Motion by curvature of networks in the plane

CARLO MANTEGAZZA

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Motion by curvature of networks in the plane – *Joint project with*

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- *Annibale Magni & Matteo Novaga, 2010 – 2014*
- *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
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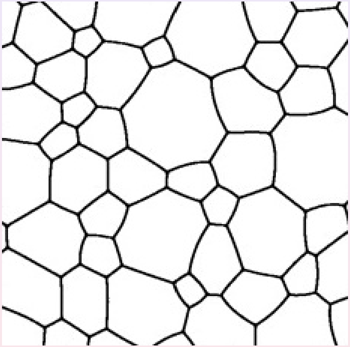

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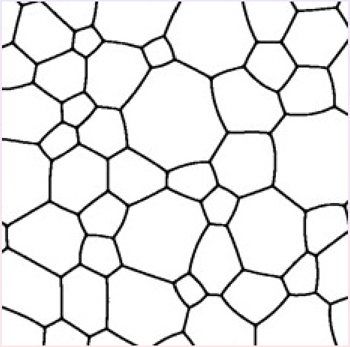

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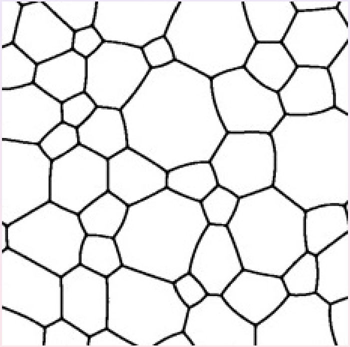

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Even if it is still possible to use several ideas from the "parametric" classical approach to mean curvature flow (differential geometry/maximum principle), some extra variational methods are needed due to the presence of the multi–points, that makes the network a singular set (possibly, the simplest).

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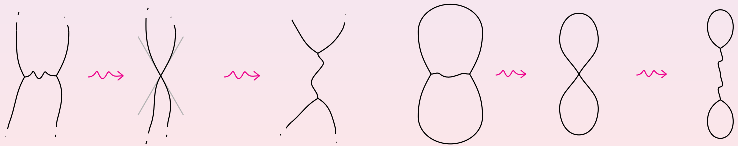

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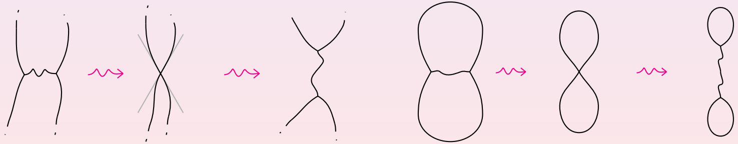
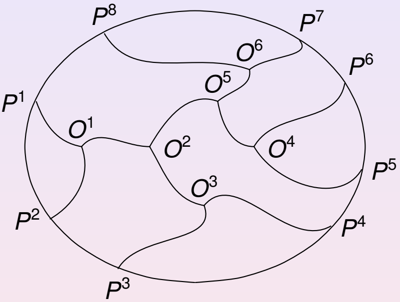


Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of “technology” from analysis and geometry is needed.

Regular networks

Let Ω be an open, regular and convex subset of $\mathbb{R}^2$.



Definition

A regular network $\mathbb{S} = \bigcup_{i=1}^{n} \sigma^i([0,1])$ *in* Ω *is a connected set described by a finite family of curves* $\sigma^i : [0,1] \to \overline{\Omega}$ *(sufficiently regular) such that:*

Regular networks

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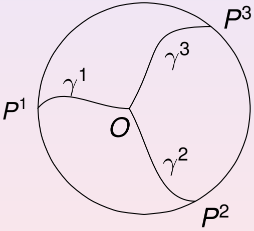
2. If a curve of the network "touches" the boundary of Ω at a fixed point $P \in \partial\Omega$, no other end–point of a curve can coincide with that point
3. All the junctions points $O^1, O^2, \dots, O^m \in \Omega$ have order three, considering $\mathbb{S}$ as a planar graph, and at each of them the three concurring curves $\{\sigma^{pi}\}_{i=1,2,3}$ meet in such a way that the external unit tangent vectors τ^{pi} satisfy $\sum_{i=1}^{3} \tau^{pi} = 0$ (the curves form three angles of 120 degrees at O^p)

Examples: The triod and the spoon

A triod $\mathbb{T}$ is a network composed only by three regular, embedded curves $\gamma^i : [0, 1] \to \overline{\Omega}$.



