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Long–time behavior results for the modified Mullins–Sekerka flow

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INTRODUCTION

Geometric evolution problems for surfaces are a fascinating topic arising from the study of models in physics and material sciences, usually used in descriptions of phase changes or flows of fluids.
In this work, following [1], we study one of the most recent of such geometric motions, namely, the *modified Mullins–Sekerka flow*. Precisely, we say that a flow of open sets with smooth boundary E_t, contained in an open set $\Omega \subseteq \mathbb{R}^n$ and with $d(E_t, \partial\Omega) > 0$ for every t in a time interval $[0, T)$, is a solution of the modified Mullins–Sekerka flow with parameter $\gamma \geq 0$, if there exists a pair of continuous functions $v, w : [0, T) \times \overline{\Omega} \to \mathbb{R}$ such that the following "mixed" system is satisfied (distributionally),

$$\begin{cases} V_t = [\partial_{\nu_t} w_t] & \text{on } \partial E_t, \\ \Delta w_t = 0 & \text{in } \Omega \setminus \partial E_t, \\ w_t = \mathrm{H}_t + 4\gamma v_t & \text{on } \partial E_t, \\ -\Delta v_t = u_{E_t} - \fint_\Omega u_{E_t}\, dx & \text{in } \Omega, \end{cases} \tag{1}$$

where $w_t = w(t, \cdot)$ and $v_t = v(t, \cdot)$, the functions ν_t, H_t, V_t are respectively, the "outer" normal, the mean curvature and the outer normal velocity of the moving boundary ∂E_t. We set $u_{E_t} = 2\chi_{E_t} - 1$ and $[\partial_{\nu_t} w_t]$ is a notation for the "jump" of the normal derivative of w_t on ∂E_t, that is $\partial_{\nu_t} w_t^+ - \partial_{\nu_t} w_t^-$, with w_t^+ and w_t^- denoting the restrictions of w_t to $\Omega \setminus \overline{E}_t$ and E_t, respectively.
We mention that the adjective "modified" comes from the introduction of the parameter $\gamma \geq 0$ in the system (1), while choosing $\gamma = 0$, we have the original flow proposed by Mullins and Sekerka in [30].
It follows that defining $M_t = \partial E_t$, which is a family of smooth hypersurfaces embedded in Ω, we can always describe the evolution, locally in space and time (and globally if the sets E_t have compact closure, see [26]), via some embedding maps $\varphi_t : M \to \Omega$ such that $\varphi_t(M) = M_t$, satisfying the evolution equation

$$\frac{\partial}{\partial t}\varphi_t = V_t \nu_t = [\partial_{\nu_t} w_t]\nu_t,$$

where M is a fixed smooth $(n-1)$–dimensional differentiable manifold.

As it is, system (1) is clearly undetermined, as the behavior of the functions w_t and v_t is not prescribed on the boundary of Ω (and this latter is possibly not bounded). One possibility to get a well–posed problem (leading to a satisfactory short time existence and uniqueness result for the flow starting with any smooth initial set, see [12]) which is actually a parabolic system of PDEs, with the above parametrization of the evolving surfaces, is to ask that Ω is bounded and that all the functions w_t and v_t are subject to homogeneous (zero) Neumann boundary conditions on $\partial\Omega$. Another possibility, which is the one we are going to discuss in our thesis, is to assume that $\Omega = \mathbb{R}^n$ (hence $\partial\Omega = \emptyset$) and that all the functions and sets involved are periodic with respect

to the standard lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$. In this case the analysis is clearly equivalent to "ambient" the problem in the n–dimensional "flat" torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (that is, $\Omega = \mathbb{T}^n$ in system (1)), making it well–posed.
We will focus on this setting, that we call the "periodic" case and only in dimension three. Anyway, all the results that we will present can be proved also in the "Neumann" three–dimensional case (see Section 4.1). Moreover, it can be shown that in the two–dimensional case the same conclusions hold analogously, while on the opposite, when the ambient dimension is higher than three, several questions are still open, up to our knowledge. By sake of completeness, in the final chapter, we will briefly discuss the "Neumann" setting and we will state the analogues of the main results in the periodic case.

The Mullins–Sekerka model has been largely studied over the last years for its importance in the analysis of pattern–forming processes such as the solidification in pure liquids. In particular, it arises in [25] as a limit of a non-local version of the Cahn–Hilliard equation, a fourth order partial differential equation proposed to describe phase separation in diblock copolymer melts (see [33]).

It is easy to see that the solutions of problem (1) evolve in such a way that the volume of the sets E_t is preserved (while it has been shown in [10] that convexity is not necessarily maintained, in contrast with the more famous *mean curvature flow*, see [26], for instance). This property is not unexpected as the modified Mullins–Sekerka flow is the $H^{-1/2}$–gradient flow (with a suitable norm on $H^{-1/2}(\partial E)$) of the following "nonlocal Area functional"

$$J(E) = \mathcal{A}_{\mathbb{T}^n}(\partial E) + \gamma \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} G(x,y) u_E(x) u_E(y)\, dx\, dy\,,$$

under the constraint that the volume $\mathrm{Vol}(E) = \mathscr{L}^n(E)$ is fixed, where

$$\mathcal{A}_{\mathbb{T}^n}(\partial E) = \int_{\partial E} d\mu$$

is the *Area functional* on the boundary hypersurface of the subsets of $\mathbb{T}^n$ (μ is the "canonical" measure associated to the Riemannian metric on ∂E, induced by the scalar product of $\mathbb{R}^n$, which coincides with the $(n-1)$–dimensional Hausdorff measure $\mathcal{H}^{n-1}$) and G is the Green function of $\mathbb{T}^n$ (see [25], for details). This means that the velocity V_t is minus the gradient of the functional J, hence the quantity $J(E_t)$ can be regarded as a natural "energy", decresing in time during the evolution.

We remark that Escher and Nishiura established in [12] a short time existence and uniqueness result for every smooth initial set $E_0 \subseteq \mathbb{T}^n$, consequently, the flow E_t exists in some time interval $[0,T)$. The purpose of this work is to show, following Acerbi, Fusco, Julin and Morini in [1], that in dimension two and three, for initial data sufficiently close to a smooth "strictly stable critical" set E for J (under a volume–constraint), the flow exists for all positive times and asymptotically "converges" exponentially fast to a "translate" of E. This result is clearly suggested by the above property of the motion of being a gradient flow.

The suitable notions of criticality and stability mentioned above can be defined in terms of the first and second variation of J. Precisely, we say that a smooth subset $E \subseteq \mathbb{T}^3$ is *critical* for J if for any smooth one–parameter family of diffeomorphisms Φ_t, such that $\mathrm{Vol}(\Phi_t(E)) = \mathrm{Vol}(E)$ and $\Phi_0 = \mathrm{Id}$, we have

$$\frac{d}{dt} J(\Phi_t(E))\Big|_{t=0} = 0.$$

We will see that this condition is equivalent to the existence of constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H} + 4\gamma v_E = \lambda \qquad \text{on } \partial E,$$

where H is the mean curvature of ∂E and v_E is the potential defined as

$$v_E(x) = \int_{\mathbb{T}^n} G(x,y) u_E(y)\, dy\,,$$

with G the Green function of the torus $\mathbb{T}^n$ and $u_E = \chi_E - \chi_{\mathbb{T}^n \setminus E}$.
The central notion of *stability* can be stated for the functional J by studying its second variation that we will compute in great detail in a more geometric "spirit" than in the papers in literature. Our first goal, at the end of Chapter 1 will be to describe its connection (which is quite involved) with the behavior of the nonlocal Area functional (under a volume–constraint) close to a smooth critical set. In particular, we will see that at a critical set E, it only depends on the normal component φ on ∂E of the infinitesimal generator of the family of diffeomorphisms $\Phi_t : \mathbb{T}^3 \to \mathbb{T}^3$, deforming E keeping its volume constant. This volume constraint on the "admissible" deformations of E implies that the functions φ must have zero integral on ∂E, hence it is natural to define a quadratic form Π_E on such space of functions which is related to the second variation of J by the following equality

$$\Pi_E(\varphi) = \frac{d^2}{dt^2} J(\partial \Phi_t(E))\Big|_{t=0} \tag{2}$$

where $\Phi_t : \mathbb{T}^n \to \mathbb{T}^n$ is a one–parameter family of diffeomorphisms satisfying $\mathrm{Vol}(\Phi_t(E)) = \mathrm{Vol}(E)$,

$$\Phi_0 = \mathrm{Id} \qquad \text{and} \qquad \frac{\partial \Phi_t}{\partial t}\Big|_{t=0} = \varphi \nu_E \quad \text{on } \partial E,$$

with ν_E the outer unit normal vector of ∂E.
Because of the *translation invariance* of the functional J, it is easy to see (by means of the formula (2)) that the form Π_E vanishes on the finite dimensional vector space given by the functions $\psi = \langle \eta, \nu_E \rangle$, for every vector $\eta \in \mathbb{R}^n$. Indeed every $\psi = \langle \eta, \nu_E \rangle$ is the normal component on ∂E of the infinitesimal generator of the family of diffeomorphisms of $\mathbb{T}^n$ which simply translate any point by the vector $t\eta$. We then say that a smooth critical set $E \subseteq \mathbb{T}^n$ is *strictly stable* if

$$\Pi_E(\varphi) > 0$$

for all non–zero functions $\varphi : \partial E \to \mathbb{R}$, with zero integral and L^2–orthogonal to every function $\psi = \langle \eta, \nu_E \rangle$.

We underline that the presence of such "natural" degenerate space of the quadratic form Π_E (or, equivalently, the translation invariance of J) is the main reason of several technical difficulties in the thesis.

In order to connect this notion to the local behavior of J around a smooth set $F \subseteq \mathbb{T}^n$, we will say that the set E is "$W^{2,p}$–close" to F, if for a $\delta > 0$ "small enough" we have $\mathrm{Vol}(E \triangle F) < \delta$, its boundary ∂E is contained in a suitable *tubular neighborhood* of ∂F and it can be described as

$$\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\}$$

for some smooth function $\psi : \partial F \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$. That is, the boundary of E is represented as the "normal graph" on ∂F of the function ψ, which is clearly a very useful way to transform the problem on *sets* into a problem on *functions*. As we said above, in the last section of Chapter 1, we will show the result in [2] that any smooth strictly stable critical set $E \subseteq \mathbb{T}^n$ is a local minimizer of J under volume constraint ("isolated" up to translation), among all smooth $W^{2,p}$–close sets $F \subseteq \mathbb{T}^n$, if $p > \max\{2, n-1\}$.

The main purpose of this work is to show that a *strictly stable* critical set is "asymptotically stable". Heuristically, one can think of a system with a "potential well", in which the strictly stable set plays the role of the stable equilibrium configuration (local minimum of the potential energy). Then, starting close to the stable set, the solutions move back to the equilibrium (asymptotically). Precisely, we will show the following main result, proved in [1]. A challenging open problem is generalizing it to arbitrary dimension, together with establishing a classification of the "periodic" strictly stable smooth critical sets.

Theorem. *Let $E \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε a suitable tubular neighborhood of E. For every $\alpha \in (0, 1/2)$ there exists $M > 0$ such that, if E_0 is a smooth set satisfying*

- $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$,
- $\mathrm{Vol}(E_0 \triangle E) \le M$,
- *the boundary of E_0 is contained in N_ε and can be represented as*
$$\partial E_0 = \{y + \psi_{E_0}(y)\nu_E(y) \,:\, y \in \partial E\},$$
for some function $\psi_{E_0} : \partial E \to \mathbb{R}$ such that $\|\psi_{E_0}\|_{C^{1,\alpha}(\partial E)} \le M$,
- $$\int_{\mathbb{T}^3} |\nabla w_{E_0}|^2 \, dx \le M$$
where $w_0 = w_{E_0}$ is the function relative to E_0, as in system (1),

then, there exists a unique smooth solution E_t of the modified Mullins–Sekerka flow (with parameter $\gamma \ge 0$) starting from E_0, which is defined for all $t \ge 0$. Moreover, $E_t \to E + \eta$ exponentially fast in $W^{5/2,2}$ as $t \to +\infty$, for some $\eta \in \mathbb{R}^3$, with the meaning that the functions $\psi_{\eta,t} : \partial E + \eta \to \mathbb{R}$ representing ∂E_t as "normal graphs" on $\partial E + \eta$, that is,

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{E+\eta}(y) \,:\, y \in \partial E + \eta\},$$

satisfy

$$\|\psi_{\eta,t}\|_{W^{5/2,2}(\partial E+\eta)} \leq Ce^{-\beta t},$$

for every $t \in [0, +\infty)$*, for some positive constants* C *and* β*.*

We remark that the line of proof in [1] that we are going to present, based on suitable energy identities and compactness arguments to establish this global existence and exponential stability result, is a new approach to manage the translation invariance of the functional J, in literature usually dealt with by means of semigroup techniques.

For the case $\gamma = 0$, a classification of the stable critical sets has been established in [39], they are *lamellae*, *balls*, 2*–tori* or *gyroids*, hence this theorem can be applied to all of them. In particular, all the lamellae, balls and 2–tori are actually strictly stable. On the contrary, for the case $\gamma > 0$, the "lamellar sets" are strictly stable if the number of interfaces is larger than some minimum value $k(\gamma)$ (see [2, 6]), however, a complete classification in this case is still missing.

Our work is organized in the following way: in Chapter 1 we study the nonlocal Area functional J and we compute its first and second variation, then we prove a sufficient condition for the local minimality with respect to $W^{2,p}$–perturbations. In Chapter 2 we introduce the modified Mullins–Sekerka flow and we show its basic properties, leading to the main theorem presented in Chapter 3. Finally, in the fourth and last chapter, we describe the "Neumann" case and discuss the classification of the (strictly) stable smooth critical sets, concluding with the connection of the nonlocal Area functional J with the so–called Ohta–Kawasaki functional and with a brief overview of some possible future work.

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1 THE NONLOCAL AREA FUNCTIONAL

In this chapter we describe the *nonlocal Area functional* and its basic properties. In particular, our main purpose will be to show a sufficient condition for the $W^{2,p}$–local minimality.

1.1 NOTATIONS AND PRELIMINARIES

In the following we denote by $\mathbb{T}^n$ the n–dimensional flat torus of unit volume which is defined as the quotient of $\mathbb{R}^n$ with respect to the equivalence relation $x \sim y \iff x - y \in \mathbb{Z}^n$ with $\mathbb{Z}^n$ the standard integer lattice of $\mathbb{R}^n$. Then, the functional space $W^{k,p}(\mathbb{T}^n)$, with $k \in \mathbb{N}$ and $p \geq 1$, can be identified with the subspace of $W^{k,p}_{loc}(\mathbb{R}^n)$ of functions that are one–periodic with respect to all coordinate directions. Similarly, $C^{k,\alpha}(\mathbb{T}^n)$, with $\alpha \in (0,1)$, denotes the space of one–periodic functions in $C^{k,\alpha}(\mathbb{R}^n)$. Finally, a set $E \subseteq \mathbb{T}^n$ is of class C^k (or smooth) if its "one–periodic extension" to $\mathbb{R}^n$ is of class C^k (or smooth,) which means that its boundary is locally a graph of a function of class C^k around every point. We will denote with $\mathrm{Vol}(E) = \mathscr{L}^n(E)$ the volume of a set $E \subseteq \mathbb{T}^n$.

Given a smooth set $E \subseteq \mathbb{T}^n$, we consider the associated potential

$$v_E(x) = \int_{\mathbb{T}^n} G(x,y) u_E(y) dy\,, \tag{1.1}$$

where G is the Green function (of the Laplacian) of the torus $\mathbb{T}^n$ and $u_E = \chi_E - \chi_{\mathbb{T}^n \setminus E}$. Precisely, G is the (distributional) solution of

$$-\Delta_x G(x,y) = \delta_y - 1 \quad \text{in } \mathbb{T}^n, \quad \text{with} \int_{\mathbb{T}^n} G(x,y)\, dx = 0, \tag{1.2}$$

for every $y \in \mathbb{T}^n$, where δ_y denotes the Dirac delta measure at $y \in \mathbb{T}^n$ (the n–torus $\mathbb{T}^n$ has unit volume).

By the properties of the Green function, v_E is the unique solution of

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } \mathbb{T}^n \\ \displaystyle\int_{\mathbb{T}^n} v_E(x)\, dx = 0 \end{cases} \tag{1.3}$$

where $m = \mathrm{Vol}(E) - \mathrm{Vol}(\mathbb{T}^n \setminus E) = 2\mathrm{Vol}(E) - 1$.

Remark 1.1. By standard elliptic regularity arguments (see [13], for instance), $v_E \in W^{2,p}(\mathbb{T}^n)$ for all $p \in [1, +\infty)$. More precisely, there exists a constant $C = C(n,p)$ such that $\|v_E\|_{W^{2,p}(\mathbb{T}^n)} \leq C$, for all $E \subseteq \mathbb{T}^n$ such that $\mathrm{Vol}(E) - \mathrm{Vol}(\mathbb{T}^n \setminus E) = m$.

We can now define the *nonlocal Area functional* (see [24, 31, 43], for instance).

Definition 1.2 (Nonlocal Area functional)**.** Given $\gamma \geq 0$, the *nonlocal Area functional* J is defined as

$$J(E) = \mathcal{A}_{\mathbb{T}^n}(\partial E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E(x)|^2 \, dx, \tag{1.4}$$

for every smooth set $E \subseteq \mathbb{T}^n$, where the function $v_E : \mathbb{T}^n \to \mathbb{R}$ is defined by formulas (1.1)–(1.3) and

$$\mathcal{A}_{\mathbb{T}^n}(\partial E) = \int_{\partial E} d\mu$$

is the *Area functional* (μ is the "canonical" measure associated to the Riemannian metric on ∂E, induced by the scalar product of $\mathbb{R}^n$, which coincides with the Hausdorff $(n-1)$–dimensional measure on ∂E).

By the properties of the potential function v_E defined by relations (1.3) and integrating by parts, we obtain the following equalities

$$\begin{aligned}
\int_{\mathbb{T}^n} |\nabla v_E(x)|^2 \, dx &= -\int_{\mathbb{T}^n} v_E(x) \Delta v_E(x) \, dx \\
&= \int_{\mathbb{T}^n} v_E(x)(u_E(x) - m) \, dx \\
&= \int_{\mathbb{T}^n} v_E(x) u_E(x) \, dx \\
&= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} G(x,y) u_E(x) u_E(y) \, dx dy,
\end{aligned} \tag{1.5}$$

hence, the functional J can be also written in the following way,

$$J(E) = \mathcal{A}_{\mathbb{T}^n}(\partial E) + \gamma \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} G(x,y) u_E(x) u_E(y) \, dx dy.$$

1.2 FIRST AND SECOND VARIATION OF THE NONLOCAL AREA FUNCTIONAL

We want to compute the *first variation* of the functional J with respect to volume–preserving variations.

Definition 1.3. Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that a vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ is *admissible* for E if the associated smooth flow $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$, defined by the system

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,x) = X(\Phi(t,x)), \\ \Phi(0,x) = x \end{cases}$$

for every $x \in \mathbb{T}^n$ and $t \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, satisfies

$$\mathrm{Vol}(E_t) = \mathrm{Vol}(E) \qquad \text{for all } t \in (-\varepsilon, \varepsilon),$$

where we set $E_t = \Phi(t, E)$.

We immediately see some properties of admissible fields X that we will need in the following.

Since $\mathrm{Vol}(E_t) = \mathrm{Vol}(\Phi(t,E)) = \mathrm{Vol}(E)$ for all $t \in (-\varepsilon, \varepsilon)$, denoting with $J\Phi(t,\cdot)$ the Jacobian of $\Phi(t,\cdot)$, by changing variables, we have

$$0 = \frac{d}{dt}\mathrm{Vol}(E_t) = \frac{d}{dt}\int_{E_t} dx = \frac{d}{dt}\int_E J\Phi(t,z)\,dz = \int_E \frac{\partial}{\partial t} J\Phi(t,z)\,dz. \tag{1.6}$$

As $J\Phi(t,z) = \det[d\Phi(t,z)]$, by means of the formula

$$\frac{d}{dt}\det A(t) = \det A(t)\,\mathrm{tr}\,[A^{-1}(t)\circ A'(t)], \tag{1.7}$$

holding for any $n\times n$ squared matrix $A(t)$ dependent on t, we obtain

$$\frac{\partial}{\partial t}J\Phi(t,z) = J\Phi(t,z)\,\mathrm{tr}\,[d\Phi(t,z)^{-1}\circ dX(\Phi(t,z))\circ d\Phi(t,z)],$$

since, by definition of Φ, we have

$$\frac{\partial}{\partial t}[d\Phi(t,z)] = d\Big(\frac{\partial}{\partial t}\Phi(t,z)\Big) = d\big[X(\Phi(t,z))\big] = dX(\Phi(t,z))\circ d\Phi(t,z).$$

Being the trace of a matrix invariant by conjugation, we conclude

$$\frac{\partial}{\partial t}J\Phi(t,z) = J\Phi(t,z))\,\mathrm{tr}\,[dX(\Phi(t,z))] = J\Phi(t,z))\,\mathrm{div}X(\Phi(t,z)), \tag{1.8}$$

hence, by equality (1.6) and the divergence theorem, it follows

$$\begin{aligned}0 = \frac{d}{dt}\mathrm{Vol}(E_t) &= \int_E \mathrm{div}X(\Phi(t,z))J\Phi(t,z)\,dz = \int_{E_t}\mathrm{div}X(x)\,dx\\ &= \int_{\partial E_t}\langle X|\nu_{E_t}\rangle\,d\mu_t\,,\end{aligned} \tag{1.9}$$

where ν_{E_t} is the *outer unit normal vector* and μ_t the canonical Riemannian measure of the smooth hypersurface ∂E_t.

Hence, if $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ is admissible for E, letting $t=0$, we have that the normal component $\varphi = \langle X|\nu_E\rangle$ of X has zero integral on ∂E. Conversely, we have the following lemma whose proof is postponed after Lemma 1.22, since the arguments are very similar.

Lemma 1.4. *Let $\varphi : \partial E \to \mathbb{R}$ a C^∞ function with zero integral. Then, there exists an admissible vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ such that $\varphi = \langle X|\nu_E\rangle$.*

We now also compute the expression of the second derivative of the volume of E_t. By means of the previous computations, we have

$$\begin{aligned}0 = \frac{d^2}{dt^2}\mathrm{Vol}(E_t) &= \frac{d}{dt}\int_{E_t}\mathrm{div}X\,dx\\ &= \frac{d}{dt}\int_E \mathrm{div}X(\Phi(t,z))J\Phi(t,z)\,dz\\ &= \int_E\Big[\langle\nabla\,\mathrm{div}X(\Phi(t,z))|X(\Phi(t,z))\rangle + (\mathrm{div}X(\Phi(t,z))^2\Big]J\Phi(t,z)\,dz\\ &= \int_{E_t}\Big[\langle\nabla\,\mathrm{div}X|X\rangle + (\mathrm{div}X)^2\Big]\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{E_t} \operatorname{div}[(\operatorname{div} X)X]\,dx \\
&= \int_{\partial E_t} \langle X | \nu_{E_t} \rangle \operatorname{div}^{\mathbb{T}^n} X \, d\mu_t \,. \qquad (1.10)
\end{aligned}
$$

By letting $t = 0$, it follows that for every vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ admissible for E, there holds

$$
\int_{\partial E} \langle X | \nu_E \rangle \operatorname{div}^{\mathbb{T}^n} X \, d\mu = 0, \qquad (1.11)
$$

where we denoted with $\operatorname{div}^{\mathbb{T}^n}$ the (standard) divergence operator in $\mathbb{T}^n$ (which is locally $\mathbb{R}^n$), in order to distinguish it by the (Riemannian) divergence operator div on the hypersurface ∂E (see Appendix A).

Given any vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ (admissible or not) with associated flow Φ as in Definition 1.3, the *first variation of J at E with respect to* Φ is then given by

$$
\frac{d}{dt} J(E_t) \Big|_{t=0} \,.
$$

We are going to compute the first variation of the nonlocal Area functional for a "general" (non necessarily volume–preserving) flow Φ and we will see that it depends only on the values of its infinitesimal generator X on ∂E, then we will restrict ourselves only to admissible vector fields X.

Since we will compute the first and the second variation of the Area functional using "geometric" notations and techniques, we refer to Appendix A for basic facts about the (Riemannian) geometry of hypersurfaces in $\mathbb{R}^n$.

In the whole thesis, we will adopt the convention of summing over the repeated indices. Moreover, when it is clear by the contest, we will write ∇ and div *for both the (Riemannian) gradient/divergence operators on a hypersurface and the standard gradient/divergence in $\mathbb{T}^n$, but these latter will be instead denoted with $\nabla^{\mathbb{T}^n}$ and* $\operatorname{div}^{\mathbb{T}^n}$ *when they will be computed at a point of a hypersurface, in order to avoid any possibility of misunderstanding. Finally, in all the estimates of the thesis, the constants C may vary from a line to another.*

Given any smooth immersion of the smooth hypersurface ∂E, boundary of a smooth set E,

$$
\psi : \partial E \to \mathbb{R}^n
$$

we can write the Area functional in the following way, using local charts (abusing a little the notation)

$$
\mathcal{A}_{\mathbb{T}^n}(\psi(\partial E)) = \int_{\partial E} d\mu = \int_{\partial E} \sqrt{\det g_{ij}(x)}\, dx
$$

where

$$
g_{ij} = \left\langle \frac{\partial \psi}{\partial x_i} \,\Big|\, \frac{\partial \psi}{\partial x_j} \right\rangle
$$

is the pull–back metric on ∂E via the map ψ.

Theorem 1.5 (First variation of J)**.** *Let $E \subseteq \mathbb{T}^n$ a smooth set and $\Phi : (-\varepsilon, \varepsilon) \times \mathbb{T}^n \to \mathbb{T}^n$ a smooth flow generated by a vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$. Then,*

$$\frac{d}{dt} J(E_t)\Big|_{t=0} = \frac{d}{dt} J(\Phi(t,E))\Big|_{t=0} = \int_{\partial E} (\mathrm{H} + 4\gamma v_E)\langle X | \nu_E\rangle \, d\mu$$

where ν_E is the outer unit normal vector *and* H *denotes the* mean curvature *of the boundary ∂E (i.e. the sum of the principal curvatures of ∂E), while the function $v_E : \mathbb{T}^n \to \mathbb{R}$ is the potential associated to E, defined by formulas* (1.1)–(1.3)*.*

Proof. We start by computing the derivative of the Area functional term of J (see [26], for instance). We let $\psi_t : \partial E \to \mathbb{T}^n$ given by

$$\psi_t(x) = \Phi(t, x)\,,$$

for $x \in \partial E$ and $t \in (-\varepsilon, \varepsilon)$, then $\psi_t(\partial E) = \partial E_t$ and $\partial_t \psi_t\big|_{t=0} = X$ at every point of ∂E.

Denoting with $g_{ij} = g_{ij}(t)$ the induced metrics (via ψ_t, as above) on the smooth hypersurfaces ∂E_t and setting $\psi_0 = \psi = \mathrm{Id}$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \psi_t}{\partial x_i} \,\Big|\, \frac{\partial \psi_t}{\partial x_j}\right\rangle\Big|_{t=0} \\
&= \left\langle \frac{\partial X}{\partial x_i} \,\Big|\, \frac{\partial \psi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j} \,\Big|\, \frac{\partial \psi}{\partial x_i}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X \,\Big|\, \frac{\partial \psi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X \,\Big|\, \frac{\partial \psi}{\partial x_i}\right\rangle - 2\left\langle X \,\Big|\, \frac{\partial^2 \psi}{\partial x_i \partial x_j}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X_\tau \,\Big|\, \frac{\partial \psi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X_\tau \,\Big|\, \frac{\partial \psi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X_\tau \,\Big|\, \frac{\partial \psi}{\partial x_k}\right\rangle \\
&\quad + 2h_{ij}\langle X \,|\, \nu_E\rangle\,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (A.2) in the last step and we denoted with $X_\tau = X - \langle X|\nu_E\rangle\nu_E$ the "tangential part" of the vector field X along the hypersurface ∂E.

Letting ω be the 1–form defined by $\omega(Y) = g(X_\tau, Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} + 2\Gamma_{ij}^k \omega_k + 2h_{ij}\langle X|\nu_E\rangle = \nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X|\nu_E\rangle\,,$$

being $\psi : \partial E \to \mathbb{T}^n$ the inclusion (identity) map of ∂E.

Hence, by formula (1.7), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\Big|_{t=0} &= \frac{\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial t} g_{ij}\big|_{t=0}}{2} \\
&= \frac{\sqrt{\det(g_{ij})}\, g^{ij}\big(\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X \,|\, \nu_E\rangle\big)}{2} \\
&= \sqrt{\det(g_{ij})}\big(\operatorname{div} X_\tau + \mathrm{H}\langle X \,|\, \nu_E\rangle\big)\,.
\end{aligned} \tag{1.12}$$

and we can conclude

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{A}_{\mathbb{T}^n}(\varphi_t(\partial E))\Big|_{t=0} &= \frac{\partial}{\partial t}\int_{\partial E} d\mu_t\Big|_{t=0} \\
&= \frac{\partial}{\partial t}\int_{\partial E}\sqrt{\det(g_{ij})}\,dx\Big|_{t=0} \\
&= \int_{\partial E}\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\Big|_{t=0}\,dx \\
&= \int_{\partial E}\big(\mathrm{div}X_\tau + \mathrm{H}\langle X\,|\,\nu_E\rangle\big)\sqrt{\det(g_{ij})}\,dx \\
&= \int_{\partial E}\big(\mathrm{div}X_\tau + \mathrm{H}\langle X\,|\,\nu_E\rangle\big)\,d\mu \\
&= \int_{\partial E}\mathrm{H}\langle X\,|\,\nu_E\rangle\,d\mu
\end{aligned} \tag{1.13}$$

where in the last step we applied the *divergence theorem* (see equation (A.1))

In order to compute the derivative of the nonlocal term, in the notations and definitions of Section 1.1, we set

$$v(t,x) = v_{E_t}(x) = \int_{\mathbb{T}^n} G(x,y)u_{E_t}(x)\,dy = \int_{E_t} G(x,y)\,dy - \int_{E_t^c} G(x,y)\,dy,$$

where $E_t^c = \mathbb{T}^n \setminus E_t$. Then,

$$\begin{aligned}
\frac{d}{dt}\Big(\int_{\mathbb{T}^n}|\nabla v_{E_t}(x)|^2\,dx\Big)\Big|_{t=0} &= \frac{d}{dt}\Big(\int_{\mathbb{T}^n}|\nabla v(t,x)|^2\,dx\Big)\Big|_{t=0} \\
&= 2\int_{\mathbb{T}^n}\nabla v_E(x)\frac{\partial}{\partial t}\nabla v(t,x)\Big|_{t=0}\,dx \\
&= 2\int_{\mathbb{T}^n}(u_E(x)-m)\frac{\partial}{\partial t}v(t,x)\Big|_{t=0}\,dx,
\end{aligned}$$

where in the last equality we used the fact that $-\Delta v_E = u_E - m$ and we integrated by parts. Now we note that

$$\frac{\partial}{\partial t}v(t,x) = \frac{\partial}{\partial t}\Big(\int_{E_t} G(x,y)\,dy\Big) - \frac{\partial}{\partial t}\Big(\int_{E_t^c} G(x,y)\,dy\Big), \tag{1.14}$$

and, by a change of variable,

$$\frac{\partial}{\partial t}\Big(\int_{E_t} G(x,y)\,dy\Big)\Big|_{t=0} = \frac{\partial}{\partial t}\Big(\int_E G(x,\Phi(t,z))J\Phi(t,z)\,dz\Big)\Big|_{t=0},$$

where $J\Phi(t,\cdot)$ is Jacobian of $\Phi(t,\cdot)$. Thus, by definition of Φ and formula (1.8), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_{E_t} G(x,y)\,dy\Big)\Big|_{t=0} &= \int_E\Big[\langle\nabla_y G(x,\Phi(t,z))|X(\Phi(t,z))\rangle \\
&\qquad + G(x,\Phi(t,z))\,\mathrm{div}X(\Phi(t,z))\Big]J\Phi(t,z)\,dz\Big|_{t=0} \\
&= \int_E\big(\langle\nabla_y G(x,y)|X(y)\rangle + G(x,y)\,\mathrm{div}X(y)\big)\,dy \\
&= \int_E \mathrm{div}_y\big(G(x,y)X(y)\big)\,dy \\
&= \int_{\partial E} G(x,y)\langle X(y)|\nu_E(y)\rangle\,d\mu(y),
\end{aligned}$$

By an analogous computation we get

$$-\frac{\partial}{\partial t}\Big(\int_{E_t^c} G(x,y)\,dy\Big)\Big|_{t=0} = \int_{\partial E} G(x,y)\langle X(y)|\nu_E(y)\rangle\,d\mu(y), \tag{1.15}$$

then, using equalities (1.1) and (1.2), we conclude

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^n}|\nabla v_{E_t}(x)|^2\,dx\,\Big|_{t=0} &= 4\int_{\mathbb{T}^n}(u_E(x)-m)\Big(\int_{\partial E} G(x,y)\langle X(y)|\nu_E(y)\rangle\,d\mu(y)\Big)\,dx\\ &= 4\int_{\partial E}\Big(\int_{\mathbb{T}^n} G(x,y)(u_E(x)-m)\,dx\Big)\langle X(y)|\nu_E(y)\rangle\,d\mu(y)\\ &= 4\int_{\partial E} v_E(y)\langle X(y)|\nu_E(y)\rangle\,d\mu(y)\,. \end{aligned}\tag{1.16}$$

Combining formulas (1.13) and (1.16), we finally obtain the conclusion. □

By Lemma 1.4, it follows that if a smooth set E satisfies

$$\int_{\partial E}(\mathrm{H}+4\gamma v_E)\langle X|\nu_E\rangle\,d\mu = 0, \tag{1.17}$$

for all admissible vector field $X \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$, then

$$\int_{\partial E}(\mathrm{H}+4\gamma v_E)\varphi\,d\mu = 0$$

for all $\varphi \in C^\infty(\partial E)$ such that $\int_{\partial E}\varphi\,d\mu = 0$, which is equivalent to say that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{H}+4\gamma v_E = \lambda \qquad \text{on } \partial E.$$

Remark 1.6. The above property, for instance, is clearly satisfied by a smooth set E which is a "minimum" of J under a volume constraint. Then, the parameter λ may be interpreted as a *Lagrange multiplier* associated with such constraint. Notice that when $\gamma = 0$, we recover the classical *constant mean curvature* condition for hypersurfaces in $\mathbb{R}^n$.

Definition 1.7. We say that a smooth set $E \subseteq \mathbb{T}^n$ is a (volume–constrained) *critical set* of J if $\mathrm{H}+4\gamma v_E$ is constant on ∂E.

We now compute the second variation of the functional J. Given a smooth set $E \subseteq \mathbb{T}^n$ and an admissible vector field X, the *second variation of J at E* with respect to the associated flow Φ is

$$\frac{d^2}{dt^2}J(E_t)\Big|_{t=0}.$$

In the following proposition we calculate the second variation of the Area functional. Then, we do the same for the nonlocal term and we conclude with the second variation of the functional J.

