Perelman, the Ricci Flow and the Poincaré Conjecture

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Poincaré and the Birth of Topology

Even if several results that today we call "topological" were already previously known, it is with Poincarè (1854–1912), *the last universalist*, that topology (*Analysis Situs*) gets its modern form.

In particular, regarding the properties of surfaces or higher dimensional spaces.

Poincaré introduced the fundamental concept of *simple connectedness*.

Surfaces – Examples

Closed surfaces (compact, without boundary, orientable):

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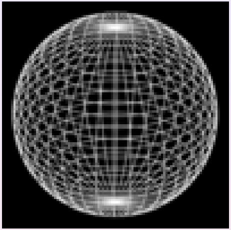

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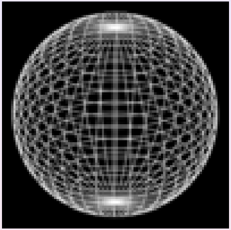

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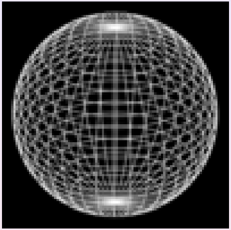

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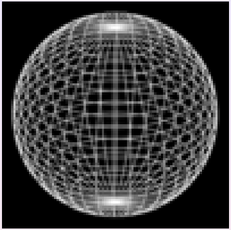


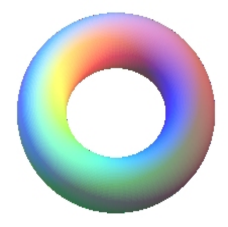

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Every closed surface is the boundary of a *bretzel*.

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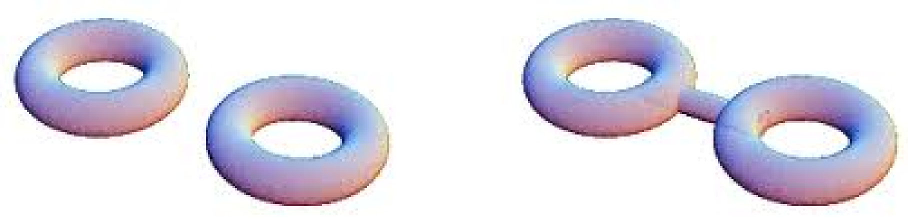

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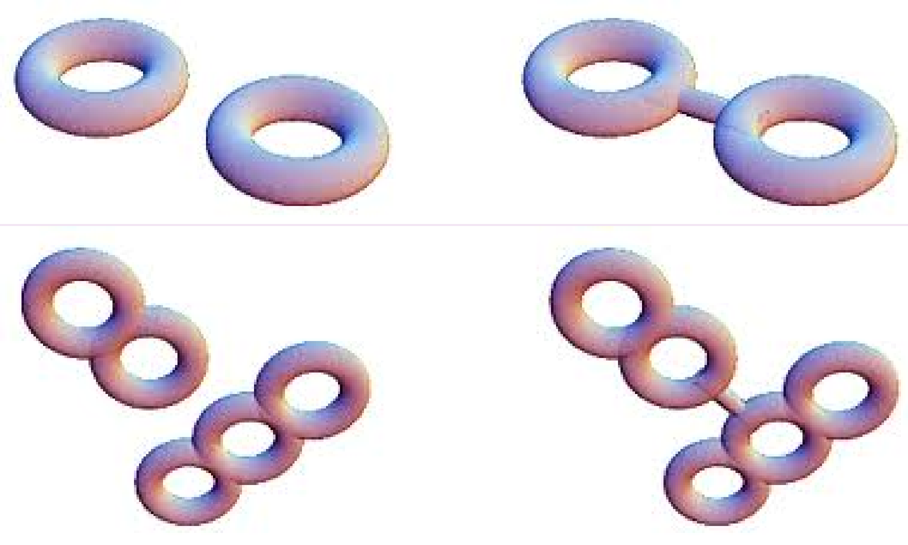