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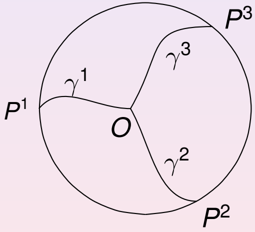
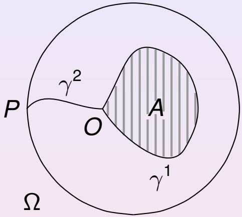
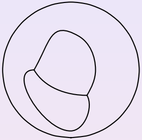
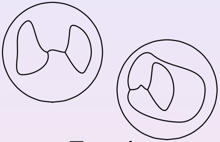
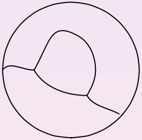
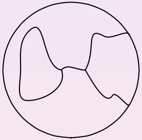
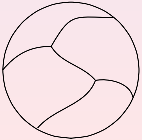




A spoon $\Gamma = \gamma^1([0, 1]) \cup \gamma^2([0, 1])$ is the union of two regular, embedded curves $\gamma^1, \gamma^2 : [0, 1] \to \overline{\Omega}$.

Examples: Networks with two triple junctions

	0 closed curves	1 closed curve	2 closed curves
0 end–points on $\partial\Omega$	 Theta		 Eyeglasses
2 end–points on $\partial\Omega$	 Lens	 Island	
4 end–points on $\partial\Omega$	 Tree		

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We say that a regular network moves by curvature if each of its time–dependent curves $\gamma^i : [0,1] \times [0,T) \to \mathbb{R}^2$ *satisfies*

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To be more precise, a family of regular networks $\mathbb{S}_t$ *is a motion by curvature in a time interval* $[0,T)$ *if the functions* $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$ *are at least* C^2 *in space and* C^1 *in time and satisfy the following system:*

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & \\ \sum \tau^i(O,t) = 0 & \text{at every 3–point} \\ \gamma^i_t = k^i\nu^i + \lambda^i\tau^i & \text{for some continuous functions } \lambda^i \end{cases} \qquad (\star)$$

with fixed end–points on $\partial\Omega$.

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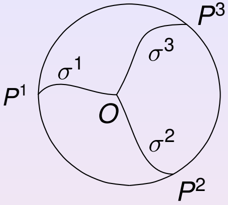
Definition

A curvature flow γ^i which satisfies

$$\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$$

will be called a special curvature flow*.*

Short time existence (Triod)



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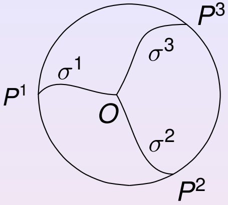

Theorem (Bronsard–Reitich, 1992 & CM–Novaga–Tortorelli, 2004)

For any initial, regular $C^{2+\alpha}$ *triod* $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0,1])$*, with* $\alpha \in (0,1)$*, which is* 2–compatible*, there exists a unique special flow in the class* $C^{2+\alpha,1+\alpha/2}([0,1]\times[0,T))$*, in a maximal time interval* $[0,T)$*.*

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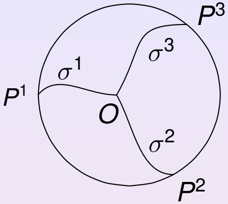

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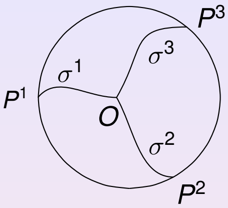

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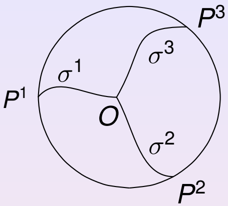

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Viceversa, if the sum of the curvatures is zero at the 3–point there is a reparametrization making the triod 2–compatible (geometric 2–compatibility).

