Evolution of planar networks of curves with multiple junctions

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Centro Ennio De Giorgi – Pisa
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Motion by curvature of planar networks

Joint project with

- *Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005*
- *Annibale Magni & Matteo Novaga, 2010 – 2014*
- *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
- *Pietro Baldi & Emanuele Haus, 2015 –*

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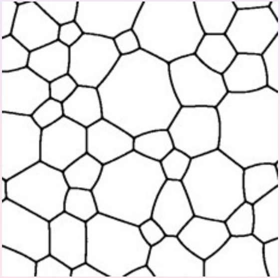

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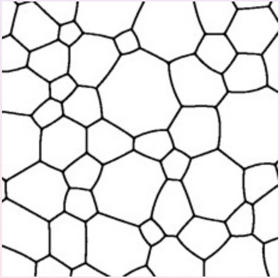

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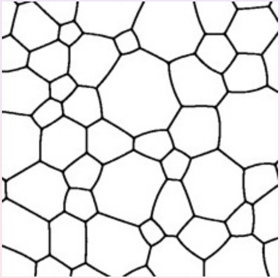

This is clearly a (toy) model for the time evolution of the interfaces of a multiphase system where the energy is given only by the area of such interfaces.

Even if it is still possible to use several ideas from the "parametric" classical approach to mean curvature flow (differential geometry/maximum principle), some extra variational methods are needed due to the presence of multi–points, that make the network a singular set (the simplest possible).

Some simple observations derived from simulations

Larger regions “eat” smaller regions. More precisely, the area of a region bounded by more than 6 edges grows, less than 6 edges it decreases.

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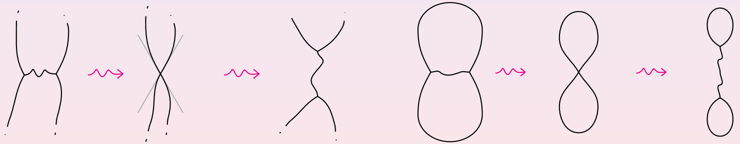
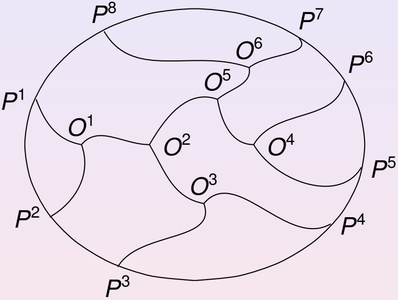
If there is no vanishing of a region, there is a collapse of only two triple junctions (when the length of a curve goes to zero) producing a 4–point in the network. Immediately after such a collapse, the network becomes again *regular* (only triple junctions, with curves forming angles of 120 degrees): a new pair of triple junctions emerges from the 4–point (standard transition).



Regular networks

Let Ω be an open, regular and convex subset of $\mathbb{R}^2$.



Definition

A regular network $\mathbb{S} = \bigcup_{i=1}^{n} \sigma^i([0,1])$ *in* Ω *is a connected set described by a finite family of curves* $\sigma^i : [0,1] \to \overline{\Omega}$ *(sufficiently regular) such that:*

Regular networks

1. the curves cannot intersect each other or self–intersect in their “interior”, but they can meet only at the end–points;

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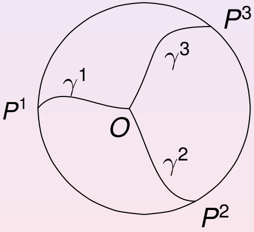
2. if a curve of the network “touches” the boundary of Ω at a fixed point $P \in \partial\Omega$, no other end–point of a curve can coincide with that point;

3. the junctions points $O^1, O^2, \dots, O^m \in \Omega$ have order three, considering $\mathbb{S}$ as a planar graph, and at each of them the three concurring curves $\{\sigma^{pi}\}_{i=1,2,3}$ meet in such a way that the external unit tangent vectors τ^{pi} satisfy $\sum \tau^{pi} = 0$ (the curves form three angles of 120 degrees at O^p).