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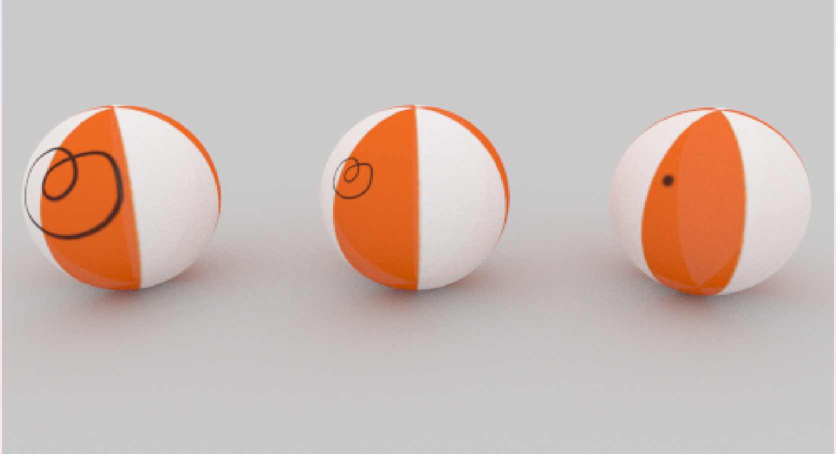

The Torus is Not Simply Connected

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Theorem

Any closed, simply connected surface is homeomorphic to the sphere (homeomorphic = deformable).

Higher Dimensional Spaces

Poincaré asked himself whether this theorem was true in dimension higher than two, in particular, in dimension three.

A 3–dimensional space is locally like our "ordinary" space (topologically), even if we could experience "strange situations" moving into it... for instance, one could exit from a door in a room and find himself entering in the same room, in a way, like walking on a circle, or on the surface of a torus one returns to the starting point. Its structure can actually be even quite more complicated.

There exist the 3–sphere, the 3–torus and several other examples analogous to the surfaces that we have seen.

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In order to have a *classification theorem* like the one for surfaces, one of the first conjectures of Poincarè was:

Poincaré Conjecture

Any closed and simply connected 3–dimensional space is homeomorphic to the 3–sphere.

Proposed in 1904 in “Rendiconti del Circolo Matematico di Palermo”.

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- $n \geq 5$ Stephen Smale (Fields Medal 1966).
- $n = 4$ Michael Freedman (Fields Medal 1986).
- $n = 3$ Grisha Perelman (Fields Medal 2006).
  Declined!!!

Geometric Flows

A possible line of proof of the conjecture is “transforming” in some way (deformation, *surgery – cut & paste*), keeping unaltered the topological properties, an hypothetical space which is a counterexample to the conjecture, in a 3–sphere.

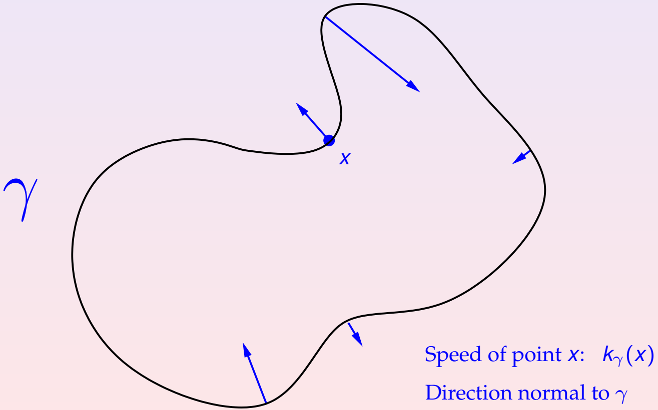
All the tries in this direction by means of topological arguments have failed for around a century.

The “winning” approach, still in this setting, happened to be deforming the space by means of evolution laws given by partial differential equations. The great advantage, with respect to the only–topological deformations, is that, by means of methods of analysis, these deformations are “quantitative”.

In general, this kind of evolutions of geometric objects are called *geometric flows*.

An Example: Curves Flowing by Curvature

Given $\gamma \subset \mathbb{R}^2$ a closed simple curve in the plane, we want that at every time, every point moves with normal velocity equal to the curvature at such point



Analytically...

Given a closed simple curve $\gamma = \gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$, we look for a smooth function $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ such that

$$\begin{cases} \dfrac{\partial \gamma}{\partial t}(\theta, t) = k(\theta, t)\vec{N}(\theta, t) \\ \\ \gamma(\cdot, 0) = \gamma_0 \end{cases}$$

for every $\theta \in \mathbb{S}^1$ and $t \in [0, T)$.