Proposition 1.8 (Second variation of $\mathcal{A}_{\mathbb{T}^n}$)**.** *Let $E \subseteq \mathbb{T}^n$, X and Φ as in Theorem 1.5. Then,*

$$\frac{d^2}{dt^2}\mathcal{A}_{\mathbb{T}^n}(\partial E_t)\Big|_{t=0} = \int_{\partial E}\Big(|\nabla\langle X|\nu_E\rangle|^2 - \langle X|\nu_E\rangle^2|B|^2\Big)\,d\mu$$
$$+ \int_{\partial E}\mathrm{H}\Big(\mathrm{H}\langle X|\nu_E\rangle^2 + \langle Z|\nu_E\rangle - 2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle + B(X_\tau, X_\tau)\Big)\,d\mu\,,$$

where $X_\tau = X - \langle X|\nu_E\rangle\nu_E$ is the tangential part *of X on ∂E, B is the second fundamental form of ∂E and*

$$Z := \frac{\partial^2}{\partial t^2}\Phi(0,\cdot) = \frac{\partial}{\partial t}X(\Phi(0,\cdot)) = dX(X).$$

Proof. We let $\psi_t = \Phi(t,\cdot)|_{\partial E}$ as in Theorem 1.5 where we showed that

$$\frac{d}{dt}\int_{\partial E_t}\sqrt{\det g_{ij}}\,dx = \int_{\partial E_t}\mathrm{H}\langle X|\nu_{E_t}\rangle\,d\mu,$$

where H is the mean curvature of ∂E_t. Consequently, we have

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \frac{d}{dt}\int_{\partial E_t}\mathrm{H}\langle X|\nu_{E_t}\rangle\sqrt{\det g_{ij}}\,dx\,\Big|_{t=0}$$

In order to simplify the notation we put $\nu = \nu_{E_t}$ and $\varphi = \langle X|\nu_{E_t}\rangle$, moreover we drop the subscript t in ψ_t, that is, we write simply ψ. To conclude we need to calculate the following derivatives

$$\frac{\partial \mathrm{H}}{\partial t}\Big|_{t=0} \tag{1.18}$$

$$\frac{\partial \varphi}{\partial t}\Big|_{t=0} \tag{1.19}$$

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} \tag{1.20}$$

We start calculating the derivative (1.20). Note that (arguing as in Theorem 1.5),

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}}\,\Big|_{t=0} = [\mathrm{div}X_\tau + \mathrm{H}\varphi]\sqrt{\det g_{ij}}\,\Big|_{t=0},$$

hence, the contribution of the term (1.20) to the second variation is given by

$$\int_{\partial E}(\varphi\mathrm{H}\,\mathrm{div}X_\tau + \varphi^2\mathrm{H}^2)\,d\mu$$

Now we can calculate

$$\frac{\partial\varphi}{\partial t}\Big|_{t=0} = \frac{\partial\langle X|\nu\rangle}{\partial t}\Big|_{t=0} = \left\langle\frac{\partial X}{\partial t}\Big|\nu\right\rangle\Big|_{t=0} + \left\langle X\Big|\frac{\partial\nu}{\partial t}\right\rangle\Big|_{t=0}$$

and using the fact that $\frac{\partial\nu}{\partial t}\big|_{t=0}$ is tangent to ∂E, we obtain

$$\left\langle X\Big|\frac{\partial\nu}{\partial t}\right\rangle\Big|_{t=0} = \left\langle X_\tau\Big|\frac{\partial\nu}{\partial t}\right\rangle\Big|_{t=0} = X_\tau^p\left\langle\frac{\partial\psi}{\partial x_p}\Big|\frac{\partial\nu}{\partial t}\right\rangle\Big|_{t=0},$$

where in the last inequality we use the notation $X_\tau = X_\tau^p \frac{\partial\psi}{\partial x_p}$. Note that, $\left\langle \frac{\partial\psi}{\partial x_p} \middle| \nu \right\rangle = 0$ for every $p \in \{1, \ldots, n-1\}$ and $t \in (-\varepsilon, \varepsilon)$, hence,

$$\begin{aligned}
0 = \frac{\partial}{\partial t} \left\langle \frac{\partial\psi}{\partial x_p} \middle| \nu \right\rangle \bigg|_{t=0} &= \left\langle \frac{\partial X}{\partial x_p} \middle| \nu \right\rangle + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \frac{\partial}{\partial x_p} \langle X | \nu \rangle - \left\langle X \middle| \frac{\partial\nu}{\partial x_p} \right\rangle + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - \left\langle X_\tau \middle| \frac{\partial\nu}{\partial x_p} \right\rangle + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q \left\langle \frac{\partial\psi}{\partial x_q} \middle| \frac{\partial\nu}{\partial x_p} \right\rangle + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q \left\langle \frac{\partial\psi}{\partial x_q} \middle| h_{pl} g^{li} \frac{\partial\psi}{\partial x_i} \right\rangle + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \frac{\partial\varphi}{\partial x_p} - X_\tau^q h_{pl} g^{li} g_{qi} + \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0}
\end{aligned}$$

and we can conclude that

$$\left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} = -\frac{\partial\varphi}{\partial x_p} + X_\tau^q h_{pq}.$$

So we obtain the following identity

$$\begin{aligned}
\frac{\partial\varphi}{\partial t} \bigg|_{t=0} &= \left\langle \frac{\partial X}{\partial t} \middle| \nu \right\rangle \bigg|_{t=0} + X_\tau^p \left\langle \frac{\partial\psi}{\partial x_p} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \\
&= \langle Z | \nu \rangle - \frac{\partial\varphi}{\partial x_p} X_\tau^p + X_\tau^p X_\tau^q h_{pq} \\
&= \langle Z | \nu \rangle - \langle X_\tau | \nabla \langle X | \nu \rangle \rangle + B(X_\tau, X_\tau)
\end{aligned}$$

and the contribution of the term (1.19) is

$$\int_{\partial E} \mathrm{H} \Big(\langle Z | \nu \rangle - \langle X_\tau | \nabla \langle X | \nu \rangle \rangle + B(X_\tau, X_\tau) \Big) \, d\mu.$$

Now we conclude calculating the term (1.18). To this aim, note that

$$\mathrm{H} = -\left\langle \frac{\partial^2\psi}{\partial x_i \partial x_j} \middle| \nu \right\rangle g^{ij}$$

hence, we need to calculate the following terms

$$\frac{\partial g^{ij}}{\partial t} \bigg|_{t=0} \tag{1.21}$$

$$\left\langle \frac{\partial^2\psi}{\partial x_i \partial x_j} \middle| \frac{\partial\nu}{\partial t} \right\rangle \bigg|_{t=0} \tag{1.22}$$

$$\left\langle \frac{\partial}{\partial t} \frac{\partial^2\psi}{\partial x_i \partial x_j} \middle| \nu \right\rangle \bigg|_{t=0} \tag{1.23}$$

We start with the term (1.21).

$$\frac{\partial g_{ij}}{\partial t}\bigg|_{t=0} = \nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X|\nu\rangle$$

where $\omega = X_\tau^\flat$. Using the fact that $g_{ij}g^{jk} = 0$, we obtain

$$0 = \frac{\partial g_{ij}}{\partial t}\bigg|_{t=0} g^{jk} + g_{ij}\frac{\partial g^{jk}}{\partial t}\bigg|_{t=0} = g^{jk}\Big(\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X|\nu\rangle\Big) + g_{ij}\frac{\partial g^{jk}}{\partial t}\bigg|_{t=0}$$

then,

$$\frac{\partial g^{pk}}{\partial t}\bigg|_{t=0} = -g^{jp}g^{ik}\Big(\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij}\langle X|\nu\rangle\Big) = -\nabla^p X_\tau^k - \nabla^k X_\tau^p - 2h^{pk}\varphi.$$

We proceed with the calculation of the term (1.22)

$$\left\langle \frac{\partial^2 \psi}{\partial x_i \partial x_j}\bigg|\frac{\partial \nu}{\partial t}\right\rangle\bigg|_{t=0} = \Gamma_{ij}^k \left\langle \frac{\partial \psi}{\partial x_k}\bigg|\frac{\partial \nu}{\partial t}\right\rangle\bigg|_{t=0} = \Gamma_{ij}^k\big(-\frac{\partial \varphi}{\partial x_k} + X_\tau^q h_{qk}\big)$$

and we conclude by calculating the term (1.23),

$$\left\langle \frac{\partial}{\partial t}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\bigg|\nu\right\rangle = \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}\bigg|\nu\right\rangle\bigg|_{t=0} = \left\langle \frac{\partial^2 (\varphi\nu)}{\partial x_i \partial x_j}\bigg|\nu\right\rangle + \left\langle \frac{\partial^2 X_\tau}{\partial x_i \partial x_j}\bigg|\nu\right\rangle.$$

Then,

$$\begin{aligned}
\left\langle \frac{\partial^2 (\varphi\nu)}{\partial x_i \partial x_j}\bigg|\nu\right\rangle &= \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \left\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j}\bigg|\nu\right\rangle \varphi \\
&= \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \left\langle \frac{\partial}{\partial x_i}\big(h_{jl}g^{lp}\frac{\partial \psi}{\partial x_p}\big)\bigg|\nu\right\rangle \varphi \\
&= \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + h_{jl}g^{lp}\left\langle \frac{\partial^2 \psi}{\partial x_i \partial x_j}\bigg|\nu\right\rangle \varphi \\
&= \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \varphi h_{jl}g^{lp}h_{ip}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \frac{\partial^2 X_\tau}{\partial x_i \partial x_j}\bigg|\nu\right\rangle &= \frac{\partial}{\partial x_i}\left\langle \frac{\partial X_\tau}{\partial x_j}\bigg|\nu\right\rangle - \left\langle \frac{\partial X_\tau}{\partial x_j}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle \frac{\partial}{\partial x_j}\Big(X_\tau^p \frac{\partial \psi}{\partial x_p}\Big)\bigg|\nu\right\rangle - \left\langle \frac{\partial X_\tau}{\partial x_j}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left[X_\tau^p \left\langle \frac{\partial^2 \psi}{\partial x_j \partial x_p}\bigg|\nu\right\rangle\right] - \left\langle \frac{\partial X_\tau}{\partial x_j}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= -\frac{\partial}{\partial x_i}(X_\tau^p h_{pj}) - \left\langle \frac{\partial X_\tau}{\partial x_j}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= -\frac{\partial}{\partial x_i}(X_\tau^p h_{pj}) - \left\langle \frac{\partial}{\partial x_j}\Big(X_\tau^p \frac{\partial \psi}{\partial x_p}\Big)\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= -\frac{\partial}{\partial x_i}(X_\tau^p h_{pj}) - X_\tau^p \left\langle \frac{\partial^2 \psi}{\partial x_j \partial x_p}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle - \frac{\partial X_\tau^p}{\partial x_j}\left\langle \frac{\partial \psi}{\partial x_p}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle \\
&= -\frac{\partial}{\partial x_i}(X_\tau^p h_{pj}) - X_\tau^p \Gamma_{jp}^k \left\langle \frac{\partial \psi}{\partial x_k}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle - \frac{\partial X_\tau^p}{\partial x_j}\left\langle \frac{\partial \psi}{\partial x_p}\bigg|\frac{\partial \nu}{\partial x_i}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial x_i}(X^p_\tau h_{pj}) - X^p_\tau \Gamma^k_{jp} h_{il} g^{lq} g_{kq} - \frac{\partial X^p}{\partial x_j} h_{il} g^{lq} g_{pq} \\
&= -\frac{\partial}{\partial x_i}(X^p_\tau h_{pj}) - X^p_\tau \Gamma^k_{jp} h_{ik} - \frac{\partial X^k}{\partial x_j} h_{ik}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial \mathrm{H}}{\partial t}\bigg|_{t=0} &= -2h_{ij}\nabla^i X^j_\tau - 2\langle X|\nu\rangle|B|^2 - g^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + g^{ij}\Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k} \\
&\quad + |B|^2\langle X|\nu\rangle - g^{ij}\Gamma^k_{ij} h_{kq} X^q_\tau + g^{ij}\frac{\partial}{\partial x_i}(X^p_\tau h_{pj}) + h_{ij}\nabla^i X^j_j \\
&= -|B|^2\langle X|\nu\rangle - h_{ij}\nabla^i X^j_\tau - \Delta\varphi + g^{ij}\left[\frac{\partial}{\partial x_i}\Big(X^p_\tau h_{pj}\Big) - \Gamma^k_{ij}\Big(X^p_\tau h_{pk}\Big)\right] \\
&= -\varphi|B|^2 - \Delta\varphi - h_{ij}\nabla^i X^j_\tau + g^{ij}\nabla_i(X^p_\tau h_{pj}) \\
&= -\varphi|B|^2 - \Delta\varphi - h_{ij}\nabla^i X^j_\tau + \operatorname{div}(X^p_\tau h_{pj}) \\
&= -\varphi|B|^2 - \Delta\varphi + \langle X_\tau|\operatorname{div} B\rangle \\
&= -\varphi|B|^2 - \Delta\varphi + \langle X_\tau|\nabla \mathrm{H}\rangle.
\end{aligned}
\tag{1.24}
$$

where in the last equality we used the Codazzi equations (A.3). We conclude that the contribution of the term (1.18) is then

$$
\int_{\partial E} \varphi\Big(-\varphi|B|^2 - \Delta\varphi + \langle X_\tau|\nabla \mathrm{H}\rangle\Big)\, d\mu.
$$

Putting all these computations together, we can finally get the second variation of the Area functional,

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}_{\mathbb{T}^n}(\partial E_t)\bigg|_{t=0} = \int_{\partial E} \Big[&-\varphi\Delta\varphi - \varphi^2|B|^2 + \varphi\langle X_\tau|\nabla \mathrm{H}\rangle + \varphi \mathrm{H}\operatorname{div} X_\tau + \varphi^2\mathrm{H}^2 \\
&+ \mathrm{H}\big(\langle Z|\nu\rangle - \langle X_\tau|\nabla\varphi\rangle + B(X_\tau, X_\tau)\big)\Big]\, d\mu
\end{aligned}
$$

Integrating by parts we obtain

$$
\int_{\partial E} \varphi\langle X_\tau|\nabla \mathrm{H}\rangle\, d\mu = -\int_{\partial E}\big[\mathrm{H}\langle X_\tau|\nabla\varphi\rangle + \mathrm{H}\varphi\operatorname{div} X_\tau\big]\, d\mu
$$

and we can conclude

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}_{\mathbb{T}^n}(\partial E_t)\bigg|_{t=0} = \int_{\partial E}\Big[&|\nabla\varphi|^2 - \varphi^2|B|^2 + \varphi^2\mathrm{H}^2 \\
&+ \mathrm{H}\big(\langle Z|\nu\rangle - 2\langle X_\tau|\nabla\varphi\rangle + B(X_\tau, X_\tau)\big)\Big]\, d\mu
\end{aligned}
$$

which is the formula we wanted to get. □

Proposition 1.9 (Second variation of the nonlocal term of J). *Let $E \subseteq \mathbb{T}^n$, X and Φ as in Theorem 1.5. Then, defining*

$$
N(t) = \int_{\mathbb{T}^n} |\nabla v_{E_t}(x)|^2\, dx\,.
$$

where $v_{E_t} : \mathbb{T}^n \to \mathbb{R}$ is the function defined by formulas (1.1)–(1.3), the following identity holds

$$N''(0) = 8\int_{\partial E}\int_{\partial E} G(x,y)\langle X|\nu(x)\rangle\langle X|\nu(y)\rangle d\mu(x)\, d\mu(y)$$
$$+ 4\int_{\partial E} \operatorname{div}^{\mathbb{T}^n}(v_E X)\langle X|\nu\rangle\, d\mu\,,$$

which gives the second variation of the nonlocal term of J.

Proof. We start noticing that, recalling the notations and definitions of Section 1.1, the function $N(t)$ can be written as

$$N(t) = \int_{\mathbb{T}^n} v(t,x)u(t,x)\, dx$$

with $v(t,x) = v_{E_t}(x)$ and $u(t,x) = u_{E_t}(x) = \chi_{E_t} - \chi_{E_t^c}$. Hence,

$$N(t) = \Big(\int_{E_t} - \int_{E_t^c}\Big) v(t,x)\, dx = \Big(\int_E - \int_{E^c}\Big) v(t,\Phi(t,z))J\Phi(t,z)\, dz$$

where we used (here and in the rest of the proof) the symbol $\big(\int_A - \int_{A^c}\big) f\, dx$ to denote the difference of the integrals of a common function $f : \mathbb{T}^n \to \mathbb{R}$ on a set $A \subseteq \mathbb{T}^n$ and on its complement $A^c = \mathbb{T}^n \setminus A$.
The first derivative of N, by definition of Φ and formula (1.8), is given by

$$\begin{aligned}
N'(t) =& \Big(\int_E - \int_{E^c}\Big)\Big[\langle\nabla v(t,\Phi(t,z))|X(\Phi(t,z))\rangle J\Phi(t,z)\\
&\qquad + v_t(t,\Phi(t,z))J\Phi(t,z) + v(t,\Phi(t,z))\frac{\partial}{\partial t}J\Phi(t,z)\Big]\, dz\\
=& \Big(\int_E - \int_{E^c}\Big)\Big[\langle\nabla v(t,\Phi(t,z))|X(\Phi(t,z))\rangle + v_t(t,\Phi(t,z))\\
&\qquad + v(t,\Phi(t,z))\operatorname{div}X(\Phi(t,z))\Big]J\Phi(t,z)\, dz\\
=& \Big(\int_{E_t} - \int_{E_t^c}\Big)\Big[\langle\nabla v(t,x)|X(x)\rangle + v_t(t,x) + v(t,x)\operatorname{div}X(x)\Big]\, dx
\end{aligned}$$

where $v_t = \frac{\partial v}{\partial t}$. Then, letting $v_i = \frac{\partial v}{\partial x_i}$, $v_{ti} = \frac{\partial^2 v}{\partial t\partial x_i}$ and $v_{ij} = \frac{\partial^2 v}{\partial x_i x_j}$, we have

$$\begin{aligned}
N''(t) =& \Big(\int_E - \int_{E^c}\Big)\Big[v_{ij}(t,\Phi(t,z))X^i(\Phi(t,z))X^j(\Phi(t,z)) + v_{tt}(t,\Phi(t,z))\\
&\qquad + 2v_{ti}(t,\Phi(t,z))X^i(\Phi(t,z))\\
&\qquad + v_i(t,\Phi(t,z))\frac{\partial X^i}{\partial x_j}(\Phi(t,z))X^j(\Phi(t,z))\\
&\qquad + 2v_t(t,\Phi(t,z))\operatorname{div}X(\Phi(t,z))\\
&\qquad + v(t,\Phi(t,z))[\operatorname{div}X(\Phi(t,z))]^2\\
&\qquad + 2v_i(t,\Phi(t,z))X^i(\Phi(t,z))\operatorname{div}X(\Phi(t,z))\\
&\qquad + v(t,\Phi(t,z))\langle\nabla\operatorname{div}X(\Phi(t,z))|X(\Phi(t,z))\rangle\Big]J\Phi(t,z)\, dz
\end{aligned}$$

hence,

$$\begin{aligned} N''(0) =\Big(\int_E - \int_{E^c}\Big)\Big[& v_{ij}(0,x)X^i(x)X^j(x) + v_{tt}(0,x) + 2v_{ti}(0,x)X^i(x) \\ & + v_i(0,x)\frac{\partial X^i}{\partial x_j}(x)X^j(x) + 2v_t(0,x)\operatorname{div}X(x) \\ & + v(0,x)[\operatorname{div}X(x)]^2 \\ & + 2v_i(0,x)X^i(x)\operatorname{div}X(x) + v(0,x)\langle\nabla\operatorname{div}X(x)|X(x)\rangle\Big]\,dx \end{aligned} \tag{1.25}$$

Now, by equations (1.14)–(1.15), there holds

$$v_t(0,x) = 2\int_{\partial E} G(x,y)\,\langle X(y)|\nu_E(y)\rangle\,d\mu(y). \tag{1.26}$$

By means of this equality, the third and fifth term of equation (1.25) become

$$\begin{aligned} & 2\Big(\int_E - \int_{E^c}\Big)\Big[v_{ti}(0,x)X^i(x) + v_t(0,x)\operatorname{div}X(x)\Big]\,dx \\ &\quad = 2\Big(\int_E - \int_{E^c}\Big)\operatorname{div}(v_t(0,x)X(x))\,dx \\ &\quad = 4\int_{\partial E} v_t(0,x)\langle X(x)|\nu_E(x)\rangle\,d\mu(x) \\ &\quad = 8\int_{\partial E}\int_{\partial E} G(x,y)\langle X(x)|\nu_E(x)\rangle\langle X(y)|\nu_E(y)\rangle\,d\mu(x)d\mu(y), \end{aligned}$$

where we applied the divergence theorem.
We then notice that

$$\begin{aligned} \operatorname{div}\big[\langle\nabla v(0,x)|X(x)\rangle X(x)\big] = & v_{ij}(0,x)X^i(x)X^j(x) + v_i(0,x)\frac{\partial X^i}{\partial x_j}(x)X^j(x) \\ & + v_i(0,x)X^i(x)\operatorname{div}X(x) \end{aligned}$$

hence, the sum of the first, fourth and half of the seventh term of equation (1.25) is given by

$$2\int_{\partial E}\langle\nabla^{\mathbb{T}^n}v_E(x)|X(x)\rangle\langle X(x)|\nu_E(x)\rangle\,d\mu(x),$$

by the divergence theorem.
From the equality

$$\begin{aligned} \operatorname{div}\big[v(0,x)\big(\operatorname{div}X(x)\big)X(x)\big] = & v_i(0,x)X^i(x)\operatorname{div}X(x) + v(0,x)\langle\nabla\operatorname{div}X(x)|X(x)\rangle \\ & + v(0,x)[\operatorname{div}X(x)]^2, \end{aligned}$$

we see that the sixth, the other half of the seventh and the last term of equation (1.25) add up to

$$2\int_{\partial E} v_E(x)\operatorname{div}^{\mathbb{T}^n}X(x)\langle X(x)|\nu_E(x)\rangle\,d\mu(x),$$

by the divergence theorem again.
Putting all these terms together, we can write

$$\begin{aligned}
N''(0) = &8\int_{\partial E}\int_{\partial E} G(x,y)\langle X(x)|\nu_E(x)\rangle\langle X(y)|\nu_E(y)\rangle\,d\mu(x)d\mu(y)\\
&+2\int_{\partial E}\langle\nabla^{\mathbb{T}^n}v_E(x)|X(x)\rangle\langle X(x)|\nu_E(x)\rangle\,d\mu(x)\\
&+2\int_{\partial E} v_E(x)\,\mathrm{div}^{\mathbb{T}^n}X(x)\langle X(x)|\nu_E(x)\rangle\,d\mu(x)\\
&+\Big(\int_E-\int_{E^c}\Big)v_{tt}(0,x)\,dx,
\end{aligned}$$

so, it remains to deal with the term $\big(\int_E-\int_{E^c}\big)v_{tt}(x,0)\,dx$. To this aim, by equation (1.14), it follows

$$\begin{aligned}
v_t(t,x) = &\Big(\int_E-\int_{E^c}\Big)\Big[\langle\nabla_y G(x,\Phi(t,z))|X(\Phi(t,z))\rangle\\
&\qquad\qquad + G(x,\Phi(t,z))\,\mathrm{div}X(\Phi(t,z))\Big]J\Phi(t,z)\,dz,
\end{aligned}$$

changing variables as before. Then, writing simply Φ for $\Phi(t,z)$ in the next formulas,

$$\begin{aligned}
&\Big(\int_E-\int_{E^c}\Big)v_{tt}(x,0)\,dx\\
&=\frac{d}{dt}\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\Big[\langle\nabla_y G(x,\Phi)|X(\Phi)\rangle + G(x,\Phi)\,\mathrm{div}X(\Phi)\Big]J\Phi\,dzdx\,\Big|_{t=0}\\
&=\frac{d}{dt}\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\mathrm{div}_y[G(x,\cdot)X](\Phi)J\Phi\,dzdx\,\Big|_{t=0}\\
&=\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\langle\nabla_y\,\mathrm{div}_y[G(x,\cdot)X](\Phi)|X(\Phi)\rangle J\Phi\,dzdx\,\Big|_{t=0}\\
&\quad+\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\mathrm{div}_y[G(x,\cdot)X](\Phi)\,\mathrm{div}X(\Phi)J\Phi\,dzdx\,\Big|_{t=0}\\
&=\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\langle\nabla_y\,\mathrm{div}_y[G(x,y)X(y)]|X(y)\rangle\,dydx\\
&\quad+\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\mathrm{div}_y[G(x,y)X(y)]\,\mathrm{div}X(y)\,dydx\\
&=\Big(\int_E-\int_{E^c}\Big)\Big(\int_E-\int_{E^c}\Big)\,\mathrm{div}_y\big(\mathrm{div}_y[G(x,y)X(y)]X(y)\big)\,dydx\,.
\end{aligned}$$

Using the divergence theorem and interchanging the order of integration, we get

$$\begin{aligned}
&\Big(\int_E-\int_{E^c}\Big)\,v_{tt}(x,0)\,dx\\
&\qquad=2\Big(\int_E-\int_{E^c}\Big)\int_{\partial E}\mathrm{div}_y^{\mathbb{T}^n}[G(x,y)X(y)]\langle X(y)|\nu_E(y)\rangle\,d\mu(y)dx\\
&\qquad=2\int_{\partial E}\langle X(y)|\nu_E(y)\rangle\,\mathrm{div}_y^{\mathbb{T}^n}\Big[X(y)\,\Big(\int_E-\int_{E^c}\Big)G(x,y)\,dx\Big]\,d\mu(y)\\
&\qquad=2\int_{\partial E}\langle X(y)|\nu_E(y)\rangle\,\mathrm{div}^{\mathbb{T}^n}[v_E(y)X(y)]\,d\mu(y)\,,
\end{aligned}$$

by the symmetry of the Green function.
Hence, we conclude

$$\begin{aligned}
N''(0) =& 8\int_{\partial E}\int_{\partial E} G(x,y)\langle X(x)|\nu_E(x)\rangle\langle X(y)|\nu_E(y)\rangle\, d\mu(x)d\mu(y)\\
&+2\int_{\partial E}\langle\nabla^{\mathbb{T}^n}v_E(x)|X(x)\rangle\langle X(x)|\nu_E(x)\rangle\, d\mu(x)\\
&+2\int_{\partial E} v_E(x)\,\mathrm{div}^{\mathbb{T}^n}X(x)\langle X(x)|\nu_E(x)\rangle\, d\mu(x)\\
&+2\int_{\partial E}\mathrm{div}^{\mathbb{T}^n}[v_E(x)X(x)]\langle X(x)|\nu_E(x)\rangle\, d\mu(x)\\
=& 8\int_{\partial E}\int_{\partial E} G(x,y)\langle X(x)|\nu_E(x)\rangle\langle X(y)|\nu_E(y)\rangle\, d\mu(x)d\mu(y)\\
&+4\int_{\partial E}\mathrm{div}^{\mathbb{T}^n}[v_E(x)X(x)]\langle X(x)|\nu_E(x)\rangle\, d\mu(x)
\end{aligned}$$

and we are done. □

Putting together Propositions 1.8 and 1.9, we then obtain the second variation of the nonlocal Area functional J.

Theorem 1.10 (Second variation of J). *Let $E\subseteq\mathbb{T}^n$, X, X_τ, Φ and v_E as in Theorem 1.5 and Propositions 1.8, 1.9. Then,*

$$\begin{aligned}
\frac{d^2}{dt^2}J(E_t)\Big|_{t=0} =& \frac{d^2}{dt^2}J(\Phi(t,E))\Big|_{t=0} = \int_{\partial E}\Big(|\nabla\langle X|\nu_E\rangle|^2-\langle X|\nu_E\rangle^2|B|^2\Big)\,d\mu\\
&+8\gamma\int_{\partial E}\int_{\partial E}G(x,y)\langle X|\nu_E(x)\rangle\langle X|\nu_E(y)\rangle\,d\mu(x)\,d\mu(y)\\
&+4\gamma\int_{\partial E}\partial_{\nu_E}v_E\langle X|\nu_E\rangle^2\,d\mu+R\,,
\end{aligned}$$

where the "remainder term" R is defined as

$$R=\int_{\partial E}(\mathrm{H}+4\gamma v_E)\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X\,d\mu-\int_{\partial E}(\mathrm{H}+4\gamma v_E)\,\mathrm{div}\big(\langle X|\nu_E\rangle X_\tau\big)\,d\mu. \tag{1.27}$$

Moreover, if E is a smooth volume–constrained critical set for J, then the remainder term R is zero and the second variation of J at E only depends on the normal component of X on ∂E, that is, on $\langle X|\nu_E\rangle$.

Proof. In Propositions 1.8 and 1.9 we showed that

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}_{\mathbb{T}^n}(\partial E_t)\Big|_{t=0} =& \int_{\partial E}\Big(|\nabla\langle X|\nu_E\rangle|^2-\langle X|\nu_E\rangle^2|B|^2\Big)\,d\mu\\
&+\int_{\partial E}\mathrm{H}\Big(\mathrm{H}\langle X|\nu_E\rangle^2+\langle Z|\nu_E\rangle-2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle+B(X_\tau,X_\tau)\Big)\,d\mu
\end{aligned}\tag{1.28}$$

and

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{\mathbb{T}^n}|\nabla v_{E_t}|^2\,dx\Big|_{t=0} =& 8\int_{\partial E}\int_{\partial E}G(x,y)\langle X|\nu_E(x)\rangle\langle X|\nu_E(y)\rangle d\mu(x)\,d\mu(y)\\
&+4\int_{\partial E}\mathrm{div}^{\mathbb{T}^n}(v_EX)\langle X|\nu_E\rangle\,d\mu\,.
\end{aligned}\tag{1.29}$$

We now claim that

$$\begin{aligned}&\mathrm{H}\langle X|\,\nu_E\rangle^2+\langle Z|\nu_E\rangle-2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle+B(X_\tau,X_\tau)\\&\qquad=\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X-\mathrm{div}(\langle X|\nu_E\rangle X_\tau)\,.\end{aligned}\tag{1.30}$$

We notice that, being every derivative of ν_E a tangent vector field,

$$\begin{aligned}\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle&=\langle\nu_E|dX(X_\tau)\rangle+\langle X|\langle X_\tau|\nabla\nu_E\rangle\rangle\\&=\langle\nu_E|dX(X_\tau)\rangle+\langle X_\tau|\langle X_\tau|\nabla\nu_E\rangle\rangle\\&=\langle\nu_E|dX(X_\tau)\rangle+B(X_\tau,X_\tau)\,.\end{aligned}$$

Therefore, recalling that $Z=dX(X)$, we have

$$\begin{aligned}&\mathrm{H}\langle X|\nu_E\rangle^2+\langle Z|\nu_E\rangle-2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle+B(X_\tau|X_\tau)\\&\quad=\mathrm{H}\langle X|\nu_E\rangle^2+\langle\nu_E|dX(X)\rangle-\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle-\langle\nu_E|dX(X_\tau)\rangle\\&\quad=\mathrm{H}\langle X|\nu_E\rangle^2+\langle\nu_E|dX(\langle X|\nu_E\rangle\nu_E)\rangle-\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle\\&\quad=\mathrm{H}\langle X|\nu_E\rangle^2+\langle X|\nu_E\rangle\langle\nu_E|dX(\nu_E)\rangle+\langle X|\nu_E\rangle\,\mathrm{div}X_\tau-\mathrm{div}(\langle X|\nu_E\rangle X_\tau)\,.\end{aligned}$$

Now we notice that, choosing an orthonormal basis $e_1,\dots,e_{n-1},e_n=\nu_E$ of $\mathbb{R}^n$ at a point $x\in\partial E$ and letting $X=X^ie_i$, we have

$$\langle e_i|\nabla^\top X^i\rangle=\langle e_i|\nabla X^i-\langle\nabla X^i|\nu_E\rangle\nu_E\rangle=\mathrm{div}^{\mathbb{T}^n}X-\langle\nu_E|dX(\nu_E)\rangle$$

where the symbol $^\top$ denotes the projection on the tangent space to ∂E. Moreover, if we choose a local parametrizazion of ∂E such that $\frac{\partial\psi}{\partial x_i}(x)=e_i$, for $i\in\{1,\dots,n-1\}$, at $x\in\partial E$ we have $e_i^j=\frac{\partial\psi^j}{\partial x_i}=g^{ij}=\delta_{ij}$ and

$$\begin{aligned}\langle e_i|\nabla^\top X^i\rangle&=\langle e_i|\nabla^\top X^i_\tau\rangle+\langle e_i|\nabla^\top(\langle X|\nu_E\rangle\nu_E^i)\rangle\\&=\langle e_i^\top|\nabla X^i_\tau\rangle+\langle X|\nu_E\rangle\langle e_i^\top|\nabla\nu_E^i\rangle\\&=\langle e_i^\top|\nabla X^i_\tau\rangle+\langle X|\nu_E\rangle\frac{\partial\psi^j}{\partial x_i}h_{jl}g^{ls}\frac{\partial\psi^i}{\partial x_s}\\&=\nabla_{e_i}X^i_\tau+\langle X|\nu_E\rangle h_{ii}\\&=\mathrm{div}X_\tau+\langle X|\nu_E\rangle\mathrm{H}\end{aligned}$$

where we used the Gauss–Weingarten relations (A.2) and the fact that the covariant derivative of a vector field along a hypersurface of $\mathbb{R}^n$ can be obtained by differentiating in Euclidean coordinates (a local extension of) the vector field and projecting the result on the tangent space to the hypersurface (see [16], for instance). Hence, we get

$$\langle\nu_E|dX(\nu_E)\rangle=\mathrm{div}^{\mathbb{T}^n}X-\langle e_i|\nabla^\top X^i\rangle=\mathrm{div}^{\mathbb{T}^n}X-\mathrm{div}X_\tau-\langle X|\nu_E\rangle\mathrm{H}$$

and it follows

$$\begin{aligned}&\mathrm{H}\langle X|\,\nu_E\rangle^2+\langle Z|\nu_E\rangle-2\langle X_\tau|\nabla\langle X|\nu_E\rangle\rangle+B(X_\tau,X_\tau)\\&\qquad=\langle X|\nu_E\rangle\,\mathrm{div}^{\mathbb{T}^n}X-\mathrm{div}(\langle X|\nu_E\rangle X_\tau)\end{aligned}$$

which is equation (1.30).
We then see that

$$\begin{aligned}
\int_{\partial E} \operatorname{div}^{\mathbb{T}^n}(v_E X)\langle X|\nu_E\rangle\, d\mu &= \int_{\partial E} \langle \nabla^{\mathbb{T}^n} v_E | X\rangle\langle X|\nu_E\rangle\, d\mu + \int_{\partial E} v_E \operatorname{div}^{\mathbb{T}^n} X\langle X|\nu_E\rangle\, d\mu \\
&= \int_{\partial E} \partial_{\nu_E} v_E \langle X|\nu_E\rangle^2\, d\mu + \int_{\partial E} \langle \nabla v_E | X_\tau\rangle\langle X|\nu_E\rangle\, d\mu + \int_{\partial E} v_E \operatorname{div}^{\mathbb{T}^n} X\langle X|\nu_E\rangle\, d\mu \\
&= \int_{\partial E} \partial_{\nu_E} v_E \langle X|\nu_E\rangle^2\, d\mu - \int_{\partial E} v_E \operatorname{div}\big(\langle X|\nu_E\rangle X_\tau\big)\, d\mu + \int_{\partial E} v_E \operatorname{div}^{\mathbb{T}^n} X\langle X|\nu_E\rangle\, d\mu
\end{aligned}$$

where in the last equality we integrated by parts. Thus, the formula in the statement of the theorem follows from this computation and equations (1.28), (1.29) and (1.30).

Now we prove that the remainder R in formula (1.27) is zero when the smooth set E is a volume–constrained critical set for J.
By the criticality condition (1.17), the remainder R is equal to

$$R = \lambda \int_{\partial E} \langle X|\nu_E\rangle \operatorname{div}^{\mathbb{T}^n} X\, d\mu - \lambda \int_{\partial E} \operatorname{div}\big(\langle X|\nu_E\rangle X_\tau\big)\, d\mu.$$

for some constant $\lambda \in \mathbb{R}$. Then, the first integral is zero by equation (1.11), as the vector field X is admissible for E and the second one is also zero by the divergence theorem (A.1). □

Remark 1.11. We note that if we have a critical set E for the *unconstrained* functional J, hence $\mathrm{H} + 4\gamma v_E = 0$ on ∂E, the remainder term is clearly zero and the second variation of J has the same form as in the constrained case.

By Theorem 1.10, the second variation of J at a critical smooth set E is a quadratic form depending only on the normal component of $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ on ∂E, that is, on $\varphi = \langle X|\nu_E\rangle$. This and the fact that the admissible vector fields $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ are in a way "characterized" by having zero integral of such normal component (see the discussion at the beginning of this section and Lemma 1.4), motivate the following definition.

Definition 1.12. Given any smooth open set $E \subseteq \mathbb{T}^n$ we define the space of (Sobolev) functions (see [5])

$$\widetilde{H}^1(\partial E) = \Big\{\varphi : \partial E \to \mathbb{R} : \varphi \in H^1(\partial E) \ \text{ and } \int_{\partial E} \varphi\, d\mu = 0\Big\},$$

and the quadratic form $\Pi_E : \widetilde{H}^1(\partial E) \to \mathbb{R}$ as

$$\begin{aligned}
\Pi_E(\varphi) = &\int_{\partial E} \Big(|\nabla\varphi|^2 - \varphi^2|B|^2\Big)\, d\mu \\
&+ 8\gamma \int_{\partial E}\int_{\partial E} G(x,y)\varphi(x)\varphi(y)\, d\mu(x)\, d\mu(y) \\
&+ 4\gamma \int_{\partial E} \partial_{\nu_E} v_E \varphi^2\, d\mu\,.
\end{aligned} \tag{1.31}$$

Remark 1.13. Letting for $\varphi \in \widetilde{H}^1(\partial E)$,

$$v_\varphi(x) = \int_{\partial E} G(x,y)\varphi(y)\, d\mu(y)\,,$$

it follows (from the properties of the Green's function) that v_φ satisfies distributionally $-\Delta v_\varphi = \varphi\mu$ in $\mathbb{T}^n$, indeed,

$$\begin{aligned}\int_{\mathbb{T}^n} \langle \nabla v_\varphi(x) | \nabla \psi(x)\rangle\, dx &= -\int_{\mathbb{T}^n} v_\varphi(x)\Delta\psi(x)\, dx \\ &= -\int_{\mathbb{T}^n}\int_{\partial E} G(x,y)\varphi(y)\Delta\psi(x)\, d\mu(y) dx \\ &= -\int_{\partial E}\varphi(y)\int_{\mathbb{T}^n} G(x,y)\Delta\psi(x)\, dx\, d\mu(y) \\ &= -\int_{\partial E}\varphi(y)\int_{\mathbb{T}^n} \Delta G(x,y)\psi(x)\, dx\, d\mu(y) \\ &= \int_{\partial E}\varphi(y)\Big[\psi(y) - \int_{\mathbb{T}^n}\psi(x)\, dx\Big]\, d\mu(y) \\ &= \int_{\partial E}\varphi(y)\psi(y)\, d\mu(y)\,,\end{aligned}$$

for all $\psi \in C^\infty(\mathbb{T}^n)$, as $\int_{\partial E}\varphi(y)\, d\mu(y) = 0$. Therefore, taking $\psi = v_\varphi$, we have

$$\int_{\mathbb{T}^n} |\nabla v_\varphi(x)|^2\, dx = \int_{\partial E}\varphi(y) v_\varphi(y)\, d\mu(y)\,,$$

hence, the following identity holds

$$\int_{\partial E}\int_{\partial E} G(x,y)\varphi(x)\varphi(y)\, d\mu(x) d\mu(y) = \int_{\partial E}\varphi(y)\, v_\varphi(y)\, d\mu(y) = \int_{\mathbb{T}^n}|\nabla v_\varphi(x)|^2\, dx\,,$$

and we can write

$$\Pi_E(\varphi) = \int_{\partial E}\Big(|\nabla\varphi|^2 - \varphi^2|B|^2\Big)\, d\mu + 8\gamma\int_{\mathbb{T}^n}|\nabla v_\varphi|^2\, dx + 4\gamma\int_{\partial E}\partial_{\nu_E} v_E \varphi^2\, d\mu\,, \tag{1.32}$$

for every $\varphi \in \widetilde{H}^1(\partial E)$.

