for any set $E \in \mathfrak{C}^{1,\alpha}_M(F)$ such that $\|\psi_E\|_{W^{2,6}(\partial F)} < \delta$, there holds

$$\Pi_E(\varphi) \geq \eta_\theta \|\varphi\|^2_{H^1(\partial E)}$$

for all $\varphi \in \widetilde{H}^1(\partial E)$ such that

$$\min_{\eta \in \mathrm{O}_F} \|\varphi - \langle \eta, \nu_E \rangle\|_{L^2(\partial E)} \geq \theta \|\varphi\|_{L^2(\partial E)}$$

and if F' is critical, $\mathrm{Vol}(F') = \mathrm{Vol}(F)$ with $\|\psi_{F'}\|_{W^{2,6}(\partial F)} < \delta$, then

$$F' = F + \eta \tag{3.13}$$

for a suitable vector $\eta \in \mathbb{R}^3$. We then assume that M_2, M_3 are small enough such that

$$\omega(2\max\{M_2, M_3\}) < \delta/2 \tag{3.14}$$

where ω is the function introduced in formula (3.4).

Step 3. (*The stopping time $\overline{T}$ is equal to the maximal time $T(E_0)$*) We show now that, by taking M_1, M_2, M_3 smaller if needed, we have $\overline{T} = T(E_0)$.

By the previous point and the suitable choice of M_2, M_3 made in its final part, formula (3.8) holds, hence we have

$$\Pi_{E_t}(\Delta \mathrm{H}_t) \geq \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial F)} \qquad \text{for all } t \in [0, \overline{T}).$$

In turn, by Lemma 2.6 and 2.9 we may estimate

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq & - \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} + \int_{\partial E_t} |B||\nabla \mathrm{H}_t|^2 |\Delta \mathrm{H}_t| \, d\mu_t \\
\leq & - \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \\
& + C\|\nabla(\Delta \mathrm{H}_t)\|^2_{L^2(\partial E_t)} \|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)} (1 + \|\mathrm{H}_t\|^3_{L^6(\partial E_t)}) \\
\leq & - \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \\
& + C\sqrt{M_3}\|\nabla(\Delta \mathrm{H}_t)\|^2_{L^2(\partial E_t)} (1 + \|\mathrm{H}_t\|^3_{L^6(\partial E_t)}) \\
\leq & - \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \\
& + C\sqrt{M_3}\|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} (1 + C\omega(\max\{M_2, M_3\}))
\end{aligned} \tag{3.15}$$

for every $t \leq \overline{T}$, where in the last step we used relations (3.6) and (3.7) (and kept into account Remark 1.1).

Noticing that from formulas (3.11) and (3.12) it follows

$$\|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)} \leq C\|\Delta \mathrm{H}_t\|_{L^2(\partial E_t)} \leq C\|\Delta \mathrm{H}_t\|_{H^1(\partial E_t)} ,$$

kepping fixed M_2 a choosing a suitably small M_3, we conclude

$$\frac{d}{dt}\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq -\frac{\sigma_\theta}{2}\|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \leq -c_0\|\nabla \mathrm{H}_t\|^2_{L^2(\partial E_t)} .$$

This argument clearly says that the quantity $\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t$ is non-increasing in time, hence, if M_2, M_3 are small enough, the inequality $\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq M_3$ is preserved during the flow. If we assume by contradiction that $\overline{T} < T(E_0)$, then it must happen that

$$\mathrm{Vol}(E_{\overline{T}} \Delta F) = 2M_1$$

or

$$\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial F)} = 2M_2 \, .$$

Before showing that this is not possible, we prove that actually the quantity $\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t$ decreases (non increases) exponentially. Indeed, integrating the differential inequality above and recalling proprierties (3.1), we obtain

$$\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq e^{-c_0 t} \int_{\partial E_0} |\nabla \mathrm{H}_{\partial E_0}|^2 \, d\mu_0 \leq M_3 e^{-c_0 t} \leq M_3 \tag{3.16}$$

for every $t \leq \overline{T}$.