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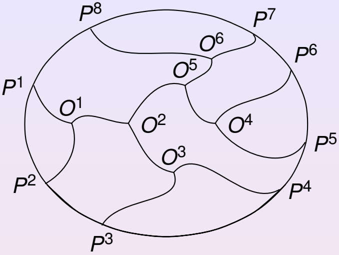

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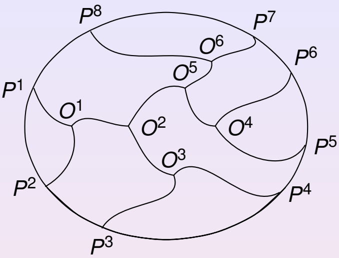

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For any initial smooth, regular network $\mathbb{S}_0$ *there exists a unique smooth special flow in a maximal time interval* $[0,T)$*.*

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$\mathbb{T}_0 \in C^{2+\alpha}$, 2–compatible
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Short time existence and uniqueness

A network is *geometrically 2–compatible* if

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- The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2–compatibility), but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. Clearly, the curvature is no more necessarily a continuous function at $t = 0$ (the maps γ^i_{xx} are not continuous up to time $t = 0$ at the triple junctions).

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- The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2–compatibility), but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. Clearly, the curvature is no more necessarily a continuous function at $t = 0$ (the maps γ^i_{xx} are not continuous up to time $t = 0$ at the triple junctions).
- The geometric uniqueness of the solution given by this theorem is an open problem.

Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

Theorem

For any initial C^2 regular network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ there exists a C^1 in space and time network flow γ^i, in a maximal time interval $[0,T)$ such that the flow $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is a smooth curvature flow for every time $t > 0$. Moreover, the flow is C^2 up to time $t = 0$, far from the triple junctions.

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- The geometric uniqueness of the solution given by this theorem is an open problem.
- General problem with uniqueness: it depends on the class of curves where we look for solutions. No proof of uniqueness in the “natural” class of C^2 in space, C^1 in time curves. Lack of maximum principle.

Short time existence for non–regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to “restart” the flow after some “collapse” (change of topological structure of the network) we really need an existence theorem also for *non–regular* networks.

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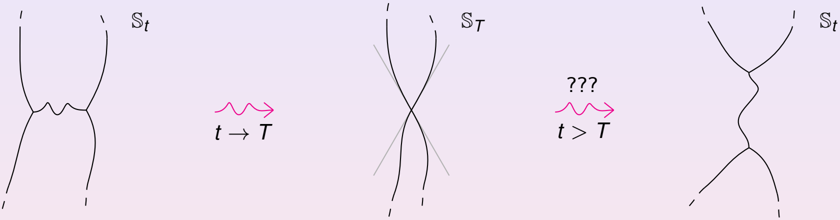

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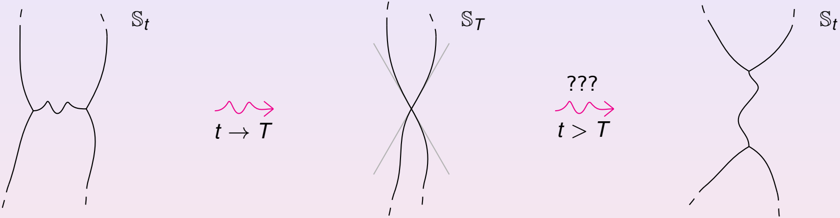

Theorem (Ilmanen–Neves–Schulze, 2014)

For any initial network of non–intersecting curves there exists a (possibly non–unique) Brakke flow by curvature in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.

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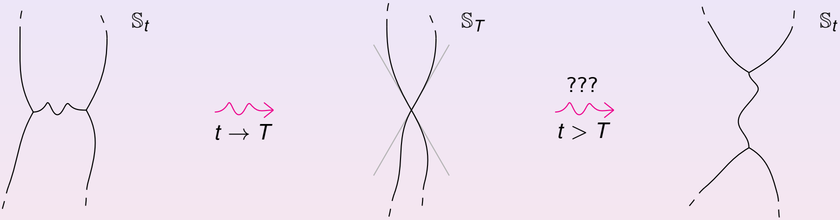

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No hope for uniqueness. Conjecturally, the flow is unique for "generic" initial networks.

Long time behavior

After discussing the existence for short time of the curvature flow, our aim is then to analyse the global (in time) behavior of the evolving networks, in particular the formation of “singularities”.