Examples: The triod and the spoon

A triod $\mathbb{T}$ is a network composed only by three regular, embedded curves $\gamma^i : [0, 1] \to \overline{\Omega}$.



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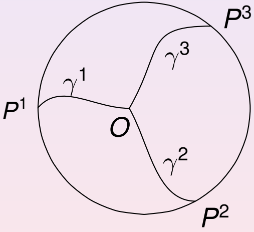
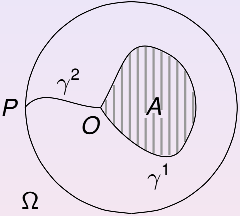
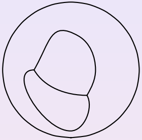
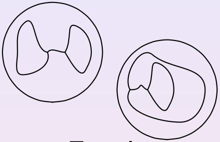
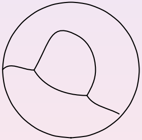
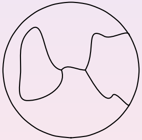
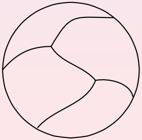




A spoon $\Gamma = \gamma^1([0,1]) \cup \gamma^2([0,1])$ is the union of two regular, embedded curves $\gamma^1, \gamma^2 : [0,1] \to \overline{\Omega}$.

Examples: Networks with two triple junctions

	0 closed curves	1 closed curve	2 closed curves
0 end–points on $\partial\Omega$	 Theta		 Eyeglasses
2 end–points on $\partial\Omega$	 Lens	 Island	
4 end–points on $\partial\Omega$	 Tree		

Motion by curvature

Definition

We say that a network moves by curvature if each of its time–dependent curves $\gamma^i : [0,1] \times (0,T) \to \mathbb{R}^2$ *satisfies*

$$\gamma^i_t(x,t)^\perp = \underline{k}^i(x,t)$$

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To be more precise, a family of networks $\mathbb{S}_t$ *is a motion by curvature (in the maximal time interval* $[0,T)$ *) if the functions* $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$ *are at least* C^2 *in space and* C^1 *in time and satisfy the following system:*

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & \\ \sum \tau^i(O,t) = 0 & \text{at every 3–point} \\ \gamma^i_t = k^i\nu^i + \lambda^i\tau^i & \text{for some continuous functions } \lambda^i \end{cases} \qquad (\star)$$

with fixed end–points on $\partial\Omega$.

Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a non–degenerate system of quasilinear parabolic equations (with several geometric properties).

Definition

A curvature flow γ^i *for the initial, regular* C^2 *network* $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ *which satisfies* $\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$ *for every* $t > 0$ *will be called a* special curvature flow *of* $\mathbb{S}_0$.

Short time existence

Theorem

For any initial, regular $C^{2+\alpha}$ *triod* $\mathbb{T}_0 = \bigcup_{i=1}^{3} \sigma^i([0,1])$*, with* $\alpha \in (0,1)$*, which is 2–compatible, there exists a unique special flow of class* $C^{2+\alpha,1+\alpha/2}([0,1]\times[0,T))$ *in a maximal time interval* $[0,T)$*.*

We say that a triod is 2–compatible if

$$\frac{\sigma^i_{xx}(0)}{|\sigma^i_x(0)|^2} = \frac{\sigma^j_{xx}(0)}{|\sigma^j_x(0)|^2} \text{ for every } i,j \in \{1,2,3\}\,,$$

which in particular implies

$$\sum k^i(O,t) = 0 \quad \text{at the 3–point.}$$

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For any initial smooth, regular network $\mathbb{S}_0$ *there exists a unique smooth special flow in a maximal time interval* $[0, T)$.

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For any initial network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ which is regular, $C^{2+\alpha}$ with $\alpha \in (0,1)$, 2–compatible, there exists a unique special flow of class $C^{2+\alpha,1+\alpha/2}([0,1] \times [0,T))$ in a maximal time interval $[0, T)$.