- $\vec{N}(\theta, t) =$ "inner" unit normal vector
- $k(\theta, t) =$ curvature of $\gamma_t = \gamma(\cdot, t)$ at the point $\gamma(\theta, t)$

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Nonlinear parabolic system of partial differential equations

Example

The Gage–Hamilton–Grayson Theorem

Theorem

Every smooth, closed, simple curve in the plane during the flow stays smooth, closed and simple. After finite time, it becomes convex, then rounder and rounder and shrinks to a single point in finite time. Rescaling the curve in order to keep the enclosed area constant, it converges to a circle.

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This is an example of a “nice” geometric flow, transforming every element of a family of geometric objects in a “canonical representative” which is a well known element. Moreover, the deformed object at every time always belongs to the same family (hence, the “family” characteristics are preserved).

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Even if this flow is possibly the simplest geometric flow, this theorem is definitely non trivial and its proof requires several ideas and techniques from both analysis and geometry.

Richard Streit Hamilton – Columbia University, NY

(1981 – Cornell University, Ithaca, NY)

The Ricci Flow

At the end of 70's – beginning of 80's, the study of Ricci and Einstein tensors from an analytic point of view gets a strong interest, for instance, in the (static) works of Dennis DeTurck. A proposal of analysis of a family of flows, among which the*Ricci flow*, was suggested by Jean–Pierre Bourguignon ("Ricci curvature and Einstein metrics", Lecture Notes in Math 838, 1981). In 1982 Richard Hamilton defines and studies the *Ricci flow*, that is, the system of PDE's

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)}$$

describing the evolution of a metric of a Riemannian manifold.

"Three–manifolds with positive Ricci curvature"
Journal of Differential Geometry **17**, 1982, pp. 255–306.

J. DIFFERENTIAL GEOMETRY
17 (1982) 255–306

THREE-MANIFOLDS WITH POSITIVE RICCI CURVATURE

RICHARD S. HAMILTON

INDEX

1. Introduction

Our goal in this paper is to prove the following result.

1.1 Main Theorem. *Let X be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then X also admits a metric of constant positive curvature.*

All manifolds of constant curvature have been completely classified by Wolf [6]. For positive curvature in dimension three there is a pleasant variety of examples, of which the best known are the lens spaces $L_{p,q}$. Wolf gives five

Received December 21, 1981.

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It is then natural to expect to get asymptotically a uniform distribution, that is, a very symmetric “geometry”, for instance like the one of a sphere.

Examples

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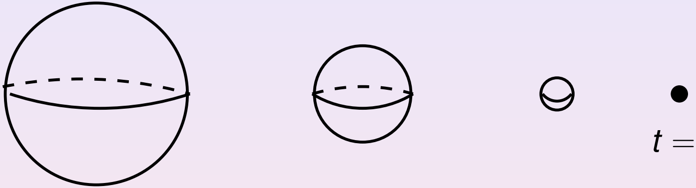
Sphere: $g(t) = (1 - 4t)g_0$.



$t = 1/4$

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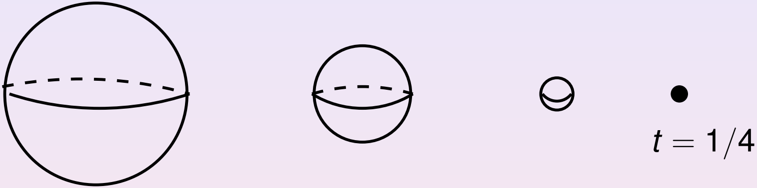
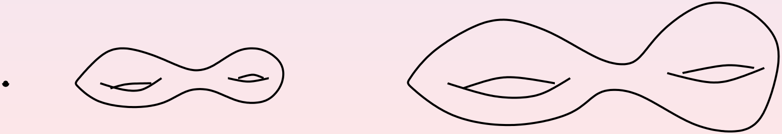


Hyperbolic surface (constant curvature -1): $g(t) = (1 + 4t)g_0$.



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If every 3–dimensional, closed and simply connected manifold admits a metric with positive Ricci tensor, then we have a proof of Poincaré conjecture.