Remark 1.14. If E is a smooth critical set and X is an admissible vector field for E with associate flow Φ, then

$$\frac{d}{dt}J(E_t)\Big|_{t=0} = \int_{\partial E}\langle X|\nu_E\rangle\, d\mu = 0$$

and

$$\frac{d^2}{dt^2}J(E_t)\Big|_{t=0} = \Pi_E(\langle X|\nu_E\rangle).$$

We observe that, by the translation invariance of the functional J, the constant vector field $X = \eta \in \mathbb{R}^n$ is clearly admissible, as the associated flow is given by $\Phi(t,x) = x + t\eta$, then $J(E_t) = J(E)$ and

$$0 = \frac{d^2}{dt^2}J(E_t)\Big|_{t=0} = \Pi_E(\langle \eta, \nu_E\rangle)\,,$$

that is, the form Π_E is zero on the vector subspace

$$T(\partial E) = \big\{ \langle \eta | \nu_E \rangle \,:\, \eta \in \mathbb{R}^n \big\} \subseteq \widetilde{H}^1(\partial E)\,.$$

of dimension less or equal than n. We can then split

$$\widetilde{H}^1(\partial E) = T(\partial E) \oplus T^{\perp}(\partial E)\,, \tag{1.33}$$

where $T^{\perp}(\partial E) \subseteq \widetilde{H}^1(\partial E)$ is the vector subspace L^2–orthogonal to $T(\partial E)$ (with respect to the measure μ on ∂E), that is,

$$\begin{aligned} T^{\perp}(\partial E) &= \Big\{ \varphi \in \widetilde{H}^1(\partial E) \,:\, \int_{\partial E} \varphi \nu_E \, d\mu = 0 \Big\} \\ &= \Big\{ \varphi \in H^1(\partial E) \,:\, \int_{\partial E} \varphi \, d\mu = 0 \ \text{ and } \int_{\partial E} \varphi \nu_E \, d\mu = 0 \Big\} \end{aligned}$$

and define the following "stability" conditions.

Definition 1.15 (Stability)**.** We say that a critical set $E \subseteq \mathbb{T}^n$ is *stable* if

$$\Pi_E(\varphi) \geq 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E)$$

and *strictly stable* if

$$\Pi_E(\varphi) > 0 \qquad \text{for all } \varphi \in T^{\perp}(\partial E) \setminus \{0\}.$$

Remark 1.16. We observe that there exists an orthonormal frame $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that

$$\int_{\partial E} \langle \nu_E | e_i \rangle \langle \nu_E | e_j \rangle \, d\mu = 0, \tag{1.34}$$

for all $i \neq j$, indeed, considering the symmetric $n \times n$–matrix $A = (a_{ij})$ with components $a_{ij} = \int_{\partial E} \nu_E^i \nu_E^j \, d\mu$, where $\nu_E^i = \langle \nu_E | \varepsilon_i \rangle$ for some basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathbb{R}^n$, we have

$$\int_{\partial E} (O\nu_E)_i (O\nu_E)_j \, d\mu = (OAO^{-1})_{ij},$$

for every $O \in SO(n)$. Choosing O such that OAO^{-1} is diagonal and setting $e_i = O^{-1}\varepsilon_i$, relations (1.34) are clearly satisfied.
Hence, the functions $\langle \nu_E | e_i \rangle$ which are not identically zero are an orthogonal basis of $T(\partial E)$. We set

$$\mathrm{I}_E = \big\{ i \in \{1, \dots, n\} \,:\, \langle \nu_E | e_i \rangle \text{ is not identically zero} \big\}$$

and

$$\mathrm{O}_E = \mathrm{Span}\{ e_i \,:\, i \in \mathrm{I}_E \}, \tag{1.35}$$

then, given any $\varphi \in \widetilde{H}^1(\partial E)$, its projection on $T^{\perp}(\partial E)$ is

$$\pi(\varphi) = \varphi - \sum_{i \in \mathrm{I}_E} \frac{\int_{\partial E} \varphi \langle \nu_E | e_i \rangle \, d\mu}{\| \langle \nu_E | e_i \rangle \|^2_{L^2(\partial E)}} \langle \nu_E | e_i \rangle\,. \tag{1.36}$$

1.3 STABILITY AND $W^{2,p}$–LOCAL MINIMALITY

From now on we will make a large use of Sobolev spaces on smooth hypersurfaces. Most of their properties hold as in $\mathbb{R}^n$, standard references are [3] in the Euclidean space and the book [5] when the ambient is a manifold.

Given a smooth set $E \subseteq \mathbb{T}^n$, for $\varepsilon > 0$ small enough, we let (d is the "Euclidean" distance on $\mathbb{T}^n$)

$$N_\varepsilon = \{x \in \mathbb{T}^n \,:\, d(x, \partial E) < \varepsilon\} \tag{1.37}$$

to be a *tubular neighborhood* of ∂E such that the *orthogonal projection map* $\pi_E : N_\varepsilon \to \partial E$ giving the (unique) closest point on ∂E and the *signed distance function* $d_E : N_\varepsilon \to \mathbb{R}$ from ∂E

$$d_E(x) = \begin{cases} d(x, \partial E) & \text{if } x \notin E \\ -d(x, \partial E) & \text{if } x \in E \end{cases} \tag{1.38}$$

are well defined and smooth in N_ε. Moreover, for every $x \in N_\varepsilon$, the projection map is given explicitly by

$$\pi_F(x) = x - \nabla d_F^2(x)/2 = x - d_F(x)\nabla d_F(x) \tag{1.39}$$

and the unit vector $\nabla d_E(x)$ is orthogonal to ∂E at the point $\pi_E(x) \in \partial E$, indeed actually $\nabla d_E(x) = \nabla d_E(\pi_E(x)) = \nu_E(\pi_E(x))$, which means that the integral curves of the vector field ∇d_E are straight segments orthogonal to ∂E.
This clearly implies that the map

$$\partial E \times (-\varepsilon, \varepsilon) \ni (y, t) \mapsto L(y, t) = y + t\nabla d_E(y) = y + t\nu_E(y) \in N_\varepsilon \tag{1.40}$$

is a smooth diffeomorphism with inverse

$$N_\varepsilon \ni x \mapsto L^{-1}(x) = (\pi_E(x), d_E(x)) \in \partial E \times (-\varepsilon, \varepsilon)\,,$$

moreover, denoting with JL its (partial and relative to the hypersurface ∂E) Jacobian, there holds

$$0 < C_1 \leq JL(y, t) \leq C_2$$

on $\partial E \times (-\varepsilon, \varepsilon)$, for a couple of constants C_1, C_2, depending on E and ε (for a proof of the existence of such tubular neighborhood and of these properties, see [27] for instance).

By means of such tubular neighborhood of a smooth set $E \subseteq \mathbb{T}^n$ and the map L, we can speak of "$W^{k,p}$–closedness" (or "$C^{k,\alpha}$–closedness") to E of another smooth set $F \subseteq \mathbb{T}^n$, asking that for some $\delta > 0$ "small enough", we have $\mathrm{Vol}(E \triangle F) < \delta$ and that ∂F is contained in a tubular neighbourhood N_ε of E, as above, described by

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\},$$

for a smooth function $\psi : \partial E \to \mathbb{R}$ with $\|\psi\|_{W^{k,p}(\partial E)} < \delta$ (resp. $\|\psi\|_{C^{k,\alpha}(\partial E)} < \delta$). That is, we are asking that the two sets E and F differ by a set of small measure and that their boundaries are "close" in $W^{k,p}$ (or $C^{k,\alpha}$).

Notice that clearly

$$\psi(y) = \pi_2 \circ L^{-1}\big(\partial E \cap \{y + \lambda\nu_E(y) \,:\, \lambda \in \mathbb{R}\}\big)\,,$$

where $\pi_2 : \partial E \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ is the projection on the second factor.
Moreover, given a sequence of smooth sets $F_i \subseteq \mathbb{T}^n$, we will write $F_i \to E$ in $W^{k,p}$ (resp. $C^{k,\alpha}$) if for every $\delta > 0$, there hold $\mathrm{Vol}(F_i \triangle E) < \delta$, the smooth boundary ∂F_i is contained in N_ε and it is described by

$$\partial F_i = \{y + \psi_i(y)\nu_E(y) \,:\, y \in \partial E\},$$

for a smooth function $\psi_i : \partial E \to \mathbb{R}$ with $\|\psi_i\|_{W^{k,p}(\partial E)} < \delta$ (resp. $\|\psi_i\|_{C^{k,\alpha}(\partial E)} < \delta$), for every $i \in \mathbb{N}$ large enough.

From now on, in all the rest of the thesis, we will refer to the volume–constrained nonlocal Area functional J, sometimes without underlining the presence of such constraint, by simplicity. Morever, with N_ε we will always denote a suitable tubular neighbourhood of a smooth set, with the above properties.

Definition 1.17. We say that a smooth set $E \subseteq \mathbb{T}^n$ is a *local minimizer* for the functional J if there exists $\delta > 0$ such that

$$J(F) \geq J(E)$$

for all smooth sets $F \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(E) = \mathrm{Vol}(F)$ and $\mathrm{Vol}(E \triangle F) < \delta$.
We say that a smooth set $E \subseteq \mathbb{T}^n$ is a *$W^{2,p}$–local minimizer* if there exists $\delta > 0$ such that

$$J(F) \geq J(E)$$

for all $F \subseteq \mathbb{T}^n$ with $\mathrm{Vol}(E) = \mathrm{Vol}(F)$, $\mathrm{Vol}(E \triangle F) < \delta$, moreover ∂F is contained in a tubular neighbourhood N_ε of E, as above and it is described by

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\},$$

for a smooth function $\psi : \partial E \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$.

We immediately see a *necessary* condition for local minimizers. Notice that a local minimizer is clearly also a $W^{2,p}$–local minimizer.

Proposition 1.18. *Let the smooth set $E \subseteq \mathbb{T}^n$ be a local minimizer of J, then E is a critical set and*

$$\Pi_E(\varphi) \geq 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E),$$

in particular E is stable.

Proof. If E is a local minimizer of J, for any admissible vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ with associated flow Φ, we have $\mathrm{Vol}(E_t) = \mathrm{Vol}(\Phi(t, E)) = \mathrm{Vol}(E)$ and for every $\delta > 0$, there clearly exists $\overline{\varepsilon} > 0$ such that for $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$ we have

$$\mathrm{Vol}(E \triangle E_t) < \delta.$$

and

$$\partial E_t = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\} \subseteq N_\varepsilon$$

for a smooth function $\psi : \partial E \to \mathbb{R}$ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$. Hence, the $W^{2,p}$–local minimality of E implies

$$J(E) \leq J(E_t),$$

for every $t \in (-\overline{\varepsilon}, \overline{\varepsilon})$. Thus,

$$0 = \frac{d}{dt} J(E_t)\Big|_{t=0} = \int_{\partial E} (\mathrm{H} + 4\gamma v_E)\langle X|\nu_E\rangle \, d\mu,$$

by Theorem 1.5, which implies that E is a critical set, by the subsequent discussion and

$$0 \leq \frac{d^2}{dt^2} J(E_t)\Big|_{t=0} = \Pi_E(\langle X|\nu_E\rangle),$$

by Theorem 1.10 and Remark 1.14.
Since by Lemma 1.4, for every smooth function $\varphi : \partial E \to \mathbb{R}$ with zero integral there exists an admissible vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ such that $\varphi = \langle X|\nu_E\rangle$, we conclude that $\Pi_E(\varphi) \geq 0$ for every $\varphi \in C^\infty(\partial E) \cap \widetilde{H}^1(\partial E)$, then the thesis follows by the density of this space in $\widetilde{H}^1(\partial E)$ (see [5]). □

The rest of this section will be devoted to show that the strict stability (see Definition 1.15) is a *sufficient* condition for the $W^{2,p}$–local minimality. Precisely, we will prove the following main theorem of this chapter.

Theorem 1.19 ($W^{2,p}$–local minimality). *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ a smooth strictly stable critical set for the nonlocal Area functional J (under a volume constraint), as in Definition 1.15, with N_ε a tubular neighbourhood of E as in formula (1.37). Then there exist constants $\delta, C > 0$ such that*

$$J(F) \geq J(E) + C[\alpha(E,F)]^2,$$

for all smooth sets $F \subseteq \mathbb{T}^n$ such that $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(F \triangle E) < \delta$, $\partial F \subseteq N_\varepsilon$ and

$$\partial F = \{y + \psi(y)\nu_E(y) \, : \, y \in \partial E\},$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, where the "distance" $\alpha(E,F)$ is defined as

$$\alpha(E,F) = \min_{\eta \in \mathbb{R}^n} \mathrm{Vol}(E \triangle (F + \eta)).$$

As a consequence, E is a $W^{2,p}$–local minimizer of J. Moreover, if F is $W^{2,p}$–close enough to E and $J(F) = J(E)$, then F is a translate of E, that is E is locally the unique $W^{2,p}$–local minimizer, up to translations.

Remark 1.20. We could have introduced the definitions of *strict* local minimizer or *strict* $W^{2,p}$–local minimizer for the nonlocal Area functional, by asking that the inequalities $J(F) \leq J(E)$ in Definition 1.17 are equalities if and only if F is a translate of E. With such notion, the conclusion of this theorem is that E is actually a strict $W^{2,p}$–local minimizer.

Remark 1.21. With some extra effort, it can be proved that in the same hypotheses of Theorem 1.19, the set F is actually a local minimizer (see [2]). Since in the analysis of the modified Mullins–Sekerka flow in the next chapter we do not need such stronger result, we omitted its proof.

We postpone the proof of this result after showing some technical lemmas. We underline that most of the difficulties are due to the presence of the degeneracy subspace $T(\partial E)$ of the form Π_E (that is, where it is zero), related to the translation invariance of the nonlocal Area functional (recall the discussion after Definition 1.12).

In the next key lemma we are going to show how to construct admissible smooth vector fields for a smooth set E, "related" to smooth sets which are $W^{2,p}$–close to it. By the same technique we then also prove Lemma 1.4 immediately after, whose proof was postponed from Section 1.2.

Lemma 1.22. *Let $E \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂E as above, in formula (1.37). For all $p > n-1$, there exist constants $\delta, C > 0$ such that if $\psi \in C^\infty(\partial E)$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$, then there exists a vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ with $\operatorname{div} X = 0$ in N_ε and the associated flow Φ satisfies*

$$\Phi(1, y) = y + \psi(y)\nu_E(y)\,, \qquad \text{for all } y \in \partial E. \tag{1.41}$$

Moreover, for every $t \in [0, 1]$

$$\|\Phi(t, \cdot) - \mathrm{Id}\|_{W^{2,p}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)}\,. \tag{1.42}$$

Finally, if $\mathrm{Vol}(E_1) = \mathrm{Vol}(E)$, then $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$ for all $t \in [-1, 1]$, that is, the vector field X is admissible.

Proof. We start considering the vector field $\widetilde{X} \in C^\infty(N_\varepsilon; \mathbb{R}^n)$ defined as

$$\widetilde{X}(x) = \xi(x)\nabla d_E(x) \tag{1.43}$$

for every $x \in N_\varepsilon$, where $d_E : N_\varepsilon \to \mathbb{R}$ is the signed distance and $\xi : N_\varepsilon \to \mathbb{R}$ is the function defined as follows: for all $y \in \partial E$, we let $f_y : (-\varepsilon, \varepsilon) \to \mathbb{R}$ to be the unique solution of the ODE

$$\begin{cases} f_y'(t) + f_y(t)\Delta d_E(y + t\nu_E(y)) = 0 \\ f_y(0) = 1 \end{cases}$$

and we set

$$\xi(x) = \xi(y + t\nu_E(y)) = f_y(t) = \exp\Big(-\int_0^t \Delta d_E(y + s\nu_E(y))\,ds\Big),$$

recalling that the map $(y, t) \mapsto x = y + t\nu_E(y)$ is a smooth diffeomorphism between $\partial E \times (-\varepsilon, \varepsilon)$ and N_ε. Notice that the function f is always positive, thus the same holds for ξ and $\xi = 1$, $\nabla d_E = \nu_E$, hence $\widetilde{X} = \nu_E$ on ∂E.

Our aim is then to prove that the smooth vector field X defined by

$$X(x) = \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\,\widetilde{X}(x) \tag{1.44}$$

for every $x \in N_\varepsilon$ and extended smoothly to all $\mathbb{T}^n$, satisfies all the properties of the statement of the lemma.

Step 1. We saw that $\widetilde{X}\big|_{\partial E} = \nu_E$, now we show that $\operatorname{div}\widetilde{X} = 0$ and analogously $\operatorname{div} X = 0$ in N_ε.
Given any $x = y + t\nu_E(y) \in N_\varepsilon$, with $y \in \partial E$, we have

$$\begin{aligned}
\operatorname{div}\widetilde{X}(x) &= \operatorname{div}[\xi(x)\nabla d_E(x)] \\
&= \langle \nabla\xi(x) | \nabla d_E(x)\rangle + \xi(x)\Delta d_E(x) \\
&= \frac{\partial}{\partial t}[\xi(y + t\nu_E(y))] + \xi(y + t\nu_E(y))\Delta d_E(y + t\nu_E(y)) \\
&= f_y'(t) + f_y(t)\Delta d_E(y + t\nu_E(y)) \\
&= 0,
\end{aligned}$$

where we used the fact that $f_y'(t) = \langle \nabla\xi(y + t\nu_E(y)) | \nu_E(y)\rangle$ and that we have $\nabla d_E(y + t\nu_E(y)) = \nu_E(y)$.
Since the function

$$x \mapsto \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu(\pi_E(x)))} = \theta(x)$$

is constant along the segments $t \mapsto x + t\nabla d_E(x)$, for every $x \in N_\varepsilon$, it follows that

$$0 = \frac{\partial}{\partial t}\big[\theta(x + t\nabla d_E(x))\big]\Big|_{t=0} = \langle \nabla\theta(x) | \nabla d_E(x)\rangle,$$

hence,

$$\operatorname{div} X = \langle \nabla\theta | \nabla d_E\rangle \xi + \theta \operatorname{div}\widetilde{X} = 0.$$

Step 2. Recalling that $\psi \in C^\infty(\partial E)$ and $p > n - 1$, we have

$$\|\psi\|_{L^\infty(\partial E)} \leq \|\psi\|_{C^1(\partial E)} \leq C_E \|\psi\|_{W^{2,p}(\partial E)},$$

by Sobolev embeddings (see [5]). Then, we can choose $\delta < \varepsilon / C_E$ such that for all $x \in \partial E$ we have that $x \pm \psi(x)\nu_E(x) \in N_\varepsilon$.
To check that equation (1.41) holds, we observe that the integral

$$\int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))} = \theta(x)$$

represents the time needed to go from $\pi_E(x)$ to $\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x))$ along the trajectory of the vector field $\widetilde{X}$, which is the segment connecting $\pi_E(x)$ and $\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x))$, of length $\psi(\pi_E(x))$, parametrized as

$$s \mapsto \pi_E(x) + s\psi(\pi_E(x))\nu_E(\pi_E(x)),$$

for $s \in [0,1]$ and which is traveled with velocity $\xi(\pi_E(x) + s\nu_E(\pi_E(x))) = f_{\pi_E(x)}(s)$. Therefore, by the above definition of $X = \theta\widetilde{X}$ and the fact that the function θ is constant along such segments, we conclude that

$$\Phi(1, y) - \Phi(0, y) = \psi(y)\nu_E(y)$$

and, equivalently,

$$\Phi(1, y) = y + \psi(y)\nu_E(y)$$

for all $y \in \partial E$.

Step 3. To establish inequality (1.42), we first show that

$$\|X\|_{W^{2,p}(N_\varepsilon)} \leq C\|\psi\|_{W^{2,p}(\partial E)} \tag{1.45}$$

for a constant $C > 0$ depending only on E and ε. This estimate will follow from the definition of X in equation (1.44) and the definition of $W^{2,p}$–norm, that is,

$$\|X\|_{W^{2,p}(N_\varepsilon)} = \|X\|_{L^p(N_\varepsilon)} + \|\nabla X\|_{L^p(N_\varepsilon)} + \|\nabla^2 X\|_{L^p(N_\varepsilon)}\,.$$

As $|\nabla d_E| = 1$ everywhere and the positive function ξ, by its definition at the beginning of the proof, satisfies $0 < C_1 \leq \xi \leq C_2$ in N_ε, for a pair of constants C_1 and C_2, we have

$$\begin{aligned}
\|X\|^p_{L^p(N_\varepsilon)} &= \int_{N_\varepsilon} \left| \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}\, \xi(x)\nabla d_E(x) \right|^p dx \\
&\leq \|\xi\|^p_{L^\infty(N_\varepsilon)} \int_{N_\varepsilon} \left| \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))} \right|^p dx \\
&\leq \frac{C_1^p}{C_2^p} \int_{N_\varepsilon} |\psi(\pi_E(x))|^p\, dx \\
&= \frac{C_1^p}{C_2^p} \int_{\partial E} \int_{-\varepsilon}^{\varepsilon} |\psi(\pi_E(y + t\nu_E(y)))|^p JL(y,t)\, dt d\mu(y) \\
&= \frac{C_1^p}{C_2^p} \int_{\partial E} |\psi(y)|^p \int_{-\varepsilon}^{\varepsilon} JL(y,t)\, dt d\mu(y) \\
&\leq C \int_{\partial E} |\psi(y)|^p\, d\mu(y) \\
&= C\|\psi\|^p_{L^p(\partial E)}\,.
\end{aligned}$$

where $L : \partial E \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ the smooth diffeomorphism defined in formula (1.40) and JL its Jacobian. Notice that the constant C depends only on E and ε.

Now we estimate the L^p–norm of ∇X. We compute

$$\begin{aligned}
\nabla X =& \frac{\nabla\psi(\pi_E(x))d\pi_E(x)}{\xi(\pi_E(x) + \psi(\pi_E(x))\nu_E(\pi_E(x)))}\, \xi(x)\nabla d_E(x) \\
&- \left[\int_0^{\psi(\pi_E(x))} \frac{\nabla\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}{\xi^2(\pi_E(x) + s\nu_E(\pi_E(x)))} d\pi_E(x)\mathrm{Id}\, ds \right] \xi(x)\nabla d_E(x) \\
&- \left[\int_0^{\psi(\pi_E(x))} \frac{\nabla\xi(\pi_E(x) + s\nu_E(\pi_E(x)))}{\xi^2(\pi_E(x) + s\nu_E(\pi_E(x)))} d\pi_E(x) s d\nu_E(\pi_E(x))\, ds \right] \xi(x)\nabla d_E(x) \\
&+ \int_0^{\psi(\pi_E(x))} \frac{ds}{\xi(\pi_E(x) + s\nu_E(\pi_E(x)))} \big(\nabla\xi(x)\nabla d_E(x) + \xi(x)\nabla^2 d_E(x)\big)
\end{aligned}$$

and we deal with the integrals in the three terms as before, changing variable by means of the function L. That is, since all the functions $d\pi_E$, $d\nu_E$, $\nabla^2 d_E$, ξ,

$1/\xi$, $\nabla\xi$ are bounded by some constants depending only on E and ε, we easily get (the constant C could vary from line to line)

$$\begin{aligned}
\|\nabla X\|^p_{L^p(N_\varepsilon)} \leq & C\int_{N_\varepsilon} |\nabla\psi(\pi_E(x))|^p\,dx + C\int_{N_\varepsilon} |\psi(\pi_E(x))|^p\,dx\\
= & C\int_{\partial E}\int_{-\varepsilon}^{\varepsilon} |\nabla\psi(\pi_E(y+t\nu_E(y)))|^p\,JL(y,t)\,dtd\mu(y)\\
& + C\int_{\partial E}\int_{-\varepsilon}^{\varepsilon} |\psi(\pi_E(y+t\nu_E(y)))|^p\,JL(y,t)\,dtd\mu(y)\\
= & C\int_{\partial E}\big(|\psi(y)|^p + |\nabla\psi(y)|^p\big)\int_{-\varepsilon}^{\varepsilon} JL(y,t)\,dtd\mu(y)\\
\leq & C\|\psi\|^p_{L^p(\partial E)} + C\|\nabla\psi\|^p_{L^p(\partial E)}\\
\leq & C\|\psi\|^p_{W^{1,p}(\partial E)}\,.
\end{aligned}$$

A very analogous estimate works for $\|\nabla^2 X\|^p_{L^p(N_\varepsilon)}$ and we obtain also

$$\|\nabla^2 X\|^p_{L^p(N_\varepsilon)} \leq C\|\psi\|^p_{W^{2,p}(\partial E)}\,,$$

hence, inequality (1.45) follows with $C = C(E,\varepsilon)$.

Applying now Lagrange theorem to every component of $\Phi(\cdot,y)$ for any $y\in\partial E$ and $t\in[0,1]$, we have

$$\Phi_i(t,y) - y_i = \Phi_i(t,y) - \Phi_i(0,y) = tX^i(\Phi(s,y))\,,$$

for every $i\in\{1,\dots,n\}$, where $s = s(y,t)$ is a suitable value in $(0,1)$. Then, it clearly follows

$$\|\Phi(t,\cdot) - \mathrm{Id}\|_{L^\infty(\partial E)} \leq C\|X\|_{L^\infty(N_\varepsilon)} \leq C\|X\|_{W^{2,p}(N_\varepsilon)} \leq C\|\psi\|_{W^{2,p}(\partial E)} \tag{1.46}$$

by estimate (1.45), with $C = C(E,\varepsilon)$ (notice that we used Sobolev embeddings, being $p > n-1$, the dimension of ∂E).

Differentiating the equations in system (1.3), we have (recall that we use the convention of summing over the repeated indices)

$$\begin{cases}\frac{\partial}{\partial t}\nabla^i\Phi_j(t,y) = \nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\\ \nabla^i\Phi_j(0,y) = \delta_{ij}\end{cases} \tag{1.47}$$

for every $i,j\in\{1,\dots,n\}$. It follows,

$$\begin{aligned}
\frac{\partial}{\partial t}\big|\nabla^i\Phi_j(t,y) - \delta_{ij}\big|^2 \leq & 2\big|(\nabla^i\Phi_j(t,y) - \delta_{ij})\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\big|\\
\leq & 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\big|\nabla^i\Phi_j(t,y) - \delta_{ij}\big|^2\\
& + 2\|\nabla X\|_{L^\infty(N_\varepsilon)}\big|\nabla^i\Phi_j(t,y) - \delta_{ij}\big|
\end{aligned}$$

hence, for almost every $t\in[0,1]$ where the following derivative exists,

$$\frac{\partial}{\partial t}\big|\nabla^i\Phi_j(t,y) - \delta_{ij}\big| \leq C\|\nabla X\|_{L^\infty(N_\varepsilon)}\big(\big|\nabla^i\Phi_j(t,y) - \delta_{ij}\big| + 1\big)\,.$$

Integrating this differential inequality, we get

$$\left|\nabla^i\Phi_j(t,y)-\delta_{ij}\right|\leq e^{tC\|\nabla X\|_{L^\infty(N_\varepsilon)}}-1\leq e^{C\|X\|_{W^{2,p}(N_\varepsilon)}}-1,$$

as $t\in[0,1]$ and where we used Sobolev embeddings again. Then, by inequality (1.45), we estimate

$$\sum_{1\leq i,j\leq n}\|\nabla^i\Phi_j(t,\cdot)-\delta_{ij}\|_{L^\infty(\partial E)}\leq C\big(e^{C\|\psi\|_{W^{2,p}(\partial E)}}-1\big)\leq C\|\psi\|_{W^{2,p}(\partial E)},\tag{1.48}$$

as $\|\psi\|_{W^{2,p}(\partial E)}\leq\delta$, for any $t\in[0,1]$ and $y\in\partial E$, with $C=C(E,\varepsilon,\delta)$.
Differentiating equations (1.47), we obtain

$$\begin{cases}\frac{\partial}{\partial t}\nabla^\ell\nabla^i\Phi_j(t,y)=\nabla^s\nabla^k X^j(\Phi(t,y))\nabla^i\Phi_k(t,y)\nabla^\ell\Phi_s(t,y)\\ \qquad\qquad\qquad\qquad+\nabla^k X^j(\Phi(t,y))\nabla^\ell\nabla^i\Phi_k(t,y)\\ \nabla^\ell\nabla^i\Phi(0,y)=0\end{cases}$$

(where we sum over s and k), for every $t\in[0,1]$, $y\in\partial E$ and $i,j,\ell\in\{1,\dots,n\}$. This is a linear *non–homogeneous* system of ODEs such that, if we control $C\|\psi\|_{W^{2,p}(\partial E)}$, the smooth coefficients in the right side multiplying the solutions $\nabla^\ell\nabla^i\Phi_j(\cdot,y)$ are uniformly bounded (as in estimate (1.48), Sobolev embeddings imply that ∇X is bounded in L^∞ by $C\|\psi\|_{W^{2,p}(\partial E)}$). Then, arguing as before, for almost every $t\in[0,1]$ where the following derivative exists, there holds

$$\begin{aligned}\frac{\partial}{\partial t}\left|\nabla^2\Phi(t,y)\right|&\leq C\|\nabla X\|_{L^\infty(N_\varepsilon)}\left|\nabla^2\Phi(t,y)\right|+C|\nabla^2X(\Phi(t,y))|\\&\leq C\delta\left|\nabla^2\Phi(t,y)\right|+C|\nabla^2X(\Phi(t,y))|\,,\end{aligned}$$

by inequality (1.45) (notice that inequality (1.48) gives an L^∞–bound on $\nabla\Phi$, *not only* in L^p, which is crucial). Thus, by means of Gronwall's lemma (see [35], for instance), we obtain the estimate

$$\left|\nabla^2\Phi(t,y)\right|\leq C\int_0^t|\nabla^2X(\Phi(s,y))|e^{C\delta(t-s)}\,ds\leq C\int_0^t|\nabla^2X(\Phi(s,y))|\,ds\,,$$

hence,

$$\begin{aligned}\|\nabla^2\Phi(t,\cdot)\|^p_{L^p(\partial E)}&\leq C\int_{\partial E}\Big(\int_0^t|\nabla^2X(\Phi(s,y))|\,ds\Big)^p\,d\mu(y)\\&\leq C\int_0^t\int_{\partial E}|\nabla^2X(\Phi(s,y))|^p\,d\mu(y)ds\\&=C\int_{N_\varepsilon}|\nabla^2X(x)|^pJL^{-1}(x)\,dx\\&\leq C\|\nabla^2X\|^p_{L^p(N_\varepsilon)}\\&\leq C\|X\|^p_{W^{2,p}(N_\varepsilon)}\\&\leq C\|\psi\|^p_{W^{2,p}(\partial E)}\,,\end{aligned}\tag{1.49}$$

by estimate (1.45), for every $t\in[0,1]$, with $C=C(E,\varepsilon,\delta)$.
Clearly, putting together inequalities (1.46), (1.48) and (1.49), we get the estimate (1.42) in the statement of the lemma.

Step 4. Finally, we remind that, by equation (1.10), we have

$$\frac{d^2}{dt^2}\mathrm{Vol}(E_t) = \int_{\partial E_t} \langle X|\nu_{E_t}\rangle \, \mathrm{div}^{\mathbb{T}^n} X \, d\mu_t,$$

hence, since by Step 1 we know that $\mathrm{div}X = 0$, we conclude $\frac{d^2}{dt^2}\mathrm{Vol}(E_t) = 0$ for all $t \in [-1, 1]$, that is, the function $t \mapsto \mathrm{Vol}(E_t)$ is linear. If $\mathrm{Vol}(E_1) = \mathrm{Vol}(E) = \mathrm{Vol}(E_0)$, it follows that $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$, for all $t \in [-1, 1]$. $\square$

With an argument similar to the one of Step 4 in this proof, we can now prove Lemma 1.4.

Proof of Lemma 1.4. Let $\varphi : \partial E \to \mathbb{R}$ a C^∞ function with zero integral, then we define the following smooth vector field in N_ε,

$$X(x) = \varphi(\pi_E(x))\widetilde{X}(x),$$

where $\widetilde{X}$ is the smooth vector field defined by formula (1.43) and we extend it to a smooth vector field $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ on the whole $\mathbb{T}^n$. Clearly, by the properties of $\widetilde{X}$ seen above,

$$\langle X(y)|\nu_E(y)\rangle = \varphi(y)\langle \widetilde{X}(y)|\nu_E(y)\rangle = \varphi(y)$$

for every $y \in \partial E$.
As the function $x \mapsto \varphi(\pi_E(x))$ is constant along the segments $t \mapsto x + t\nabla d_E(x)$, for every $x \in N_\varepsilon$, it follows, as in Step 1 of the previous proof, that $\mathrm{div}X = 0$ in N_ε. Then, arguing as in Step 4, the function $t \mapsto \mathrm{Vol}(E_t)$ is linear, for t in some interval $(-\delta, \delta)$. Since, by equation (1.9), there holds

$$\frac{d}{dt}\mathrm{Vol}(E_t)\Big|_{t=0} = \int_{\partial E} \langle X|\nu_E\rangle \, d\mu = \int_{\partial E} \varphi \, d\mu = 0,$$

such function $t \mapsto \mathrm{Vol}(E_t)$ must actually be constant.
Hence, $\mathrm{Vol}(E_t) = \mathrm{Vol}(E)$, for all $t \in (-\delta, \delta)$ and X is admissible. $\square$

The next lemma gives a technical estimate needed in the proof of Theorem 1.19.

Lemma 1.23. *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ a strictly stable critical set for the (volume–constrained) functional J. Then, in the hypoteses and notation of Lemma 1.22 there exist constants δ, $C > 0$ such that if $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ then $|X| \leq C|\langle X|\nu_{E_t}\rangle|$ on ∂E_t and*

$$\|\nabla X\|_{L^2(\partial E_t)} \leq C\|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)} \tag{1.50}$$

(here ∇ is the covariant derivative along E_t), for all $t \in [0, 1]$, where $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ is the smooth vector field defined in formula (1.44).