Theorem [Bronsard–Reitich]:
$\mathbb{T}_0 \in C^{2+\alpha}$, 2–compatible
special flow $\gamma_t = \gamma_{xx}/|\gamma_x|^2$
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there exists a unique solution
$\mathbb{T}_t \in C^{2+\alpha,1+\alpha/2}([0,1] \times [0,T))$.

well–posedness of the system $\dashrightarrow$

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$\mathbb{S}_0 \in C^{2+\alpha}$, 2–compatible
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Theorem

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Short time existence and uniqueness

We say that a network is geometrically 2–compatible if

$$\sum k^i(O,t) = 0 \quad \text{at every 3–point.}$$

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Proposition

For any initial network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ *which is regular,* $C^{2+\alpha}$ *with* $\alpha \in (0,1)$, geometrically *2–compatible, there exists a* geometrically unique *solution* γ^i *of Problem* $(\star)$ *in the class of maps* $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ *in a maximal positive time interval* $[0,T)$*.*

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Short time existence and uniqueness

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Theorem

For any initial C^2 *regular network* $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ *there exists a solution* γ^i *of Problem* $(\star)$ *in a maximal time interval* $[0,T)$.
Such flow $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ *is a smooth flow for every time* $t > 0$.

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Such flow $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is a smooth flow for every time $t>0$.

- The relevance of this theorem is that the initial network is not required to satisfy any additional condition (compatibility condition), but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. In particular, it is not necessary that the sum of the three curvatures at a 3–point is zero.
- The geometric uniqueness of the solution γ^i found in this theorem is an open problem.

Long time behavior

Our aim is now to analyse the global (in time) behavior of the evolving networks, with particular attention to singularity formation.

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Theorem

If $T < +\infty$ is the maximal time interval of smooth existence of the curvature flow of a network $\mathbb{S}_t$, then at least one of the following holds:

1. *the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
2. *the curvature is not bounded as $t \to T$.*

Evolution of a triod

Theorem

If none of the lengths of the three curves of the evolving triod goes to zero, the flow is smooth for all times and the triod converges (asymptotically) to the Steiner configuration connecting the three endpoints.

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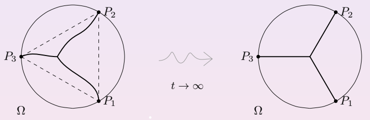

If all the angles of the triangle with vertices the three end–points on the boundary are less than 120 degrees and the initial triod is contained in the triangle, then the triod converges in infinite time to the Steiner configuration connecting the three points.

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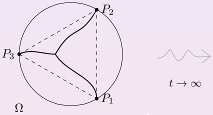

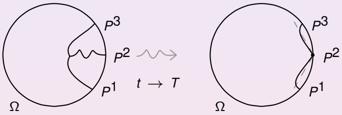

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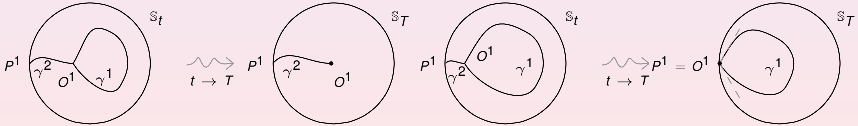
If the fixed end–points on the boundary form a triangle with an angle of more than 120 degrees, then the length of a curve goes to zero in finite time.

Evolution of a spoon

Theorem

The maximal time of existence of a smooth evolution of a spoon is finite and one of the following situations occurs:

1. *the closed loop shrinks to a point in a finite time (asymptotically approaching the shape of a Brakke spoon) and the maximum of the curvature of the network goes to $+\infty$, as $t \to T$;*
2. *the "open" curve vanishes and there is a 2–point formation on the boundary of the domain of evolution, but the curvature remains bounded.*

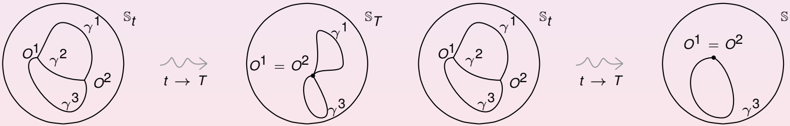


Evolution of a Theta (double cell)

Theorem

The maximal time of existence of a smooth evolution of a Theta is finite and one of the following situations occurs:

1. *the length of a curve that connects the two 3–points goes to zero as $t \to T$ and the curvature remains bounded;*
2. *the length of the curves composing one of the loops go to zero as $t \to T$ and the maximum of the curvature goes to $+\infty$.*



In any case the network cannot completely vanish shrinking to a point as $t \to T$.