Then, we assume that $\mathrm{Vol}(E_{\overline{T}} \Delta F) = 2M_1$ or $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})} = 2M_2$. Recalling formula (3.2) and denoting by X_t the velocity field of the flow (see Definition 2.1 and the subsequent discussion), we compute

$$\begin{aligned}
\frac{d}{dt} D(E_t) &= \frac{d}{dt} \int_{E_t} d_F \, dx = \int_{E_t} \mathrm{div}(d_F X_t) \, dx = \int_{\partial E_t} d_E \langle X_t, \nu_t \rangle \, d\mu_t \\
&= \int_{\partial E_t} d_F \, \Delta \mathrm{H}_t \, d\mu_t - \int_{\partial E_t} \langle \nabla d_F, \nabla \mathrm{H}_t \rangle \, d\mu_t \\
&\leq C \|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)} \leq C\sqrt{M_3} \mathrm{e}^{-c_0 t/2} \, ,
\end{aligned}$$

for all $t \leq \overline{T}$, where the last inequality clearly follows from inequality (3.16).

By integrating this differential inequality over $[0, \overline{T})$ and recalling estimate (3.3), we get

$$\begin{aligned}
\mathrm{Vol}(E_{\overline{T}} \Delta F) &\leq C \|\psi_{\overline{T}}\|_{L^2(\partial E_{\overline{T}})} \leq C \sqrt{D(E_{\overline{T}})} \\
&\leq C \sqrt{D(E_0) + C\sqrt{M_3}} \leq C \sqrt[4]{M_3} \, ,
\end{aligned} \tag{3.17}$$

as $D(E_0) \leq M_1$, provided that M_1, M_3 are chosen suitably small. This shows that $\mathrm{Vol}(E_{\overline{T}} \Delta F) = 2M_1$ cannot happen if we chose $C\sqrt[4]{M_3} \leq M_1$. By arguing as in Lemma 2.13 (keeping into account inequality (3.6) and formula (3.4)), we can see that the L^2–estimate (3.17) implies a $W^{2,6}$–bound on $\psi_{\overline{T}}$ with a constant going to zero, keeping fixed M_2, as $\int_{\partial E_t} |\nabla \mathrm{H}_{\overline{T}}|^2 \, d\mu_t \to 0$, hence, by estimate (3.16), as $M_3 \to 0$. Then, by Sobolev embeddings, the same holds for $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})}$, hence, if M_3 is small enough, we have a contradiction with $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})} = 2M_2$.

Thus, $\overline{T} = T(E_0)$ and

$$\mathrm{Vol}(E_t \Delta F) \leq C\sqrt[4]{M_3} \, , \quad \|\psi_t\|_{C^{1,\alpha}(\partial E_t)} \leq 2M_2 \, , \quad \int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t \leq M_3 e^{-c_0 t} \, , \tag{3.18}$$

for every $t \in [0, T(E_0))$, by choosing M_1, M_2, M_3 small enough.

Step 4. (*Long time existence*) We now show that, by taking M_1, M_2, M_3 smaller if needed, we have $T(E_0) = +\infty$, that is, the flow exists for all times.

We assume by contradiction that $T(E_0) < +\infty$ and we notice that, by computation (3.15) and the fact that $\overline{T} = T(E_0)$, we have

$$\frac{d}{dt}\int_{\partial E_t} |\nabla \mathrm{H}_t|^2 \, d\mu_t + \sigma_\theta \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \leq 0$$

for all $t \in [0, T(E_0))$. Integrating this differential inequality over the interval $[T(E_0) - T/2, T(E_0) - T/4]$, where T is given by Proposition 2.4, as we said at the beginning of the proof, we obtain

$$\begin{aligned}\sigma_\theta \int_{T(E_0)-T/2}^{T(E_0)-T/4} \|\Delta \mathrm{H}_t\|^2_{H^1(\partial E_t)} \, dt \leq & \int_{\partial E_{T(E_0)-\frac{T}{2}}} |\nabla \mathrm{H}|^2 \, d\mu_{T(E_0)-\frac{T}{2}} \\ & - \int_{\partial E_{T(E_0)-\frac{T}{4}}} |\nabla \mathrm{H}|^2 \, d\mu_{T(E_0)-\frac{T}{4}} \\ & \leq M_3 \,,\end{aligned}$$

where the last inequality follows from estimate (3.18). Thus, by the mean value theorem there exists $\overline{t} \in (T(E_0) - T/2, T(E_0) - T/4)$ such that

$$\|\Delta \mathrm{H}_{\overline{t}}\|^2_{H^1(\partial E_{\overline{t}})} \leq \frac{4M_3}{T\sigma_\theta} \,.$$