Proof. Fixed $\varepsilon > 0$, from inequality (1.42) it follows that there exist $\delta > 0$ such that if $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ there holds

$$|\nu_{E_t}(\Phi(t, y)) - \nu_E(y)| \leq \varepsilon$$

for every $y \in \partial E$, hence, as $\nabla d_E = \nu_E$ on ∂E, we have

$$|\nabla d_E(\Phi^{-1}(t,x)) - \nu_{E_t}(x)| = |\nu_E(\Phi^{-1}(t,x)) - \nu_{E_t}(x)| \leq \varepsilon$$

for every $x \in \partial E_t$. Then, if $\|\psi\|_{W^{2,p}(\partial E)}$ is small enough, $\Phi^{-1}(t,\cdot)$ is close to the identity, thus

$$|\nabla d_E(\Phi^{-1}(t,x)) - \nabla d_E(x)| \leq \varepsilon$$

on ∂E_t and we conclude

$$\|\nabla d_E - \nu_{E_t}\|_{L^\infty(\partial E_t)} \leq 2\varepsilon\,.$$

We estimate $X_{\tau_t} = X - \langle X|\nu_{E_t}\rangle\nu_{E_t}$ (recall that $X = \langle X|\nabla d_E\rangle\nabla d_E$),

$$\begin{aligned}
|X_{\tau_t}| &= |X - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&= |\langle X|\nabla d_E\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&= |\langle X|\nabla d_E\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nabla d_E + \langle X|\nu_{E_t}\rangle\nabla d_E - \langle X|\nu_{E_t}\rangle\nu_{E_t}| \\
&\leq |\langle X|(\nabla d_E - \nu_{E_t})\rangle\nabla d_E| + |\langle X|\nu_{E_t}\rangle(\nabla d_E - \nu_{E_t})| \\
&\leq 2|X|\,|\nabla d_E - \nu_{E_t}| \\
&\leq 4\varepsilon|X|\,,
\end{aligned}$$

then

$$|X_{\tau_t}| \leq 4\varepsilon|X_{\tau_t} + \langle X|\nu_{E_t}\rangle\nu_{E_t}| \leq 4\varepsilon|X_{\tau_t}| + |\langle X|\nu_{E_t}\rangle|\,,$$

hence,

$$|X_{\tau_t}| \leq C|\langle X|\nu_{E_t}\rangle|\,. \tag{1.51}$$

We now estimate its covariant derivative ∇ along E_t, that is,

$$\begin{aligned}
|\nabla X_{\tau_t}| &= |\nabla X - \nabla(\langle X|\nu_{E_t}\rangle\nu_{E_t})| \\
&= |\nabla(\langle X|\nabla d_E\rangle\nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle\nu_{E_t})| \\
&= |\nabla(\langle X|\nabla d_E\rangle\nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle\nabla d_E) \\
&\quad + \nabla(\langle X|\nu_{E_t}\rangle\nabla d_E) - \nabla(\langle X|\nu_{E_t}\rangle\nu_{E_t})| \\
&\leq |\nabla(\langle X|(\nabla d_E - \nu_{E_t})\rangle\nabla d_E)| + |\nabla(\langle X|\nu_{E_t}\rangle(\nabla d_E - \nu_{E_t}))| \\
&\leq C\varepsilon\big[|\nabla X| + |\nabla\langle X|\nu_{E_t}\rangle|\big] + C|X|\big[|\nabla(\nabla d_E)| + |\nabla\nu_{E_t}|\big] \\
&\leq C\varepsilon\big[|\nabla(\langle X|\nu_{E_t}\rangle\nu_{E_t} + X_{\tau_t})| + |\nabla\langle X|\nu_{E_t}\rangle|\big] \\
&\quad + C\big(|\langle X|\nu_{E_t}\rangle| + |X_{\tau_t}|\big)\,\big[|\nabla^2 d_E| + |\nabla\nu_{E_t}|\big]
\end{aligned}$$

hence, using inequality (1.51) and arguing as above, there holds

$$|\nabla X_{\tau_t}| \leq C|\nabla\langle X|\nu_{E_t}\rangle| + C|\langle X|\nu_{E_t}\rangle|\big[|\nabla^2 d_E| + |\nabla\nu_{E_t}|\big]\,.$$

Then, we get

$$\begin{aligned}
\|\nabla X_{\tau_t}\|^2_{L^2(\partial E_t)} &\leq C\|\nabla\langle X|\nu_{E_t}\rangle\|^2_{L^2(\partial E_t)} \\
&\qquad + C\int_{\partial E_t} |\langle X|\nu_{E_t}\rangle|^2\big[|\nabla^2 d_E| + |\nabla\nu_{E_t}|\big]^2\,d\mu \\
&\leq C\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)} \\
&\qquad + C\|\langle X|\nu_{E_t}\rangle\|^2_{L^{\frac{2p}{p-2}}(\partial E_t)}\big\||\nabla^2 d_E| + |\nabla\nu_{E_t}|\big\|^2_{L^p(\partial E_t)} \\
&\leq C\,\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}
\end{aligned}$$

where in the last inequality we used as usual Sobolev embeddings, as $p > \max\{2, n-1\}$.
Considering the covariant derivative of $X = X_{\tau_t} + \langle X|\nu_{E_t}\rangle\nu_{E_t}$, by means of this estimate, the trivial one

$$\|\nabla\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \leq \|\langle X|\nu_{E_t}\rangle\|_{H^1(\partial E_t)}$$

and inequality (1.51), we obtain estimate (1.50). □

We now show that any smooth set E sufficiently $W^{2,p}$–close to another smooth set F, can be "translated" by a vector $\eta \in \mathbb{R}^n$ such that $\partial E - \eta = \{y + \varphi(y)\nu_F(y) \,:\, y \in \partial F\}$, for a function $\varphi \in C^\infty(\partial F)$ having a suitable small "projection" on $T(\partial F)$ (see the definitions and the discussion at the end of the previous section).

Lemma 1.24. *Let $p > n-1$ and $F \subseteq \mathbb{T}^n$ a smooth set with a tubular neighbourhood N_ε as above, in formula (1.37). For any $\tau > 0$ there exist constants $\delta, C > 0$ such that if another smooth set $E \subseteq \mathbb{T}^n$ satisfies $\mathrm{Vol}(E \triangle F) < \delta$ and $\partial E = \{y + \psi(y)\nu_F(y) \,:\, y \in \partial F\} \subseteq N_\varepsilon$ for a function $\psi \in C^\infty(\mathbb{R})$ with $\|\psi\|_{W^{2,p}(\partial F)} < \delta$, then there exist $\eta \in \mathbb{R}^n$ and $\varphi \in C^\infty(\partial F)$ with the following properties:*

$$\partial E - \eta = \{y + \varphi(y)\nu_F(y) \,:\, y \in \partial F\}\,,$$

$$|\eta| \leq C\|\psi\|_{W^{2,p}(\partial F)}, \qquad \|\varphi\|_{W^{2,p}(\partial F)} \leq C\|\psi\|_{W^{2,p}(\partial F)}$$

and

$$\Big|\int_{\partial F} \varphi\nu_F \, d\mu\Big| \leq \tau\|\varphi\|_{L^2(\partial F)}\,.$$

Proof. We let d_F to be the signed distance function from ∂F. We underline that, throughout all the proof, the various constants will be all independent of $\psi : \partial F \to \mathbb{R}$.

We recall that in Remark 1.16 we saw that there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that the functions $\langle\nu_F, e_i\rangle$ are orthogonal in $L^2(\partial F)$, that is,

$$\int_{\partial F} \langle\nu_F, e_i\rangle\langle\nu_F, e_j\rangle \, d\mu = 0, \tag{1.52}$$

for all $i \neq j$ and we let I_F to be the set of the indices $i \in \{1, \ldots, n\}$ such that $\|\langle\nu_F, e_i\rangle\|_{L^2(\partial F)} > 0$. Given a smooth function $\psi : \partial F \to \mathbb{R}$, we set $\eta = \sum_{i=1}^n \eta_i e_i$, where

$$\eta_i = \begin{cases} \frac{1}{\|\langle\nu_F, e_i\rangle\|^2_{L^2(\partial F)}} \int_{\partial F} \psi(x)\langle\nu_F(x), e_i\rangle \, d\mu & \text{if } i \in \mathrm{I}_F, \\ \eta_i = 0 & \text{otherwise.} \end{cases} \tag{1.53}$$

Note that, from Hölder inequality, it follows

$$|\eta| \leq C_1\|\psi\|_{L^2(\partial F)}\,. \tag{1.54}$$

Step 1. Let $T_\psi : \partial F \to \partial F$ be the map

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta)\,.$$

It is easily checked that there exists $\varepsilon_0 > 0$ such that if

$$\|\psi\|_{W^{2,p}(\partial F)} + |\eta| \le \varepsilon_0 \le 1\,, \tag{1.55}$$

then T_ψ is a smooth diffeomorphism, moreover,

$$\|JT_\psi - 1\|_{L^\infty(\partial F)} \le C\|\psi\|_{C^1(\partial F)} \tag{1.56}$$

and

$$\|T_\psi - \mathrm{Id}\|_{W^{2,p}(\partial F)} + \|T_\psi^{-1} - \mathrm{Id}\|_{W^{2,p}(\partial F)} \le C(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|)\,. \tag{1.57}$$

Therefore, setting $\widehat{E} = E - \eta$, we have

$$\partial\widehat{E} = \{z + \varphi(z)\nu_F(z)\ :\ z \in \partial F\}$$

for some function φ, which is linked to ψ by the following relation: for all $y \in \partial F$ we let $z = z(y) \in \partial F$ such that

$$y + \psi(y)\nu_F(y) - \eta = z + \varphi(z)\nu_F(z)\,,$$

then

$$T_\psi(y) = \pi_F(y + \psi(y)\nu_F(y) - \eta) = \pi_F(z + \varphi(z)\nu_F(z)) = z,$$

that is, $y = T_\psi^{-1}(z)$ and

$$\begin{aligned}\varphi(z) &= \varphi(T_\psi(y))\\ &= d_F(z + \varphi(z)\nu_F(z))\\ &= d_F(y + \psi(y)\nu_F(y) - \eta)\\ &= d_F(T_\psi^{-1}(z) + \psi(T_\psi^{-1}(z))\nu_F(T_\psi(y)) - \eta).\end{aligned}$$

Thus, using inequality (1.57), we have

$$\|\varphi\|_{W^{2,p}(\partial F)} \le C_2\big(\|\psi\|_{W^{2,p}(\partial F)} + |\eta|\big), \tag{1.58}$$

for some constant $C_2 > 1$. We now estimate

$$\begin{aligned}\int_{\partial F} \varphi(z)\nu_F(z)\,d\mu(z) &= \int_{\partial F} \varphi(T_\psi(y))\nu_F(T_\psi(y))JT_\psi(y)\,d\mu(y)\\ &= \int_{\partial F} \varphi(T_\psi(y))\nu_F(T_\psi(y))\,d\mu(y) + R_1\,,\end{aligned} \tag{1.59}$$

where

$$\begin{aligned}|R_1| &= \left|\int_{\partial F} \varphi(T_\psi(y))\nu_F(T_\psi(y))\,[J_{n-1}\nabla T_\psi(y) - 1]\,d\mu(y)\right|\\ &\le C_3\|\psi\|_{C^1(\partial F)}\|\varphi\|_{L^2(\partial F)}\,,\end{aligned} \tag{1.60}$$

by inequality (1.56).
On the other hand,

$$
\begin{aligned}
\int_{\partial F} &\varphi(T_\psi(y))\nu_F(T_\psi(y))\, d\mu(y) \\
&= \int_{\partial F} \big[y + \psi(y)\nu_F(y) - \eta - T_\psi(y)\big]\, d\mu(y) \\
&= \int_{\partial F} \big[y + \psi(y)\nu_F(y) - \eta - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\, d\mu(y) \\
&= \int_{\partial F} \big\{\psi(y)\nu_F(y) - \eta + \big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\big\}\, d\mu(y) \\
&= \int_{\partial F} (\psi(y)\nu_F(y) - \eta)\, d\mu(y) + R_2\,,
\end{aligned}
\tag{1.61}
$$

where

$$
\begin{aligned}
R_2 &= \int_{\partial F} \big[\pi_F(y) - \pi_F(y + \psi(y)\nu_F(y) - \eta)\big]\, d\mu(y) \\
&= -\int_{\partial F} d\mu(y) \int_0^1 \nabla\pi_F(y + t(\psi(y)\nu(y) - \eta))(\psi(y)\nu_F(y) - \eta)\, dt \\
&= -\int_{\partial F} \nabla\pi_F(y)(\psi(y)\nu_F(y) - \eta)\, d\mu(y) + R_3\,.
\end{aligned}
\tag{1.62}
$$

In turn, recalling inequality (1.54), we get

$$
\begin{aligned}
|R_3| &\leq \int_{\partial F} d\mu(y) \int_0^1 |\nabla\pi_F(y + t(\psi(y)\nu_F(y) - \eta)) - \nabla\pi_F(y)|\, |\psi(y)\nu_F(y) - \eta|\, dt \\
&\leq C_4 \|\psi\|^2_{L^2(\partial F)}\,.
\end{aligned}
\tag{1.63}
$$

Since in N_ε, by equation (1.39), we have $\pi_F(x) = x - d_F(x)\nabla d_F(x)$, it follows

$$
\frac{\partial \pi^i_F}{\partial x_j}(x) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(x)\frac{\partial d_F}{\partial x_j}(x) - d_F(x)\frac{\partial^2 d_F}{\partial x_i \partial x_j}(x),
$$

thus, for all $y \in \partial F$

$$
\frac{\partial \pi^i_F}{\partial x_j}(y) = \delta_{ij} - \frac{\partial d_F}{\partial x_i}(y)\frac{\partial d_F}{\partial x_j}(y)\,.
$$

From this identity and equalities (1.59), (1.61) and (1.62), we conclude

$$
\int_{\partial F} \varphi(z)\nu_F(z)\, d\mu(z) = \int_{\partial F} \big[\psi(x)\nu_F(x) - \langle \eta \,|\, \nu_F(x)\rangle \nu_F(x)\big]\, d\mu(x) + R_1 + R_3\,.
$$

As the integral at the right–hand side vanishes by relations (1.52) and (1.53), estimates (1.60) and (1.63) imply

$$
\begin{aligned}
\Big|\int_{\partial F} \varphi(y)\nu_F(y)\, d\mu(y)\Big| &\leq C_3 \|\psi\|_{C^1(\partial F)} \|\varphi\|_{L^2(\partial F)} + C_4 \|\psi\|^2_{L^2(\partial F)} \\
&\leq C \|\psi\|_{C^1(\partial F)} \big(\|\varphi\|_{L^2(\partial F)} + \|\psi\|_{L^2(\partial F)}\big) \\
&\leq C_5 \|\psi\|^{1-\vartheta}_{W^{2,p}(\partial F)} \|\psi\|^{\vartheta}_{L^2(\partial F)} \big(\|\varphi\|_{L^2(\partial F)} + \|\psi\|_{L^2(\partial F)}\big)\,,
\end{aligned}
\tag{1.64}
$$

where in the last passage we used a well–known interpolation inequality, with $\vartheta \in (0,1)$ depending only on $p > n-1$ (see [5, Theorem 3.70]).

Step 2. The previous estimate does not allow to conclude directly, but we have to rely on the following iteration procedure. Fix any number $K > 1$ and assume that $\delta \in (0,1)$ is such that (possibly considering a smaller τ)

$$\tau + \delta < \varepsilon_0/2, \qquad C_2\delta(1+2C_1) \leq \tau, \qquad 2C_5\delta^{\vartheta}K \leq \delta\,. \tag{1.65}$$

Given ψ, we set $\varphi_0 = \psi$ and we denote by η^1 the vector defined as in (1.53). We set $E_1 = E - \eta^1$ and denote by φ_1 the function such that $\partial E_1 = \{x + \varphi_1(x)\nu_F(x) \,:\, x \in \partial F\}$. As before, φ_1 satsfies

$$y + \varphi_0(y)\nu_F(y) - \eta^1 = z + \varphi_1(z)\nu_F(z)\,.$$

Since $\|\psi\|_{W^{2,p}(\partial F)} \leq \delta$ and $|\eta| \leq C_1\|\psi\|_{L^2(\partial F)}$, by inequalities (1.54), (1.58) and (1.65) we have

$$\|\varphi_1\|_{W^{2,p}(\partial F)} \leq C_2\delta(1+C_1) \leq \tau\,. \tag{1.66}$$

Using again that $\|\psi\|_{W^{2,p}(\partial F)} < \delta < 1$, by estimate (1.64) we obtain

$$\Big|\int_{\partial F}\varphi_1(y)\nu_F(y)\,d\mu(y))\Big| \leq C_5\|\varphi_0\|^{\vartheta}_{L^2(\partial F)}\big(\|\varphi_1\|_{L^2(\partial F)} + \|\varphi_0\|_{L^2(\partial F)}\big)\,,$$

where we have $\|\varphi_0\|_{L^2(\partial F)} \leq \delta$.
We now distinguish two cases.
If $\|\varphi_0\|_{L^2(\partial F)} \leq K\|\varphi_1\|_{L^2(\partial F)}$, from the previous inequality and (1.65), we get

$$\begin{aligned}\Big|\int_{\partial F}\varphi_1(y)\nu_F(y)\,d\mu(y)\Big| &\leq C_5\delta^{\vartheta}\big(\|\varphi_1\|_{L^2(\partial F)} + \|\varphi_0\|_{L^2(\partial F)}\big)\\ &\leq 2C_5\delta^{\vartheta}K\|\varphi_1\|_{L^2(\partial F)}\\ &\leq \delta\|\varphi_1\|_{L^2(\partial F)}\,,\end{aligned}$$

thus, the conclusion follows with $\eta = \eta^1$.
In the other case,

$$\|\varphi_1\|_{L^2(\partial F)} \leq \frac{\|\varphi_0\|_{L^2(\partial F)}}{K} \leq \frac{\delta}{K} \leq \delta\,. \tag{1.67}$$

We then repeat the whole procedure: we denote by η^2 the vector defined as in formula (1.53) with ψ replaced by φ_1, we set $E_2 = E_1 - \eta^2 = E - \eta^1 - \eta^2$ and we consider the corresponding φ_2 which satisfies

$$w + \varphi_2(w)\nu_F(w) = z + \varphi_1(z)\nu_F(z) - \eta^2 = y + \varphi_0(y)\nu_F(y) - \eta^1 - \eta^2\,.$$

Since

$$\begin{aligned}\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \eta^2| &\leq \delta + C_1\delta + C_1\|\varphi_1\|_{L^2(\partial F)}\\ &\leq \delta + C_1\delta\Big(1 + \frac{1}{K}\Big) \leq C_2\delta(1+2C_1) \leq \tau\,,\end{aligned}$$

the map $T_{\varphi_0}(y) = \pi_F(y + \varphi_0(y)\nu_F(y) - (\eta^1 + \eta^2))$ is a diffeomorphism thanks to formula (1.55) (having chosen τ and δ small enough).

Thus, by applying inequalities (1.58) (with $\eta = \eta^1 + \eta^2$), (1.54), (1.65) and (1.67), we get

$$\|\varphi_2\|_{W^{2,p}(\partial F)} \leq C_2\big(\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \eta^2|\big) \leq C_2\delta\Big(1 + C_1 + \frac{C_1}{K}\Big) \leq \tau\,,$$

as $K > 1$, analogously to conclusion (1.66). On the other hand, by estimates (1.54), (1.66) and (1.67),

$$\|\varphi_1\|_{W^{2,p}(\partial F)} + \eta^2 \leq C_2\delta(1 + C_1) + C_1\frac{\delta}{K} \leq C_2\delta(1 + 2C_1) \leq \tau\,,$$

hence, also the map $T_{\varphi_1}(x) = \pi_F(x + \varphi_1(x)\nu_F(x) - \eta^2)$ is a diffeomorphism satisfying inequalities (1.55) and (1.56). Therefore, arguing as before, we obtain

$$\Big|\int_{\partial F} \varphi_2(y)\nu_F(y)\,d\mu(y)\Big| \leq C_5\|\varphi_1\|^{\vartheta}_{L^2(\partial F)}\big(\|\varphi_2\|_{L^2(\partial F)} + \|\varphi_1\|_{L^2(\partial F)}\big)\,.$$

Since $\|\varphi_1\|_{L^2(\partial F)} \leq \delta$ by inequality (1.67), if $\|\varphi_1\|_{L^2(\partial F)} \leq K\|\varphi_2\|_{L^2(\partial F)}$ the conclusion follows with $\eta = \eta^1 + \eta^2$. Otherwise, we iterate the procedure observing that

$$\|\varphi_2\|_{L^2(\partial F)} \leq \frac{\|\varphi_1\|_{L^2(\partial F)}}{K} \leq \frac{\|\varphi_0\|_{L^2(\partial F)}}{K^2} \leq \frac{\delta}{K^2}\,.$$

This construction leads to three (possibly finite) sequences η^n, E_n and φ_n such that

$$\begin{cases} E_n = E - \eta^1 - \cdots - \eta^n, \qquad |\eta^n| \leq \frac{C_1\delta}{K^{n-1}} \\ \|\varphi_n\|_{W^{2,p}(\partial F)} \leq C_2\big(\|\varphi_0\|_{W^{2,p}(\partial F)} + |\eta^1 + \cdots + \eta^n|\big) \leq C_2\delta(1 + 2C_1) \\ \|\varphi_n\|_{L^2(\partial F)} \leq \frac{\delta}{K^n} \\ \partial E_n = \{x + \varphi_n(x)\nu_F(x)\,:\, x \in \partial F\} \end{cases}$$

If for some $n \in \mathbb{N}$ we have $\|\varphi_{n-1}\|_{L^2(\partial F)} \leq K\|\varphi_n\|_{L^2(\partial F)}$, the construction stops, since, arguing as before,

$$\Big|\int_{\partial F} \varphi_n(y)\nu_F(y)\,d\mu(y)\Big| \leq \delta\|\varphi_n\|_{L^2(\partial F)}$$

and conclusion follows with $\eta = \eta^1 + \cdots + \eta^n$ and $\varphi = \varphi_n$. Otherwise, the iteration continues indefinitely and we reach the conclusion with

$$\eta = \sum_{n=1}^{\infty} \eta^n, \qquad \varphi = 0\,,$$

(notice that the series is converging) which actually means that $E = \eta + F$, hence the thesis is obvious. $\square$

We are now ready to prove the main theorem of this chapter.

Proof of Theorem 1.19.
Step 1. We first want to show that

$$m_0 = \inf\left\{\Pi_E(\varphi) \,:\, \varphi \in T^\perp(\partial E), \|\varphi\|_{H^1(\partial E)} = 1\right\} > 0. \tag{1.68}$$

To this aim, we consider a minimizing sequence φ_i for the above infimum and we assume that $\varphi_i \rightharpoonup \varphi_0$ weakly in $H^1(\partial E)$, then $\varphi_0 \in T^\perp(\partial E)$ (since it is a closed subspace of $H^1(\partial E)$) and if $\varphi_0 \neq 0$, there holds

$$m_0 = \lim_{i\to+\infty} \Pi_E(\varphi_i) \geq \Pi_E(\varphi_0) > 0$$

due to the strict stability of E and the lower semicontinuity of Π_E (recall formula (1.31) and the fact that the weak convergence in $H^1(\partial E)$ implies strong convergence in $L^2(\partial E)$ by Sobolev embeddings). On the other hand, if instead $\varphi_0 = 0$, again by the strong convergence of $\varphi_i \to \varphi_0$ in $L^2(\partial E)$, by looking at formula (1.31), we have

$$m_0 = \lim_{i\to\infty} \Pi_E(\varphi_i) = \lim_{i\to\infty} \int_{\partial E} |\nabla\varphi_i|^2\, d\mu = \lim_{i\to\infty} \|\varphi_i\|^2_{H^1(\partial E)} = 1$$

since $\|\varphi_i\|_{L^2(\partial E)} \to 0$.
Step 2. Now we show that there exists a constant $\delta_1 > 0$ such that if E is like in the statement and $\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\}$, with $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta_1$, and $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, then

$$\inf\left\{\Pi_F(\varphi) \,:\, \varphi \in \widetilde{H}^1(\partial F), \|\varphi\|_{H^1(\partial F)} = 1, \left|\int_{\partial F} \varphi\nu_F\, d\mu\right| \leq \delta_1\right\} \geq \frac{m_0}{2}. \tag{1.69}$$

We argue by contradiction assuming that there exists a sequence of sets F_i with $\partial F_i = \{y + \psi_i(y)\nu_E(y) \,:\, y \in \partial E\}$ with $\|\psi_i\|_{W^{2,p}(\partial E)} \to 0$ and $\mathrm{Vol}(F_i) = \mathrm{Vol}(E)$, and a sequence of functions $\varphi_i \in \widetilde{H}^1(\partial F_i)$ with $\|\varphi_i\|_{H^1(\partial F_i)} = 1$ and $\int_{\partial F_i} \varphi_i \nu_{F_i}\, d\mu_i \to 0$, such that

$$\Pi_{F_i}(\varphi_i) < \frac{m_0}{2}.$$

We then define the following sequence of smooth functions

$$\widetilde{\varphi}_i(y) = \varphi_i(y + \psi_i(y)\nu_E(y)) - \fint_{\partial E} \varphi_i(y + \psi_i(y)\nu_E(y))\, d\mu(y) \tag{1.70}$$

which clearly belong to $\widetilde{H}^1(\partial E)$. Setting $\theta_i(y) = y + \psi_i(y)\nu_E(y)$, as $p > \max\{2, n-1\}$, by the Sobolev embeddings, $\theta_i \to \mathrm{Id}$ in $C^{1,\alpha}$ and $\nu_{F_i} \circ \theta_i \to \nu_E$ in $C^{0,\alpha}(\partial E)$, hence, the sequence $\widetilde{\varphi}_i$ is bounded in $H^1(\partial E)$ and if $\{e_k\}$ is the special orthonormal basis found in Remark 1.16, we have $\langle \nu_{F_i} \circ \theta_i | e_k\rangle \to \langle \nu_E, e_k\rangle$ uniformly for all $k \in \{1, \ldots, n\}$. Thus,

$$\int_{\partial E} \widetilde{\varphi}_i \langle \nu_E | \varepsilon_i\rangle\, d\mu \to 0,$$

as $i \to \infty$, indeed,

$$\int_{\partial E} \widetilde{\varphi}_i \langle \nu_E | e_k\rangle\, d\mu - \int_{\partial E} \widetilde{\varphi}_i \langle \nu_{F_i} \circ \theta_i | e_k\rangle\, d\mu \to 0$$

and
$$\int_{\partial E} \widetilde{\varphi}_i \langle \nu_{F_i} \circ \theta_i | e_k \rangle \, d\mu = \int_{\partial F_i} \varphi_i \langle \nu_{F_i} | e_k \rangle \, J\theta_i^{-1} d\mu_i \to 0,$$
as the Jacobians (notice that $J\theta_i$ are Jacobians "relative" to the hypersurface ∂E) $J\theta_i^{-1} \to 1$ uniformly and we assumed $\int_{\partial F_i} \varphi_i \nu_{F_i} \, d\mu_i \to 0$.
Hence, using expression (1.36), for the projection map π on $T^{\perp}(\partial E)$, it follows
$$\|\pi(\widetilde{\varphi}_i) - \widetilde{\varphi}_i\|_{H^1(\partial E)} \to 0$$
as $i \to \infty$ and
$$\lim_{i\to\infty} \|\pi(\widetilde{\varphi}_i)\|_{H^1(\partial E)} = \lim_{i\to\infty} \|\widetilde{\varphi}_i\|_{H^1(\partial E)} = \lim_{i\to\infty} \|\varphi_i\|_{H^1(\partial F_i)} = 1, \tag{1.71}$$
since $\|\varphi_i\|_{W^{2,p}(\partial E)} \to 0$, thus $\|\varphi_i\|_{C^{1,\alpha}(\partial E)} \to 0$, by looking at the definition of the functions $\widetilde{\varphi}_i$ in formula (1.70).
Note now that the $W^{2,p}$– convergence of F_i to E (computing similarly to Remark (A.2) in Appendix A, the second fundamental form $B_{\partial F_i}$ of ∂F_i is "morally" the Hessian of φ_i) implies
$$B_{\partial F_i} \circ \theta_i \to B_{\partial E} \qquad \text{in } L^p(\partial E)\,,$$
as $i \to \infty$, then, by Sobolev embeddings again (in particular $H^1(\partial E) \hookrightarrow L^q(\partial E)$ for any $q \in [1, 2^*)$, with $2^* = 2(n-1)/(n-3)$ which is larger than 2) and the $W^{2,p}$–convergence of F_i to E, we get
$$\int_{\partial F_i} |B_{\partial F_i}|^2 \varphi_i^2 \, d\mu_i - \int_{\partial E} |B_{\partial E}|^2 \widetilde{\varphi}_i^2 \, d\mu \to 0\,.$$
Standard elliptic estimates for the problem (1.3) (see [13], for instance) imply the convergence of the potentials
$$v_{F_i} \to v_E \ \text{ in } C^{1,\beta}(\mathbb{T}^n) \text{ for all } \beta \in (0,1),$$
for $i \to \infty$, hence arguing as before,
$$\int_{\partial F_i} \partial_{\nu_{F_i}} v_{F_i} \varphi_i^2 \, d\mu_i - \int_{\partial E} \partial_{\nu_E} v_E \widetilde{\varphi}_i^2 \, d\mu \to 0\,.$$
Setting, as in Remark 1.13,
$$\begin{aligned} v_{E,\widetilde{\varphi}_i}(x) &= \int_{\partial E} G(x,y) \widetilde{\varphi}_i(y) \, d\mu(y) \\ &= \int_{\partial E} G(x,y) \varphi_i(\theta_i(y)) \, d\mu(y) - m_i \int_{\partial E} G(x,y) \, d\mu(y)\,, \end{aligned}$$
where $m_i = \fint_{\partial E} \varphi_i(y + \psi_i(y)\nu_E(y)) \, d\mu(y) \to 0$, as $i \to \infty$, and
$$v_{F_i,\varphi_i}(x) = \int_{\partial F_i} G(x,z) \varphi_i(z) \, d\mu_i(z) = \int_{\partial E} G(x, \theta_i(y)) \varphi_i(\theta_i(y)) J\theta_i(y) \, d\mu(y)\,,$$
it is easy to check (see [2, pages 537–538], for details) that
$$\int_{\mathbb{T}^n} |\nabla v_{F_i,\varphi_i}|^2 \, dx - \int_{\mathbb{T}^n} |\nabla v_{E,\widetilde{\varphi}_i}|^2 \, dx \to 0\,.$$

Finally, recalling expression (1.32), we conclude

$$\Pi_{F_i}(\varphi_i) - \Pi_E(\widetilde{\varphi}_i) \to 0\,,$$

since we have

$$\|\varphi_i\|_{L^2(\partial F_i)} - \|\widetilde{\varphi}_i\|_{L^2(\partial E)} \to 0\,,$$

which easily follows again by looking at the definition of the functions $\widetilde{\varphi}_i$ in formula (1.70) and taking into account that $\|\varphi_i\|_{C^{1,\alpha}(\partial E)} \to 0$, hence limits (1.71) imply

$$\|\nabla\varphi_i\|_{L^2(\partial F_i)} - \|\nabla\widetilde{\varphi}_i\|_{L^2(\partial E)} \to 0\,.$$

By the previous conclusion $\|\pi(\widetilde{\varphi}_i) - \widetilde{\varphi}_i\|_{H^1(\partial E)} \to 0$ and Sobolev embeddings, it this then straightforward, arguing as above, to get also

$$\Pi_E(\widetilde{\varphi}_i) - \Pi_E(\pi(\widetilde{\varphi}_i)) \to 0,$$

hence,

$$\Pi_{F_i}(\varphi_i) - \Pi_E(\pi(\widetilde{\varphi}_i)) \to 0.$$

Since we assumed that $\Pi_{F_i}(\varphi_i) < m_0/2$, we conclude that for $i \in \mathbb{N}$, large enough there holds

$$\Pi_E(\pi(\widetilde{\varphi}_i)) \leq \frac{m_0}{2} < m_0,$$

which is a contraddiction to Step 1, as $\pi(\widetilde{\varphi}_i) \in T^\perp(\partial E)$.

Step 3. Let us fix F such that $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(F \triangle E) < \delta$ and

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\} \subseteq N_\varepsilon,$$

with $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ where $\delta > 0$ is smaller than δ_1 given by Step 2.
Taking a possibly smaller $\delta > 0$, we consider the field X and the associated flow Φ found in Lemma 1.22. Hence, $\operatorname{div} X = 0$ in N_ε and $\Phi(1, y) = y + \psi(y)\nu_E(y)$, for all $y \in \partial E$, that is, $\Phi(1, \partial E) = \partial F \subseteq N_\varepsilon$ which implies $E_1 = \Phi(1, E) = F$. Then X is an admissible smooth vector field, as $\mathrm{Vol}(E_1) = \mathrm{Vol}(E) = \mathrm{Vol}(F)$, by the last part of such lemma.
By Lemma 1.24, choosing an even smaller $\delta > 0$ if necessary, possibly replacing F with a translate $F - \sigma$ for some $\eta \in \mathbb{R}^n$ if needed, we can assume that

$$\left|\int_{\partial E} \psi\,\nu_E\,d\mu\right| \leq \frac{\delta_1}{2}\|\psi\|_{L^2(\partial E)}. \tag{1.72}$$

Letting $E_t = \Phi_t(E)$, we now claim that

$$\left|\int_{\partial E_t} \langle X|\nu_{E_t}\rangle\nu_{E_t}\,d\mu\right| \leq \delta_1\|\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \qquad \forall t \in [0,1]. \tag{1.73}$$

To this aim, we write

$$\begin{aligned}
\int_{\partial E_t} \langle X|\nu_{E_t}\rangle\nu_{E_t}\,d\mu &= \int_{\partial E} \langle X\circ\Phi_t|\nu_{E_t}\circ\Phi_t\rangle(\nu_{E_t}\circ\Phi_t)\,J\Phi_t\,d\mu \\
&= \int_{\partial E} \langle X\circ\Phi_t|\nu_E\rangle\nu_E\,d\mu + R_1 \\
&= \int_{\partial E} \langle X(x)|\nu_E\rangle\nu_E\,d\mu + R_1 + R_2 \\
&= \int_{\partial E} \psi\nu_E\,d\mu + R_1 + R_2 + R_3
\end{aligned}$$

with R_1, R_2 and R_3 appropriate.
By the definition of X in formula (1.44) (in the proof of Lemma 1.22), the bounds $0 < C_1 \leq \xi \leq C_2$ and $\|J(\pi_E \circ \Phi_t)^{-1}\|_{L^\infty(\partial E)} \leq C_3$ (by inequality (1.42) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we have $\|\Phi(t,\cdot) - \mathrm{Id}\|_{C^{1,\alpha}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)} \leq C\delta$), the following inequality holds

$$
\begin{aligned}
\int_{\partial E} |X(\Phi(t,x))|\,d\mu &= \\
&= \int_{\partial E} \left| \int_0^{\psi(\pi_E(\Phi(t,x)))} \frac{\xi(\Phi(t,x))\nabla d_E(\Phi(t,x))}{\xi(\Phi(t,x) + s\nu(\pi_E(\Phi(t,x))))}\,ds \right| d\mu \\
&\leq C \int_{\partial E} |\psi(\pi_E(\Phi(t,x)))|\,d\mu \\
&= \int_{\partial E} |\psi(z)| J(\pi_E \circ \Phi_t)^{-1}(z)\,d\mu(z) \\
&\leq C\|\psi\|_{L^2(\partial E)}. \qquad (1.74)
\end{aligned}
$$

for every $t \in [0,1]$.
We want now to prove that for every $\bar{\varepsilon} > 0$, choosing a suitably small $\delta > 0$ we have the estimate

$$|R_1| + |R_2| + |R_3| \leq \varepsilon\|\psi\|_{L^2(\partial E)}. \qquad (1.75)$$

First,

$$
\begin{aligned}
R_1 &= \int_{\partial E} \langle X \circ \Phi_t | \nu_{E_t} \circ \Phi \rangle \nu_{E_t} \circ \Phi_t [J\Phi_t - 1]\,d\mu \\
&\quad + \int_{\partial E} \langle X \circ \Phi_t | \nu_{E_t} \circ \Phi_t \rangle \nu_{E_t} \circ \Phi_t\,d\mu - \int_{\partial E} \langle X \circ \Phi_t, \nu_E \rangle \nu_E\,d\mu \\
&= \int_{\partial E} \langle X \circ \Phi_t | \nu_{E_t} \circ \Phi_t \rangle \nu_{E_t} \circ \Phi_t\,[J\Phi_t - 1]\,d\mu \\
&\quad + \int_{\partial E} \langle X \circ \Phi_t | \nu_{E_t} \circ \Phi_t - \nu_E \rangle \nu_E\,d\mu \\
&\quad + \int_{\partial E} \langle X \circ \Phi_t | \nu_{E_t} \circ \Phi_t \rangle (\nu_{E_t} \circ \Phi_t - \nu_E)\,d\mu \\
&\leq \int_{\partial E} |X \circ \Phi_t|\,\|J\Phi_t - 1\|_{L^\infty(\partial E)}\,d\mu \\
&\quad + \int_{\partial E} |X \circ \Phi_t|\,\|\nu_E - \nu_{E_t} \circ \Phi_t\|_{L^\infty(\partial E)}\,d\mu\,,
\end{aligned}
$$

then, since by equality (1.41), it follow that for every $t \in [0,1]$ the two terms

$$\|\nu_E - \nu_{E_t} \circ \Phi(t,x)\|_{L^\infty(\partial E)} \qquad \text{and} \qquad \|J\Phi_t - 1\|_{L^\infty(\partial E)}$$

can be made (uniformly in $t \in [0,1]$) small as we want, if $\delta > 0$ is small enough, by using inequality (1.74), we obtain

$$|R_1| \leq \bar{\varepsilon}\|\psi\|_{L^2(\partial E)}/3.$$

Then we estimate, by means of inequality (1.41) and where $s = s(t, y) \in [t, 1]$,

$$
\begin{aligned}
|R_2| &\leq \int_{\partial E} |X(\Phi(t,x)) - X(\Phi(1,x))| + |X(\Phi(1,x)) - X(x)|\, d\mu \\
&\leq \int_{\partial E} |X(\Phi(t,x)) - X(\Phi(1,x))| + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial E)} \\
&= \int_{\partial E} (1-t) |\nabla X(\Phi_s(y))| \left| \frac{\partial \Phi_s}{\partial t}(y) \right| d\mu(y) + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial E)} \\
&\leq \int_{\partial E} |\nabla X(\Phi(s,x))| |\Phi(t,x) - \Phi(1,x)| + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial E)} \\
&\leq C \|\nabla X\|_{L^\infty(N_\varepsilon)} C \|\psi\|_{L^2(\partial E)} + \|\nabla X\|_{L^2(N_\varepsilon)} \|\psi\|_{L^2(\partial E)},
\end{aligned}
$$

where in the last inequality we use equation (1.74). Hence, using equality (1.45) and Sobolev embeddings, as $p > \max\{2, n-1\}$, we get

$$
|R_2| \leq C \|\psi\|_{W^{2,p}(\partial E)} \|\psi\|_{L^2(\partial E)},
$$

then, since $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, we obtain

$$
|R_2| < \bar{\varepsilon} \|\psi\|_{L^2(\partial E)} / 3,
$$

if δ_2 is small enough.
Arguing similarly, recalling the definition of X given by formula (1.44), we also obtain $|R_3| \leq \bar{\varepsilon} \|\psi\|_{L^2(\partial E)}$, hence estimate (1.75) follows. We can then conclude that, for $\delta > 0$ small enough, we have

$$
\begin{aligned}
\left| \int_{\partial E_t} \langle X | \nu_{E_t} \rangle \nu_{E_t}\, d\mu_t \right| &\leq \left| \int_{\partial E} \psi \nu_E\, d\mu \right| + \bar{\varepsilon} \|\psi\|_{L^2(\partial E)} \\
&\leq \Big(\frac{\delta_1}{2} + \bar{\varepsilon} \Big) \|\psi\|_{L^2(\partial E)}
\end{aligned}
$$

for any $t \in [0,1]$, where in the last inequality we used the assumption (1.72), thus choosing $\bar{\varepsilon} = \delta_1/4$ we get

$$
\left| \int_{\partial E_t} \langle X | \nu_{E_t} \rangle \nu_{E_t}\, d\mu_t \right| \leq \frac{3\delta_1}{4} \|\psi\|_{L^2(\partial E)}.
$$

Along the same line, it is then easy to prove that

$$
\|\langle X | \nu_{E_t} \rangle\|_{L^2(\partial E_t)} \geq (1 - \varepsilon) \|\psi\|_{L^2(\partial E)}, \tag{1.76}
$$

for any $t \in [0,1]$, hence claim (1.73) follows.
As a consequence, since $\langle X | \nu_{E_t} \rangle \in \widetilde{H}^1(\partial E_t)$, being X admissible for E_t (recalling computation 1.9) and ∂E_t can be described as a graph over ∂E with a function with small norm in $W^{2,p}(\partial E)$ (by estimate (1.42) of Lemma 1.22 and arguing as in Remark (A.2) in Appendix A), we can apply Step 2 with $F = E_t$ to the function $\langle X | \nu_{E_t} \rangle / \|\langle X | \nu_{E_t} \rangle\|_{H^1(\partial E_t)}$, concluding

$$
\Pi_{E_t}(\langle X | \nu_{E_t} \rangle) \geq \frac{m_0}{2} \|\langle X | \nu_{E_t} \rangle\|_{H^1(\partial E_t)}. \tag{1.77}
$$

By means of Lemma 1.23, for $\delta > 0$ small enough, we now show the following inequality on ∂E_t (here div is the divergence operator and $X_{\tau_t} = X - \langle X|\nu_{E_t}\rangle\nu_{E_t}$ is a tangent vector field on ∂E_t), for any $t \in [0,1]$,

$$\begin{aligned}
\|\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial E_t)} &= \|\operatorname{div} X_{\tau_t}\langle X|\nu_{E_t}\rangle + \langle X_{\tau_t}|\nabla\langle X,\nu_{E_t}\rangle\rangle\|_{L^{\frac{p}{p-1}}(\partial E_t)} \\
&\leq C\|\nabla X_{\tau_t}\|_{L^2(\partial E_t)}\|\langle X|\nu_{E_t}\rangle\|_{L^{\frac{2p}{p-2}}(\partial E_t)} \\
&\quad + C\|X_{\tau_t}\|_{L^{\frac{2p}{p-2}}(\partial E_t)}\|\nabla\langle X|\nu_{E_t}\rangle\|_{L^2(\partial E_t)} \\
&\leq C\|X\|_{H^1(\partial E_t)}\|X\|_{L^{\frac{2p}{p-2}}(\partial E_t)} \\
&\leq C\|X\|^2_{H^1(\partial E_t)} \\
&\leq C\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)},
\end{aligned} \tag{1.78}$$

where we used the Sobolev embedding $H^1(\partial E_t) \hookrightarrow L^{\frac{2p}{p-2}}(\partial E_t)$, as $p > \max\{2, n-1\}$.