Long time behavior

We have seen examples in which the length of at least a curve goes to zero and examples in which at the same time the length of a curve goes to zero and the curvature of the network is unbounded.

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BUT there are NO examples in which the lengths of all the curves of the network remain uniformly far away from zero during the evolution and the curvature is unbounded, as $t \to T$.

Conjecture

If no length of the curves of the network goes to zero as $t \to T$, then T is not a singular time (maximal time of smooth existence).

We are able to show the following:

Theorem

If no length goes to zero and the ”Multiplicity–One Conjecture” below is valid, then the curvature is bounded. Hence, T is not a singular time.

“Multiplicity–One Conjecture” (M1)

Every possible limit of rescaled networks is a network with multiplicity one.

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If no length goes to zero and the ”Multiplicity–One Conjecture” below is valid, then the curvature is bounded. Hence, T is not a singular time.

“Multiplicity–One Conjecture” (M1)

Every possible limit of rescaled networks is a network with multiplicity one.

To explain where we need this conjecture, we give a sketch of the proof of this theorem. The strategy is based on a blow–up technique.

Huisken's dynamical rescaling

We rescale the flow in its maximal time interval $[0, T)$ of smooth existence as follows:

Definition

Fixed $x_0 \in \mathbb{R}^2$, we define the rescaled flow of networks as

$$\widetilde{\mathbb{S}}_{x_0,s} = \frac{\mathbb{S}_{t(s)} - x_0}{\sqrt{2(T - t(s))}}$$

where $s \in [-1/2 \log T, +\infty)$ and $s(t) = -\frac{1}{2} \log (T - t)$.

Rescaled monotonicity formula

Proposition

Let $x_0 \in \mathbb{R}^2$, *set* $\widetilde{\rho}(x) = e^{-\frac{|x|^2}{2}}$ *and* $\widetilde{P}^r(s) = \frac{P^r - x_0}{\sqrt{2(T - t(s))}}$. *For every* $s \in [-1/2 \log T, +\infty)$ *the following identity holds*

$$\frac{d}{ds}\int_{\widetilde{\mathbb{S}}_{x_0,s}} \widetilde{\rho}(x)\, d\sigma = -\int_{\widetilde{\mathbb{S}}_{x_0,s}} |\,\underline{\widetilde{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\, d\sigma + \sum_{r=1}^{l} \Big[\Big\langle\, \widetilde{P}^r(s) \,\Big|\, \tau(P^r, t(s))\Big\rangle - \widetilde{\lambda}(P^r, s)\Big]\, \widetilde{\rho}(\widetilde{P}^r(s)).$$

Rescaled monotonicity formula

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Integrating between s_1 and s_2 with $-1/2 \log T \leq s_1 \leq s_2 < +\infty$ we get

$$\int_{s_1}^{s_2}\int_{\widetilde{\mathbb{S}}_{x_0,s}} |\,\underline{\widetilde{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\,d\sigma\,ds = \int_{\widetilde{\mathbb{S}}_{x_0,s_1}} \widetilde{\rho}(x)\,d\sigma - \int_{\widetilde{\mathbb{S}}_{x_0,s_2}} \widetilde{\rho}(x)\,d\sigma + \sum_{r=1}^{l}\int_{s_1}^{s_2}\Big[\Big\langle\,\widetilde{P}^r(s)\,\Big|\,\tau(P^r, t(s))\Big\rangle - \widetilde{\lambda}(P^r, s)\Big]\,\widetilde{\rho}(\widetilde{P}^r(s))\,ds\,.$$