Bad Examples: “Neckpinch”

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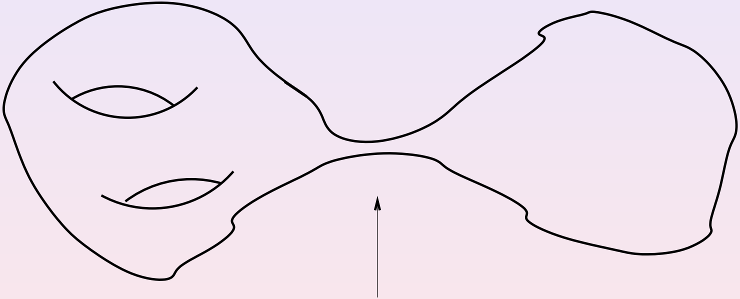

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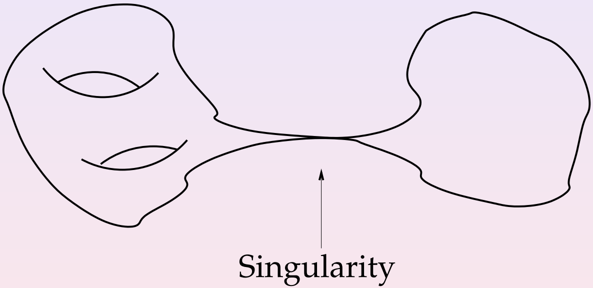

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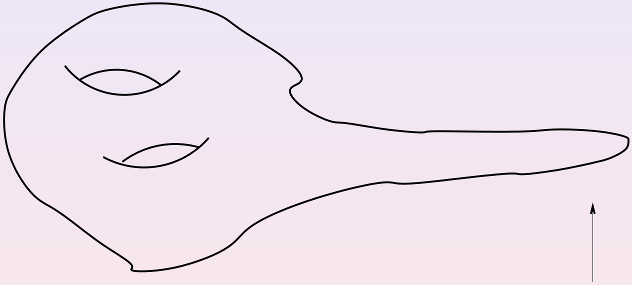

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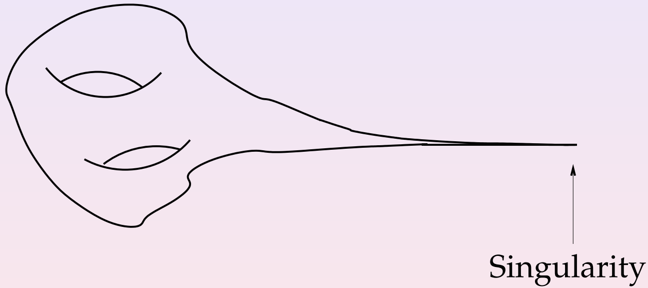

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- If the singularity is not a “spherical collapse”, we try to get the maximum of quantitative information about what is happening to the manifold. To this aim, we need to “classify” all the possible singularities.

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- Show that, assuming the simple connectedness of the initial manifold, after a finite number of steps (and in finite time) this procedure ends leaving only a family of collapsing 3–spheres.
- Recovering the initial manifold by "inverting" the surgery steps, we conclude that the initial manifold actually was a 3–sphere too (topologically), hence proving Poincaré conjecture.

The Classification of Singularities

Conjecture (R. Hamilton)

The behavior of the manifold at a singularity can only be one of the three we have seen: collapsing sphere, neckpinch and formation of a cusp.

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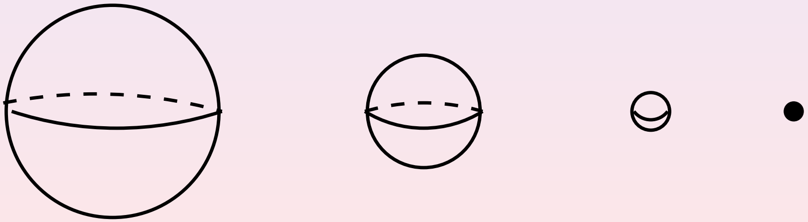

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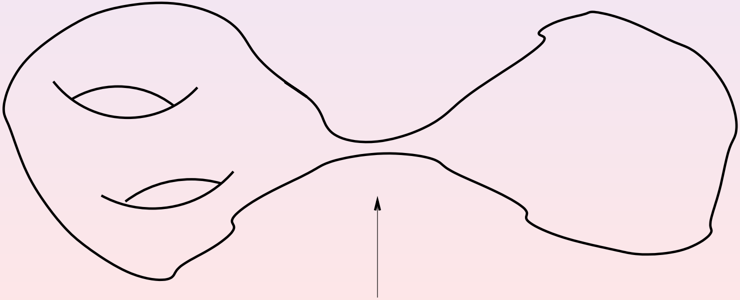