Then, we compute (here H_t is the mean curvature of ∂E_t and v_{E_t} is the potential relative to E_t, defined by formula (1.1))

$$\begin{aligned}
J(F) - J(E) &= J(E_1) - J(E) \\
&= \int_0^1 (1-t)\frac{d^2}{dt^2}J(E_t)\,dt \\
&= \int_0^1 (1-t)\Pi_{E_t}(\langle X|\nu_{E_t}\rangle)\,dt \\
&\quad - \int_0^1 (1-t)\int_{\partial E_t}(4\gamma v_{E_t} + \mathrm{H}_t)\operatorname{div}_{\partial E_t}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu\Big)\,dt.
\end{aligned}$$

by Theorem 1.10, the definition of Π_{E_t} in formula (1.31) and taking into account that $\operatorname{div} X = 0$ in N_ε.

Hence, by estimate (1.77), we have (recall that $4\gamma v_E + \mathrm{H} = 4\gamma v_{E_0} + \mathrm{H}_0 = \lambda$ constant, as E is a critical set)

$$\begin{aligned}
J(F) - J(E) &\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&\quad - \int_0^1 (1-t)\int_{\partial E_t}(\mathrm{H}_t + 4\gamma v_{E_t})\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu_t\,dt \\
&= \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&\quad - \int_0^1 (1-t)\int_{\partial E_t}(\mathrm{H}_t + 4\gamma v_{E_t} - \lambda)\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\,d\mu_t\,dt \\
&\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt \\
&\quad - \int_0^1 (1-t)\|\mathrm{H}_t + 4\gamma v_{E_t} - \lambda\|_{L^p(\partial E_t)}\|\operatorname{div}(X_{\tau_t}\langle X|\nu_{E_t}\rangle)\|_{L^{\frac{p}{p-1}}(\partial E_t)}\,dt
\end{aligned}$$

$$\geq \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt$$
$$-C\int_0^1 (1-t)\|\mathrm{H}_t + 4\gamma v_{E_t} - \lambda\|_{L^p(\partial E_t)}\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt,$$

by estimate (1.78). If $\delta > 0$ is sufficiently small, as E_t is $W^{2,p}$–close to E (recall again Remark (A.2) in Appendix A and the definition of v_{E_t} in formula (1.1)), we have $\|\mathrm{H}_t + 4\gamma v_{E_t} - \lambda\|_{L^p(\partial E_t)} < m_0/4C$, hence

$$J(F) - J(E) \geq \frac{m_0}{4}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt.$$

Then, we can conclude the proof of the theorem with the following series of inequalities, holding for a suitably small $\delta > 0$ as in the statement,

$$\begin{aligned}
J(F) &\geq J(E) + \frac{m_0}{2}\int_0^1 (1-t)\|\langle X|\nu_{E_t}\rangle\|^2_{H^1(\partial E_t)}\,dt\\
&\geq J(E) + C\|\langle X|\nu_E\rangle\|^2_{L^2(\partial E)}\\
&\geq J(E) + C\|\psi\|^2_{L^2(\partial E)}\\
&\geq J(E) + C[\mathrm{Vol}(E\triangle F)]^2\\
&\geq J(E) + C[\alpha(E,F)]^2,
\end{aligned}$$

where the first inequality is due to the $W^{2,p}$–closedness of E_t to E, the second one by the very expression (1.44) of the vector field X on ∂E,

$$|\langle X(y)|\nu_E(y)\rangle| = \Big|\int_0^{\psi(y)} \frac{ds}{\xi(y + s\nu_E(y))}\Big| \leq C|\psi(y)|,$$

the third follows by a straighfoward computation (involving the map L defined by formula (1.40) and its Jacobian), as ∂E is a "normal graph" over ∂F with ψ as "height function", finally the last one simply by the definition of the "distance" α, recalling that we possibly translated the "original" set F by a vector $\eta \in \mathbb{R}^n$, at the beginning of this step. □

We conclude this chapter by proving two propositions that will be used later. The first one says that when a set is sufficiently $W^{2,p}$–close to a strictly stable critical set of the functional J, then the quadratic form (1.31) remains uniformly positive definite (on the orthogonal complement of its degeneracy subspace, see the discussion at the end of the previous section).

Proposition 1.25. *Let $p > \max\{2, n-1\}$ and $E \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε a tubular neighbourhood of E, as in formula (1.37). Then, for every $\theta \in (0,1]$ there exist $\sigma_\theta, \delta > 0$ such that if a smooth set $F \subseteq \mathbb{T}^n$ is $W^{2,p}$–close to E, that is, $\mathrm{Vol}(F\triangle E) < \delta$ and $\partial F \subseteq N_\varepsilon$ with*

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\}$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, there holds

$$\Pi_F(\varphi) \geq \sigma_\theta\|\varphi\|^2_{H^1(\partial F)}, \tag{1.79}$$

for all $\varphi \in \widetilde{H}^1(\partial F)$ *satisfying*

$$\min_{\eta \in \mathrm{O}_E} \|\varphi - \langle \eta | \nu_F \rangle\|_{L^2(\partial F)} \geq \theta \|\varphi\|_{L^2(\partial F)},$$

where O_E *is defined by formula* (1.35).

Proof.
Step 1. We first claim that the strict stability of E implies

$$\Pi_E(\varphi) > 0 \qquad \text{for all } \varphi \in \widetilde{H}(\partial E) \setminus T(\partial E). \tag{1.80}$$

To this aim we observe that from formula (1.1) and the properties of the Green function, we get

$$\begin{aligned} \nabla v_E(x) &= \int_{\mathbb{T}^n} \nabla_x G(x,y) u_E(x)\, dy \\ &= \int_E \nabla_x G(x,y)\, dy - \int_{E^c} \nabla_x G(x,y)\, dy \\ &= -\int_E \nabla_y G(x,y)\, dy + \int_{E^c} \nabla_y G(x,y)\, dy \\ &= -2 \int_{\partial E} G(x,y) \nu_E(y)\, d\mu(y)\,, \end{aligned} \tag{1.81}$$

where in the last passage we applied the divergence theorem.
By means of formula (A.4)

$$\Delta \nu_E = \nabla \mathrm{H} - |B|^2 \nu_E\,,$$

since E (being critical) satisfies $\mathrm{H} + 4\gamma v_E = \lambda$ for some constant $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} -\Delta \nu_E - |B|^2 \nu_E &= \nabla(4\gamma v_E - \lambda) \\ &= \nabla^{\mathbb{T}^n}(4\gamma v_E - \lambda) - \partial_{\nu_E}(4\gamma v_E - \lambda) \\ &= -4\gamma(\partial_{\nu_E} v_E)\nu_E - 8\gamma \int_{\partial E} G(x,y)\nu_E(y)\, d\mu(y) \end{aligned}$$

on ∂E, by formula (1.81).
This equation can be written as $L(\nu_i) = 0$, for every $i \in \{1, \dots, n\}$, where L is the self–adjoint, linear operator defined as

$$L(\varphi) = -\Delta\varphi - |B|^2\varphi + 4\gamma \partial_{\nu_E} v_E \varphi + 8\gamma \int_{\partial E} G(x,y)\varphi(y)\, d\mu(y)\,,$$

then, if we "decompose" a smooth function $\varphi \in \widetilde{H}(\partial E) \setminus T(\partial E)$ as $\varphi = \psi + \langle \eta | \nu_E \rangle$, for some $\eta \in \mathbb{R}^n$ and $\psi \in T^\perp(\partial E) \setminus \{0\}$, we have (recalling formula (1.31))

$$\begin{aligned} \Pi_E(\varphi) &= \int_{\partial E} \langle L(\varphi) | \varphi \rangle\, d\mu \\ &= \int_{\partial E} \langle L(\psi) | \psi \rangle\, d\mu + 2 \int_{\partial E} \langle L(\langle \eta | \nu_E \rangle), \psi \rangle\, d\mu + \int_{\partial E} \langle L(\langle \eta | \nu_E \rangle) | \langle \eta | \nu_E \rangle \rangle\, d\mu \\ &= \Pi_E(\psi)\,. \end{aligned}$$

By approximation with smooth functions, we conclude that this equality holds for every function in $\widetilde{H}(\partial E)\setminus T(\partial E)$, hence $\Pi_E(\varphi)=\Pi_E(\psi)>0$ for every $\varphi\in\widetilde{H}(\partial E)\setminus T(\partial E)$, by the strict stability assumption on E.

We now show that for every $\theta\in(0,1]$ there holds

$$m_\theta=\inf\Big\{\,\Pi_E(\varphi)\,:\,\varphi\in\widetilde{H}^1(\partial E)\,,\|\varphi\|_{H^1(\partial E)}=1\\ \text{and }\min_{\eta\in\mathrm{O}_E}\|\varphi-\langle\eta|\nu_E\rangle\|_{L^2(\partial E)}\geq\theta\|\varphi\|_{L^2(\partial E)}\Big\}>0\,. \tag{1.82}$$

Indeed, let φ_i be a minimizing sequence for this infimum and assume that $\varphi_i\rightharpoonup\varphi_0\in\widetilde{H}^1(\partial E)$ weakly in $H^1(\partial E)$.
If $\varphi_0\neq 0$, as the weak convergence in $H^1(\partial E)$ implies strong convergence in $L^2(\partial E)$ by Sobolev embeddings, for every $\eta\in\mathrm{O}_E$ we have

$$\|\varphi_0-\langle\eta|\nu_E\rangle\|_{L^2(\partial E)}=\lim_{i\to\infty}\|\varphi_i-\langle\eta|\nu_E\rangle\|_{L^2(\partial E)}\geq\lim_{i\to\infty}\theta\|\varphi_i\|_{L^2(\partial E)}=\theta\|\varphi_0\|_{L^2(\partial E)},$$

hence,

$$\min_{\eta\in\mathrm{O}_E}\|\varphi_0-\langle\eta|\nu_E\rangle\|_{L^2(\partial E)}\geq\theta\|\varphi_0\|_{L^2(\partial E)}>0,$$

thus, we conclude $\varphi_0\in\widetilde{H}^1(\partial E)\setminus T(\partial E)$ and

$$m_\theta=\lim_{i\to\infty}\Pi_E(\varphi_i)\geq\Pi_E(\varphi_0)>0\,,$$

where the last inequality follows from estimate (1.80).
If $\varphi_0=0$, then again by the strong convergence of $\varphi_i\to\varphi_0$ in $L^2(\partial E)$, by looking at formula (1.31), we have

$$m_\theta=\lim_{i\to\infty}\Pi_E(\varphi_i)=\lim_{i\to\infty}\int_{\partial E}|\nabla\varphi_i|^2\,d\mu=\lim_{i\to\infty}\|\varphi_i\|^2_{H^1(\partial E)}=1$$

since $\|\varphi_i\|_{L^2(\partial E)}\to 0$.
Step 2. In order to finish the proof it is enough to show the existence of some $\delta>0$ such that if $\mathrm{Vol}(F\triangle E)<\delta$ and $\partial F=\big\{y+\psi(y)\nu_E(y)\,:\,y\in\partial E\big\}$ with $\|\psi\|_{W^{2,p}(\partial E)}<\delta$, then

$$\inf\Big\{\,\Pi_F(\varphi)\,:\,\varphi\in\widetilde{H}^1(\partial F)\,,\|\varphi\|_{H^1(\partial F)}=1\\ \text{and }\min_{\eta\in\mathrm{O}_E}\|\varphi-\langle\eta|\nu_F\rangle\|_{L^2(\partial F)}\geq\theta\|\varphi\|_{L^2(\partial F)}\Big\}\geq\sigma_\theta=\frac{1}{2}\min\{m_{\theta/2},1\}\,, \tag{1.83}$$

where $m_{\theta/2}$ is defined by formula (1.82), with $\theta/2$ in place of θ.
Assume by contradiction that there exist a sequence of smooth sets $F_i\subseteq\mathbb{T}^n$, with $\partial F_i=\{y+\psi_i(y)\nu_E(y)\,:\,y\in\partial E\}$ and $\|\psi_i\|_{W^{2,p}(\partial E)}\to 0$, and a sequence $\varphi_i\in\widetilde{H}^1(\partial F_i)$, with $\|\varphi_i\|_{H^1(\partial F_i)}=1$ and $\min_{\eta\in\mathrm{O}_E}\|\varphi_i-\langle\eta|\nu_{F_i}\rangle\|_{L^2(\partial F_i)}\geq\theta\|\varphi_i\|_{L^2(\partial F_i)}$, such that

$$\Pi_{F_i}(\varphi_i)<\sigma_\theta\leq m_{\theta/2}/2\,. \tag{1.84}$$

Let us suppose first that $\lim_{i\to\infty}\|\varphi_i\|_{L^2(\partial F_i)}=0$ and observe that by Sobolev embeddings $\|\varphi_i\|_{L^q(\partial F_i)}\to 0$ for some $q>2$, thus, since the functions ψ_i are uniformly bounded in $W^{2,p}(\partial E)$ for $p > \max\{2, n-1\}$, recalling formula (1.31), it is easy to see that

$$\lim_{i\to\infty} \Pi_{F_i}(\varphi_i) = \lim_{i\to\infty} \int_{\partial F_i} |\nabla\varphi_i|^2 \, d\mu_i = \lim_{i\to\infty} \|\varphi_i\|^2_{H^1(\partial F_i)} = 1\,,$$

which is a contradiction with assumption (1.84).
Hence, we may assume that

$$\lim_{i\to\infty} \|\varphi_i\|_{L^2(\partial F_i)} > 0. \tag{1.85}$$

The idea now is to write every φ_i as a function on ∂E. We define the functions $\widetilde{\varphi}_i(\partial E) \to \mathbb{R}$, given by

$$\widetilde{\varphi}_i(y) = \varphi_i\big(y + \psi_i(y)\nu_E(y)\big) - \fint_{\partial E} \varphi_i\big(y + \psi_i(y)\nu_E(y)\big)\, d\mu(y)\,,$$

for every $y \in \partial E$.
As $\psi_i \to 0$ in $W^{2,p}(\partial E)$, we have in particular that

$$\widetilde{\varphi}_i \in \widetilde{H}^1(\partial E)\,, \qquad \|\widetilde{\varphi}_i\|_{H^1(\partial E)} \to 1 \qquad \text{and} \qquad \frac{\|\widetilde{\varphi}_i\|_{L^2(\partial E)}}{\|\varphi_i\|_{L^2(\partial F_i)}} \to 1\,,$$

moreover, note also that $\nu_{F_i}(\cdot + \psi_i(\cdot)\nu_E(\cdot)) \to \nu_E$ in $W^{1,p}(\partial E)$ and thus in $C^{0,\alpha}(\partial E)$ for a suitable $\alpha \in (0,1)$, depending on p, by Sobolev embeddings. Using this fact and taking into account the third limit above and inequality (1.85), one can easily show that

$$\liminf_{i\to\infty} \frac{\min_{\eta\in\mathrm{O}_E} \|\widetilde{\varphi}_i - \langle\eta|\nu_E\rangle\|_{L^2(\partial E)}}{\|\widetilde{\varphi}_i\|_{L^2(\partial E)}} \geq \liminf_{i\to\infty} \frac{\min_{\eta\in\mathrm{O}_E} \|\varphi_i - \langle\eta|\nu_{F_i}\rangle\|_{L^2(\partial F_i)}}{\|\varphi_i\|_{L^2(\partial E_i)}} \geq \theta\,.$$

Hence, for $i \in \mathbb{N}$ large enough, we have

$$\|\widetilde{\varphi}_i\|_{H^1(\partial E)} \geq 3/4 \qquad \text{and} \qquad \min_{\eta\in\mathrm{O}_E} \|\widetilde{\varphi}_i - \langle\eta|\nu_E\rangle\|_{L^2(\partial E)} \geq \frac{\theta}{2}\|\widetilde{\varphi}_i\|_{L^2(\partial E)}\,,$$

then, in turn, by Step 1, we infer

$$\Pi_E(\widetilde{\varphi}_i) \geq \frac{9}{16} m_{\theta/2}\,. \tag{1.86}$$

Arguing now exactly like in the final part of Step 2 in the proof of Theorem 1.19, we have that all the terms of $\Pi_{F_i}(\varphi_i)$ are asymptotically close to the corresponding terms of $\Pi_E(\widetilde{\varphi}_i)$, thus

$$\Pi_{F_i}(\varphi_i) - \Pi_E(\widetilde{\varphi}_i) \to 0\,,$$

which is a contradiction, by inequalities (1.84) and (1.86). This establishes inequality (1.83) and concludes the proof. □

The following final result of this chapter states the fact that close to a strictly stable critical set there are no other critical sets (up to translations).

Proposition 1.26. *Let p and $E \subseteq \mathbb{T}^n$ be as in Proposition 1.25. Then, there exists $\delta > 0$ such that if $E' \subseteq \mathbb{T}^n$ is a smooth critical set with $\mathrm{Vol}(E') = \mathrm{Vol}(E)$, $\mathrm{Vol}(E \triangle E') < \delta$, $\partial E' \subseteq N_\varepsilon$ and*

$$\partial E' = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\}$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$, then E' is a translate of E.

Proof. In Step 3 of the proof of Theorem 1.19, it is shown that under these hypotheses on E and E', if $\delta > 0$ is small enough, we may find a small vector $\eta \in \mathbb{R}^n$ and an admissible smooth vector field X such that the associated flow Φ satisfies $\Phi(0, E) = E$, $\Phi(1, E) = E' - \eta$ and

$$\frac{d^2}{dt^2} J(\Phi(t, E)) \geq C[\mathrm{Vol}(E \triangle (E' - \eta))]^2 \,,$$

for all $t \in [0, 1]$, where C is a positive constant independent of E'.
Assume that E' is a smooth critical set as in the statement, which is not a translate of E, then $\frac{d}{dt} J(\Phi(t, E))\big|_{t=0} = 0$, but from the above formula it follows $\frac{d}{dt} J(\Phi(t, E))\big|_{t=1} > 0$, which implies that $E' - \eta$ cannot be critical, hence neither E', which is a contradiction. Indeed, $-X$ is an admissible vector field for $E' - \eta$ with an associate flow Ψ satisfying $\Psi(s, E' - \eta) = \Phi(1 - s, E)$, for every $s \in [0, 1]$, hence

$$\frac{d}{ds} J(\Psi(s, E' - \eta))\bigg|_{s=0} = \frac{d}{ds} J(\Phi(1 - s, E))\bigg|_{s=0} = -\frac{d}{dt} J(\Phi(t, E))\bigg|_{t=1} < 0 \,,$$

showing that $E' - \eta$ is not critical. $\square$

2

THE MODIFIED MULLINS–SEKERKA FLOW

In this chapter we introduce the *modified Mullins–Sekerka flow* and we describe its properties, leading to a long time existence result stating that the flow starting from a smooth subset of $\mathbb{T}^3$ "close enough" to a strictly stable set, exists smooth for all times.

We will make use several times of fractional Sobolev spaces*, about which we refer to Appendix B, for the basic facts. As for the "standard" Sobolev spaces (with integer order) they can be defined on smooth hypersurfaces (by standard localization/partition of unity technique), keeping most of their properties, in particular all the propositions of Appendix B hold for them.*

2.1 DEFINITION AND BASIC PROPERTIES

We start with the notion of smooth flow.

Definition 2.1 (Smooth flows of sets)**.** Let $E_t \subseteq \mathbb{T}^n$ for $t \in [0,T)$ be a one-parameter family of sets, we say that it is a *smooth flow* if there exists a smooth *reference set* $F \subseteq \mathbb{T}^n$ and a map $\Psi \in C^\infty(\mathbb{T}^n \times (0,T);\mathbb{T}^n)$ such that $\Psi_t = \Psi(\cdot,t)$ is a smooth diffeomorphism from $\mathbb{T}^n$ to $\mathbb{T}^n$ and $E_t = \Psi_t(F)$ for all $t \in [0,T)$.

The *velocity* of the motion of any point $x = \Psi_t(y)$ of the set E_t, with $y \in F$, is then given by

$$X_t(x) = \frac{\partial \Psi_t}{\partial t}(y),$$

hence,

$$\frac{\partial \Psi_t}{\partial t}(y) = X_t(\Psi_t(y)),$$

for every $y \in F$. Notice that, in general, the smooth vector field X_t is not independent of t, so it is not the infinitesimal generator of the flow Ψ, but we will see, in the computations in the sequel, that it will behave similarly to the (time–independent) vector fields X used to compute the first and second variation in the previous chapter.
When $x \in \partial E_t$, we define the *outer normal velocity* of the flow of the boundaries, which are smooth hypersurfaces of $\mathbb{T}^n$, as

$$V_t(x) = \langle X_t(x) | \nu_{E_t}(x)\rangle,$$

for every $t \in [0,T)$, where ν_{E_t} is the outer normal vector to E_t.

Before giving the definition of the modified Mullins–Sekerka flow we need some notations. Given a smooth set $E \subseteq \mathbb{T}^n$ and $\gamma \geq 0$, we denote by w_E the unique solution in $H^1(\mathbb{T}^n)$ of the following problem

$$\begin{cases} \Delta w_E = 0 & \text{in } \mathbb{T}^n \setminus \partial E \\ w_E = \mathrm{H} + 4\gamma v_E & \text{on } \partial E, \end{cases} \tag{2.1}$$

where v_E is the potential introduced in (1.3) and H is the mean curvature of ∂E. Moreover, we denote by w_E^+ and w_E^- the restrictions $w_E|_{E^c}$ and $w_E|_E$, respectively. Finally, denoting as usual by ν_E the outer unit normal to E, we set

$$[\partial_{\nu_E} w_E] = \partial_{\nu_E} w_E^+ - \partial_{\nu_E} w_E^- = -(\partial_{\nu_{E^c}} w_E^+ + \partial_{\nu_E} w_E^-)\,.$$

that is the jump of the normal derivative of w_E on ∂E.

Definition 2.2 (Modified Mullins–Sekerka flow)**.** Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that a smooth flow E_t such that $E_0 = E$, is a *modified Mullins–Sekerka flow with parameter* $\gamma \geq 0$, on the time interval $[0, T)$ and with initial datum E, if the outer normal velocity V_t of the moving boundaries ∂E_t is given by

$$V_t = [\partial_{\nu_t} w_t] \qquad \text{on } \partial E_t \text{ for all } t \in [0, T), \tag{2.2}$$

where $w_t = w_{E_t}$ (with the above definitions) and we used the simplified notation $\partial_{\nu_t} w_t$ in place of $\partial_{\nu_{E_t}} w_{E_t}$.

Remark 2.3. The adjective "modified" comes from the introduction of the parameter $\gamma \geq 0$ in the problem, while considering $\gamma = 0$, we have the original flow proposed by Mullins and Sekerka in [30].

Parametrizing the smooth hypersurfaces $M_t = \partial E_t$ of $\mathbb{T}^n$ by some smooth embeddings $\psi_t : M \to \mathbb{T}^n$ such that $\psi_t(M) = \partial E_t$ (here M is a fixed smooth differentiable $(n-1)$–dimensional manifold and the map $(t, p) \mapsto \psi(t, p) = \psi_t(p)$ is smooth), the geometric evolution law (2.2) can be expressed equivalently as

$$\left\langle \frac{\partial \psi_t}{\partial t} \,\middle|\, \nu_t \right\rangle = [\partial_{\nu_t} w_t], \tag{2.3}$$

where we denoted with ν_t the outer unit normal to $M_t = \partial E_t$.
Moreover, as the moving hypersurfaces $M_t = \partial E_t$ are compact, it is always possible to smoothly reparametrize them with maps (that we still call) ψ_t such that

$$\frac{\partial \psi_t}{\partial t} = [\partial_{\nu_t} w_t] \nu_t\,, \tag{2.4}$$

describing such flow. This follows by the *invariance by tangential perturbations of the velocity*, shared by the flow due to its geometric nature and can be proved following the line in Section 1.3 of [26], where the analogous property is shown in full detail for the (more famous) mean curvature flow. Roughly speaking, the tangential component of the velocity of the points of the moving hypersurfaces, does not affect the global "shape" during the motion.

Like the nonlocal Area functional J (see Definition 1.2), the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{T}^n$ (or $\mathbb{R}^n$). Moreover, if $\psi : [0, T) \times M \to \mathbb{T}^n$ is a modified Mullins–Sekerka flow of hypersurfaces, in the sense of equation (2.3) and $\Phi : [0, T) \times M \to M$ is a time–dependent family of smooth diffeomorphisms of M, then it is easy to check that the reparametrization $\widetilde{\psi} : [0, T) \times M \to \mathbb{T}^n$ defined as $\widetilde{\psi}(t, p) = \psi(t, \Phi(t, p))$ is still a modified Mullins–Sekerka flow (again in the sense of equation (2.3)). This property can be reread as "the flow is invariant under reparametrization", suggesting that the really relevant objects are actually the subsets $M_t = \psi_t(M)$ of $\mathbb{T}^n$.

We show now that the volume of the sets E_t is preserved during the evolution. We remark that instead, other geometric properties shared for instance by the mean curvature flow (see [26, Chapter 2]), like convexity are not necessarily maintained (see [10]), neither there holds the so–called "comparison property" asserting that if two initial sets are one contained in the other, they stay so during the two respective flows.

This volume–preserving property can be easily proved, arguing as in computation (1.9), indeed, if $E_t = \Psi_t(F)$ is a modified Mullins–Sekerka flow, described by $\Psi \in C^\infty([0,T) \times \mathbb{T}^n; \mathbb{T}^n)$, with an associated smooth vector field X_t satisfying

$$\frac{\partial \Psi_t}{\partial t}(y) = X_t(\Psi_t(y)) \,,$$

we have

$$\begin{aligned}
0 = \frac{d}{dt}\mathrm{Vol}(E_t) &= \int_F \frac{\partial}{\partial t} J\Psi_t(y)\, dy = \int_F \operatorname{div} X_t(\Psi(t,y)) J\Psi(t,y)\, dy \\
&= \int_{E_t} \operatorname{div} X_t(x)\, dx = \int_{\partial E_t} \langle X | \nu_t \rangle \, d\mu_t = \int_{\partial E_t} V_t \, d\mu_t \\
&= \int_{\partial E_t} [\partial_{\nu_t} w_t] \, d\mu_t = \int_{\partial E_t} \big(\partial_{\nu_t} w_t^+ - \partial_{\nu_t} w_t^-\big)\, d\mu_t = 0
\end{aligned}$$

where μ_t is in the canonical measure induced on ∂E_t by the flat metric of $\mathbb{T}^n$ and the last equality follows from the divergence theorem A.1 and the fact that w_t is harmonic in $\mathbb{T}^n \setminus \partial E_t$.

Another important property of the modified Mullins–Sekerka flow is that it can be regarded as the $H^{-1/2}$–gradient flow of the functional J under the constraint that the volume is fixed, that is, the outer normal velocity V_t is minus such $H^{-1/2}$–gradient of the functional J (see [25]). For a smooth set $E \subseteq \mathbb{T}^n$, we let the space $\widetilde{H}^{-1/2}(\partial E) \subseteq L^2(\partial E)$ to be the dual of $\widetilde{H}^{1/2}(\partial E)$ (the functions in $H^{1/2}(\partial E)$ with zero integral, see Appendix B) with the Gagliardo $H^{1/2}$–seminorm

$$\|u\|^2_{\widetilde{H}^{1/2}(\partial E)} = [u]^2_{H^{1/2}(\partial E)} = \int_{\partial E}\int_{\partial E} \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\, d\mu(x) d\mu(y)$$

(it is a norm for $\widetilde{H}^{1/2}(\partial E)$ since the functions in it have zero integral) and the pairing between $\widetilde{H}^{1/2}(\partial E)$ and $\widetilde{H}^{-1/2}(\partial E)$ simply being the integral of the product of the functions on ∂E.

We define the linear operator $\Delta_{\partial E}$ on the smooth functions u with zero integral on ∂E as follows: we consider the unique smooth solution w of the problem

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{T}^n \setminus \partial E \\ w = u & \text{on } \partial E \end{cases}$$

and we denote by w^+ and w^- the restrictions $w|_{E^c}$ and $w|_E$, respectively, then we set

$$\Delta_{\partial E} u = \partial_\nu w^+ - \partial_\nu w^- = [\partial_\nu w] \,,$$

which is another smooth function on ∂E with zero integral. Then, we have

$$\int_{\mathbb{T}^n} |\nabla w|^2\, dx = \int_{E\cup E^c} \operatorname{div}(w\nabla w)\, dx = -\int_{\partial E} u\Delta_{\partial E} u\, d\mu$$

and such quantity turns out to be a norm equivalent to the one given by the Gagliardo seminorm on $\widetilde{H}^{1/2}(\partial E)$ above (this is related to the theory of trace spaces, mentioned at the end of Appendix B, for which we refer to [3, 14]), see [25]. Hence, it induces the dual norm

$$\|v\|^2_{\widetilde{H}^{-1/2}(\partial E)} = \int_{\partial E} v(-\Delta_{\partial E})^{-1} v\, d\mu$$

for every smooth function $v \in \widetilde{H}^{-1/2}(\partial E)$. By polarization, we have the $\widetilde{H}^{-1/2}(\partial E)$–scalar product between a pair of smooth functions $u, v : \partial E \to \mathbb{R}$ with zero integral,

$$\langle u|v\rangle_{\widetilde{H}^{-1/2}(\partial E)} = \int_{\partial E} u(-\Delta_{\partial E})^{-1} v\, d\mu\,.$$

This scalar product, extended to the whole space $\widetilde{H}^{-1/2}(\partial E)$, make it a Hilbert space (see [17]), hence, by *Riesz representation theorem*, there exists a function $\nabla^{\widetilde{H}^{-1/2}}_{\partial E} J \in \widetilde{H}^{-1/2}(\partial E)$ such that, for every smooth function $v \in \widetilde{H}^{-1/2}(\partial E)$, there holds

$$\begin{aligned}\int_{\partial E} v(\mathrm{H} + 4\gamma v_E)\, d\mu = \delta J_{\partial E}(v) &= \langle v, |\nabla^{\widetilde{H}^{-1/2}}_{\partial E} J\rangle_{\widetilde{H}^{-1/2}(\partial E)}\\ &= \int_{\partial E} v(-\Delta_{\partial E})^{-1}\nabla^{\widetilde{H}^{-1/2}}_{\partial E} J\, d\mu\,,\end{aligned}$$

by Theorem 1.5, where v_E is the potential introduced in (1.3) and H is the mean curvature of ∂E.
Then, by the *fundamental lemma of calculus of variations*, we conclude

$$(-\Delta_{\partial E})^{-1}\nabla^{\widetilde{H}^{-1/2}}_{\partial E} J = \mathrm{H} + 4\gamma v_E + c\,,$$

for a constant $c \in \mathbb{R}$, that is, recalling the definition of w_E in problem (2.1) and of $\Delta_{\partial E}$,

$$\nabla^{\widetilde{H}^{-1/2}}_{\partial E} J = -\Delta_{\partial E}(\mathrm{H} + 4\gamma v_E) = -[\partial_{\nu_E} w_E]\,.$$

It clearly follows that the outer normal velocity of the moving boundaries of a surface diffusion flow $V_t = [\partial_{\nu_t} w_t]$ is minus the $\widetilde{H}^{-1/2}$–gradient of the volume–constrained functional J.

Remark 2.4. In the case $\gamma = 0$, the functional J is simply the Area functional on the boundary of the sets. It is then interesting to note that the (classical, unmodified) Mullins–Sekerka flow is its $H^{-1/2}$–gradient flow (under the constraint that the volume is fixed), while it is easy to see that the mean curvature flow, where $V_t = -\mathrm{H}_t$, is its L^2–gradient flow (without constraints). Moreover, considering its H^{-1}–gradient flow under a volume constraint, we get the so–called *surface diffusion flow* (see [11], for instance), where $V_t = \Delta_t \mathrm{H}_t$, see [17].

We now state a short time existence and uniqueness result of the modified Mullins–Sekerka flow starting from a smooth hypersurface, proved by Escher and Nishiura in [12]. It deals with the flow in the whole space $\mathbb{R}^n$, but it is straightforward to adapt the same arguments to our case when the ambient is the flat torus $\mathbb{T}^n$.

Given a smooth set $F \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂F, as in formula (1.37), for any $M \in (0, \varepsilon/2)$ (recall the discussion at the beginning of Section 1.3 about our notion of "closedness" of sets), we denote by $\mathfrak{C}^1_M(F)$, the class of all smooth sets $E \subseteq F \cup N_\varepsilon$ such that $\mathrm{Vol}(E \triangle F) \leq M$ and

$$\partial E = \{x + \psi_E(x)\nu_F(x) : x \in \partial F\}\,, \tag{2.5}$$

for some $\psi_E \in C^\infty(\partial F)$, with $\|\psi_E\|_{C^1(\partial F)} \leq M$ (hence, $\partial E \subseteq N_\varepsilon$). For every $k \in \mathbb{N}$ and $\alpha \in (0,1)$, we also denote by $\mathfrak{C}^{k,\alpha}_M(F)$ the collection of sets $E \in \mathfrak{C}^1_M(F)$ such that $\|\psi_E\|_{C^{k,\alpha}(\partial F)} \leq M$.

Theorem 2.5 (Short time existence and uniqueness). *Let $F \subseteq \mathbb{T}^n$ be a smooth set and N_ε a tubular neighborhood of ∂F, as in formula (1.37), Then, for every $\alpha \in (0,1)$ and $M \in (0, \varepsilon/2)$ small enough, there exists $T = T(F, M, \alpha) > 0$ such that if $E_0 \in \mathfrak{C}^{2,\alpha}_M(F)$ there exists a unique smooth modified Mullins–Sekerka flow with parameter $\gamma \geq 0$, starting from E_0, in the time interval $[0, T)$.*

In the next chapter we will show that for special "initial" sets, the flow exists for all times and we will study its long time behavior.