Sketch of the proof

Letting $s_1 = -1/2 \log T$ and $s_2 \to +\infty$ in the rescaled monotonicity formula, we get (the last term is uniformly bounded)

$$\int_{-1/2\log T}^{+\infty} \int_{\widetilde{\mathbb{S}}_{x_0,s}} |\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, ds < +\infty\,,$$

which implies, for a subsequence of rescaled times s_j,

$$\lim_{j\to+\infty} \int_{\widetilde{\mathbb{S}}_{x_0,s_j}} |\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma = 0\,.$$

Then, by a standard compactness argument the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,s_j}$ (possibly after reparametrization) converges (up to subsequence) weakly in $W^{2,2}_{loc}$ and strongly in $C^{1,\alpha}_{loc}$, to a (possibly empty) limit $\widetilde{\mathbb{S}}_\infty$ (possibly with multiplicity) which satisfies the *shrinkers* equation

$$\underline{k} + x^\perp = 0.$$

Shrinkers

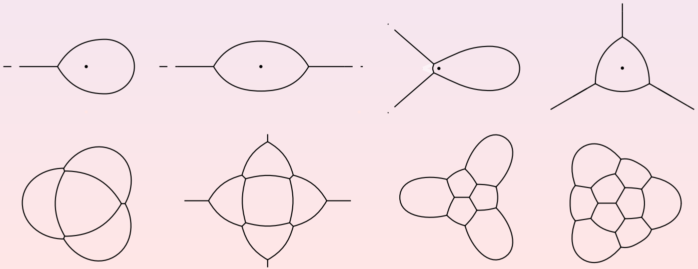
Definition

A regular C^2 open network $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ is called a regular shrinker *if at every point $x \in \mathbb{S}$ there holds*

$$\underline{k} + x^\perp = 0.$$

This is called the shrinkers equation*.*

The name comes from the fact that if $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ is a shrinker, then the evolution given by $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i(I_i, t)$ where $\gamma^i(x,t) = \sqrt{-2t}\,\sigma^i(x)$ is a self–similarly shrinking curvature flow in the time interval $(-\infty, 0)$ with $\mathbb{S} = \mathbb{S}_{-1/2}$.



Sketch of the proof

Assuming that all lengths of the evolving network do not go to zero, then all the curves of $\widetilde{\mathbb{S}}_\infty$ have infinite length and must be pieces of straight lines from the origin. Hence there are only three possibilities for the blow–up limit $\widetilde{\mathbb{S}}_\infty$:

- a straight line (with multiplicity);
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- White's theorem in the case of a straight line;
- White's theorem and a reflection argument in the case of an half line;
- the results by Magni – Mantegazza – Tortorelli or Ilmanen – Neves – Schulze for an infinite flat triod,

it follows that the original network $\mathbb{S}_t$ has bounded curvature as $t \to T$.

Sketch of the proof

Thanks to this previous

Theorem

If $T < +\infty$ is the maximal time interval of smooth existence of the curvature flow $\mathbb{S}_t$, then:

1. *either the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
2. *or the curvature is not bounded as $t \to T$.*

we conclude that T cannot be a singular time and we are done.

Hence, assuming (from now on) the truth of the

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Then, there are two possible situations:

- The curvature stays bounded.
- The curvature is unbounded as $t \to T$.

In the smooth case (curves, hypersurfaces) there are no singularities with bounded curvature. Indeed the maximum of curvature always satisfies $k_{\max}(t) \geq 1/\sqrt{2(T-t)}$.

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Singularities are actually "classified" by the rate the curvature goes to $+\infty$, as $t \to T$:

- *Type I* – The maximum of the curvature is of order $1/\sqrt{T-t}$.
- *Type II* – The maximum of the curvature is of higher order (on a subsequence of times $t_i \to T$).

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For networks there are actually singularities (collapse of curves) with bounded curvature, *Type 0* singularities (copyright of Tom Ilmanen).

Collapse with bounded curvature

The analysis consists in understanding the possible limit networks that can arise as $t \to T$ (and finding out how to continue the flow, if possible).