Thus, by Lemma 2.8

$$\begin{aligned}\|\nabla^2 \mathrm{H}_{\overline{t}}\|^2_{L^2(\partial E_{\overline{t}})} \leq & \, C \|\Delta \mathrm{H}_{\overline{t}}\|^2_{L^2(\partial E_{\overline{t}})} (1 + \|\mathrm{H}_{\overline{t}}\|^4_{L^4(\partial E_{\overline{t}})}) \\ \leq & \, C M_3 (1 + \omega^4 (2 \max\{M_2, M_3\}))\end{aligned}$$

where in the last inequality we also used the curvature bounds provided by formula (3.7). In turn, for $p \in \mathbb{R}$ large enough, we get

$$[\mathrm{H}_{\overline{t}}]^2_{C^{0,\alpha}(\partial E_{\overline{t}})} \leq C \|\nabla \mathrm{H}_{\overline{t}}\|^2_{L^p(\partial E_{\overline{t}})} \leq C \|\nabla \mathrm{H}_{\overline{t}}\|^2_{H^1(\partial E_{\overline{t}})} \leq C M_3 (M_2, M_3) \,,$$

where $[\cdot]_{C^{0,\alpha}(\partial E_{\overline{t}})}$ stands for the α–Hölder seminorm on $\partial E_{\overline{t}}$ and in the last inequality we used the previous estimate.

By means of Schauder estimates (as Calderón–Zygmund inequality implied estimate (2.13)), it is possible to show (see [5]) that there exists a constant $C > 0$ depending only on F, M, α and $p > 1$ such that for every $E \in \mathfrak{C}^{1,\alpha}_M(F)$, choosing even smaller M_1, M_2, M_3, there holds

$$\|B\|_{C^{0,\alpha}(\partial E)} \leq C (1 + \|\mathrm{H}\|_{C^{0,\alpha}(\partial E)}) \,.$$

Thus, if we choose M_3 sufficiently small, by the above discussion (and Remark 1.1, as before), we can conclude that $E_{\overline{t}} \in \mathfrak{C}^{2,\alpha}_M(F)$. Therefore, the maximal time of existence of the classical solution starting from $E_{\overline{t}}$ is at least T, which means that the flow E_t can be continued beyond $T(E_0)$, which is a contradiction.

Step 5. (*Convergence, up to subsequences, to a translate of F*) Let $t_n \to +\infty$, then, by estimates (3.18), the sets E_{t_n} satisfy the hypotheses of Lemma 2.13, hence, up to a (not relabeled) subsequence we have that there exists a critical set $F' \in \mathfrak{C}^{1,\alpha}_M(F)$ such that $E_{t_n} \to F'$ in $W^{3,2}$. Due to formulas (3.4) and (3.14) we also have $\|\psi_{F'}\|_{W^{2,6}(\partial F)} \leq \delta$ and $F' = F + \eta$ for some (small) $\eta \in \mathbb{R}^3$ (equality (3.13)).

Step 6. *(Exponential convergence of the full sequence)* Consider now

$$D_\eta(E) = \int_{E\Delta(F+\eta)} \operatorname{dist}(x, \partial F + \eta)\, dx\,.$$

The very same calculations performed in Step 3 show that

$$\left|\frac{d}{dt} D_\eta(E_t)\right| \leq C\|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)} \leq C\sqrt{M_3}e^{-c_0t/2}$$

for all $t \geq 0$, moreover, by means of the previous step, it follows $\lim_{t\to+\infty} D_\eta(E_t) = 0$. In turn, by integrating this differential inequality and writing

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{F+\eta}(y) : y \in \partial F + \eta\}\,,$$

we get

$$\|\psi_{\eta,t}\|^2_{L^2(\partial F+\eta)} \leq C D_\eta(E_t) \leq \int_t^{+\infty} C\sqrt{M_3}e^{-c_0s/2}\, ds \leq C\sqrt{M_3}e^{-c_0t/2}\,.$$