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The previous results provide a smooth flow of regular networks (satisfying the compatibility conditions of every order) in a maximal time interval $[0, T)$. We will discuss later on the case $T = +\infty$, now we assume that T is finite and investigate what is happening to the moving network as $t \to T$.

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Theorem

If $T < +\infty$ is the maximal time of smooth existence of the curvature flow of a network $\mathbb{S}_t$, then at least one of the following holds:

1. *the lenght of at least one curve of $\mathbb{S}_t$ goes to zero on a sequence $t_n \to T$,*
2. *the curvature is not bounded as $t \to T$.*

Gagliardo–Nirenberg estimates

Theorem (Niremberg, "On elliptic partial...", Ann. SNS 13, 1959 Section 3, pp. 257–263)

Let γ be a C^∞, regular curve in $\mathbb{R}^2$ with finite lenght L. If u is a C^∞ function defined on γ and $m \geq 1$, $p \in [2, +\infty]$, we have the estimates

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{L^{m\sigma}} \|u\|_{L^2}$$

for every $n \in \{0, \dots, m-1\}$ where

$$\sigma = \frac{n + 1/2 - 1/p}{m}$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ.
In particular, if $p = +\infty$,

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{L^{m\sigma}} \|u\|_{L^2} \qquad \text{with} \qquad \sigma = \frac{n + 1/2}{m}$$

Evolution of a triod

Theorem

If none of the lenghts of the three curves of an evolving triod goes to zero, the flow is smooth for all times and the triod converges (asymptotically) to the Steiner configuration connecting the three endpoints.

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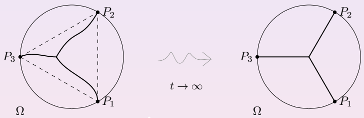

If all the angles of the triangle with vertices the three end–points on the boundary are less than 120 degrees and the initial triod is contained in the triangle, then the triod converges in infinite time to the Steiner configuration connecting the three points.

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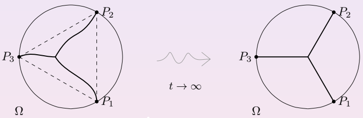

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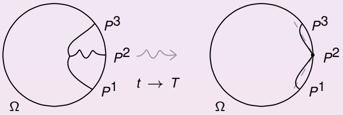


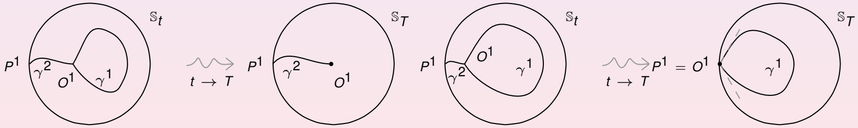
If the fixed end–points on the boundary form a triangle with an angle of more than 120 degrees, then the lenght of a curve goes to zero in finite time.

Evolution of a spoon

Theorem

The maximal time of existence of a smooth evolution of a spoon is finite and one of the following situations occurs:

1. *the closed loop shrinks to a point in a finite time and the maximum of the curvature goes to $+\infty$, as $t \to T$;*
2. *the "open" curve vanishes and there is a 2–point formation on the boundary of the domain of evolution, but the curvature remains bounded.*

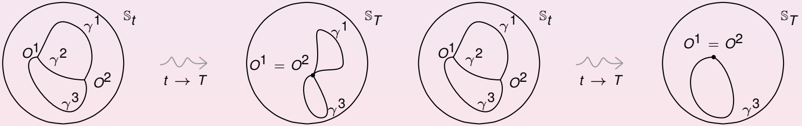


Evolution of a “theta” (double cell)

Theorem

The maximal time of existence of a smooth evolution of a “theta” is finite and one of the following situations occurs:

1. *the lenght of a curve that connects the two* 3*–points goes to zero as* $t \to T$ *and the curvature remains bounded;*
2. *the lenght of the curves composing one of the loops go to zero as* $t \to T$ *and the maximum of the curvature goes to* $+\infty$*.*



In any case the network cannot completely vanish shrinking to a point as $t \to T$ (not easy to show).