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If **M1** *is true, every* interior *vertex of such limit network either is a regular triple junction or it is a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* $120/60$ *degrees among them.*

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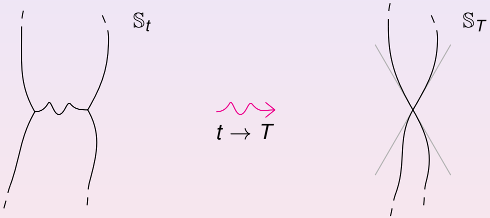

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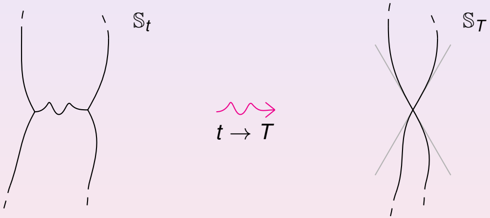

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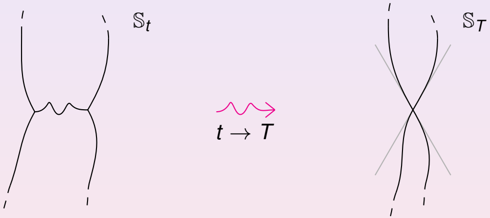

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Being the limit network possibly non–regular since it can have 4–junctions, the short time existence theorem does not apply.

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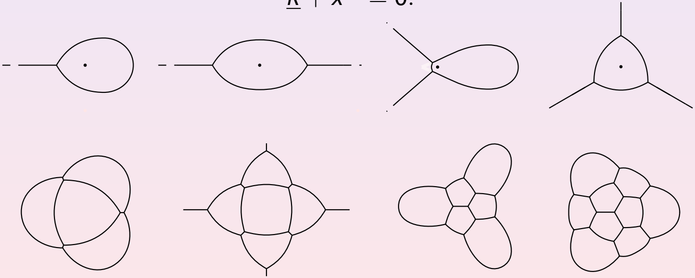

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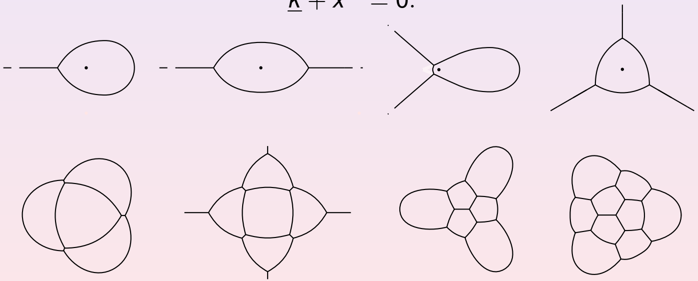

Then, the (local) structure (topology) of the evolving network plays an important role in the analysis.

Tree–like evolving networks

Theorem

If **M1** *holds and the network is a* tree *(no loops), the curvature is uniformly bounded during the flow, hence the only "singularities" are given by the collapse of a curve with only two triple junctions going to collide.*

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- *Type 0* – singularities actually exist.
- This result can be localized (if the network is locally a tree around a singular point, the curvature is locally bounded).
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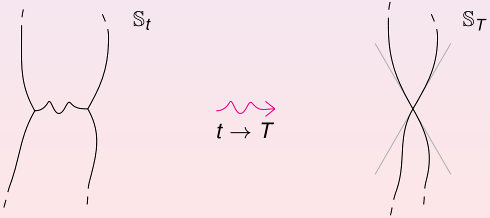

This can happen if and only if the curvature is (locally) bounded.

Collapse with unbounded curvature

By the previous analysis, unbounded curvature as $t \to T$ implies that in the blow–up limit we find a shrinker with regions. These regions are the “memory” of the collapsing region in the non–rescaled flow and the unbounded lines gives the limit tangents of the concurring curves at the point of collapse.

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Such a unique *limit non–regular network has bounded curvature.*

Restarting the flow after a singularity

In the case of bounded curvature or if the previous "uniqueness of the limit network" conjecture holds (even without the bounded curvature conjecture), assuming always **M1**, we are able to restart the flow (of a possibly non–regular network).

Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

For any initial network of non–intersecting curves there exists a (possibly non–unique) Brakke flow by curvature *in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.*

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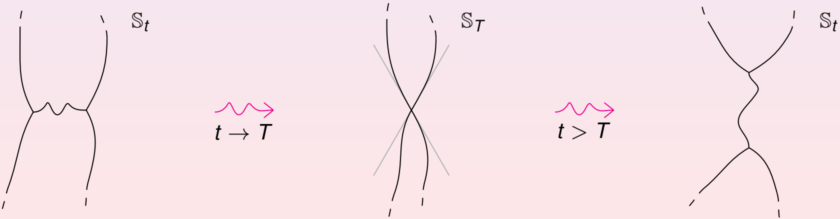

Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities. In order to study its asymptotic behavior, we need that it is defined for every positive time and this requires that the singular times do not “accumulate”.

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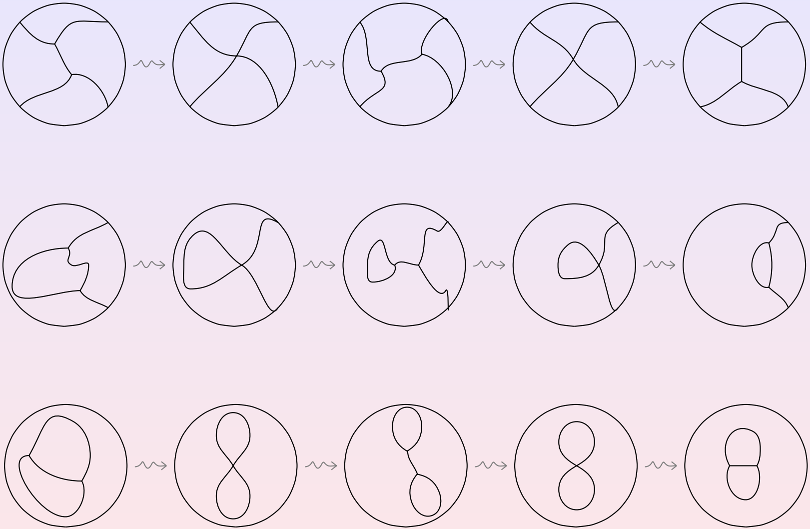
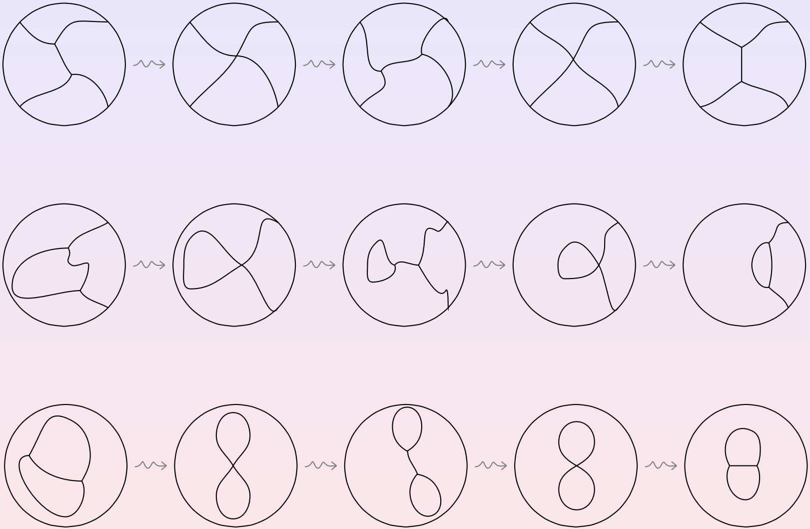
The number of singular times is finite *(at least for a tree–like network).*

In such case the flow is definitely smooth and the evolving network converges (asymptotically) to a Steiner configuration connecting the fixed endpoints.





Possible infinite “oscillations” via standard transitions from a shape to another and viceversa.

Open problems and research directions

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Main Open Problem – “Multiplicity–One Conjecture” (M1)

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Theorem (CM, M. Novaga, A. Pluda – 2015)

- *If during the flow the triple–junctions stay uniformly far each other, then* **M1** *is true.*
- *If the initial network has at most two triple junctions, then* **M1** *is true.*

Analysis of singularity formation for some flows of networks with "few" triple junctions can then be made rigorous (under uniqueness of the limit hypothesis in some cases).