Since by the previous steps $\|\psi_{\eta,t}\|_{W^{2,6}(\partial F+\eta)}$ is bounded, we infer from this inequality, Sobolev embeddings and standard interpolation estimates that also $\|\psi_{\eta,t}\|_{C^{1,\beta}(\partial F+\eta)}$ decays exponentially for $\beta \in (0, 2/3)$. Denoting the average of H_t on ∂E_t by $\overline{\mathrm{H}}_t$, as by estimates (3.11) and (3.16), we have that

$$\begin{aligned}
\|\mathrm{H}_t(\cdot + \psi_{\eta,t}(\cdot)\nu_{F+\eta}(\cdot)) - \overline{\mathrm{H}}_t\|_{H^1(\partial F+\eta)}& \\
&\leq C\|\mathrm{H}_t - \overline{\mathrm{H}}_t\|_{H^1(\partial E_t)}\|\psi_{\eta,t}\|_{C^1(\partial F+\eta)}\\
&\leq C\|\nabla \mathrm{H}_t\|_{L^2(\partial E_t)}\\
&\leq C\sqrt{M_3}e^{-c_0t/2}\,.
\end{aligned}$$

It follows that

$$\|[\mathrm{H}_t(\cdot + \psi_{\eta,t}(\cdot)\nu_{F+\eta}(\cdot)) - \overline{\mathrm{H}}_t] - [\mathrm{H}_{\partial F+\eta} - \overline{\mathrm{H}}_{\partial F+\eta}]\|_{H^1(\partial F+\eta)} \to 0 \tag{3.19}$$

exponentially fast, as $t \to +\infty$, where $\overline{\mathrm{H}}_{\partial F+\eta}$ stands for the average of $\mathrm{H}_{\partial F+\eta}$ on $\partial F + \eta$.

Since $E_t \to F + \eta$ (up to a subsequence) in $W^{3,2}$, it is easy to check that $|\overline{\mathrm{H}}_t - \overline{\mathrm{H}}_{\partial F+\eta}| \leq C\|\psi_{\eta,t}\|_{C^1(\partial F+\eta)}$ which decays exponentially, therefore, thanks to limit (3.19), we have

$$\|\mathrm{H}_t(\cdot + \psi_{\eta,t}(\cdot)\nu_{F+\eta}(\cdot)) - \mathrm{H}_{\partial F+\eta}\|_{H^1(\partial F+\eta)} \to 0$$

exponentially fast.

The conclusion then follows arguing as at the end of Step 4. □

4 SOME CONNECTED TOPICS AND RESEARCH LINES

This last chapter deals with some problems related to our work. In particular, we briefly discuss a perturbed version of the surface diffusion flow and the modified Mullins–Sekerka flow, which is studied in [1] using the same arguments and techniques proposed before. Then, we give some information about the classification of the stable critical sets. Finally, we describe some possible lines of research.

4.1 THE SURFACE DIFFUSION FLOW WITH ELASTICITY

We introduce the *surface diffusion flow with elasticity* and we give an asymptotic stability result in the three–dimensional Euclidean space, for more details, see [10] and the references therein. About the fractional Sobolev spaces, we refer to [23] for definitions and basic facts.

We define a "nonlocal" energy functional, where the nonlocality is given by an "elasticity" term. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with a smooth boundary and $F \subseteq \Omega$ a smooth set such that its closure $\overline{F}$ is a compact subset of Ω. With ν_F we will denote the outer unit normal vector to ∂F.

In all this section we will always consider sets $F \subseteq \Omega$ with this property saying that F is "compactly contained" in Ω.

Let $u_F \in H^1(\Omega \setminus F, \mathbb{R}^3)$ be the unique solution of the following PDE system

$$\begin{cases} \operatorname{div} \mathbb{C}E(u_F) = 0 & \text{in } \Omega \setminus F \\ \mathbb{C}E(u_F)\nu_F = 0 & \text{on } \partial F \cup \partial\Omega \setminus \partial_D\Omega \\ u_F = \omega_0 & \text{on } \partial_D\Omega \end{cases}$$

where $\mathbb{C}$ is the *elasticity tensor* acting on 3×3–matrices,

$$E(u_F) = \frac{Du_F + Du_F^\top}{2}$$

is the *elastic stress associated* to the "displacement function" u_F, ω_0 is a fixed function in $H^{\frac{1}{2}}(\partial\Omega)$ and $\partial_D\Omega$ is a relatively open subset of $\partial\Omega$ ($Du_F^\top$ is the transpose matrix of Du_F).
We then define the following modification of the *Area functional*

$$\mathcal{J}(F) = \mathcal{A}(\partial F) + \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) \cdot E(u_F)\, dx\,, \tag{4.1}$$

under a constraint of fixed volume, where the dot denotes the scalar product between matrices, that is $A \cdot B = \operatorname{tr} AB$.