Long time behavior

We have seen examples in which the lenght of at least a curve goes to zero while the curvature is bounded and examples in which at the same time the lenght of a curve goes to zero and the curvature of the network is unbounded.

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But NO examples in which the lenghts of all the curves of the network remain uniformly bounded away from zero during the evolution and the curvature is unbounded, as $t \to T$.

Conjecture

If no lenght of the curves of the network goes to zero on any sequence of times $t_n \to T$*, then* T *is not a singular time (maximal time of smooth existence).*

Long time behavior

We are actually able to show the following:

Theorem

If no lenght goes to zero on any sequence of times $t_n \to T$ and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence, T is not a singular time and the flow is smooth.

"Multiplicity–One Conjecture" (M1)

Every possible limit of rescaled networks of the flow is a network with multiplicity one.

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To show this theorem and in general to understand the nature of the singularities of the flow, we will employ a blow–up technique where the validity of such conjecture will play a key role.

Huisken's dynamical rescaling – Blow–up technique

We rescale the smooth flow $\mathbb{S}_t$ of regular networks in its maximal time interval $[0, T)$ of smooth existence as follows:

Definition

Fixed $x_0 \in \mathbb{R}^2$, we define the "rescaled" flow as

$$\widetilde{\mathbb{S}}_{x_0,\tau} = \frac{\mathbb{S}_{t(\tau)} - x_0}{\sqrt{2(T - t(\tau))}}$$

where $\tau \in [-1/2 \log T, +\infty)$ and $\tau(t) = -\frac{1}{2} \log (T - t)$.

We denote with σ the arclenght measure on the rescaled networks.

Rescaled monotonicity formula

Proposition

Let $x_0 \in \mathbb{R}^2$*, for every* $\tau \in [-1/2 \log T, +\infty)$ *the following identity holds*

$$\frac{d}{d\tau} \int_{\widetilde{\mathbb{S}}_{x_0,\tau}} e^{-\frac{|x|^2}{2}}\, d\sigma = - \int_{\widetilde{\mathbb{S}}_{x_0,\tau}} |\underline{\widetilde{k}} + x^{\perp}|^2 e^{-\frac{|x|^2}{2}}\, d\sigma \, + \widetilde{BT}(\tau)\,.$$

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Integrating between τ_1 and τ_2 with $-1/2 \log T \leq \tau_1 \leq \tau_2 < +\infty$ we get

$$\int_{\tau_1}^{\tau_2} \int\limits_{\widetilde{\mathbb{S}}_{x_0,\tau}} |\,\underline{\widetilde{k}} + x^{\perp}|^2 e^{-\frac{|x|^2}{2}}\, d\sigma\, d\tau = \int\limits_{\widetilde{\mathbb{S}}_{x_0,\tau_1}} e^{-\frac{|x|^2}{2}}\, d\sigma - \int\limits_{\widetilde{\mathbb{S}}_{x_0,\tau_2}} e^{-\frac{|x|^2}{2}}\, d\sigma + \int_{\tau_1}^{\tau_2} \widetilde{BT}(\tau)\, d\tau$$

moreover, the last integral is uniformly bounded independently of τ_1, τ_2.

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Putting $\tau_1 = -1/2\log T$ and sending τ_2 to $+\infty$, we conclude

$$\int_{-1/2\log T}^{+\infty}\int_{\widetilde{\mathbb{S}}_{x_0,\tau}} |\underline{\widetilde{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}}\,d\sigma\,d\tau \leq \int_{\widetilde{\mathbb{S}}_{x_0,-1/2\log T}} e^{-\frac{|x|^2}{2}}\,d\sigma + C < +\infty\,.$$

Blow–up limits – Skrinkers

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The limit shrinker is not unique!!!

Regular shrinkers

Definition

A smooth regular open (no boundary points) network $\mathbb{S}$ *is called a* regular shrinker *if at every point* $x \in \mathbb{S}$ *there holds*

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The name comes from the fact that the evolution given by $\mathbb{S}_t = \sqrt{-2t}\,\mathbb{S}$ is a self–similarly shrinking curvature flow in the time interval $(-\infty, 0)$ with $\mathbb{S} = \mathbb{S}_{-1/2}$.