2.2 TECHNICAL LEMMAS

In this section we prove some technical lemmas necessary for the proof of the global existence result. In the following, in order to simplify the notation, for a smooth set $E_t \subseteq \mathbb{T}^n$ we will write ν_t for ν_{E_t}, ∂_{ν_t} in place of $\partial_{\nu_{E_t}}$ and w_t for the function $w_{E_t} \in H^1(\mathbb{T}^n)$ uniquely defined by problem (2.1). Moreover, we will also denote with v_t the smooth potential function v_{E_t} associated to E_t by formula (1.3).

We start with the following lemma holding in all dimensions.

Lemma 2.6 (Energy identities). *Let $E_t \subseteq \mathbb{T}^n$ be a modified Mullins–Sekerka flow as in Definition (2.2). Then, the following identities hold:*

$$\frac{d}{dt} J(E_t) = -\int_{\mathbb{T}^n} |\nabla w_t|^2\, dx\,, \tag{2.6}$$

and

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^n} |\nabla w_t|^2\, dx = -\Pi_{E_t}\big([\partial_{\nu_t} w_t]\big) + \frac{1}{2}\int_{\partial E_t} \big(\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-\big)[\partial_{\nu_t} w_t]^2\, d\mu_t\,, \tag{2.7}$$

where Π_{E_t} is the quadratic form defined in formula (1.31).

Proof. Let ψ_t the smooth family of maps describing the flow as in formula (2.4). By formula (1.12), where X is the smooth (velocity) vector field $X_t = \frac{\partial \psi_t}{\partial t} =$

$[\partial_{\nu_t} w_t]\nu_t$ along ∂E_t, hence $X_\tau = X_t - \langle X_t, \nu_t\rangle \nu_t = 0$ (as usual ν_t is the outer normal unit vector of ∂E_t), following the computation in the proof of Theorem (1.5), we have

$$\frac{d}{dt} J(E_t) = \int_{\partial E_t} (\mathrm{H}_t + 4\gamma v_t)\langle X_t | \nu_t\rangle \, d\mu_t = \int_{\partial E_t} w_t [\partial_{\nu_t} w_t] \, d\mu_t = -\int_{\mathbb{T}^n} |\nabla w_t|^2 \, dx\,,$$

where the last equality follows integrating by parts, as w_t is harmonic in $\mathbb{T}^n \setminus \partial E_t$. This establishes relation (2.6).

In order to get identity (2.7), we compute

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{E_t} |\nabla w_t^-|^2 \, dx &= \frac{1}{2}\int_{\partial E_t} |\nabla^{\mathbb{T}^n} w_t^-|^2 \, \langle X_t | \nu_t\rangle \, d\mu_t + \frac{1}{2}\int_{E_t} \frac{d}{dt} |\nabla w_t^-|^2 \, dx \\
&= \frac{1}{2}\int_{\partial E_t} |\nabla^{\mathbb{T}^n} w_t^-|^2 [\partial_{\nu_t} w_t] \, d\mu + \int_{E_t} \nabla \partial_t w_t^- \nabla w_t^- \, d\mu_t \\
&= \frac{1}{2}\int_{\partial E_t} |\nabla^{\mathbb{T}^n} w_t^-|^2 [\partial_{\nu_t} w_t] \, d\mu + \int_{\partial E_t} \partial_t w_t^- \partial_{\nu_t} w_t^- \, d\mu_t, \qquad (2.8)
\end{aligned}$$

where we interchanged time and space derivatives and we applied the divergence theorem, taking into account that w_t^- is harmonic in E_t.
Then, we need to compute $\partial_t w_t^-$ on ∂E_t. We know that

$$w_t^- = \mathrm{H}_t + 4\gamma v_t\,.$$

on ∂E_t, hence, (totally) differentiating in time this equality, we get

$$\partial_t w_t^- + \big\langle \nabla^{\mathbb{T}^n} w_t^- \big| X_t \big\rangle = \partial_t \mathrm{H}_t + 4\gamma \partial_t v_t + 4\gamma \big\langle \nabla^{\mathbb{T}^n} v_t \big| X_t \big\rangle\,,$$

that is,

$$\begin{aligned}
\partial_t w_t^- + [\partial_{\nu_t} w_t] \partial_{\nu_t} w_t^- &= \partial_t \mathrm{H}_t + 4\gamma \partial_t v_t + 4\gamma [\partial_{\nu_t} w_t] \partial_{\nu_t} v_t \\
&= -\,|B_t|^2 [\partial_{\nu_t} w_t] - \Delta_t [\partial_{\nu_t} w_t] + 4\gamma \partial_t v_t + 4\gamma [\partial_{\nu_t} w_t] \partial_{\nu_t} v_t\,,
\end{aligned}$$

where we used computation (1.24).
Therefore from equations (2.8) and (1.26) we get

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{E_t} |\nabla w_t^-|^2 \, dx = &-\int_{\partial E_t} \partial_{\nu_t} w_t^- \, \Delta_t [\partial_{\nu_t} w_t] \, d\mu_t - \int_{\partial E_t} \partial_{\nu_t} w_t^- \, |B_t|^2 \, [\partial_{\nu_t} w_t] \, d\mu_t \\
&+ 8\gamma \int_{\partial E_t}\int_{\partial E_t} G(x,y) \, \partial_{\nu_t} w_t^-(x) \, [\partial_{\nu_t} w_t](y) \, d\mu_t(x) d\mu_t(y) \\
&+ 4\gamma \int_{\partial E_t} \partial_{\nu_t} v_t \, \partial_{\nu_t} w_t^- \, [\partial_{\nu_t} w_t] \, d\mu_t \\
&+ \frac{1}{2}\int_{\partial E_t} |\nabla^{\mathbb{T}^n} w_t^-|^2 [\partial_{\nu_t} w_t] \, d\mu_t - \int_{\partial E_t} (\partial_{\nu_t} w_t^-)^2 [\partial_{\nu_t} w_t] \, d\mu_t\,.
\end{aligned}$$

Computing analogously for w_t^+ in E^c and adding the two results, we get

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^n} |\nabla w_t|^2\,dx = & \int_{\partial E_t} [\partial_{\nu_t} w_t]\,\Delta_t[\partial_{\nu_t} w_t]\,d\mu_t + \int_{\partial E_t} |B_t|^2\,[\partial_{\nu_t} w_t]^2\,d\mu_t \\
& - 8\gamma \int_{\partial E_t}\int_{\partial E_t} G(x,y)\,[\partial_{\nu_t} w_t](x)\,[\partial_{\nu_t} w_t](y)\,d\mu_t(x) d\mu_t(y) \\
& - 4\gamma \int_{\partial E_t} \partial_{\nu_t} v_t\,[\partial_{\nu_t} w_t]^2\,d\mu_t \\
& + \int_{\partial E_t} \big((\partial_{\nu_t} w_t^+)^2 - (\partial_{\nu_t} w_t^-)^2\big)[\partial_{\nu_t} w_t]\,d\mu_t \\
& - \frac{1}{2}\int_{\partial E_t} \big(|\nabla^{\mathbb{T}^n} w_t^+|^2 - |\nabla^{\mathbb{T}^n} w_t^-|^2\big)[\partial_{\nu_t} w_t]\,d\mu_t \\
= & - \Pi_{E_t}\big([\partial_{\nu_t} w_t]\big) + \frac{1}{2}\int_{\partial E_t} \big(\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-\big)[\partial_{\nu_t} w_t]^2\,d\mu_t\,,
\end{aligned}$$

where we integrated by parts the very first term of the right hand side, recalled Definition (1.31) and in the last step we used the identity

$$|\nabla^{\mathbb{T}^n} w_t^+|^2 - |\nabla^{\mathbb{T}^n} w_t^-|^2 = (\partial_{\nu_t} w_t^+)^2 - (\partial_{\nu_t} w_t^-)^2 = (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]\,.$$

Hence, also equation (2.7) is proved. □

From now on, we restrict ourselves to the three–dimensional case, that is, we will consider smooth subsets of $\mathbb{T}^3$ with boundaries which then are smooth embedded (2–dimensional) surfaces.

In the estimates in the following series of lemmas, we will be interested in having uniform constants for the families $\mathfrak{C}^{1,\alpha}_M(F)$, given a smooth set $F \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂F as in formula (1.37), for any $M \in (0, \varepsilon/2)$ and $\alpha \in (0,1)$. This is guaranteed if the constants in the Sobolev, Gagliardo–Nirenberg interpolation and Calderón–Zygmung inequalities, relative to all the smooth hypersurfaces ∂E boundaries of the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$, are uniform, as it is proved in detail in [8].

We remind that in all the inequalities, the constants C may vary from one line to another.

The next lemma provides some boundary estimates for harmonic functions.

Lemma 2.7 (Boundary estimates for harmonic functions)**.** *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E \in \mathfrak{C}^{1,\alpha}_M(F)$. Let $f \in C^\alpha(\partial E)$ with zero integral on ∂E and let $u \in H^1(\mathbb{T}^3)$ be the (distributional) solution of*

$$-\Delta u = f\mu\big|_{\partial E}$$

with zero integral on $\mathbb{T}^3$. Let $u^- = u|_E$ and $u^+ = u|_{E^c}$ and assume that u^- and u^+ are of class C^1 up to the boundary ∂E. Then, for every $1 < p < +\infty$ there exists a constant $C = C(F, M, \alpha, p) > 0$, such that:

(i) $$\|u\|_{L^p(\partial E)} \leq C\|f\|_{L^p(\partial E)}$$

(ii) $$\|\partial_{\nu_E} u^+\|_{L^2(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^2(\partial E)} \leq C\|u\|_{H^1(\partial E)}$$

(iii) $$\|\partial_{\nu_E} u^+\|_{L^p(\partial E)} + \|\partial_{\nu_E} u^-\|_{L^p(\partial E)} \leq C\|f\|_{L^p(\partial E)}$$

(iv) $$\|u\|_{C^{0,\beta}(\partial E)} \leq C\|f\|_{L^p(\partial E)}$$

for all $\beta \in (0, \frac{p-2}{p})$, with C depending also on β.

Moreover, if $f \in H^1(\partial E)$, then for every $2 \leq p < +\infty$ there exists a constant $C = C(F, M, \alpha, p) > 0$, such that

$$\|f\|_{L^p(\partial E)} \leq C\|f\|^{(p-1)/p}_{H^1(\partial E)} \|u\|^{1/p}_{L^2(\partial E)} .$$

Proof. We are not going to underline it every time, but it is easy to check that all the constants that will appear in the proof will depend with only on F, M, α and sometimes p, recalling the previous discussion about the "uniform" inequalities holding for the families of sets $\mathfrak{C}^{1,\alpha}_M(F)$.

(i) Recalling Remark 1.13, we have

$$u(x) = \int_{\partial E} G(x, y) f(y)\, d\mu(y).$$

It is well known that it is always possible to write $G(x, y) = h(x - y) + r(x - y)$ where $h : \mathbb{R} \to \mathbb{R}$ is smooth away from 0, one–periodic and $h(t) = \frac{1}{4\pi|t|}$ in a neighborhood of 0, while $r : \mathbb{R} \to \mathbb{R}$ is smooth and one–periodic. The conclusion then follows since for $v(x) = \int_{\partial E} \frac{f(y)}{|x-y|}\, d\mu(y)$ there holds

$$\|v\|_{L^p(\partial E)} \leq C\|f\|_{L^p(\partial E)} ,$$

with $C = C(F, M, \alpha, p) > 0$.

(ii) We are going to adapt the proof of [22] to the periodic setting. First observe that since u is harmonic in $E \subseteq \mathbb{T}^3$ we have

$$\operatorname{div}\big(2\langle \nabla u | x\rangle \nabla u - |\nabla u|^2 x + u\nabla u\big) = 0. \tag{2.9}$$

Moreover, there exist constants $r > 0$, C_0 and $N \in \mathbb{N}$, depending only on F, M, α, such that we may cover ∂E with N balls $B_r(x_k)$, with every $x_k \in F$ and

$$\frac{1}{C_0} \leq \langle x | \nu_E(x)\rangle \leq C_0 \qquad \text{for } x \in \partial E \cap B_{2r}(x_k) . \tag{2.10}$$

for every that $E \in \mathfrak{C}^{1,\alpha}_M(F)$.

If then $0 \leq \varphi_k \leq 1$ is a smooth function with compact support in $B_{2r}(x_k)$ such that $\varphi_k \equiv 1$ in $B_r(x_k)$ and $|\nabla \varphi_k| \leq C/r$, by integrating the function

$$\operatorname{div}\big(\varphi_k \big(2\langle \nabla u | x\rangle \nabla u - |\nabla u|^2 x + u\nabla u\big)\big)$$

in E and using equality (2.9), we get

$$\begin{aligned}\int_E \langle \nabla \varphi_k |\, 2\,\langle \nabla u | x\rangle\, \nabla u - |\nabla u|^2 x + u\nabla u\rangle\, dx \\ &= \int_E \operatorname{div}\big(\varphi_k (2\,\langle \nabla u | x\rangle\, \nabla u - |\nabla u|^2 x + u\nabla u)\big)\, dx \\ &= \int_{\partial E} \big(2\varphi_k \langle \nabla^{\mathbb{T}^3} u | x\rangle \partial_{\nu_E} u - \varphi_k |\nabla^{\mathbb{T}^3} u|^2 \langle x | \nu_E\rangle + \varphi_k u \partial_{\nu_E} u\big)\, d\mu,\end{aligned}$$

hence,

$$\begin{aligned}\int_E \big\langle \nabla\varphi_k \big| 2\big\langle \nabla^{\mathbb{T}^3} u|x\big\rangle \nabla u - |\nabla u|^2 x + u\nabla u\big\rangle\,dx &- \int_{\partial E} \varphi_k u \partial_{\nu_E} u^-\,d\mu \\ &- 2\int_{\partial E} \varphi_k \langle \nabla u|x\rangle \partial_{\nu_E} u^-\,d\mu \\ &= -\int_{\partial E} \varphi_k |\nabla^{\mathbb{T}^3} u^-|^2 \langle x|\nu_E\rangle\,d\mu + 2\int_{\partial E} \varphi_k |\partial_{\nu_E} u^-|^2 \langle x|\nu_E\rangle\,d\mu \\ &= \int_{\partial E} \varphi_k |\partial_{\nu_E} u^-|^2 \langle x|\nu_E\rangle\,d\mu - \int_{\partial E} \varphi_k |\nabla u|^2 \langle x|\nu_E\rangle\,d\mu\,.\end{aligned}$$

Using the Poincaré inequality on the torus $\mathbb{T}^3$ (recall that u has zero integral) and estimate (2.10), this inequality implies

$$\begin{aligned}\int_{\partial E\cap B_r(x_k)} |\partial_{\nu_E} u|^2\,d\mu &\le C\int_{\partial E} (u^2 + |\nabla u|^2)\,d\mu + C\int_{\mathbb{T}^3} (u^2 + |\nabla u|^2)\,dx \\ &\le C\int_{\partial E} (u^2 + |\nabla u|^2)\,d\mu + C\int_{\mathbb{T}^3} |\nabla u|^2\,dx\,.\end{aligned}$$

Putting together all the above estimates and repeating the argument on E^c, we get

$$\int_{\partial E} (|\partial_{\nu_E} u^-|^2 + |\partial_{\nu_E} u^+|^2)\,d\mu \le C\int_{\partial E} (u^2 + |\nabla u|^2)\,d\mu + C\int_{\mathbb{T}^3} |\nabla u|^2\,dx\,.$$

The thesis then follows by observing that

$$\int_{\mathbb{T}^3} |\nabla u|^2\,dx = \int_{\partial E} u(\partial_{\nu_E} u^- - \partial_{\nu_E} u^+)\,d\mu\,.$$

(iii) Let us define

$$Kf(x) = \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x,y)|\nu_E(x)\big\rangle f(y)\,d\mu(y)\,.$$

We want to show that

$$\|Kf\|_{L^p(\partial E)} \le C\|f\|_{L^p(\partial E)}. \tag{2.11}$$

By the decomposition recalled at the point (i), we have $\nabla_x^{\mathbb{T}^3} G(x,y) = \nabla_x^{\mathbb{T}^3}[h(x-y)] + \nabla_x^{\mathbb{T}^3}[r(x-y)]$, where $\nabla_x^{\mathbb{T}^3}[h(x-y)] = -\frac{1}{4\pi}\frac{x-y}{|x-y|^3}$, for $|x-y|$ small enough and $\nabla_x^{\mathbb{T}^3}[r(x-y)]$ is smooth. Thus, by a standard partition of unity argument we may localize the estimate and reduce to show that if $\varphi \in C_c^{1,\alpha}(\mathbb{R}^2)$ and $U \subseteq \mathbb{R}^2$ is a bounded domain setting $\Gamma = \{(x', \varphi(x'))\,:\,x' \in U\} \subseteq \mathbb{R}^3$ and

$$Tf(x) = \int_\Gamma \frac{\langle x-y|\nu_E(x)\rangle}{|x-y|^3} f(y)\,d\mu(y)$$

for every $x \in \Gamma$, where ν_E is the "upper" normal to the graph Γ, then $Tf(x)$ is well defined at every $x \in \Gamma$ and

$$\|Tf\|_{L^p(\Gamma)} \le C\|f\|_{L^p(\Gamma)}\,.$$

In order to show this we observe that we may write

$$Tf(x) = \int_U \frac{\varphi(x') - \varphi(y') - \langle \nabla\varphi(x') | x' - y' \rangle}{(|x'-y'|^2 + [\varphi(x') - \varphi(y')]^2)^{3/2}} f(y', \varphi(y'))\, dy'.$$

where we used the fact that

$$\Gamma = \{(x', y') \,:\, y' - \varphi(x') = F(x', y') = 0\}$$

and then that

$$\nu_E = \frac{\nabla F}{|\nabla F|} = \frac{(-\nabla\varphi(x'), 1)}{\sqrt{1 + |\nabla\varphi(x')|^2}}.$$

Therefore,

$$\begin{aligned} |Tf(x)| &\leq C \int_U \frac{|x'-y'|^{1+\alpha}}{(|x'-y'|^2 + [\varphi(x') - \varphi(y')]^2)^{3/2}} |f(y', \varphi(y'))|\, dy' \\ &\leq C \int_U \frac{|f(y', \varphi(y'))|}{|x'-y'|^{2-\alpha}}\, dy'. \end{aligned}$$

Thus, inequality (2.11) follows from a standard convolution estimate.

For $x \in E$ we have

$$\nabla u(x) = \int_{\partial E} \nabla_x^{\mathbb{T}^3} G(x, y) f(y)\, d\mu(y),$$

hence, for $x \in \partial E$ there holds

$$\langle \nabla u(x - t\nu_E(x)) |\, \nu_E(x) \rangle = \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x - t\nu_E(x), y) |\, \nu_E(x) \big\rangle f(y)\, d\mu(y).$$

We claim that

$$\partial_{\nu_E} u^-(x) = \lim_{t\to 0+} \langle \nabla u(x - t\nu_E(x)) |\, \nu_E(x) \rangle = Kf(x) + \frac{1}{2} f(x), \tag{2.12}$$

for every $x \in \partial E$, then the result follows from inequality (2.11) and this limit, together with the analogous identity for $\partial_{\nu_E} u^+(x)$.
To show equality (2.12) we first observe that

$$\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) |\, \nu_E(y) \big\rangle\, d\mu(y) = 1 - \mathrm{Vol}(E) \qquad \text{if } x \in E \setminus \partial E \tag{2.13}$$

$$\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) |\, \nu_E(y) \big\rangle\, d\mu(y) = 1/2 - \mathrm{Vol}(E) \qquad \text{if } x \in \partial E. \tag{2.14}$$

Indeed, using Definition (1.3), we have

$$\begin{aligned} \Delta v_E(x) &= \int_E \Delta_x G(x, y)\, dy - \int_{E^c} \Delta_x G(x, y)\, dy \\ &= -2 \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) |\, \nu_E(y) \big\rangle\, d\mu(y) \\ &= 2\mathrm{Vol}(E) - 1 - u_E(x), \end{aligned}$$

then,

$$\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) |\, \nu_E(y) \big\rangle\, d\mu(y) = 1/2 - \mathrm{Vol}(E) + u_E(x)/2,$$

which clearly implies equation (2.13). Equality (2.14) instead follows by an approximation argument, after decomposing the Green function as at the beginning of the proof of point (i), $G(x,y) = h(x-y) + r(x-y)$, with $h(t) = \frac{1}{4\pi|t|}$ in a neighborhood of 0 and $r : \mathbb{R} \to \mathbb{R}$ a smooth function.
Therefore, we may write, for $x \in \partial E$ and $t > 0$ (remind that ν_E is the *outer* unit normal vector, hence $x - t\nu_E(x) \in E$),

$$\begin{aligned}\langle \nabla u(x - t\nu_E(x)) \,|\, \nu_E(x)\rangle = &\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x - t\nu_E(x), y) | \nu_E(x) \big\rangle (f(y) - f(x))\, d\mu(y)\\ &+ f(x) \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x - t\nu_E(x), y) | \nu_E(x) - \nu_E(y) \big\rangle \, d\mu(y)\\ &+ f(x)(1 - \mathrm{Vol}(E)),\end{aligned} \tag{2.15}$$

by equality (2.13).
Let us now prove that

$$\begin{aligned}\lim_{t\to 0^+} &\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x - t\nu_E(x), y) \,|\, \nu_E(x) \big\rangle (f(y) - f(x))\, d\mu(y)\\ &= \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) \,|\, \nu_E(x) \big\rangle (f(y) - f(x))\, d\mu(y),\end{aligned}$$

observing that since ∂E is of class $C^{1,\alpha}$ then for $|t|$ sufficiently small we have

$$|x - y - t\nu_E(x)| \geq \frac{1}{2}|x - y| \qquad \text{for all } y \in \partial E\,. \tag{2.16}$$

Then, in view of the decomposition of $\nabla_x G$ above, it is enough show that

$$\begin{aligned}\lim_{t\to 0^+} &\int_{\partial E} \frac{\langle x - y - t\nu_E(x) \,|\, \nu_E(x)\rangle}{|x - y - t\nu_E(x)|^3} (f(y) - f(x))\, d\mu(y)\\ &= \int_{\partial E} \frac{\langle x - y \,|\, \nu_E(x)\rangle}{|x - y|^3} (f(y) - f(x))\, d\mu(y)\,,\end{aligned}$$

which follows from the dominated convergence theorem, after observing that due to the α–Hölder continuity of f and to inequality (2.16), the absolute value of both integrands can be estimated from above by $C/|x - y|^{2-\alpha}$ for some constant $C > 0$.
Arguing analogously, we also get

$$\begin{aligned}\lim_{t\to 0^+} &\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x - t\nu_E(x), y) | \nu_E(x) - \nu_E(y) \big\rangle \, d\mu(y)\\ &= \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) | \nu_E(x) - \nu_E(y) \big\rangle \, d\mu(y)\,.\end{aligned}$$

Then, letting $t \to 0^+$ in equality (2.15), for every $x \in \partial E$, we obtain

$$\begin{aligned}\lim_{t\to 0^+} \langle \nabla u(x - t\nu_E(x)) | \nu_E(x)\rangle = &\int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) | \nu_E(x) \big\rangle (f(y) - f(x))\, d\mu(y)\\ &+ f(x) \int_{\partial E} \big\langle \nabla_x^{\mathbb{T}^3} G(x, y) | \nu_E(x) - \nu_E(y) \big\rangle \, d\mu(y)\\ &+ f(x)(1 - \mathrm{Vol}(E))\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial E} \langle \nabla_x^{\mathbb{T}^3} G(x,y) | \nu_E(x) \rangle f(y)\, d\mu(y) \\
&\quad - f(x) \int_{\partial E} \langle \nabla_x^{\mathbb{T}^3} G(x,y) | \nu_E(y) \rangle \, d\mu(y) \\
&\quad + f(x)(1 - \mathrm{Vol}(E)) \\
&= Kf(x) + f(x)(\mathrm{Vol}(E) - 1/2) + f(x)(1 - \mathrm{Vol}(E)) \\
&= Kf(x) + \frac{1}{2} f(x),
\end{aligned}
$$

where we used equality (2.14), then limit (2.12) holds and the thesis follows.
(iv) Fixed $p > 2$ and $\beta \in (0, \frac{p-2}{p})$, as before, due to the properties of the Green's function, it is sufficient to establish the statement for the function

$$v(x) = \int_{\partial E} \frac{f(y)}{|x-y|}\, d\mu(y)\,.$$

For x_1, $x_2 \in \partial E$ we have

$$|v(x_1) - v(x_2)| \le \int_{\partial E} |f(y)| \frac{\big||x_1 - y| - |x_2 - y|\big|}{|x_1 - y|\,|x_2 - y|}\, d\mu(y)\,.$$

In turn, by an elementary inequality, we have

$$\frac{\big||x_1 - y| - |x_2 - y|\big|}{|x_1 - y|\,|x_2 - y|} \le C(\beta) \frac{\big||x_1 - y|^{1-\beta} + |x_2 - y|^{1-\beta}\big|}{|x_1 - y|\,|x_2 - y|} |x_1 - x_2|^\beta\,,$$

thus, by Hölder inequality we have

$$
\begin{aligned}
|v(x_1) - v(x_2)| &\le C(\beta) \int_{\partial E} |f(y)| \frac{\big||x_1 - y|^{1-\beta} + |x_2 - y|^{1-\beta}\big|}{|x_1 - y|\,|x_2 - y|}\, d\mu(y)\ |x_1 - x_2|^\beta \\
&\le C'(\beta) \|f\|_{L^p} |x_1 - x_2|^\beta\,,
\end{aligned}
$$

where we set

$$C'(\beta) = 2C(\beta) \biggl(\sup_{z_1,\, z_2 \in \partial E} \int_{\partial E} \frac{1}{|z_1 - y|^{\beta p'}\, |z_2 - y|^{p'}}\, d\mu(y) \biggr)^{1/p'}\,,$$

with $p' = p/(p-1)$.
For the second part of the lemma, we start by observing that

$$\|f\|_{L^2(\partial E)} \le C \|f\|_{H^1(\partial E)}^{1/2} \|f\|_{H^{-1}(\partial E)}^{1/2}.$$

If $p > 2$ we have, by Gagliardo–Nirenberg interpolation inequalities (see [5, Theorem 3.70]),

$$\|f\|_{L^p(\partial E)} \le C \|f\|_{H^1(\partial E)}^{(p-2)/p} \|f\|_{L^2(\partial E)}^{2/p}.$$

Therefore, by combining the two previous inequalities we get that, for $p \ge 2$, there holds

$$\|f\|_{L^p(\partial E)} \le C \|f\|_{H^1(\partial E)}^{(p-1)/p} \|f\|_{H^{-1}(\partial E)}^{1/p}.$$

Hence, the thesis follows once we show

$$\|f\|_{H^{-1}(\partial E)} \le C \|u\|_{L^2(\partial E)}.$$

To this aim, let us fix $\varphi \in H^1(\partial E)$ and with a little abuse of notation denote its harmonic extension to $\mathbb{T}^3$ still by φ. Then, by integrating by parts twice and by point (ii), we get

$$\begin{aligned}\int_{\partial E} \varphi f \, d\mu &= -\int_{\partial E} \varphi \Delta u \, d\mu \\ &= -\int_{\partial E} u[\partial_{\nu_E} \varphi] \, d\mu \\ &\leq \|u\|_{L^2(\partial E)} \big\| [\partial_{\nu_E} \varphi] \big\|_{L^2(\partial E)} \\ &\leq \|u\|_{L^2(\partial E)} \big(\|\partial_{\nu_E} \varphi^+\|_{L^2(\partial E)} + \|\partial_{\nu_E} \varphi^-\|_{L^2(\partial E)} \big) \\ &\leq C \|u\|_{L^2(\partial E)} \|\varphi\|_{H^1(\partial E)}.\end{aligned}$$

Therefore,

$$\|f\|_{H^{-1}(\partial E)} = \sup_{\|\varphi\|_{H^1(\partial E)} \leq 1} \int_{\partial E} \varphi f \, d\mu \leq C \|u\|_{L^2(\partial E)}$$

and we are done. □

For any smooth set $E \subseteq \mathbb{T}^3$, the fractional Sobolev space $W^{s,p}(\partial E)$, usually obtained via local charts and partitions of unity, has an equivalent definition considering directly the *Gagliardo $W^{s,p}$–seminorm* of a function $f \in L^p(\partial E)$, for $s \in (0,1)$, as follows

$$[f]^p_{W^{s,p}(\partial E)} = \int_{\partial E} \int_{\partial E} \frac{|f(x) - f(y)|^p}{|x-y|^{2+sp}} \, d\mu(x) d\mu(y)$$

and setting $\|f\|_{W^{s,p}(\partial E)} = \|f\|_{L^p(\partial E)} + [f]_{W^{s,p}(\partial E)}$ (see Appendix B). As it is customary, we set $[f]_{H^s(\partial E)} = [f]_{W^{s,2}(\partial E)}$ and $H^s(\partial E) = W^{s,2}(\partial E)$.

Then, it can be shown that for all the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$, given a smooth set $F \subseteq \mathbb{T}^3$ and a tubular neighborhood N_ε of ∂F as in formula (1.37), for any $M \in (0, \varepsilon/2)$ and $\alpha \in (0,1)$, the constants giving the equivalence between this norm above and the "standard" norm of $W^{s,p}(\partial E)$ can be chosen to be uniform, independent of E. Moreover, as for the "usual" (with integer order) Sobolev spaces, all the constants in the embeddings of the fractional Sobolev spaces (in particular the ones in the propositions of Appendix B) are also uniform for this family. This is related to the possibility, due to the closedness in $C^{1,\alpha}$ and the graph representation, of "localizing" and using partitions of unity "in a single common way" for all the smooth hypersurfaces ∂E boundaries of the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$, see [8] for details.

Then, we have the following technical lemma.

Lemma 2.8. *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E \in \mathfrak{C}^{1,\alpha}_M(F)$. For every $\beta \in [0, 1/2)$, there exists a constant $C = C(F, M, \alpha, \beta)$ such that if $f \in H^{\frac{1}{2}}(\partial E)$ and $g \in W^{1,4}(\partial E)$, then*

$$[fg]_{H^{\frac{1}{2}}(\partial E)} \leq C[f]_{H^{\frac{1}{2}}(\partial E)} \|g\|_{L^\infty(\partial E)} + C\|f\|_{L^{\frac{4}{1+\beta}}(\partial E)} \|g\|^{\beta}_{L^\infty(\partial E)} \|\nabla g\|^{1-\beta}_{L^4(\partial E)}.$$

Proof. We estimate with Hölder inequality, noticing that $6\beta/(1+\beta) < 2$, as $\beta \in [0, 1/2)$, hence there exists $\delta > 0$ such that $(6\beta + \delta)/(1+\beta) < 2$,

$$\begin{aligned}
[fg]^2_{H^{\frac{1}{2}}(\partial E)} &\leq 2[f]^2_{H^{\frac{1}{2}}(\partial E)} \|g\|^2_{L^\infty(\partial E)} + 2\int_{\partial E}\int_{\partial E} |f(y)|^2 \frac{|g(x)-g(y)|^2}{|x-y|^3}\, d\mu(x) d\mu(y) \\
&\leq 2[f]^2_{H^{\frac{1}{2}}(\partial E)} \|g\|^2_{L^\infty(\partial E)} \\
&\quad + C\int_{\partial E}\int_{\partial E} \frac{|f(y)|^2}{|x-y|^{3\beta+\delta/2}} \frac{|g(x)-g(y)|^{2(1-\beta)}}{|x-y|^{3(1-\beta)-\delta/2}} \|g\|^{2\beta}_{L^\infty(\partial E)}\, d\mu(x) d\mu(y) \\
&\leq 2[f]^2_{H^{\frac{1}{2}}(\partial E)} \|g\|^2_{L^\infty(\partial E)} \\
&\quad + C\Big(\int_{\partial E} |f(y)|^{\frac{4}{1+\beta}} \int_{\partial E} \frac{1}{|x-y|^{\frac{6\beta+\delta}{1+\beta}}}\, d\mu(x) d\mu(y)\Big)^{(1+\beta)/2} \|g\|^{2\beta}_{L^\infty(\partial E)} \\
&\qquad \cdot \Big(\int_{\partial E}\int_{\partial E} \frac{|g(x)-g(y)|^4}{|x-y|^{6-\frac{\delta}{1-\beta}}}\, d\mu(x) d\mu(y)\Big)^{(1-\beta)/2} \\
&\leq 2[f]^2_{H^{\frac{1}{2}}(\partial E)} \|g\|^2_{L^\infty(\partial E)} \\
&\quad + C\Big(\int_{\partial E} |f(y)|^{\frac{4}{1+\beta}}\, d\mu(y)\Big)^{(1+\beta)/2} \|g\|^{2\beta}_{L^\infty(\partial E)}\, [g]^{2(1-\beta)}_{W^{1-\frac{\delta}{4(1-\beta)},4}(\partial E)} \\
&\leq 2[f]^2_{H^{\frac{1}{2}}(\partial E)} \|g\|^2_{L^\infty(\partial E)} + C\|f\|^2_{L^{\frac{4}{1+\beta}}(\partial E)} \|g\|^{2\beta}_{L^\infty(\partial E)} \|\nabla g\|^{2(1-\beta)}_{L^4(\partial E)},
\end{aligned}$$

where in the last inequality we applied Proposition B.1 (extended to the fractional Sobolev spaces on ∂E). Hence the thesis follows noticing that all the constants C above depend only on F, M, α and β, by the previous discussion, before the lemma. □

As a corollary we have the following estimate.

Lemma 2.9. *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E \in \mathfrak{C}^{1,\alpha}_M(F)$. Then, for M small enough, there holds*

$$\|\psi_E\|_{W^{\frac{5}{2},2}(\partial F)} \leq C(F, M, \alpha)\big(1 + \|\mathrm{H}\|^2_{H^{\frac{1}{2}}(\partial E)}\big),$$

where H is the mean curvature of ∂E and the function ψ_E is defined by formula (2.5).

Proof. By a standard localization/partition of unity/straightening argument, we may reduce ourselves to the case where the function ψ_E is defined in a disk $D \subseteq \mathbb{R}^2$ and $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$. Fixed a smooth cut–off function φ with compact support in D and equal to one on a smaller disk $D' \subseteq D$, we have (see Remark A.2)

$$\Delta(\varphi\psi_E) - \frac{\nabla^2(\varphi\psi_E)\nabla\psi_E\nabla\psi_E}{1+|\nabla\psi_E|^2} = \varphi\mathrm{H}\sqrt{1+|\nabla\psi_E|^2} + R(x, \psi_E, \nabla\psi_E),$$

where the remainder term $R(x, \psi_E, \nabla\psi_E)$ is a smooth Lipschitz function. Then, using Lemma 2.8 with $\beta = 0$ and recalling that $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$, we estimate

$$\begin{aligned}[\Delta(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \leq C(F,M,\alpha)\Big(& M^2[\nabla^2(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \\ & + [\mathrm{H}]_{H^{\frac{1}{2}}(\partial E)}(1+\|\nabla\psi_E\|_{L^\infty(D)}) \\ & + \|\mathrm{H}\|_{L^4(\partial E)}(1+\|\psi_E\|_{W^{2,4}(D)}) \\ & + 1 + \|\psi_E\|_{W^{2,4}(D)}\Big).\end{aligned}$$

We now use the fact that, by a simple integration by part argument, if u is a smooth function with compact support in $\mathbb{R}^2$, there holds

$$[\Delta u]_{H^{\frac{1}{2}}(\mathbb{R}^2)} = [\nabla^2 u]_{H^{\frac{1}{2}}(\mathbb{R}^2)},$$

hence,

$$\begin{aligned}[\nabla^2(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} &= [\Delta(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \\ &\leq C(F,M,\alpha)\Big(M^2[\nabla^2(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \\ &\qquad + [\mathrm{H}]_{H^{\frac{1}{2}}(\partial E)}(1+\|\nabla\psi_E\|_{L^\infty(D)}) \\ &\qquad + \|\mathrm{H}\|_{L^4(\partial E)}(1+\|\psi_E\|_{W^{2,4}(D)}) \\ &\qquad + 1 + \|\psi_E\|_{W^{2,4}(D)}\Big),\end{aligned}$$

then, if M is small enough, we have

$$[\nabla^2(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)})(1+\|\mathrm{Hess}\,\psi_E\|_{L^4(D)}), \tag{2.17}$$

as

$$\|\mathrm{H}\|_{L^4(\partial E)} \leq C(F,M,\alpha)\|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)}, \tag{2.18}$$

by Proposition B.2 with $q = 4$, $s = 1/2$ and $p = 2$.
By the Calderón–Zygmund estimates (holding uniformly for every hypersurface ∂E, with $E \in \mathfrak{C}^{1,\alpha}_M(F)$, see [8]),

$$\|\mathrm{Hess}\,\psi_E\|_{L^4(D)} \leq C(F,M,\alpha)(\|\psi_E\|_{L^4(D)} + \|\Delta\psi_E\|_{L^4(D)}) \tag{2.19}$$

and the expression of the mean curvature (Remark A.2)

$$\mathrm{H} = \frac{\Delta\psi_E}{\sqrt{1+|\nabla\psi_E|^2}} - \frac{\mathrm{Hess}\,\psi_E(\nabla\psi_E\nabla\psi_E)}{(\sqrt{1}+|\nabla\psi_E|)^3}.$$

we obtain

$$\begin{aligned}\|\Delta\psi_E\|_{L^4(D)} &\leq 2M\|\mathrm{H}\|_{L^4(\partial E)} + M^2\|\mathrm{Hess}\,\psi_E\|_{L^4(D)} \\ &\leq 2M\|\mathrm{H}\|_{L^4(\partial E)} + C(F,M,\alpha)M^2(\|\psi_E\|_{L^4(D)} + \|\Delta\psi_E\|_{L^4(D)})\end{aligned}$$

hence, possibly choosing a smaller M, we conclude

$$\|\Delta\psi_E\|_{L^4(D)} \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{L^4(\partial E)}) \leq C(F,M,\alpha)(1+\|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)}),$$

again by inequality (2.18).
Thus, by estimate (2.19), we get

$$\|\mathrm{Hess}\,\psi_E\|_{L^4(D)} \leq C(F, M, \alpha)(1 + \|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)}), \tag{2.20}$$

and using this inequality in estimate (2.17),

$$[\nabla^2(\varphi\psi_E)]_{H^{\frac{1}{2}}(D)} \leq C(F, M, \alpha)(1 + \|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)})^2,$$

hence,

$$[\nabla^2\psi_E]_{H^{\frac{1}{2}}(D')} \leq C(F, M, \alpha)(1 + \|\mathrm{H}\|_{H^{\frac{1}{2}}(\partial E)})^2 \leq C(F, M, \alpha)(1 + \|\mathrm{H}\|^2_{H^{\frac{1}{2}}(\partial E)}).$$

The inequality in the statement of the lemma then easily follows by this inequality, estimate (2.20) and $\|\psi_E\|_{C^{1,\alpha}(D)} \leq M$, with a standard covering argument. □

We are now ready to prove the last lemma of this section.