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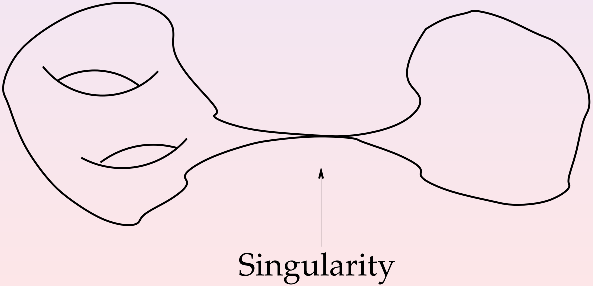

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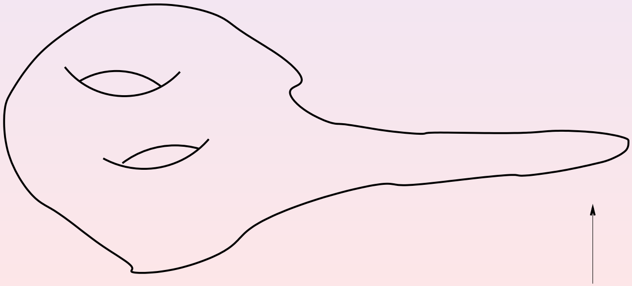

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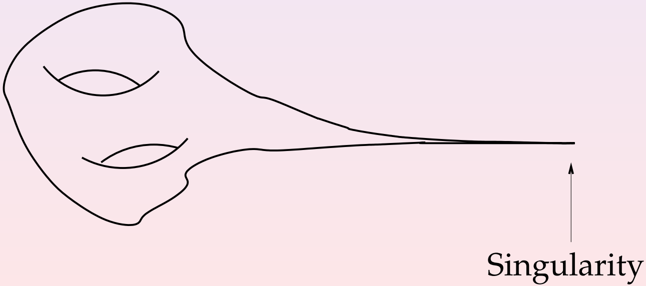


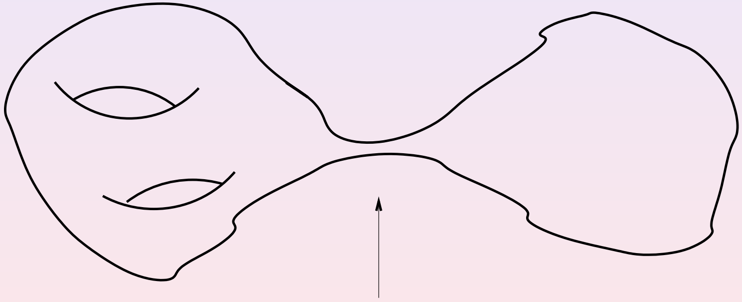
The Surgery Procedure

If the classification conjecture is true, in the neckpinch and cusp cases it is necessary to develop a "quantitative surgery procedure" which is good enough to allow us to show that in finite time and after a finite number of "operations" we get a "final" family of only 3–spheres.

Despite some positive partial results, the lack of a complete proof of the classification conjecture and of the associated quantitative estimates was an obstacle to have an effective procedure.

The Surgery Procedure – Neckpinch

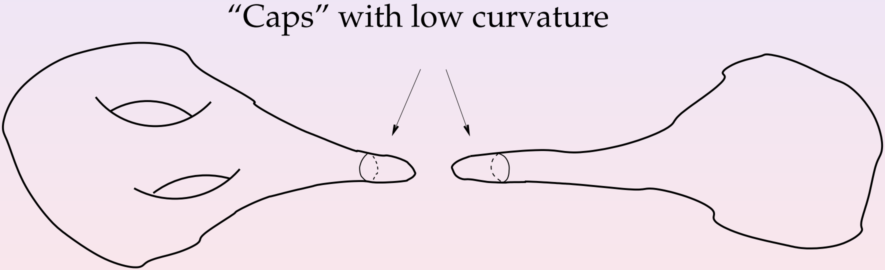
Before:



High curvature

The Surgery Procedure – Neckpinch

After:



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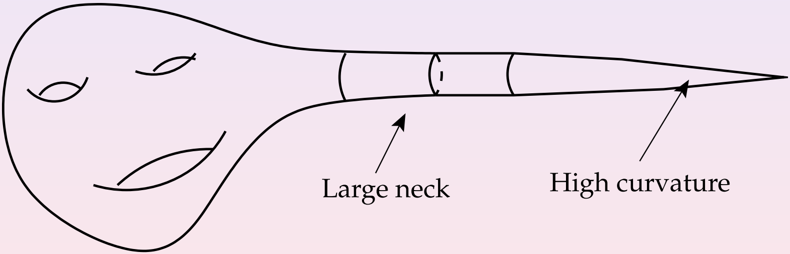