We can compute the first and second variations of $\mathcal{J}$, along the lines of Chapter 1.

Proposition 4.1. *Let F be a smooth set compactly contained in Ω, $X \in C_c^\infty(\Omega, \mathbb{R}^3)$ and Φ_t for $t \in (-\varepsilon, \varepsilon)$ the associated flow, as in formula (1.9). Then,*

$$\frac{d}{dt}\mathcal{J}(\Phi_t(F))\Big|_{t=0} = \int_{\partial F} \Big(\mathrm{H} - \frac{1}{2}\,\mathbb{C}E(u_F)\cdot E(u_F)\Big)\varphi\, d\mu$$

where $\varphi = \langle X, \nu_F\rangle$ on ∂F. If in addition $\operatorname{div} X = 0$ in a neighborhood of ∂F we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0} &= \int_{\partial F} |\nabla\varphi|^2 - |B|^2\varphi^2\, d\mu - \int_{\Omega\setminus F} \mathbb{C}E(u_\varphi)\cdot E(u_F)\, dx \\ &\quad - \frac{1}{2}\int_{\partial F} \langle \nabla(\mathbb{C}E(u_F)\cdot E(u_F)\varphi^2), \nu_F\rangle\, d\mu \\ &\quad - \int_{\partial F}\Big(\mathrm{H} - \frac{1}{2}\,\mathbb{C}E(u_F)\cdot E(u_F)\Big)\operatorname{div}(\varphi X_\tau)\, d\mu,\end{aligned}$$

where the function $u_\varphi \in H^1(\Omega\setminus F, \mathbb{R}^3)$ is the unique solution of

$$\int_{\Omega\setminus F} \mathbb{C}E(u_\varphi)\cdot E(\psi)\, dx = -\int_{\partial F} \langle \operatorname{div}(\varphi\,\mathbb{C}E(u_F)), \psi\rangle\, d\mu$$

with $u_\varphi = 0$ on $\partial_D\Omega$, for all $\psi \in H^1(\Omega\setminus F, \mathbb{R}^3)$ such that $\psi = 0$ on $\partial_D\Omega$.

We fix a smooth set $G \subseteq \Omega$ and a tubular neighborhood N_ε of ∂G, as in formula (1.36), with $\pi : N_\varepsilon \to \partial G$ the associated smooth orthogonal projection. As before, we denote by $\mathfrak{C}^{k,\alpha}_M(G)$ the class of smooth sets F with $\mathrm{Vol}(F \triangle G) < M$ and whose boundary is a normal graph over ∂G with a function whose $C^{k,\alpha}$–norm is smaller that M.
Let $G_1, \dots, G_m$ be the connected components of G, with smooth boundaries $\Gamma_{G,1} = \partial G_1, \dots, \Gamma_{G,m} = \partial G_m$. For M small, every $F \in \mathfrak{C}^{1,\alpha}_M(G)$ is C^1-diffeomorphic to G, thus, ∂F has the same number m of connected components $\Gamma_{F,1}, \dots, \Gamma_{F,m}$, which can be numbered in such a way that, for every $i \in \{1, \dots, m\}$, we have

$$\Gamma_{F,i} = \{y + h_{F,i}(y)\nu_G(y) \,:\, y \in \Gamma_{G,i}\}\,,$$

for suitable functions $h_{F_i} \in C^{1,\alpha}(\partial G_i)$ and the respectively enclosed sets F_i are diffeomorpic to G_i.

Also in this case, we are interested in volume–preserving variations, in the following sense.