Only 1 triple junction

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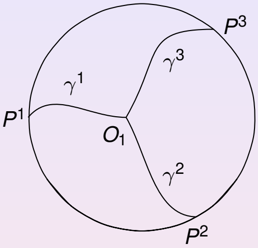
The Triod – A. Magni, CM, M. Novaga, V. Tortorelli



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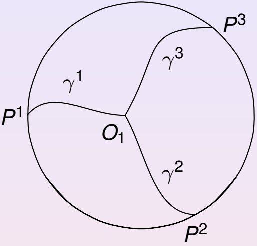
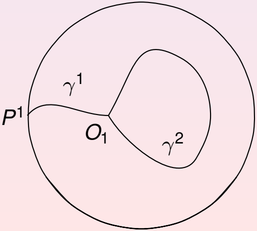


The Spoon – A. Pluda



2 triple junctions – CM, M. Novaga, A. Pluda

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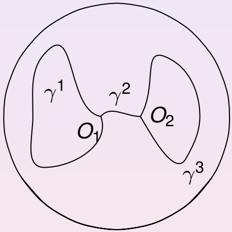

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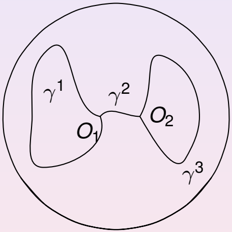
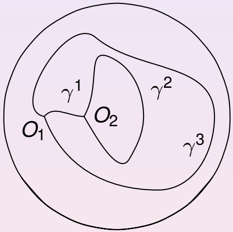
The Eyeglasses and... the Broken Eyeglasses





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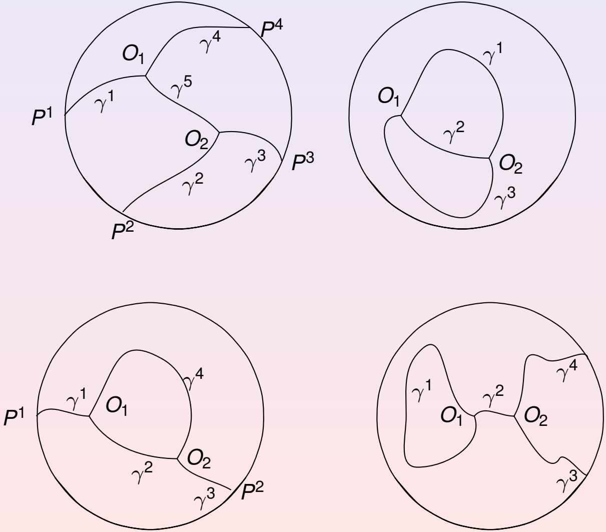
The “Steiner”, Theta, Lens and Island



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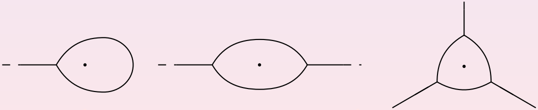

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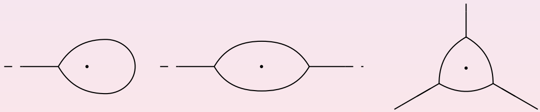

Generically only regions with at most three edges can collapse.

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Study of the motion by mean curvature of 2–dimensional interfaces in $\mathbb{R}^3$ (*double–bubble*), for instance.

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- Short time existence/uniqueness for special initial interfaces by Depner – Garcke – Kohsaka
- Short time existence/estimates for special initial interfaces by Schulze – White

Open problems and research directions – Higher dimensions

Study of the motion by mean curvature of 2–dimensional interfaces in $\mathbb{R}^3$ (*double–bubble*), for instance.

- Short time existence/uniqueness for special initial interfaces by Depner – Garcke – Kohsaka
- Short time existence/estimates for special initial interfaces by Schulze – White
- Basic computations for the geometric quantities and estimates with Bellettini – Magni – Novaga (*Work in progress*)

Thanks for your attention