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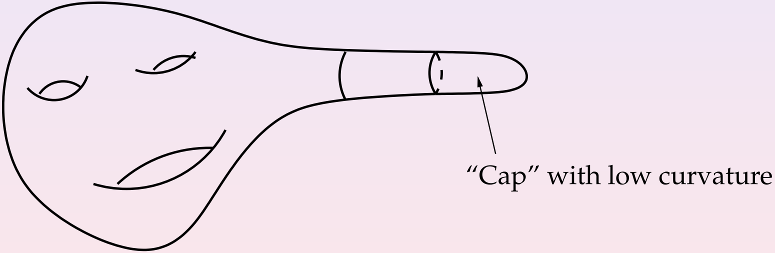
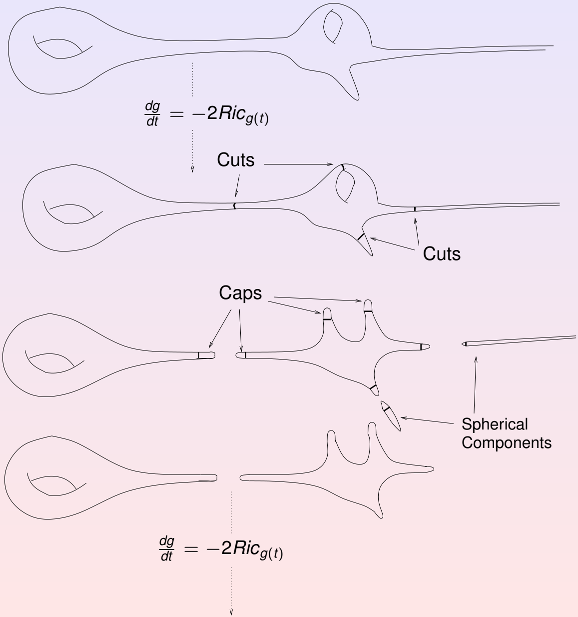
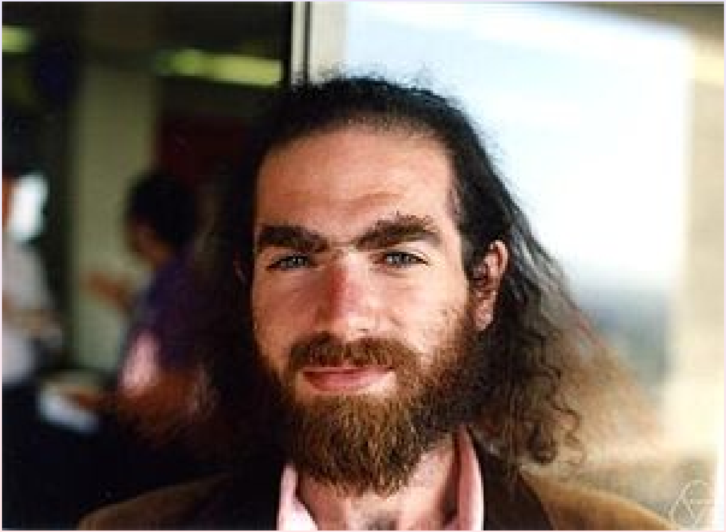


The Ricci Flow with Surgery in Action



Grigori Yakovlevich Perelman
Steklov Institute, St. Petersburg



The Preprints on ArXiv – 2002/2003

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- *Finite extinction time for the solutions to the Ricci flow on certain three–manifolds.*