Definition 4.2. Let F be a smooth set compactly contained in Ω. We say that a vector field $X \in C_c^\infty(\Omega, \mathbb{R}^3)$ is *admissible for F*, if there exists $\varepsilon_0 \in (0,1)$ such that

$$\mathrm{Vol}(\Phi_t(F_i)) = \mathrm{Vol}(F_i) \quad \text{for } t \in (-\varepsilon_0, \varepsilon_0) \text{ and } i = 1, \dots, m\,,$$

where Φ is the flow associated to X.

We then give the definition of *critical* and *stable* sets.

Definition 4.3. We say that a smooth set $F \subseteq \Omega$, with connected components $F_1, \dots, F_m$, is critical for the functional $\mathcal{J}$ if there exist constants $\lambda_1, \dots, \lambda_m$ such that

$$\mathrm{H} - \frac{1}{2}\mathbb{C}E(u_F) \cdot E(u_F) = \lambda_i \qquad \text{on } \Gamma_{F,i}$$

for every $i = 1, \dots, m$.

Note that if F is a smooth (local) minimizer of $\mathcal{J}$ under a volume constraint, then there exists a constant λ such that

$$\mathrm{H} - \frac{1}{2}\mathbb{C}E(u_F) \cdot E(u_F) = \lambda \qquad \text{on } \partial F\,,$$

which is a stronger condition than the criticality one above.

When F is critical, the formula for the second variation in Proposition (4.1) reduces to

$$\begin{aligned}\widetilde{\Pi}_F(\varphi) &= \frac{d^2}{dt^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0}\\ &= \int_{\partial F} |\nabla\varphi|^2 - |B|^2\varphi^2\, d\mu - \int_{\Omega\setminus F} \mathbb{C}E(u_\varphi)\cdot E(u_F)\, dx\\ &\quad - \frac{1}{2}\int_{\partial F} \langle \nabla(\mathbb{C}E(u_F)\cdot E(u_F)\varphi^2), \nu_F\rangle\, d\mu,\end{aligned}$$

where $\varphi = \langle X, \nu_F\rangle$ on ∂F.

Definition 4.4. Let $F \subseteq \Omega$ be a smooth critical set. We say that F is stable if

$$\widetilde{\Pi}_F(\psi) \geq 0 \qquad \text{for all } \psi \in \widetilde{H}^1(\partial F)$$

and it is strictly stable if

$$\widetilde{\Pi}_F(\psi) > 0 \qquad \text{for all } \psi \in \widetilde{H}^1(\partial F) \setminus \{0\}\,,$$

where $\widetilde{H}^1(\partial F)$ is defined as in formula (1.30).

Definition 4.5 (Surface diffusion flow with elasticity)**.** Let $F \subseteq \Omega$ be a smooth set. We say that a smooth flow F_t for $t \in [0, T)$, with $F_0 = F$, is a *surface diffusion flow with elasticity* starting from F, if the outer normal velocity V_t of the moving boundaries ∂F_t, defined as in formula (2.1), is given by

$$V_t = \Delta_t\Big(\mathrm{H} - \frac{1}{2}\mathbb{C}E(u_{F_t}) \cdot E(u_{F_t})\Big) \qquad \text{for all } t \in [0, T)\,. \tag{4.2}$$

By the work in [10], it is possible to prove a short time existence and uniqueness of a solution starting from any smooth set. The strategy proposed by Fusco, Julin e Morini is based on the idea of thinking the elastic contribution as a "forcing term" and using a fixed point argument in a suitably chosen function space.

To conclude, we analyze the behavior of the flow when the initial set is close to a smooth strictly stable critical set G.

Theorem 4.6 ([10, Theorem 5.1])**.** *Let $G \subseteq \Omega$ be a smooth strictly stable critical set, in the sense of Definition 4.4. There exists $M > 0$ such that if $F_0 \in \mathfrak{C}^3_M(G)$, then the unique solution F_t of the flow* (4.2) *starting from F_0 is defined for all times $t > 0$.*
Moreover $F_t \to F_\infty$ exponentially fast, where F_∞ is the unique smooth critical set "close" to G such that $\mathrm{Vol}(F_{\infty,i}) = \mathrm{Vol}(F_{0,i})$ for $i = 1, \dots, m$. In particular, if $\mathrm{Vol}(F_{0,i}) = \mathrm{Vol}(G_i)$ for $i = 1, \dots, m$, then $F_t \to G$ exponentially fast ($G_i, F_{0,i}, F_{\infty,i}$ denote the connected components respectively of G, F_0, F_∞).