Lemma 2.10 (Compactness). *Let $F \subseteq \mathbb{T}^3$ be a smooth set and $E_n \subseteq \mathfrak{C}^{1,\alpha}_M(F)$ a sequence of smooth sets such that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{T}^3} |\nabla w_{E_n}|^2\, dx < +\infty\,,$$

where w_{E_n} are the functions associated to E_n by problem (2.1).
Then, if $\alpha \in (0, 1/2)$ and M is small enough, there exists a smooth set $F' \in \mathfrak{C}^1_M(F)$ such that, up to a (non relabeled) subsequence, $E_n \to F'$ in $W^{2,p}$ for all $1 \leq p < 4$ (recall the definition of convergence of sets at the beginning of Section 1.3).
Moreover, if

$$\int_{\mathbb{T}^3} |\nabla w_{E_n}|^2\, dx \to 0\,,$$

then F' is critical for the volume–constrained nonlocal Area functional J and the convergence $E_n \to F'$ is in $W^{\frac{5}{2},2}$.

Proof. Throughout all the proof we write w_n, H_n, and v_n instead of w_{E_n}, $\mathrm{H}_{\partial E_n}$, and v_{E_n}, respectively. Moreover, we denote by $\widehat{w}_n = \fint_{\mathbb{T}^3} w_n\, dx$ and we set $\widetilde{w}_n = \fint_{\partial E_n} w_n\, d\mu_n$ and $\widetilde{\mathrm{H}}_n = \fint_{\partial E_n} \mathrm{H}_n\, d\mu_n$.

First, we recall that

$$w_n = \mathrm{H}_n + 4\gamma v_n \quad \text{on } \partial E_n \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \|v_n\|_{C^{1,\alpha}(\mathbb{T}^3)} < +\infty\,, \tag{2.21}$$

by standard elliptic estimates. We want to show that

$$\|w_n - \widetilde{w}_n\|^2_{H^{\frac{1}{2}}(\partial E_n)} \leq \|w_n - \widehat{w}_n\|^2_{H^{\frac{1}{2}}(\partial E_n)}. \tag{2.22}$$

To this aim, we recall that for every constant a

$$\|w_n - a\|^2_{L^2(\partial E_n)} = \|w_n\|^2_{L^2(\partial E_n)} + a^2\mathcal{A}_{\mathbb{T}^3}(\partial E_n) - 2a\int_{\partial E_n} w_n\, d\mu_n$$

then,

$$\frac{d}{dt}\|w_n - a\|^2_{L^2(\partial E_n)} = 2a\mathcal{A}_{\mathbb{T}^3}(\partial E_n) - 2\int_{\partial E_n} w_n \, d\mu_n .$$

The above equality vanishes if and only if $a = \fint_{\partial E_n} w_n \, d\mu_n$, hence,

$$\|w_n - \widetilde{w}_n\|_{L^2(\partial E_n)} = \min_{a\in\mathbb{R}}\|w_n - a\|_{L^2(\partial E_n)}$$

and inequality (2.22) follows by the definition of $\|\cdot\|_{H^{\frac{1}{2}}(\partial E_n)}$ (see Appendix B) and the observation on the Gagliardo seminorms just before Lemma 2.8.
Then, from the *trace inequality* (see [13]), which holds with a "uniform" constant $C = C(F, M, \alpha)$, for all the sets $E \in \mathfrak{C}^{1,\alpha}_M(F)$ (see [8]), we obtain

$$\|w_n - \widetilde{w}_n\|^2_{H^{\frac{1}{2}}(\partial E_n)} \leq \|w_n - \widehat{w}_n\|^2_{H^{\frac{1}{2}}(\partial E_n)} \leq C\int_{\mathbb{T}^3} |\nabla w_n|^2 \, dx < C < +\infty \quad (2.23)$$

with a constant C independent of $n \in \mathbb{N}$.
We claim now that

$$\sup_{n\in\mathbb{N}} \|\mathrm{H}_n\|_{H^{\frac{1}{2}}(\partial E_n)} < +\infty. \quad (2.24)$$

To see this note that by the uniform $C^{1,\alpha}$–bounds on ∂E_n, we may find a fixed solid cylinder of the form $C = D \times (-L, L)$, with $D \subseteq \mathbb{R}^2$ a ball centered at the origin and functions f_n, with

$$\sup_{n\in\mathbb{N}} \|f_n\|_{C^{1,\alpha}(\overline{D})} < +\infty \, , \quad (2.25)$$

such that $\partial E_n \cap C = \{(x', x_n) \in D \times (-L, L) : x_n = f_n(x')\}$ with respect to a suitable coordinate frame (depending on $n \in \mathbb{N}$).

$$\begin{aligned}\int_D (\mathrm{H}_n - \widetilde{\mathrm{H}}_n) \, dx' + \widetilde{\mathrm{H}}_n \mathrm{Area}(D) &= \int_D \mathrm{div}\Big(\frac{\nabla_{x'} f_n}{\sqrt{1 + |\nabla_{x'} f_n|^2}}\Big) \, dx' \\ &= \int_{\partial D} \frac{\nabla_{x'} f_n}{\sqrt{1 + |\nabla_{x'} f_n|^2}} \cdot \frac{x'}{|x'|} \, d\sigma \, .\end{aligned}$$

where σ is the canonical (standard) measure on the circle ∂D.
Hence, recalling the uniform bound (2.25) and the fact that $\|\mathrm{H}_n - \widetilde{\mathrm{H}}_n\|_{H^{\frac{1}{2}}(\partial E_n)}$ are equibounded thanks to inequalities (2.21) and (2.23), we get that $\widetilde{\mathrm{H}}_n$ are also equibounded (by a standard "localization" argument, "uniformly" applied to all the hypersurfaces ∂E_n). Therefore, the claim (2.24) follows.
By applying the Sobolev embedding theorem on each connected component of ∂F, we have that

$$\|\mathrm{H}_n\|_{L^p(\partial E_n)} \leq C\|\mathrm{H}_n\|_{H^{\frac{1}{2}}(\partial E_n)} < C < +\infty \qquad \text{for all } p \in [1, 4].$$

for a constant C independent of $n \in \mathbb{N}$.
Now, by means of Calderón–Zygmund estimates, it is possible to show (see [8]) that there exists a constant $C > 0$ depending only on F, M, α and $p > 1$ such that for every $E \in \mathfrak{C}^{1,\alpha}_M(F)$, there holds

$$\|B\|_{L^p(\partial E)} \leq C(1 + \|\mathrm{H}\|_{L^p(\partial E)}) \, . \quad (2.26)$$

Then, if we write

$$\partial E_n = \{y + \psi_n(y)\nu_F(y) : y \in \partial F\},$$

we have $\sup_{n\in\mathbb{N}} \|\psi_n\|_{W^{2,p}(\partial F)} < +\infty$, for all $p \in [1, 4]$ (taking into account Remark A.2). Thus, by the Sobolev compact embedding $W^{2,p}(\partial F) \hookrightarrow C^{1,\alpha}(\partial F)$, up to a subsequence (not relabeled), there exists a set $F' \in \mathfrak{C}^{1,\alpha}_M(F)$ such that

$$\psi_n \to \psi_{F'} \text{ in } C^{1,\alpha}(\partial F) \quad \text{and} \quad v_n \to v_{F'} \text{ in } C^{1,\beta}(\mathbb{T}^3)$$

for all $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$.
From estimate (2.24) and Lemma 2.9 (possibly choosing a smaller M), we have then that the functions ψ_n are bounded in $W^{\frac{5}{2},2}(\partial F)$. Hence, possibly passing to another subsequence (again not relabeled), we conclude that $E_n \to F'$ in $W^{2,p}$ for every $p \in [1, 4)$, by the Sobolev compact embedding in Proposition B.2 with $q \in [1, 4)$, $s = 1/2$ and $p = 2$ there, applied to $\operatorname{Hess}\psi_n$.

If moreover we have

$$\int_{\mathbb{T}^3} |\nabla w_n|^2\, dx \to 0$$

then the above arguments yield the existence of $\lambda \in \mathbb{R}$ and a subsequence (not relabeled) such that $w_n\big(\,\cdot + \psi_n(\cdot)\nu_F(\cdot)\big) \to \lambda$ in $H^{\frac{1}{2}}(\partial F)$. In turn,

$$\mathrm{H}_n\big(\,\cdot + \psi_n(\cdot)\nu_F(\cdot)\big) \to \lambda - 4\gamma v_{F'}\big(\,\cdot + \psi_{F'}(\cdot)\nu_F(\cdot)\big) = \mathrm{H}\big(\,\cdot + \psi_{F'}(\cdot)\nu_F(\cdot)\big)$$

in $H^{\frac{1}{2}}(\partial F)$, where H is the mean curvature of F'. Hence F' is critical.
To conclude the proof we then only need to show that ψ_n converge to $\psi = \psi_{F'}$ in $W^{\frac{5}{2},2}(\partial F)$.
Fixed $\delta > 0$, arguing as in the proof of Lemma 2.9, we reduce ourselves to the case where the functions ψ_n are defined on a disk $D \subseteq \mathbb{R}^2$, are bounded in $W^{\frac{5}{2},2}(D)$, converge in $W^{2,p}(D)$ for all $p \in [1, 4)$ to $\psi \in W^{\frac{5}{2},2}(D)$ and $\|\nabla\psi\|_{L^\infty(D)} \leq \delta$. Then, fixed a smooth cut–off function φ with compact support in D and equal to one on a smaller disk $D' \subseteq D$, we have

$$\begin{aligned}
\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|\nabla\psi_n|^2}} - \frac{\Delta(\varphi\psi)}{\sqrt{1+|\nabla\psi|^2}} &= (\nabla^2(\varphi\psi_n) - \nabla^2(\varphi\psi))\frac{\nabla\psi\nabla\psi}{(1+|\nabla\psi|^2)^{3/2}} \\
&\quad + \nabla^2(\varphi\psi_n)\Big(\frac{\nabla\psi_n\nabla\psi_n}{(1+|\nabla\psi_n|^2)^{3/2}} - \frac{\nabla\psi\nabla\psi}{(1+|\nabla\psi|^2)^{3/2}}\Big) \\
&\quad + \varphi(\mathrm{H}_n - \mathrm{H}) + R(x, \psi_n, \nabla\psi_n) - R(x, \psi, \nabla\psi),
\end{aligned}$$

where R is a smooth Lipschitz function. Then, using Lemma 2.8 with $\beta \in (0, 1/2)$, an argument similar to the one in the proof of Lemma 2.9 shows that

$$\begin{aligned}
\Big[\frac{\Delta(\varphi\psi_n)}{\sqrt{1+|\nabla\psi_n|^2}} - \frac{\Delta(\varphi\psi)}{\sqrt{1+|\nabla\psi|^2}}\Big]_{H^{\frac{1}{2}}(D)} &\leq C(M)\Big(\delta^2[\nabla^2(\varphi\psi_n) - \nabla^2(\varphi\psi)]_{H^{\frac{1}{2}}(D)} \\
&\quad + \|\nabla^2(\varphi\psi_n) - \nabla^2(\varphi\psi)\|_{L^{\frac{4}{1+\beta}}(D)}\|\nabla\psi\|^{\beta}_{L^\infty(D)}\|\nabla^2\psi\|^{1-\beta}_{L^4(D)} + \\
&\quad + [\nabla^2(\varphi\psi_n)]_{H^{\frac{1}{2}}(D)}\|\nabla\psi_n - \nabla\psi\|_{L^\infty(D)} \\
&\quad + \|\nabla^2(\varphi\psi_n)\|_{L^{\frac{4}{1+\beta}}(D)}\|\nabla\psi_n - \nabla\psi\|^{\beta}_{L^\infty(D)}(\|\nabla^2\psi_n\|_{L^4(D)} + \|\nabla^2\psi\|_{L^4(D)})^{1-\beta} \\
&\quad + \|\mathrm{H}_n - \mathrm{H}\|_{H^{\frac{1}{2}}(D)} + \|\psi_n - \psi\|_{W^{2,2}(D)}\Big).
\end{aligned}$$

Using Lemma 2.8 again to estimate $[\Delta(\varphi\psi_n) - \Delta(\varphi\psi)]_{H^{\frac{1}{2}}(D)}$ with the seminorm on the left hand side of the previous inequality and arguing again as in the proof of Lemma 2.9, we finally get

$$[\nabla^2(\varphi\psi_n) - \nabla^2(\varphi\psi)]_{H^{\frac{1}{2}}(D)} \leq C(M)\Big(\|\psi_n - \psi\|_{W^{2,\frac{4}{1+\beta}}(D)} + \|\nabla\psi_n - \nabla\psi\|^{\beta}_{L^\infty(D)} + \|\mathrm{H}_n - \mathrm{H}\|_{H^{\frac{1}{2}}(D)}\Big)\,,$$

hence,

$$[\nabla^2\psi_n - \nabla^2\psi]_{H^{\frac{1}{2}}(D')} \leq C(M)\Big(\|\psi_n - \psi\|_{W^{2,\frac{4}{1+\beta}}(D')} + \|\nabla\psi_n - \nabla\psi\|^{\beta}_{L^\infty(D')} + \|\mathrm{H}_n - \mathrm{H}\|_{H^{\frac{1}{2}}(D')}\Big)\,,$$

from which the conclusion follows, by the first part of the lemma with $p = 4/(1+\beta) < 4$ and a standard covering argument. □

3

GLOBAL EXISTENCE AND BEHAVIOR OF SOLUTIONS

We are ready to prove the long time existence result. We will follow the proof in [1], by means of the lemmas proved in the previous chapter. As described in the introduction, the following theorem shows that a strictly stable critical set is in a way like the equilibrium configuration of a system at the bottom of potential well. Indeed we are going to show that under the modified Mullins–Sekerka flow, a smooth set starting close to a smooth strictly stable critical set, asymptotically moves back to a translate of such set.

Theorem 3.1. *Let $E \subseteq \mathbb{T}^n$ be a smooth strictly stable critical set with N_ε (with $\varepsilon < 1$) a tubular neighbourhood of E, as in formula (1.37). For every $\alpha \in (0, 1/2)$ there exists $M > 0$ such that, if E_0 is a smooth set in $\mathfrak{C}^{1,\alpha}_M(E)$ satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$ and*

$$\int_{\mathbb{T}^3} |\nabla w_{E_0}|^2\,dx \le M$$

where $w_0 = w_{E_0}$ is the function relative to E_0 as in problem (2.1), then the unique smooth solution E_t of the modified Mullins–Sekerka flow (with parameter $\gamma \ge 0$) starting from E_0, given by Theorem 2.5, is defined for all $t \ge 0$. Moreover, $E_t \to E + \eta$ exponentially fast in $W^{5/2,2}$ as $t \to +\infty$ (recall the definition of convergence of sets at the beginning of Section 1.3), for some $\eta \in \mathbb{R}^3$, with the meaning that the functions $\psi_{\eta,t} : \partial E + \eta \to \mathbb{R}$ representing ∂E_t as "normal graphs" on $\partial E + \eta$, that is,

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{E+\eta}(y) \,:\, y \in \partial E + \eta\},$$

satisfy

$$\|\psi_{\eta,t}\|_{W^{5/2,2}(\partial E+\eta)} \le Ce^{-\beta t},$$

for every $t \in [0, +\infty)$, for some positive constants C and β.

Remark 3.2. We already said that the property of a set E_0 to belong to $\mathfrak{C}^{1,\alpha}_M(E)$ is a "closedness" in L^1 of E_0 and E, and in $C^{1,\alpha}$ of their boundaries. The extra condition in the theorem on the L^2–smallness of the gradient of w_0 (see the second part of Lemma 2.10 and its proof) implies that the quantity $\mathrm{H}_0 + 4\gamma v_0$ on ∂E_0 is "close" to be constant, as it is the analogous quantity for the set E (or actually for any critical set). Notice that this is a second order condition for the boundary of E_0, in addition to the first order one $E_0 \in \mathfrak{C}^{1,\alpha}_M(E)$.

Proof of Theorem 3.1. Throughout the whole proof C will denote a constant depending only on E, M and α, whose value may vary from line to line.

Assume that the modified Mullins–Sekerka flow E_t is defined for t in the maximal time interval $[0, T(E_0))$, where $T(E_0) \in (0, +\infty]$ and let the moving boundaries ∂E_t be represented as "normal graphs" on ∂E as

$$\partial E_t = \{y + \psi_t(y)\nu_E(y) \,:\, y \in \partial E\},$$

for some smooth functions $\psi_t : \partial E \to \mathbb{R}$. As before we set $\nu_t = \nu_{E_t}$, $v_t = v_{E_t}$ and $w_t = w_{E_t}$.

We recall that, by Theorem 2.5, for every $F \in \mathfrak{C}^{2,\alpha}_M(E)$, the flow is defined in the time interval $[0, T)$, with $T = T(E, M, \alpha) > 0$.

We show the theorem for the smooth sets $E_0 \subseteq \mathbb{T}^3$ satisfying

$$\mathrm{Vol}(E_0 \triangle E) \leq M_1, \quad \|\psi_0\|_{C^{1,\alpha}(\partial E)} \leq M_2 \quad \text{and} \quad \int_{\mathbb{T}^3} |\nabla w_0|^2 \, dx \leq M_3 \,,$$

for some positive constants M_1, M_2, M_3, then we get the thesis by setting $M = \min\{M_1, M_2, M_3\}$.

For any set $F \in \mathfrak{C}^{1,\alpha}_M(E)$ we introduce the following quantity

$$D(F) = \int_{F \Delta E} d(x, \partial E) \, dx = \int_F d_E \, dx - \int_E d_E \, dx, \tag{3.1}$$

where d_E is the signed distance function defined in formula (1.38). We observe that

$$\mathrm{Vol}(F \Delta E) \leq C \|\psi_F\|_{L^1(\partial E)} \leq C \|\psi_F\|_{L^2(\partial E)}$$

for a constant C depending only on E and, as $F \subseteq N_\varepsilon$,

$$D(F) \leq \int_{F \Delta E} \varepsilon \, dx \leq \varepsilon \mathrm{Vol}(F \Delta E).$$

Moreover,

$$\begin{aligned}
\|\psi_F\|^2_{L^2(\partial E)} &= 2 \int_{\partial E} \int_0^{|\psi_F(y)|} t \, dt \, d\mu(y) \\
&= 2 \int_{\partial E} \int_0^{|\psi_F(y)|} d(L(y,t), \partial E) \, dt \, d\mu(y) \\
&= 2 \int_{E \Delta F} d(x, \partial E) \, JL^{-1}(x) \, dx \\
&\leq C D(F).
\end{aligned}$$

where $L : \partial E \times (-\varepsilon, \varepsilon) \to N_\varepsilon$ the smooth diffeomorphism defined in formula (1.40) and JL its Jacobian. As we already said, the constant C depends only on E and ε. This clearly implies

$$\mathrm{Vol}(F \Delta E) \leq C \|\psi_F\|_{L^1(\partial E)} \leq C \|\psi_F\|_{L^2(\partial E)} \leq C \sqrt{D(F)} \,. \tag{3.2}$$

Hence, by this discussion, the initial smooth set $E_0 \in \mathfrak{C}^{1,\alpha}_M(E)$ satisfies $D(E_0) \leq M \leq M_1$ (having chosen $\varepsilon < 1$).

By rereading the proof of Lemma 2.10, it follows that for M_2, M_3 small enough, if $\|\psi_F\|_{C^{1,\alpha}(\partial E)} \leq M_2$ and

$$\int_{\mathbb{T}^3} |\nabla w_F|^2 \, dx \leq M_3 \,,$$

then

$$\|\psi_F\|_{W^{2,3}(\partial E)} \leq \omega(\max\{M_2, M_3\}) \,, \tag{3.3}$$

where $s \mapsto \omega(s)$ is a positive nondecreasing function (defined on $\mathbb{R}$) such that $\omega(s) \to 0$ as $s \to 0^+$. This clearly implies (recalling Remark A.2)

$$\|\nu_F\|_{W^{1,3}(\partial F)} \leq \omega'(\max\{M_2, M_3\}), \tag{3.4}$$

for a function ω' with the same properties of ω. Both ω and ω' only depend on E and α, for M small enough.

We split the proof of the theorem into steps.

Step 1. (*Stopping–time*) Let $\overline{T} \leq T(E_0)$ be the maximal time such that

$$\mathrm{Vol}(E_t \triangle E) \leq 2M_1, \quad \|\psi_t\|_{C^{1,\alpha}(\partial E)} \leq 2M_2 \quad \text{and} \quad \int_{\mathbb{T}^3} |\nabla w_t|^2 \, dx \leq 2M_3, \tag{3.5}$$

for all $t \in [0, \overline{T})$. Hence,

$$\|\psi_t\|_{W^{2,3}(\partial E)} \leq \omega(2\max\{M_2, M_3\})$$

for all $t \in [0, \overline{T}')$, as in formula (3.3). Note that such a maximal time is clearly positive, by the hypotheses on E_0.

We claim that by taking M_1, M_2, M_3 small enough, we have $\overline{T} = T(E_0)$.

Step 2. (*Estimate of the translational component of the flow*) We want to see that there exists a small constant $\theta > 0$ such that

$$\min_{\eta \in \mathrm{O}_E} \| [\partial_{\nu_t} w_t] - \langle \eta \,|\, \nu_t \rangle \|_{L^2(\partial E_t)} \geq \theta \big\| [\partial_{\nu_t} w_t] \big\|_{L^2(\partial E_t)} \qquad \text{for all } t \in [0, \overline{T}), \tag{3.6}$$

where O_E is defined by formula (1.35).

If M is small enough, clearly there exists a constant $C_0 = C_0(E, M, \alpha) > 0$ such that, for every $i \in \mathrm{I}_E$, we have $\|\langle e_i | \nu_t \rangle\|_{L^2(\partial E_t)} \geq C_0 > 0$, holding $\|\langle e_i | \nu_E \rangle\|_{L^2(\partial E)} > 0$. It is then easy to show that the vector $\eta_t \in \mathrm{O}_E$ realizing such minimum is unique and satisfies

$$[\partial_{\nu_t} w_t] = \langle \eta_t \,|\, \nu_t \rangle + g, \tag{3.7}$$

where $g \in L^2(\partial E_t)$ is a function L^2–orthogonal (with respect to the measure μ_t on ∂E_t) to the vector subspace of $L^2(\partial E_t)$ spanned by $\langle e_i | \nu_t \rangle$, with $i \in \mathrm{I}_E$, where $\{e_1, \ldots, e_3\}$ is the orthonormal basis of $\mathbb{R}^3$ given by Remark 1.16. Moreover, the inequality

$$|\eta_t| \leq C \big\| [\partial_{\nu_t} w_t] \big\|_{L^2(\partial E_t)} \tag{3.8}$$

holds, with a constant C depending only on E, M and α.

We now argue by contradiction, assuming $\|g\|_{L^2(\partial E_t)} < \theta \big\| [\partial_{\nu_t} w_t] \big\|_{L^2(\partial E_t)}$. First, by formula (1.5) and the translation invariance of the functional J, we have

$$0 = \frac{d}{ds} J(E_t + s\eta_t) \Big|_{s=0} = \int_{\partial E_t} (\mathrm{H}_t + 4\gamma v_t) \langle \eta_t \,|\, \nu_t \rangle \, d\mu_t = \int_{\partial E_t} w_t \langle \eta_t \,|\, \nu_t \rangle \, d\mu_t \,.$$

It follows that, by multiplying equality (3.7) by $w_t - \widehat{w}_t$, with $\widehat{w}_t = \fint_{\mathbb{T}^3} w_t\, dx$ and integrating over ∂E_t, we get

$$\begin{aligned}
\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx &= -\int_{\partial E_t} w_t[\partial_{\nu_t} w_t]\, d\mu_t \\
&= -\int_{\partial E_t} (w_t - \widehat{w}_t)[\partial_{\nu_t} w_t]\, d\mu_t \\
&= -\int_{\partial E_t} (w_t - \widehat{w}_t) g\, d\mu_t \\
&\le \theta \|w_t - \widehat{w}_t\|_{L^2(\partial E_t)} \big\|[\partial_{\nu_t} w_t]\big\|_{L^2(\partial E_t)}.
\end{aligned}$$

Note that in the second and the third equality above we have used the fact that $[\partial_{\nu_t} w_t]$ and ν_t have zero integral on ∂E_t.
By the trace inequality (see [13]), we have

$$\|w_t - \widehat{w}_t\|^2_{L^2(\partial E_t)} \le \|w_t - \widehat{w}_t\|^2_{H^{\frac{1}{2}}(\partial E_t)} \le C \int_{\mathbb{T}^3} |\nabla w_t|^2\, dx\,, \tag{3.9}$$

hence, by the previous estimate, we conclude

$$\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \le C\theta^2 \big\|[\partial_{\nu_t} w_t]\big\|^2_{L^2(\partial E_t)}. \tag{3.10}$$

Let us denote with $f : \mathbb{T}^3 \to \mathbb{R}$ the harmonic extension of $\langle \eta_t \,|\, \nu_t \rangle$ to $\mathbb{T}^3$, we then have

$$\|\nabla f\|_{L^2(\mathbb{T}^3)} \le C \|\langle \eta_t \,|\, \nu_t \rangle\|_{H^{\frac{1}{2}}(\partial E_t)} \le C|\eta_t| \|\nu_t\|_{W^{1,3}(\partial E_t)} \le C\big\|[\partial_{\nu_t} w_t]\big\|_{L^2(\partial E_t)}\,, \tag{3.11}$$

where the first inequality comes by standard elliptic estimates (holding with a constant $C = C(E, M, \alpha) > 0$, see [8] for details), the second is trivial and the last one follows by inequalities (3.4) and (3.8).
Thus, by equality (3.7) and estimates (3.10) and (3.11), we get

$$\begin{aligned}
\|\langle \eta_t \,|\, \nu_t \rangle\|^2_{L^2(\partial E_t)} &= \int_{\partial E_t} [\partial_{\nu_t} w_t] \langle \eta_t \,| \nu_t \rangle\, d\mu \\
&= -\int_{\mathbb{T}^3} \langle \nabla w_t \,|\, \nabla f \rangle\, dx \\
&\le \Big(\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \Big)^{1/2} \Big(\int_{\mathbb{T}^3} |\nabla f|^2\, dx \Big)^{1/2} \\
&\le C\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{L^2(\partial E_t)}\,.
\end{aligned}$$

If then $\theta > 0$ is chosen so small that $C\theta + \theta^2 < 1$ in the last inequality, then we have a contradiction with equality (3.7) and the fact that $\|g\|_{L^2(\partial E_t)} < \theta \big\|[\partial_{\nu_t} w_t]\big\|_{L^2(\partial E_t)}$, as they imply (by L^2–orthogonality) that

$$\|\langle \eta_t \,|\, \nu_t \rangle\|^2_{L^2(\partial E_t)} > (1 - \theta^2) \big\|[\partial_{\nu_t} w_t]\big\|^2_{L^2(\partial E_t)}\,.$$

All this argument shows that for such a choice of θ condition (3.6) holds.
By Propositions 1.25 and 1.26, there exist positive constants σ_θ and δ with the

following properties: for any set $F \in \mathfrak{C}^{1,\alpha}_M(E)$ such that $\|\psi_F\|_{W^{2,3}(\partial E)} < \delta$, there holds

$$\Pi_F(\varphi) \geq \sigma_\theta \|\varphi\|^2_{H^1(\partial F)}$$

for all $\varphi \in \widetilde{H}^1(\partial F)$ such that $\min_{\eta \in \mathrm{O}_E} \|\varphi - \langle \eta \,|\, \nu_F \rangle\|_{L^2(\partial F)} \geq \theta \|\varphi\|_{L^2(\partial F)}$ and if E' is critical, $\mathrm{Vol}(E') = \mathrm{Vol}(E)$ with $\|\psi_{E'}\|_{W^{2,3}(\partial E)} < \delta$, then

$$E' = E + \eta \tag{3.12}$$

for a suitable vector $\eta \in \mathbb{R}^3$. We then assume that M_2, M_3 are small enough such that

$$\omega(2\max\{M_2, M_3\}) < \delta/2 \tag{3.13}$$

where ω is the function introduced in formula (3.3).

Step 3. (*The stopping time $\overline{T}$ is equal to the maximal time $T(E_0)$*) We show now that, by taking M_1, M_2, M_3 smaller if needed, we have $\overline{T} = T(E_0)$.

By the previous point and the suitable choice of M_2, M_3 made in its final part, formula (3.6) holds, hence we have

$$\Pi_{E_t}([\partial_{\nu_t} w_t]) \geq \sigma_\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E)} \qquad \text{for all } t \in [0, \overline{T}).$$

In turn, by Lemma 2.6 we may estimate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}^3} |\nabla w_t|^2\,dx\right) \leq -\,\sigma_\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)} + \frac{1}{2}\int_{\partial E_t} (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2\,d\mu_t$$

for every $t \leq \overline{T}$.
It is now easy to see that

$$\Delta w_t = [\partial_{\nu_t} w_t]\mu_t\,,$$

then, by point (iii) of Lemma 2.7, we estimate the last term as

$$\begin{aligned}\int_{\partial E_t} (\partial_{\nu_t} w_t^+ + \partial_{\nu_t} w_t^-)[\partial_{\nu_t} w_t]^2\,d\mu_t &\leq C\int_{\partial E_t} (|\partial_{\nu_t} w_t^+|^3 + |\partial_{\nu_t} w_t^-|^3)\,d\mu_t \\ &\leq C\int_{\partial E_t} \big|[\partial_{\nu_t} w_t]\big|^3\,d\mu_t\,,\end{aligned}$$

thus, the last estimate in the statement of Lemma 2.7 implies

$$\big\|[\partial_{\nu_t} w_t]\big\|_{L^3(\partial E_t)} \leq C\big\|[\partial_{\nu_t} w_t]\big\|^{2/3}_{H^1(\partial E_t)} \|w_t - \widehat{w}_t\|^{1/3}_{L^2(\partial E_t)}.$$

Therefore, combining the last three estimates, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^3} |\nabla w_t|^2\,dx &\leq -\,2\sigma_\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)} + C\|w_t - \widehat{w}_t\|_{L^2(\partial E_t)} \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)} \\ &\leq -\,\sigma_\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)},\end{aligned} \tag{3.14}$$

for every $t \in [0, \overline{T})$, where in the last inequality we used the trace inequality (3.9)

$$\|w_t - \widehat{w}_t\|^2_{L^2(\partial E_t)} \leq \|w_t - \widehat{w}_t\|^2_{H^{\frac{1}{2}}(\partial E_t)} \leq C\int_{\mathbb{T}^3} |\nabla w_t|^2\,dx \leq 2CM_3,$$

possibly choosing a smaller M such that $2CM_3 < \sigma_\theta$.

This argument clearly says that the quantity $\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx$ is nonincreasing in time, hence, if M_2, M_3 are small enough, the inequality $\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \leq 2M_3$ is preserved during the flow. If we assume by contradiction that $\overline{T} < T(E_0)$, then it must happen that $\mathrm{Vol}(E_{\overline{T}} \triangle E) = 2M_1$ or $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E)} = 2M_2$. Before showing that this is not possible, we prove that actually the quantity $\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx$ decreases (non increases) exponentially.

Computing as in the previous step,

$$\begin{aligned}
\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx &= -\int_{\partial E_t} w_t [\partial_{\nu_t} w_t]\, d\mu_t \\
&= -\int_{\partial E_t} (w_t - \widehat{w}_t)[\partial_{\nu_t} w_t]\, d\mu_t \\
&\leq \|w_t - \widehat{w}_t\|_{L^2(\partial E_t)} \big\|[\partial_{\nu_t} w_t]\big\|_{L^2(\partial E_t)} \\
&\leq C \left(\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \right)^{1/2} \big\|[\partial_{\nu_t} w_t]\big\|_{L^2(\partial E_t)},
\end{aligned}$$

where we used again the trace inequality (3.9). Then,

$$\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \leq C \big\|[\partial_{\nu_t} w_t]\big\|^2_{L^2(\partial E_t)} \leq C \|[\partial_{\nu_t} w_t]\|^2_{H^1(\partial E_t)},$$

and combining this inequality with estimate (3.14), we obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \leq -c_0 \int_{\mathbb{T}^3} |\nabla w_t|^2\, dx,$$

for every $t \leq \overline{T}$ and for a suitable constant $c_0 \geq 0$. Integrating this differential inequality, we get

$$\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \leq e^{-c_0 t} \int_{\mathbb{T}^3} |\nabla w_0|^2\, dx \leq M_3 e^{-c_0 t} \leq M_3\,, \tag{3.15}$$

for every $t \leq \overline{T}$.

Then, we assume that $\mathrm{Vol}(E_{\overline{T}} \triangle E) = 2M_1$ or $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})} = 2M_2$. Recalling formula (3.1) and denoting by X_t the velocity field of the flow (see Definition 2.1 and the subsequent discussion), we compute

$$\begin{aligned}
\frac{d}{dt} D(E_t) = \frac{d}{dt} \int_{E_t} d_E\, dx &= \int_{E_t} \mathrm{div}(d_E X_t)\, dx = \int_{\partial E_t} d_E \langle X_t | \nu_t \rangle\, d\mu_t \\
= \int_{\partial E_t} d_E [\partial_{\nu_t} w_t]\, d\mu_t &= -\int_{\mathbb{T}^3} \langle \nabla h \,|\, \nabla w_t \rangle\; dx\,,
\end{aligned}$$

where h denotes the harmonic extension of d_E to $\mathbb{T}^3$. Note that, by standard elliptic estimates and the properties of the signed distance function d_E, we have

$$\|\nabla h\|_{L^2(\mathbb{T}^3)} \leq C \|d_E\|_{C^{1,\alpha}(\partial E)} \leq C = C(E)\,,$$

then, by the previous equality and formula (3.15), we get

$$\frac{d}{dt} D(E_t) \leq C \|\nabla w_t\|_{L^2(\mathbb{T}^3)} \leq C \sqrt{M_3}\, e^{-c_0 t/2}\,,$$

for every $t \leq \overline{T}$. By integrating this differential inequality over $[0, \overline{T})$ and recalling estimate (3.2), we get

$$\mathrm{Vol}(E_{\overline{T}} \triangle E) \leq C \|\psi_{\overline{T}}\|_{L^2(\partial E_{\overline{T}})} \leq C\sqrt{D(E_{\overline{T}})} \leq C\sqrt{D(E_0) + C\sqrt{M_3}} \leq C\sqrt[4]{M_3}\,, \tag{3.16}$$

as $D(E_0) \leq M_1$, provided that M_1, M_3 are chosen suitably small. This shows that $\mathrm{Vol}(E_{\overline{T}} \triangle E) = 2M_1$ cannot happen if we chose $C\sqrt[4]{M_3} \leq M_1$.
By arguing as in Lemma 2.10 (keeping into account inequality (3.5) and formula (3.3)), we can see that the L^2–estimate (3.16) implies a $W^{2,3}$–bound on $\psi_{\overline{T}}$ with a constant going to zero, keeping fixed M_2, as $\int_{\mathbb{T}^3} |\nabla w_{\overline{T}}|^2\, dx \to 0$, hence, by estimate (3.15), as $M_3 \to 0$. Then, by Sobolev embeddings, the same holds for $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})}$, hence, if M_3 is small enough, we have a contradiction with $\|\psi_{\overline{T}}\|_{C^{1,\alpha}(\partial E_{\overline{T}})} = 2M_2$.

Thus, $\overline{T} = T(E_0)$ and

$$\mathrm{Vol}(E_t \triangle E) \leq C\sqrt[4]{M_3}\,, \qquad \|\psi_t\|_{C^{1,\alpha}(\partial E_t)} \leq 2M_2\,, \qquad \int_{\mathbb{T}^3} |\nabla w_t|^2\, dx \leq M_3 e^{-c_0 t}\,, \tag{3.17}$$

for every $t \in [0, T(E_0))$, by choosing M_1, M_2, M_3 small enough.