The entropy formula for the Ricci flow and its geometric applications

Grisha Perelman*

March 27, 2013

Introduction

1. The Ricci flow equation, introduced by Richard Hamilton [H 1], is the evolution equation $\frac{d}{dt}g_{ij}(t) = -2R_{ij}$ for a riemannian metric $g_{ij}(t)$. In his seminal paper, Hamilton proved that this equation has a unique solution for a short time for an arbitrary (smooth) metric on a closed manifold. The evolution equation for the metric tensor implies the evolution equation for the curvature tensor of the form $Rm_t = \triangle Rm + Q$, where Q is a certain quadratic expression of the curvatures. In particular, the scalar curvature R satisfies $R_t = \triangle R + 2|\mathrm{Ric}|^2$, so by the maximum principle its minimum is non-decreasing along the flow. By developing a maximum principle for tensors, Hamilton [H 1,H 2] proved that Ricci flow preserves the positivity of the Ricci tensor in dimension three and of the curvature operator in all dimensions; moreover, the eigenvalues of the Ricci tensor in dimension three and of the curvature operator in dimension four are getting pinched pointwisely as the curvature is getting large. This observation allowed him to prove the convergence results: the evolving metrics (on a closed manifold) of positive Ricci curvature in dimension three, or positive curvature operator

*St.Petersburg branch of Steklov Mathematical Institute, Fontanka 27, St.Petersburg 191011, Russia. Email: perelman@pdmi.ras.ru or perelman@math.sunysb.edu ; I was partially supported by personal savings accumulated during my visits to the Courant Institute in the Fall of 1992, to the SUNY at Stony Brook in the Spring of 1993, and to the UC at Berkeley as a Miller Fellow in 1993-95. I'd like to thank everyone who worked to make those opportunities available to me.