In order to show this asymptotic exponential stability result, it is enough to adapt to this situation the methods used for the surface diffusion flow. The rough idea is to look at the asymptotic behavior of the quantity

$$\int_{\partial F_t} \left|\nabla\Big(\mathrm{H} - \frac{1}{2}\mathbb{C}E(u_{F_t}) \cdot E(u_{F_t})\Big)\right|^2 d\mu_t \,,$$

and to show that it is decreasing and vanishes exponentially fast, as $t \to +\infty$. A crucial role in this analysis is played by energy identities (similar to the identities proven in Lemma 2.6) and by the estimates of the k–order derivatives of the solution, depending only on the initial set.

4.2 THE MODIFIED MULLINS–SEKERKA FLOW

Another interesting problem related to our work is the study of the *modified Mullins–Sekerka flow* in the three–dimensional flat torus, also carried out in [1].

We introduce the following *nonlocal Area functional* J, also known as *sharp–interface Ohta–Kawasaki energy*, which was first proposed in [24] to describe phase separation in diblock copolymer melts.

Let $E \subseteq \mathbb{T}^n$ be a smooth set, we consider the associated potential

$$v_E(x) = \int_{\mathbb{T}^n} G(x, y) u_E(y) dy \,, \tag{4.3}$$

where G is the Green function (of the Laplacian) of the torus $\mathbb{T}^n$ and $u_E = \chi_E - \chi_{\mathbb{T}^n \setminus E}$.
Given $\gamma \geq 0$, we define the (volume–constrained) functional

$$J(E) = \mathcal{A}(\partial E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E(x)|^2 \, dx \,, \tag{4.4}$$

which has the Euler–Lagrange equation

$$\mathrm{H} + 4\gamma v_E = \lambda \qquad \text{on } \partial E$$

for a constant $\lambda \in \mathbb{R}$.
We observe that for this energy functional there hold the same definitions

and results that we proved in Chapter 1 for the surface diffusion flow. In particular, we can show, following [2], that any smooth critical set for J with positive second variation (that is, *strictly stable* in an analogous sense to Definition (1.9)) is a $W^{2,p}$–local minimizer, for all $p > 2$.

Then, to any smooth set $E \subseteq \mathbb{T}^n$ we associate the function w_E which is the unique solution in $H^1(\mathbb{T}^n)$ of the problem

$$\begin{cases} \Delta w_E = 0 & \text{in } \mathbb{T}^n \setminus \partial E \\ w_E = \mathrm{H} + 4\gamma v_E & \text{on } \partial E \end{cases}$$

where v_E is the potential introduced in formula (4.3) and we denote by w_E^+ and w_E^- the restrictions $w_E|_{\mathbb{T}^n\setminus E}$ and $w_E|_E$, and we set

$$[\partial_{\nu_E} w_E] = \partial_{\nu_E} w_E^+ - \partial_{\nu_E} w_E^- = -(\partial_{\nu_{\mathbb{T}^e\setminus E}} w_E^+ + \partial_{\nu_E} w_E^-)\,,$$

that is, $[\partial_{\nu_E} w_E]$ is the *jump* of the normal derivative of w_E on ∂E.

Definition 4.7 (Modified Mullins–Sekerka flow)**.** Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that a smooth flow E_t, with $E_0 = E$, is a *modified Mullins–Sekerka flow (with parameter $\gamma \geq 0$)* on the interval $[0, T)$ with initial datum E, if the outer normal velocity V_t of the moving boundaries ∂E_t is given by

$$V_t = [\partial_{\nu_{E_t}} w_{E_t}] \quad \text{on } \partial E_t \text{ for all } t \in [0, T)\,.$$

Notice that the adjective “modified” is due to the parameter γ, when it is positive. Indeed, if $\gamma = 0$ the potential v_E becomes irrelevant, the functional J becomes the Area functional and we recover the “classical” Mullins–Sekerka flow, which was studied in [22] (it can be shown that this latter can be regarded as the $H^{-1/2}$–gradient flow of the Area functional under the constraint that the volume is fixed, see [18]).