Step 4. (*Long time existence*) We now show that, by taking M_1, M_2, M_3 smaller if needed, we have $T(E_0) = +\infty$, that is, the flow exists for all times.
We assume by contradiction that $T(E_0) < +\infty$ and we recall that, by estimate (3.14) and the fact that $\overline{T} = T(E_0)$, we have

$$\frac{d}{dt}\int_{\mathbb{T}^3} |\nabla w_t|^2\, dx + \sigma_\theta \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)} \leq 0$$

for all $t \in [0, T(E_0))$. Integrating this differential inequality over the interval $[T(E_0) - T/2, T(E_0) - T/4]$, where T is given by Theorem 2.5, as we said at the beginning of the proof, we obtain

$$\begin{aligned}\sigma_\theta \int_{T(E_0)-T/2}^{T(E_0)-T/4} \big\|[\partial_{\nu_t} w_t]\big\|^2_{H^1(\partial E_t)}\, dt &\leq \int_{\mathbb{T}^3} |\nabla w_{T(E_0)-\frac{T}{2}}|^2\, dx - \int_{\mathbb{T}^3} |\nabla w_{T(E_0)-\frac{T}{4}}|^2\, dx \\ &\leq M_3\,,\end{aligned}$$

where the last inequality follows from estimate (3.17). Thus, by the mean value theorem there exists $\bar{t} \in (T(E_0) - T/2, T(E_0) - T/4)$ such that

$$\big\|[\partial_{\nu_{\bar{t}}} w_{\bar{t}}]\big\|^2_{H^1(\partial E_t)} \leq \frac{4M_3}{T\sigma_\theta}\,.$$

Note that for any smooth set $F \subseteq \mathbb{T}^3$, we have $\|v_F\|_{C^1(\mathbb{T}^3)} \leq L$, for some "absolute" constant L and that w_F is constant, then, since $H^1(\partial E_{\bar{t}})$ embeds into $L^p(\partial E_{\bar{t}})$ for all $p > 1$, by Lemma 2.7, we in turn infer that

$$\begin{aligned}&[\mathrm{H}_{\bar{t}}(\cdot + \psi_{\bar{t}}(\cdot)\,\nu_E(\cdot)) - \mathrm{H}_E]^2_{C^{0,\alpha}(\partial E)} \\ &\qquad \leq C[w_{\bar{t}}(\cdot + \psi_{\bar{t}}(\cdot)\nu_E(\cdot)) - w_E]^2_{C^{0,\alpha}(\partial E)} \\ &\qquad\quad + C[v_{\bar{t}}(\cdot + \psi_{\bar{t}}(\cdot)\nu_E(\cdot)) - v_{\bar{t}}]^2_{C^{0,\alpha}(\partial E)} + C[v_{\bar{t}} - v_E]^2_{C^{0,\alpha}(\partial E)} \\ &\qquad \leq C[w_{\bar{t}}]^2_{C^{0,\alpha}(\partial E_{\bar{t}})} \|\psi_{\bar{t}}\|^2_{C^{1,\alpha}(\partial F)} + CL^2 \|\psi_{\bar{t}}\|^2_{C^{1,\alpha}(\partial F)} + C\|u_{\bar{t}} - u_E\|^2_{L^2(\mathbb{T}^3)} \\ &\qquad \leq C\frac{M_3}{T\sigma_\theta} + CL^2 \|\psi_{\bar{t}}\|^2_{C^{1,\alpha}(\partial E)} + C\mathrm{Vol}(E_{\bar{t}} \triangle E)^2.\end{aligned}$$

where $[\cdot]_{C^{0,\alpha}(\partial E_{\bar{t}})}$ and $[\cdot]_{C^{0,\alpha}(\partial E)}$ stand for the α–Hölder seminorms on $\partial E_{\bar{t}}$ and ∂E, respectively and remind that $v_{\bar{t}}, v_E$ are the potentials, defined by formula (1.1), associated to $u_{\bar{t}} = \chi_{E_{\bar{t}}} - \chi_{\mathbb{T}^n \setminus E_{\bar{t}}}$ and $u_E = \chi_E - \chi_{\mathbb{T}^n \setminus E}$.
By means of Schauder estimates (as Calderón–Zygmund inequality implied estimate (2.26)), it is possible to show (see [8]) that there exists a constant $C > 0$ depending only on E, M, α and $p > 1$ such that for every $F \in \mathfrak{C}^{1,\alpha}_M(E)$, choosing even smaller M_1, M_2, M_3, there holds

$$\|B\|_{C^{0,\alpha}(\partial F)} \leq C(1 + \|\mathrm{H}\|_{C^{0,\alpha}(\partial F)})\,.$$

Hence, by the above discussion (and Remark A.2, as before), we can conclude that $E_{\bar{t}} \in \mathfrak{C}^{2,\alpha}_M(E)$. Therefore, the maximal time of existence of the classical solution starting from $E_{\bar{t}}$ is at least T, which means that the flow E_t can be continued beyond $T(E_0)$, which is a contradiction.

Step 5. (*Convergence, up to subsequences, to a translate of* E) Let $t_n \to +\infty$, then, by estimates (3.17), the sets E_{t_n} satisfy the hypotheses of Lemma 2.10, hence, up to a (not relabeled) subsequence we have that there exists a critical set $E' \in \mathfrak{C}^{1,\alpha}_M(E)$ such that $E_{t_n} \to E'$ in $W^{\frac{5}{2},2}$. Due to formulas (3.3) and (3.13) we also have $\|\psi_{E'}\|_{W^{2,3}(\partial E)} \leq \delta$ and $E' = E + \eta$ for some (small) $\eta \in \mathbb{R}^3$ (equality (3.12)).

Step 6. (*Exponential convergence of the full sequence*) Consider now

$$D_\eta(F) = \int_{F\Delta(E+\eta)} \mathrm{dist}\,(x, \partial E + \eta)\, dx\,.$$

The very same calculations performed in Step 3 show that

$$\left|\frac{d}{dt} D_\eta(E_t)\right| \leq C\|\nabla w_t\|_{L^2(\mathbb{T}^3)} \leq C\sqrt{M_3}e^{-c_0 t/2}$$

for all $t \geq 0$, moreover, by means of the previous step, it follows $\lim_{t\to+\infty} D_\eta(E_t) = 0$. In turn, by integrating this differential inequality and writing

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{E+\eta}(y) : y \in \partial E + \eta\}\,,$$

we get

$$\|\psi_{\eta,t}\|^2_{L^2(\partial E+\eta)} \leq C D_\eta(E_t) \leq \int_t^{+\infty} C\sqrt{M_3}e^{-c_0 s/2}\, ds \leq C\sqrt{M_3}e^{-c_0 t/2}\,. \tag{3.18}$$

Since by the previous steps $\|\psi_{\eta,t}\|_{W^{2,3}(\partial E+\eta)}$ is bounded, we infer from this inequality and interpolation estimates that also $\|\psi_{\eta,t}\|_{C^{1,\beta}(\partial E+\eta)}$ decays exponentially for all $\beta \in (0, 1/3)$. Then, setting $p = \frac{2}{1-\beta}$, we have, by estimates (3.18) and (3.2) (and standard elliptic estimates),

$$\begin{aligned}\|v_t - v_{E+\eta}\|_{C^{1,\beta}(\mathbb{T}^3)} &\leq C\|v_t - v_{E+\eta}\|_{W^{2,p}(\mathbb{T}^3)} \leq C\|u_t - u_{E+\eta}\|_{L^p(\mathbb{T}^3)}\\ &\leq C\mathrm{Vol}(E_t\triangle(E+\eta))^{1/p} \leq C\|\psi_{\eta,t}\|^{1/p}_{L^2(\partial E+\eta)}\\ &\leq CM_3^{1/4p}e^{-c_0 t/4pt}\end{aligned} \tag{3.19}$$

for all $\beta \in (0, 1/3)$. Denoting the average of w_t on ∂E_t by $\overline{w}_t$, as by estimates (3.9) and (3.15) (recalling the argument to show inequality (2.22)), we have that

$$\begin{aligned}\|w_t(\,\cdot + \psi_{\eta,t}(\cdot)\nu_{E+\eta}(\cdot)) - \overline{w}_t\|_{H^{\frac{1}{2}}(\partial E+\eta)} &\leq C\|w_t - \overline{w}_t\|_{H^{\frac{1}{2}}(\partial E_t)}\|\psi_{\eta,t}\|_{C^1(\partial E+\eta)} \\ &\leq C\|\nabla w_t\|_{L^2(\mathbb{T}^3)} \\ &\leq C\sqrt{M_3}e^{-c_0 t/2}\,.\end{aligned}$$

It follows, taking into account inequality (3.19), that

$$\big\|[\mathrm{H}_t(\,\cdot + \psi_{\eta,t}(\cdot)\nu_{E+\eta}(\cdot)) - \overline{\mathrm{H}}_t] - [\mathrm{H}_{\partial E+\eta} - \overline{\mathrm{H}}_{\partial E+\eta}]\big\|_{H^{\frac{1}{2}}(\partial E+\eta)} \to 0 \tag{3.20}$$

exponentially fast, as $t \to +\infty$, where $\overline{\mathrm{H}}_t$ and $\overline{\mathrm{H}}_{\partial E+\eta}$ stand for the averages of H_t on ∂E_t and of $\mathrm{H}_{\partial E+\eta}$ on $\partial E+\eta$, respectively.
Since $E_t \to E+\eta$ (up to a subsequence) in $W^{\frac{5}{2},2}$, it is easy to check that $|\overline{\mathrm{H}}_t - \overline{\mathrm{H}}_{\partial E+\eta}| \leq C\|\psi_{\eta,t}\|_{C^1(\partial E+\eta)}$ which decays exponentially, therefore, thanks to limit (3.20), we have

$$\big\|\mathrm{H}_t(\,\cdot + \psi_{\eta,t}(\cdot)\nu_{E+\eta}(\cdot)) - \mathrm{H}_{\partial E+\eta}\big\|_{H^{\frac{1}{2}}(\partial E+\eta)} \to 0$$

exponentially fast.
The conclusion then follows arguing as at the end of Step 4. □

4 RELATED RESULTS AND RESEARCH DIRECTIONS

In this final chapter we give an overview of the "Neumann" case for the modified Mullins–Sekerka flow and we discuss the classification of the strictly stable sets. We also present the Ohta–Kawasaki functional and the link between its minimizers and the $W^{2,p}$–local minimality result, Theorem 1.19. We then conclude with some possible research directions.

4.1 A BRIEF OVERVIEW OF THE NEUMANN CASE

Let Ω be a smooth bounded open subset of $\mathbb{R}^n$. As before we consider the nonlocal Area functional

$$J_N(E) = \mathcal{A}_\Omega(\partial E) + \gamma \int_\Omega |\nabla v_E|^2 \, dx \,,$$

for every $E \subseteq \Omega$ with $\partial E \cap \partial \Omega = \emptyset$, where $\gamma \geq 0$ is a real parameter and v_E is the potential defined as follows, similarly to problem (1.3),

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } \Omega \\ \dfrac{\partial v_E}{\partial \nu_E} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega v_E \, dx = 0 \end{cases}$$

with $m = \fint_\Omega u_E \, dx$, $u_E = \chi_E - \chi_{\Omega \setminus E}$ and ν_E the outer unit normal to E.
As in formula (1.5), we have

$$\int_\Omega |\nabla v_E|^2 \, dx = \int_\Omega \int_\Omega G(x,y) u_E(x) u_E(y) \, dx dy \,,$$

where G is the (distributional) solution of

$$\begin{cases} -\Delta_x G(x,y) = \delta_y - \frac{1}{\mathrm{Vol}(\Omega)} & \text{for every } x \in \Omega \\ \langle \nabla_x G(x,y) | \nu_E(x) \rangle = 0 & \text{for every } x \in \partial\Omega \\ \displaystyle\int_\Omega G(x,y) \, dx = 0 \end{cases}$$

for every $y \in \Omega$.
Note that, unlike the "periodic" case (when the ambient is the torus $\mathbb{T}^n$), the functional J_N is not translation invariant, therefore several arguments simplify. The calculus of the first and second variations of J_N, under a volume constraint, is exactly the same as for J, then we say that a smooth set $E \subseteq \Omega$, with $\partial E \cap \partial\Omega = \emptyset$, is a *critical set*, if it satisfies the Euler–Lagrange equation

$$\mathrm{H} + 4\gamma v_E = \lambda \qquad \text{on } \partial E,$$

for a constant $\lambda \in \mathbb{R}$, instead, since J_N is not translation invariant, the spaces $T(\partial E)$, $T^{\perp}(\partial E)$, and the decomposition (1.33) are no longer needed and, defining the same quadratic form Π_E as in formula (1.31), we say that a smooth critical set E is *strictly stable* if

$$\Pi_E(\varphi) > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E) \setminus \{0\}.$$

We say that $E \subseteq \Omega$ is a *local minimizer* if there exists a $\delta \geq 0$ such that

$$J_N(F) \geq J_N(E),$$

for all $F \subseteq \Omega$, $\partial F \cap \partial\Omega = \emptyset$, $\mathrm{Vol}(F) = \mathrm{Vol}(E)$ and $\mathrm{Vol}(E \triangle F) \leq \delta$. Then, as in the periodic case, we have a local minimality result with respect to small $W^{2,p}$–perturbations. Precisely, the following counterpart to Theorem 1.19 holds (see [23]).

Theorem 4.1. *Let $p > \max\{2, n-1\}$ and $E \subseteq \Omega$ a smooth strictly stable critical set for the nonlocal Area functional J_N (under a volume constraint) with N_ε a tubular neighbourhood of E as in formula (1.37). Then there exist constants $\delta, C > 0$ such that*

$$J_N(F) \geq J_N(E) + C[\mathrm{Vol}(E \triangle F)]^2 \,,$$

for all smooth sets $F \subseteq \mathbb{T}^n$ such that $\mathrm{Vol}(F) = \mathrm{Vol}(E)$, $\mathrm{Vol}(F \triangle E) < \delta$, $\partial F \subseteq N_\varepsilon$ and

$$\partial F = \{y + \psi(y)\nu_E(y) \,:\, y \in \partial E\},$$

for a smooth ψ with $\|\psi\|_{W^{2,p}(\partial E)} < \delta$.
As a consequence, E is a $W^{2,p}$–local minimizer of J_N (as defined above). Moreover, if F is $W^{2,p}$–close enough to E and $J_N(F) = J_N(E)$, then $F = E$, that is, E is locally the unique $W^{2,p}$–local minimizer.

Sketch of the proof. Following the line of proof of Theorem 1.19, since the functional is not translation invariant we do not need Lemma 1.24 and inequality (1.69), proved in Step 2 of the proof of Theorem 1.19, simplifies to

$$\inf\Big\{\Pi_F(\varphi) \,:\, \varphi \in \widetilde{H}^1(\partial F)\,, \|\varphi\|_{H^1(\partial F)} = 1\Big\} \geq \frac{m_0}{2}\,,$$

where m_0 is the constant defined in formula (1.68). The proof of this inequality then goes exactly as there.
Coming to Step 3 of the proof of Theorem 1.19, we do not need inequality (1.72), thus we do not need to replace F by a suitable translated set $F - \eta$. Instead, we only need to observe that inequality (1.76) is still satisfied and the rest of the proof remains unchanged. □

The short time existence and uniqueness result 2.5, proved in [12] in any dimensions, holds also in the "Neumann" case for the modified Mullins–Sekerka flow with parameter $\gamma \geq 0$, obtained (as in Definition 2.2) by letting the outer normal velocity V_t of the moving boundaries given by

$$V_t = [\partial_{\nu_t} w_t] \qquad \text{on } \partial E_t \text{ for all } t \in [0, T),$$

where $\nu_t = \nu_{E_t}$ and $w_t = w_{E_t}$ is the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} \Delta w_{E_t} = 0 & \text{in } \Omega \setminus \partial E_t \\ w_{E_t} = \mathrm{H} + 4\gamma v_{E_t} & \text{on } \partial E_t, \end{cases}$$

with v_{E_t} the potential defined above and, as before, $[\partial_{\nu_t} w_t]$ is the jump of the outer normal derivative of w_{E_t} on ∂E_t.

Then, we conclude by stating the following analogue of Theorem 3.1.

Theorem 4.2. *Let Ω be an open smooth subset of $\mathbb{R}^3$ and let $E \subseteq \Omega$ be a smooth strictly stable critical set with $\partial E \cap \partial\Omega = \emptyset$ and N_ε (with $\varepsilon < 1$) a tubular neighborhood of ∂E, as in formula (1.37). Then, for every $\alpha \in (0, 1/2)$ there exists $M > 0$ such that, if E_0 is a smooth set in $\mathfrak{C}^{1,\alpha}_M(E)$ satisfying $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$ and*

$$\int_\Omega |\nabla w_{E_0}|^2\, dx \leq M$$

where $w_0 = w_{E_0}$ is the function relative to E_0 as in problem (2.1) (with Ω in place of $\mathbb{T}^3 \setminus \partial E$), then, the unique smooth solution E_t to the Mullins–Sekerka flow (with parameter $\gamma \geq 0$) starting from E_0, given by Theorem 2.5, is defined for all $t \geq 0$. Moreover, $E_t \to E$ in $W^{5/2,2}$ exponentially fast as $t \to +\infty$ (recall the definition of convergence of sets at the beginning of Section 1.3), for some $\eta \in \mathbb{R}^3$, with the meaning that the functions $\psi_{\eta,t} : \partial E \to \mathbb{R}$ representing ∂E_t as "normal graphs" on ∂E, that is,

$$\partial E_t = \{y + \psi_{\eta,t}(y)\nu_{E+\eta}(y) \,:\, y \in \partial E\},$$

satisfy

$$\|\psi_{\eta,t}\|_{W^{5/2,2}(\partial E)} \leq Ce^{-\beta t},$$

for every $t \in [0, +\infty)$, for some positive constants C and β.

The proof of this result is similar to the one of Theorem 3.1 and actually it is simpler since we do not need the argument used in Step 2 of such proof, where we controlled the translational component of the flow. Note also that in the statement of Lemma 1.25, in this case, inequality (1.79) holds for all $\varphi \in \widetilde{H}^1(\partial F)$. Finally, observe that under the hypotheses of Proposition 1.26 we may actually conclude that $E' = E$, that is, there are no other critical sets close to E.

The assumption that ∂E does not touch the boundary of Ω may seem restrictive, however we remark that in two and three dimensions there are examples of strictly stable critical sets which consist of either a single or multiple "almost spherical" sets well contained in Ω. The precise conditions on the parameters m, γ and $\mathrm{Vol}(\Omega)$ under which these strictly stable sets exist are given in [36, 37, 38]. Other examples of local minimizers well contained in Ω are given in [7].

4.2 THE CLASSIFICATION OF THE STABLE CRITICAL SETS

We are going to discuss the class of sets to which Theorem 3.1 can be applied. In the three–dimensional case, for the Area functional (that is $\gamma = 0$), the stable critical sets in $\mathbb{T}^3$ has been fully classified in [39]. Indeed, it has been proved that the stable critical sets are *balls*, *cylinders*, *gyroids* or *lamellae*.

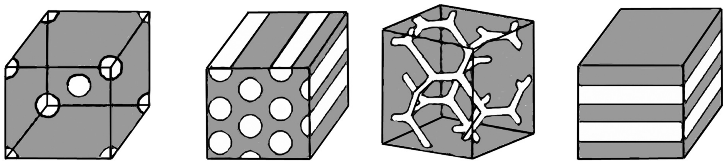

Figure 1: From left to right: balls, cylinders, gyroids and lamellae.

It is easy to see that balls, cylinders and lamellae are actually also strictly stable, while the strict stability of gyroids holds only in some cases (see [18, 19, 40]).

For the case $\gamma > 0$ a complete classification of the stable periodic structures is still missing. However, it has been shown that lamellar configuration are strictly stable if the number of interfaces is larger then some minimum value $k(\gamma)$, where $k(\gamma) \to +\infty$ as $\gamma \to +\infty$ (see [6]).

It is worth to mention what is shown in [2] about the minimizers configurations. They proved that if a horizontal strip L is the unique global minimizer of the Area functional in $\mathbb{T}^n$, then it is also the unique global minimizer of the nonlocal functional J under a volume constraint, provided that γ is sufficiently small. Precisely, the following result holds.

Theorem 4.3. *Assume that $L \subseteq \mathbb{T}^n$ is the unique, up to rigid motions, global minimizer of the Area functional, under a volume constraint. Then the same set is also the unique global minimizer of the nonlocal Area functional (1.4), provided that $\gamma > 0$ is sufficiently small.*

Moreover, in the two–dimensional case, in [21] it has been proved that if the volume parameter m satisfies $m < 1 - 2/\pi$, then the lamellae are the unique global minimizers of the Area functional in $\mathbb{T}^2$ (under a volume constraint). Hence, by Theorem 4.3, for small $\gamma > 0$, any set realizing

$$\min\left\{\mathcal{A}_{\mathbb{T}^2}(\partial E) + \gamma \int_{\mathbb{T}^2} |\nabla v_E(x)|^2\,dx \,:\, E \subseteq \mathbb{T}^2, \mathrm{Vol}(E) = \frac{m+1}{2}\right\}$$

is a lamella.

In the three–dimensional case, the global minimality of lamellae has been shown in [20] for the case $m = 0$ (that is, among the sets $E \subseteq \mathbb{T}^3$ with $\mathrm{Vol}(E) = 1/2$). Moreover, in [2], the authors proved that this conclusion still holds for m sufficiently close to 0. As before, from Theorem 4.3 we then have the following result.

Theorem 4.4. *Let n=3. There exists $m_0 > 0$ and $\gamma_0 > 0$ such that for $|m| < m_0$ and $\gamma < \gamma_0$, any solution of*

$$\min\left\{\mathcal{A}_{\mathbb{T}^3}(\partial E) + \gamma \int_{\mathbb{T}^2} |\nabla v_E(x)|^2\,dx \,:\, E \subseteq \mathbb{T}^3, \mathrm{Vol}(E) = \frac{m+1}{2}\right\}$$

is a lamella.

We conclude mentioning that in [2] it is shown that lamellae with multiple strips are local minimizers of the functional J, if the number of strips is large enough.

4.3 THE OHTA–KAWASAKI FUNCTIONAL

The Ohta–Kawasaki functional, first proposed in the modeling of microphase separation for diblock copolymer melts in [33], is defined as follows,

$$\begin{aligned} \mathcal{E}_\varepsilon(u) = {} & \varepsilon \int_\Omega |\nabla u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega (u^2-1)^2\,dx \\ & + \gamma \int_\Omega \int_\Omega G(x,y)(u(x)-m)(u(y)-m)\,dx\,dy, \end{aligned} \tag{4.1}$$

where u is any function in $H^1(\Omega)$, $m = \fint_\Omega u\,dx$, G is the Green's function of $-\Delta$ in Ω and $\gamma \geq 0$ is a fixed parameter.

We recall that, as proved in [2], the $W^{2,p}$–local minimality for the functional J (Theorem 1.19) implies its L^1–local minimality (see Remark 1.21). Moreover, it is well known that the functionals $\mathcal{E}_\varepsilon$ Γ–converge in L^1 to the nonlocal Area functional J (see [29] and see [28] for the definition and the properties of the Γ–convergence). We then state a result that links the local minimizers of the functional J with the local minimizers of the Ohta–Kawasaki functional (4.1). Fixed $m \in (-1,1)$, we say that a function $u \in H^1(\mathbb{T}^n)$ is an *isolated local minimizer* for the functionals $\mathcal{E}_\varepsilon$ with prescribed volume m, if $\fint_{\mathbb{T}^n} u\,dx = m$ and there exists a constant $\delta > 0$ such that

$$\begin{aligned} \mathcal{E}_\varepsilon(u) < \mathcal{E}_\varepsilon(w) \qquad & \text{for all } w \in H^1(\mathbb{T}^n) \text{ with } \int_{\mathbb{T}^n} w\,dx = m \\ & \text{and } 0 < \min_{\tau \in \mathbb{R}^n} \|u - w(\cdot + \tau)\|_{L^1(\mathbb{T}^n)} \leq \delta. \end{aligned}$$

The Γ–convergence and the L^1–local minimality discussed above then imply the following theorem.

Theorem 4.5. *Let $E \subseteq \mathbb{T}^n$ be a strictly stable critical set for the functional J and $u_E = \chi_E - \chi_{\mathbb{T}^n \setminus E}$. Then, there exists $\varepsilon_0 > 0$ and a family u_ε, with $\varepsilon < \varepsilon_0$, of local minimizers of $\mathcal{E}_\varepsilon$ with prescribed volume $m = \fint_{\mathbb{T}^n} u\,dx$, such that $u_\varepsilon \to u$ in $L^1(\mathbb{T}^n)$ as $\varepsilon \to 0$.*

An analogous result holds in the "Neumann" case.

4.4 SOME RESEARCH LINES

It would be very interesting and challenging trying to generalize all the results to arbitrary dimension, larger than three. Moreover, as we said in the previous section, the complete classification of the sets to which the global existence and stability result can be applied, is still missing.

Another possible line of research is the study (in every dimension) of the "singular perturbations" of the flow, that is, adding to the nonlocal Area functional an extra "energy" term, such as

$$\varepsilon \int_{\partial E} |\mathrm{H}|^p \, d\mu, \qquad \varepsilon \int_{\partial E} |B|^p \, d\mu, \qquad \text{or} \qquad \varepsilon \int_{\partial E} |\nabla^k B|^2 \, d\mu,$$

where ε is a small positive parameter.
This perturbation should regularize the evolutions arising from considering the gradient flows of the perturbed functionals, giving better global existence properties (in particular, when the dimension is larger than three), while keeping the models still interesting for the applications to physical phenomena. Moreover, from the mathematics point of view, it would be interesting to analyze the behavior of the solutions of the perturbed flows and to determine under which hypotheses they converge to the solutions of the original flow, when the parameter $\varepsilon > 0$ of the singular perturbation converges to zero.

APPENDICES

A

GEOMETRY OF HYPERSURFACES

In this appendix we introduce some basic notations and facts about hypersurfaces in Euclidean spaces, possible references are [16, 26, 34].

The main objects we will consider are $(n-1)$–dimensional, complete hypersurfaces immersed in $\mathbb{R}^n$, that is, pairs (M, ψ) where M is an $(n-1)$–dimensional, smooth manifold with empty boundary and $\psi : M \to \mathbb{R}^n$ is a smooth immersion (the rank of the differential $d\psi$ is equal to n everywhere on M).

The manifold M gets in a natural way a metric tensor g turning it into a Riemannian manifold (M, g) by pulling back the standard scalar product of $\mathbb{R}^n$ with the immersion map ψ.

Taking local coordinates around $p \in M$, we have local bases of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and 1–forms $\{dx_j\}$.

We will denote the vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, the 1–forms by $\omega = \omega_j$, that is, $\omega = \omega_j dx_j$ and a general mixed tensor by $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$, where the indices refer to the local basis.

Sometimes we will consider tensors along M viewing it as a submanifold of $\mathbb{R}^n$ via the map ψ, in such case we will use the Greek indices to denote the components of the tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^n$, for instance, given a vector field X along M, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

The metric g of M extended to tensors is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l} ,$$

where g_{ij} is the matrix of the coefficients of g in local coordinates and g^{ij} is its inverse matrix. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T, T)} .$$

The scalar product of $\mathbb{R}^n$ will be denoted by $\langle \cdot \,|\, \cdot \rangle$. As the metric g is obtained by pulling it back via ψ, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (d\psi^* \langle \cdot \,|\, \cdot \rangle)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial \psi}{\partial x_i} \,\middle|\, \frac{\partial \psi}{\partial x_j} \right\rangle .$$

The canonical measure induced by the metric g is given in a coordinate chart by $\mu = \sqrt{G}\, \mathscr{L}^n$ where $G = \det(g_{ij})$ and $\mathscr{L}^{n-1}$ is the standard Lebesgue measure of $\mathbb{R}^{n-1}$.

The induced covariant derivative on (M, g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k , \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji} \omega_k ,$$

where the Christoffel symbols Γ^i_{jk} are expressed by the formula,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right).$$

The covariant derivative ∇T of a general tensor $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$ will be denoted by $\nabla_s T^{i_1 \dots i_k}_{j_1 \dots j_l} = (\nabla T)^{i_1 \dots i_k}_{s j_1 \dots j_l}$ (we recall that such extension of the covariant derivative is uniquely defined on the full tensor algebra by imposing the Leibniz rule and the commutativity with any metric contraction).
$\nabla^m T$ will stand for the m–th iterated covariant derivative of T.

The gradient ∇f of a function and the divergence $\operatorname{div} X$ of a tangent vector field X at a point $p \in M$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_p M$$

and

$$\operatorname{div} X = \operatorname{tr} \nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k .$$

The Laplacian ΔT of a tensor T is given by

$$\Delta T = g^{ij} \nabla_i \nabla_j T .$$

If X is a smooth vector field with compact support on M, as $\partial M = \emptyset$, the following *divergence theorem* holds

$$\int_M \operatorname{div} X \, d\mu = 0 , \tag{A.1}$$

for every tangent vector field X on M which clearly implies, in particular,

$$\int_M \Delta f \, d\mu = 0$$

for every smooth function $f : M \to \mathbb{R}$ with compact support.

Since ψ is locally an embedding in $\mathbb{R}^n$, at every point $p \in M$ we can define up to a sign a unit normal vector $\nu(p)$. Locally, we can always choose ν in order that it is smooth.
If the hypersurface M is compact and embedded, that is, the map ψ is one–to-one, the *inside* of M is easily defined and we will consider ν to be the *outer pointing* unit normal vector at every point of M. In this case the vector field $\nu : M \to \mathbb{R}^n$ is globally smooth.

The *second fundamental form* $B = h_{ij}$ of M is the symmetric 2–form defined as follows,

$$h_{ij} = - \left\langle \nu \,\middle|\, \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\rangle$$

and the *mean curvature* H is the trace of B, that is $\mathrm{H} = g^{ij} h_{ij}$. Despite its name, H is the *sum* of the eigenvalues of the second fundamental form, not their average mean (some few authors actually define H/n as the mean curvature).

Remark A.1. Notice that since the unit normal ν is defined up to a sign, the same is true for B and H. Instead, the *vector valued second fundamental form* $h_{ij}\nu$, which is a 2–form with values in $\mathbb{R}^n$, and the *mean curvature vector* $\mathrm{H}\nu$ are uniquely defined.

The linear map $W_p : T_pM \to T_pM$ given by $W_p(v) = h^i_j(p)v^j \frac{\partial}{\partial x_i}$ is called the *Weingarten operator* and its eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ the *principal curvatures* at the point $p \in M$. It is easy to see that $\mathrm{H} = \lambda_1 + \cdots + \lambda_n$ and $|B|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.

Remark A.2. If the hypersurface $M \subseteq \mathbb{R}^n$ is locally the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, that is, $\psi(x) = (x, f(x))$, we have

$$g_{ij} = \delta_{ij} + f_i f_j\,, \qquad \nu = \frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{(\sqrt{1 + |\nabla f|^2})^3} = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)$$

where $f_i = \partial_i f$ and $\mathrm{Hess} f$ is the Hessian of the function f.

If the hypersurface $M \subseteq \mathbb{R}^n$ is locally the *zero set* of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, with $\nabla f \neq 0$ on such level set, we have

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right).$$

The following Gauss–Weingarten relations will be fundamental,

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial \psi}{\partial x_k} - h_{ij}\nu\,, \qquad \frac{\partial \nu}{\partial x_j} = h_{jl} g^{ls} \frac{\partial \psi}{\partial x_s}\,. \tag{A.2}$$

Actually, they express the fact that $\nabla^M = \nabla^{\mathbb{R}^n} + B\nu$. We recall that considering M locally as a regular submanifold of $\mathbb{R}^n$, we have $\nabla^M_X Y = (\nabla^{\mathbb{R}^n}_X \widetilde{Y})^M$ where the sign ^{M} denotes the projection on the tangent space to M and $\widetilde{Y}$ is a local extension of the field Y in a local neighborhood $\Omega \subseteq \mathbb{R}^n$ of $\psi(M)$.
Notice that, by these relations, it follows

$$\Delta\psi = g^{ij}\nabla^2_{ij}\psi = g^{ij}\left(\frac{\partial^2\psi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial \psi}{\partial x_k}\right) = g^{ij} h_{ij}\nu = \mathrm{H}\nu\,.$$

Moreover, we will also need the following symmetry property of the covariant derivative of B, called *Codazzi equations*,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}\,. \tag{A.3}$$

Finally, we have the formula

$$\Delta\nu = \nabla \mathrm{H} - |B|^2\nu\,, \tag{A.4}$$

indeed, computing in normal coordinates at a point $x \in M$, by the above Gauss–Weingarten relations, we have

$$\begin{aligned}\Delta\nu &= g^{ij}\Big(\frac{\partial^2\nu}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\nu}{\partial x_k}\Big)\\ &= g^{ij}\frac{\partial}{\partial x_i}\Big(h_{jl}g^{ls}\frac{\partial\psi}{\partial x_s}\Big)\\ &= g^{ij}\nabla_i h_{jl}g^{ls}\frac{\partial\psi}{\partial x_s} + g^{ij}h_{jl}g^{ls}\frac{\partial^2\psi}{\partial x_i\partial x_s}\\ &= g^{ij}\nabla_l h_{ij}g^{ls}\frac{\partial\psi}{\partial x_s} - g^{ij}h_{jl}g^{ls}h_{is}\nu\\ &= \nabla \mathrm{H} - |B|^2\nu\,,\end{aligned}$$

since all Γ^k_{ij} and $\frac{\partial}{\partial x_i}g^{jk}$ are zero at $x \in M$ and we used Codazzi equations (A.3).

B

FRACTIONAL SOBOLEV SPACES

In this appendix we introduce the basic facts and properties of the *fractional Sobolev spaces*, mainly following [32], other classical books on the subject are [3, 9, 41].

Let Ω be an open subset of $\mathbb{R}^n$. For any real $s > 0$ and for any $p \in [1, +\infty)$, we want to define the fractional Sobolev spaces $W^{s,p}(\Omega)$. In the literature, this spaces are also called *Aronszajn, Gagliardo* or *Slobodeckij spaces*, by the names of the ones who introduced them, almost simultaneously in [4], [15] and [42]).

Fixing the fractional order s in $(0,1)$, for any $p \in [1, +\infty)$, we define $W^{s,p}(\Omega)$ as follows,

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \, : \, \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$$

which is an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p},$$

where the term

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p}$$

is the so–called *Gagliardo $W^{s,p}$–seminorm* of u.

It is worth noticing that, as in the classical case with s integer, the space $W^{s',p}$ is continuously embedded in $W^{s,p}$ when $s \leq s'$, as next result points out.

Proposition B.1. *Let $p \in [1, +\infty)$ and $0 < s \leq s' < 1$. Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then,*

$$[u]_{W^{s,p}(\Omega)} \leq C [u]_{W^{s',p}(\Omega)}, \qquad \|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s',p}(\Omega)},$$

for some suitable positive constant $C = C(n, s, s', p)$. In particular,

$$L^p(\Omega) = W^{0,p}(\Omega) \subseteq W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Moreover, if $\Omega \subseteq \mathbb{R}^n$ has a smooth boundary, for every $s \in (0,1)$, we have

$$[u]_{W^{s,p}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \qquad \|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)},$$

and

$$W^{s,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

We define the *fractional critical exponent* associated to p,

$$p^* = p^*(n,s) = \frac{np}{n-sp}$$

(as when $s \in \mathbb{N}$), then the following embedding result holds.

Proposition B.2. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary. Then, there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$, we have*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p,p^*]$, that is, the space $W^{s,p}(\Omega)$ is continuosly embedded in $L^q(\Omega)$ for any $q \in [p,p^*]$. Such embedding is compact if $q \in [p,p^*)$.
If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$, is continuously embedded in $L^q(\Omega)$ for any $q \in [1,p^*]$ and such embedding is compact if $q \in [1,p^*)$.

When $s > 1$ and it is not an integer we write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$, the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$W^{s,p}(\Omega) = \left\{u \in W^{m,p}(\Omega) \,:\, D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ such that } |\alpha| = m\right\}$$

which is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)}\right)^{1/p}.$$

If s is an integer, the space $W^{s,p}(\Omega)$ is defined as usual.
The following proposition extends Proposition B.1.

Proposition B.3. *Let $p \in [1,+\infty)$ and $0 < s < s'$ Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary, then $W^{s',p}(\Omega)$ is continuosly embedded in $W^{s,p}(\Omega)$.*

As in the classical case with s integer, any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. It means, denoting with $W_0^{s,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$, that

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)\,,$$

but $W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$ for a general $\Omega \subseteq \mathbb{R}^n$. Furthermore, the same inclusions stated in Propositions B.1 and B.3 also hold for the spaces $W_0^{s,p}(\Omega)$.

For $s < 0$ and $p \in (1,+\infty)$, we define $W^{s,p}(\Omega)$ as the dual space of $W_0^{-s,q}(\Omega)$ where $1/p + 1/q = 1$.

Remark B.4. Finally, it is worth mentioning that the fractional Sobolev spaces play an important role in the *trace theory*. For instance, for any $p \in (1,+\infty)$, assuming that the open set $\Omega \subseteq \mathbb{R}^n$ is sufficiently smooth, the space of the traces Tu on $\partial\Omega$ of functions u in $W^{1,p}(\Omega)$ is characterized by $\|Tu\|_{W^{1-1/p,p}(\Omega)} < +\infty$ (see [14]).

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