STAR-SHAPEDNESS UNDER MEAN CURVATURE FLOW

CARLO MANTEGAZZA

ABSTRACT. We show that a star–shaped curve in the plane remains star–shaped moving by curvature.

1. SIMPLE CURVES IN THE PLANE MOVING BY CURVATURE

Given a closed $C^1$ curve $\gamma : S^1 \rightarrow \mathbb{R}^2$ we say that it is regular if $\gamma_\theta = \frac{\partial \gamma}{\partial \theta}$ is never zero. It is then well defined its unit tangent vector $\tau = \gamma_\theta / |\gamma_\theta|$. We define its unit normal vector as $\nu = R\tau = R\gamma_\theta / |\gamma_\theta|$, where $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the counterclockwise rotation centered in the origin of angle $\pi/2$. If the curve $\gamma$ is $C^2$ and regular, its curvature vector is well defined as $k = \frac{\tau_\theta}{|\gamma_\theta|} = \frac{d\tau}{d\theta} / |\gamma_\theta|$. The curvature of $\gamma$ is then given by $k = \langle k | \nu \rangle$, as $k = k\nu$. The arclength parameter $s$ of the curve $\gamma$ is given by $s = s(\theta) = \int_0^\theta |\gamma_\theta(\xi)| d\xi$. Notice that $\partial_s = |\gamma_\theta|^{-1} \partial_\theta$, then $\tau = \partial_s \gamma$ and $k = \partial_s \tau = \partial_{ss}^2 \gamma$.

With these notations, a circle covered counterclockwise, has positive curvature and $\nu$ is the inner unit normal vector.

Defining the quantity $Q = \langle \gamma | \nu \rangle$, it is easy to see that a simple curve $\gamma$ in the plane is star–shaped with respect to the origin of $\mathbb{R}^2$ if and only if $Q \leq 0$ at every point of the curve.

Consider then a smooth, regular, simple, closed curve $\gamma : S^1 \rightarrow \mathbb{R}^2$, star–shaped with respect to the origin, and its motion by curvature, given by $\gamma_t = k = k\nu$. It is well known that such a curve evolves smoothly by curvature, it remains embedded, becomes convex and shrinks to a point in finite time, becoming asymptotically rounder and rounder, see [1–7].

Date: July 13, 2010.
We compute the evolution equation for the quantity $Q$ during the flow. As $\partial_t \tau = k_s \nu$ and $\partial_s \nu = \partial_t R \tau = -k_s \tau$, we have
\[
\partial_t Q - \partial_{ss}^2 Q = \langle \gamma_t | \nu \rangle + \langle \gamma | \partial_t \nu \rangle - \langle \gamma_{ss} | \nu \rangle - \langle \gamma | \partial_s^2 \nu \rangle - 2\langle \gamma_s | \partial_s \nu \rangle
\]
\[
= k - k_s \langle \gamma | \tau \rangle - k + \langle \gamma | k_s \tau + k^2 \nu \rangle + 2k
\]
\[
= 2k + Qk^2
\]
and
\[
\partial_s Q = \langle \gamma_s | \nu \rangle + \langle \gamma | \partial_s \nu \rangle = \langle \tau | \nu \rangle - k \langle \gamma | \tau \rangle = -k \langle \gamma | \tau \rangle.
\]

If the initial curve $\gamma$ is star–shaped with respect to the origin $O \in \mathbb{R}^2$, we have $Q \leq 0$, then, if at some time $t$ and point $\gamma(\theta, t)$, we have $Q = 0$, at such point there holds
\[
0 = \partial_s Q = -k \langle \gamma | \tau \rangle
\]
and
\[
0 \leq \partial_t Q - \partial_{ss}^2 Q = 2k + Qk^2 = 2k.
\]
If $O \neq \gamma(\theta, t)$, for instance if the open region bounded by the curve $\gamma$ at time $t$ contains the origin of $\mathbb{R}^2$, we have $\gamma(\theta, t) \neq 0$, and being $Q = \langle \gamma | \nu \rangle = 0$ at such point, there must be $\langle \gamma | \tau \rangle \neq 0$, hence, $k$ is zero. It follows that $\partial_t Q - \partial_{ss}^2 Q = 0$ and we can conclude by the maximum principle that $Q$ stays nonpositive during the flow, hence the curve remains star–shaped with respect to the origin.

Since the flow by curvature is translation invariant, we have the following conclusion.

**Theorem 1.1.** An initial smooth, regular, simple closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, star–shaped with respect to a point $P$, remains star–shaped under its motion by curvature, till the point $P$ is contained in the open region of the plane bounded by the evolving curve.

**Problem.** Despite this theorem, we do not know if the property of star–shapedness (in general, not with respect to a fixed point) is actually preserved under the motion by curvature of a simple closed curve in the plane.

2. **Higher Dimension**

By a little bit more involved, but straightforward, computation (see [8] for the formulas), it can be seen that for an $n$–dimensional hypersurface $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ moving by curvature, the analogous quantity
\[
Q = \langle \varphi | \nu \rangle,
\]
satisfies the parabolic equation
\[
\partial_t Q - \Delta Q = 2H + Q|A|^2
\]
and
\[
\nabla Q = -G(\pi^M \varphi),
\]
where $A$ is the second fundamental form, $G$ is the Gauss operator of $M$ and $\pi^M$ is the projection on the tangent space to $M$ (thinking of $M_t = \varphi(M, t)$ as a submanifold of $\mathbb{R}^{n+1}$ and identifying any tangent space $T_p M_t$ with a hyperplane in $\mathbb{R}^{n+1}$).
Unfortunately, if \( n \geq 2 \), we cannot repeat the argument above, since the nullity of the gradient of \( Q \) at \( p \in M_t \) only says that the second fundamental form of \( M_t \) has a null eigenvalue in the direction \( \pi^M_p \varphi \in T_pM_t \), while we would need that \( H \leq 0 \) at such point \( p \in M \) where \( Q = 0 \) (notice that \( Q = 0 \) implies that \( \varphi \in T_pM_t \), hence \( \pi^M_p \varphi = \varphi \)).

Actually, even if we are not able to construct an explicit example where the star–shapedness of a surface is lost during its motion by mean curvature, it is very likely that this can happen.

REFERENCES


CARLO MANTEGAZZA, SCUOLA NORMALE SUPERIORE DI PISA, P.ZZA DEI CAVALIERI 7, 56126 PISA, ITALY

E-mail address: c.mantegazza@sns.it