The Emails between Vitali Kapovitch and Perelman

```
    Date: Wed, 20 Nov 2002 11:46:49 +0300 (MSK)
    From: Grigory Perelman <perelman@euclid.pdmi.ras.ru>
Reply-To: Grigory Perelman <perelman@euclid.pdmi.ras.ru>
 Subject: Re: geometrization
      To: Vitali Kapovitch <vitali@math.ucsb.edu>

That’s correct.
Grisha

On Tue, 19 Nov 2002, Vitali Kapovitch wrote:

> Hi Grisha,
> Sorry to bother you but a lot of people are asking me
> about your preprint "The entropy formula for the Ricci...".
> Do I understand it correctly that while you can not yet
> do all the steps in the Hamilton program you can do enough
> so that using some collapsing results you can prove
> geometrization?
>
> Vitali
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Perelman’s Results

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- Using these, he shows (not completely but enough to have a surgery procedure) the conjecture of classification of singularities.
- He finds new estimates on the geometric quantities during the formation of a singularity.
- He modifies the existing Hamilton's surgery procedure in order to make it effective and proves that after finite time and a finite number of surgery operations one gets a final finite family of only 3–spheres.

After the Publication Online of Perelman's Works...

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- Bruce Kleiner and John Lott immediately realize the importance of Perelman's work and start writing down some explicative notes, filling the missing technical details and expanding the most difficult parts and arguments.

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- In June 2006 the Asian Journal of Mathematics publishes (on "paper") a work of Zhu Xi–Ping of (Zhongshan University, China) and Huai–Dong Cao (Lehigh University, US) with a complete proof of Poincaré conjecture.

After the Publication Online of Perelman's Works...

- In July 2006 John Morgan (Columbia University) and Gang Tian (MIT) publish online on ArXiv (now a standard book) the paper "Ricci Flow and the Poincaré Conjecture" containing a complete and detailed version of Perelman's proof. This work and the subsequent awarding at the *International Congress of Mathematicians* in Madrid in August of the same year, of the Fields Medal to Perelman (who declined it), mark the formal and substantial acceptation of the mathematical community of his proof of the Poincaré conjecture.

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- Up to now, no mistakes or gaps were found in Perelman's proof. Moreover, a modified and simplified version (in some steps) of the proof was presented in 2007 by L. Bessières, G. Besson, M. Boileau, S. Maillot and J. Porti.

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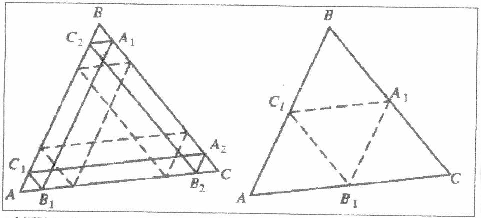
- Perelman resigned from his position at the Steklov Institute in Saint Petersburg and declared his intention to stop doing mathematics.
  His three fundamental papers were never published, they are available on the ArXiv preprint server at http://arxiv.org.

Metro Newspaper in Rome – “CityRoma” 9/1/2004

CityRoma venerdì 9 gennaio 2004 I FATTI DELLA VITA

Un russo risolve la Congettura di Poincaré Il mistero matematico durava da 100 anni



Lo studioso francese Jules-Henri Poincaré (1854-1912) e la rappresentazione grafica della sua Congettura, risalente al 1904. (FOTO AP)

SAN FRANCISCO (California, Usa) - Il mondo matematico è in fermento: il mistero della Congettura di Poincaré sullo studio degli spazi tridimensionali è stato forse risolto 100 anni dopo la sua prima formulazione. Manca ancora l'ufficialità, ma gli studi del russo Grigori "Grisha" Perelman stanno via via guadagnando credito da quando, nel novembre 2002, ne è iniziata la verifica da parte delle massime autorità matematiche.

Il caso non è semplice perché la congettura non è mai stata dimostrata neanche dal suo ideatore, lo studioso francese Jules-Henri Poincaré, il quale nel 1904 arrivò ad elaborare un metodo per applicare facilmente le regole di calcolo per le misurazioni bidimensionali (altezza-larghezza) a quelle tridimensionali (altezza-larghezza-profondità). Il metodo funziona ma resta astratto, non essendo dimostrato matematicamente, per cui solo la sua applicazione ai problemi più complessi della materia può dire se è esatto o meno. Più volte, in passato, soluzioni proposte da insigni studiosi sono tramontate alla prova dei fatti. La teoria di Perelman si rifà alle *correnti di Ricci* e alla geometria differenziale: "Sono studi molto complicati, con molte parti variabili. Ci vuole tempo ed è facile perdere il filo", ammette John Morgan, docente della Columbia University.

A complicare le cose c'è il carattere riservato dello studioso russo, che solo un anno fa è uscito dalla semi-reclusione nella quale si era rinchiuso da otto anni e ha esposto le sue scoperte ad alcuni college statunitensi. Perelman, matematico dell'Istituto Steklav dell'Accademia russa delle scienze, ha anche rifiutato finora il milione di dollari messo in palio dal *Clay Mathematics Institute* di Cambridge, Massachusetts, per la soluzione di ognuno dei sette più grandi misteri matematici. La condizione per il premio Fields, una sorta di Nobel matematico, è infatti che la soluzione sia pubblicata su un giornale scientifico, cosa finora evitata da Perelman.

La soluzione della Congettura di Poincaré sarebbe utile soprattutto nello studio dell'universo, ma non avrebbe applicazioni nella vita di tutti i giorni. (AP)

Thurston Conjecture

An even more general conjecture describing the structure of *all* the 3–dimensional manifolds was stated by William P. Thurston (30 October 1946 – 21 August 2012), awarded with the Fields Medal in 1982.

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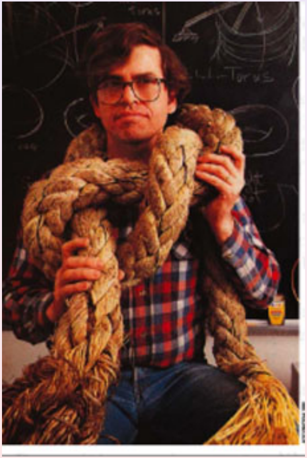

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- W. Thurston gave a partial proof.

Thurston Conjecture

Also this conjecture can be solved by means of the deformation method based on the Ricci flow, like the Poincaré conjecture. Hence, it is natural to ascribe also its proof to the work of Perelman, completed in detail in the papers:

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- Laurent Bessières, Gérard Besson, Michel Boileau, Sylvain Maillot and Joan Porti, *Geometrisation of 3–Manifolds*, 2007.
- John Morgan and Gang Tian, *Completion of the Proof of the Geometrization Conjecture*, 2008.

The $1/4$–Pinched Sphere Theorem (Differentiable)

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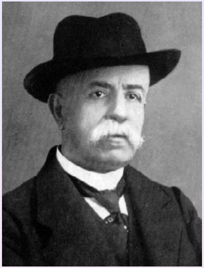
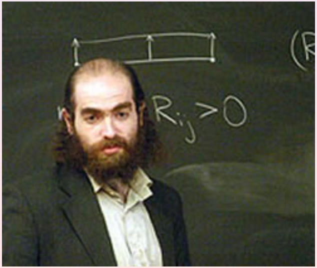
Proved in 2007 by Simon Brendle and Richard Schoen by means of Ricci flow.

The Main Characters...





Thanks...

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Grazie dell'attenzione

Thanks for your attention