Also for this situation, analogously to the surface diffusion flow, a short time and uniqueness result was established by Escher and Nishiura in [7] and, in the three–dimensional case, following [1], it is possible to show that if the initial set $E \subseteq \mathbb{T}^3$ is sufficiently close to a smooth strictly stable critical set F, then the modified Mullins–Sekerka flow, starting from E, exists for all time and converge exponentially fast to a translate of F.

4.3 THE CLASSIFICATION OF THE STABLE CRITICAL SETS

We discuss now a while the class of initial sets to which Theorem 3.1 can be applied, hence, “dynamically exponentially stable” for the surface diffusion flow. In the three–dimensional case, the smooth stable “periodic” critical sets are classified, a first description was given by Ros in [26], where it is shown that in a three–dimensional flat torus $\mathbb{T}^3$, for the volume–constrained Area functional, they are *balls*, 2*–tori*, *gyroids* or *lamellae*.

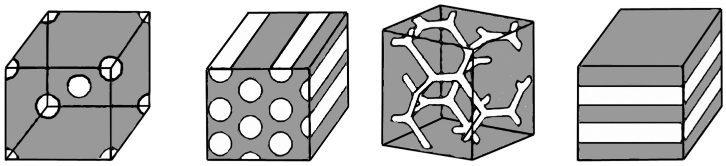

Figure 1: From left to right: balls, cylinders, gyroids and lamellae.

Notice that the lamellae are a finite collection of parallel planar 2–tori, where 2–tori are simply a quotient of a circular cylinders. The surfaces in the first three classes are actually strictly stable, hence, it is possible to apply the aforementioned theorem. While in [14, 15, 27] the authors establish the strict stability of gyroids only in some cases. To give an example, we refer to [15] where Grosse–Brauckmann and Wohlgemuth showed the strictly stability of the gyroids that are fixed with respect to translations. We remind that the gyroids, that were discovered by the crystallographer Schoen in the 1970 (see [28]), are the unique non–trivial embedded members of the family of the Schwarz P surfaces and then conjugate to the D surfaces, that are the simplest and most well–known triply–periodic minimal surfaces (see [27]).

It is worth mentioning, without going into detail, that instead, for the functional (4.4) a complete classification of the stable periodic structures is still missing.

4.4 POSSIBLE FUTURE RESEARCH DIRECTIONS

A natural continuation of the research presented in the thesis is trying to generalize the results to dimension $n > 3$, which is at the moment an open, absolutely not easy problem. Another challenging research line (actually, relevant for physics) is modeling the evolution of epitaxially strained elastic films, that is, the growth of a thin layer on the surface of a crystal so that the layer has the same structure as the underlying crystal. The proposed physical models are driven by laws similar to the surface diffusion flow with elasticity, seen above, with extra (regularizing) curvature terms (as in [9]). In order to give an example, we define the following energy functional,

$$\mathcal{J}_\varepsilon(F) = \mathcal{A}(\partial F) + \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F)\cdot E(u_F)\,dx + \frac{\varepsilon}{p}\int_{\partial F} |\mathrm{H}|^p\,d\mu$$

where ε is a positive parameter and $p > 2$, which is a “singular perturbation” of the energy defined in formula (4.1) and we consider the associated (gradient) evolution law

$$V_t = \Delta_t\Big(\mathrm{H} - \frac{1}{2}\mathbb{C}E(u_{F_t})\cdot E(u_{F_t})\Big) - \varepsilon\Delta_t\Delta_t(|\mathrm{H}|^{p-2}\mathrm{H}) + \mathrm{LOT}$$

giving the motion of the boundary ∂F_t (LOT stands for “lower order terms”).

This problem was studied in dimensions two and three by Fonseca, Fusco, Leoni and Morini in [8] and [9], respectively. We aim to study such flows in higher dimensions, possibly with different singular perturbations, for instance, of higher order as

$$\mathcal{J}_\varepsilon(F) = \mathcal{A}(\partial F) + \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F)\cdot E(u_F)\,dx + \varepsilon\int_{\partial F} |\nabla^m B|^2\,d\mu$$

and also to investigate what happens when the perturbation "goes to zero", trying to show that the associated perturbed gradient flows "converge" in some sense to the original one (surface diffusion flow with elasticity), as the perturbation term gets smaller and smaller.

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