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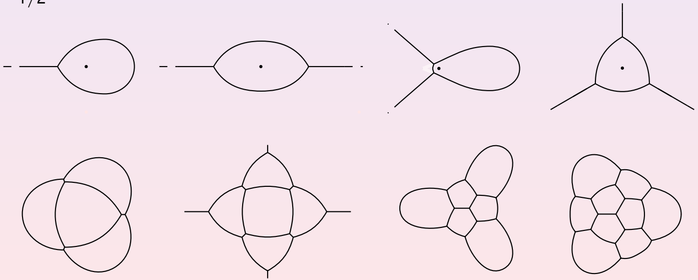

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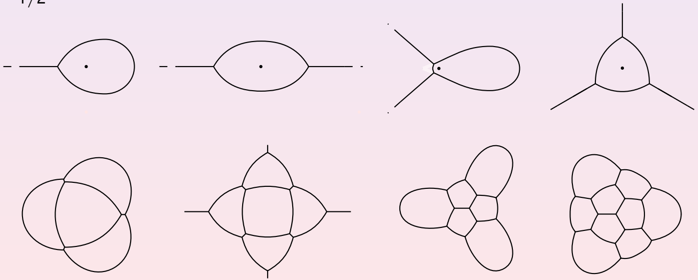

The unit circle, a straight line or an infinite regular straight triod are regular shrinkers also.

Sketch of the proof

Theorem

If no lenght goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence, T is not a singular time and the flow is smooth.

"Multiplicity–One Conjecture" (**M1**)

Every possible limit of rescaled networks of the flow is a network with multiplicity one.

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We conclude that $\widetilde{\mathbb{S}}_\infty$ is composed only by lines and halflines for the origin of $\mathbb{R}^2$.

Sketch of the proof

Since the lenghts of the converging sequence to $\widetilde{\mathbb{S}}_\infty$ are going to $+\infty$, two triple junctions cannot going to collide, then, by the C^1_{loc}–convergence, there are only the following possibilities for the blow–up limit shrinker $\widetilde{\mathbb{S}}_\infty$:

- the empty set
- a straight line
- a halfline
- an infinite regular straight triod

all with multiplicity one (notice they all have zero curvature).

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If we get an empty set, it means that the evolving network $\mathbb{S}_t$ "stays far" from the point x_0. In the other cases, we can then apply

- White's theorem in the case of a straight line

Sketch of the proof

Since the lenghts of the converging sequence to $\widetilde{\mathbb{S}}_\infty$ are going to $+\infty$, two triple junctions cannot going to collide, then, by the C^1_{loc}–convergence, there are only the following possibilities for the blow–up limit shrinker $\widetilde{\mathbb{S}}_\infty$:

- the empty set
- a straight line
- a halfline
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all with multiplicity one (notice they all have zero curvature).

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to conclude that in a neighbourhood of x_0 the curvature of the networks $\mathbb{S}_t$ is bounded uniformly in time, for $t \in [0, T)$.

Sketch of the proof

Since the previous conclusion holds for all $x_0 \in \mathbb{R}^2$, it follows that the curvature is globally bounded along the flow, which is a contradiction.

Remember indeed that

Theorem

If $T < +\infty$ is the maximal time of smooth existence of the curvature flow $\mathbb{S}_t$, then:

1. *either the lenght of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
2. *or the curvature is not bounded as $t \to T$.*

$\square$

Long time behavior

Hence, assuming (from now on) the truth of the

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Then, there are two possible situations:

- The curvature stays bounded along the flow
- The curvature is unbounded as $t \to T$

Singularity types

In the smooth case of a closed curve evolving by curvature there are no singularities with bounded curvature. Indeed the maximum of the modulus of the curvature always satisfies

$$|k|_{\max}(t) \geq 1/\sqrt{2(T-t)}$$

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Singularities are then "classified" by the rate the curvature goes to $+\infty$, as $t \to T$:

- *Type I* – The maximum of the modulus of the curvature is of order $1/\sqrt{T-t}$, as $t \to T$.
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For the flow of networks there are actually singularities (collapse of curves) with bounded curvature, that we can call *Type 0* singularities (copyright of Tom Ilmanen).

Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to $+\infty$. Moreover, only "isolated" curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

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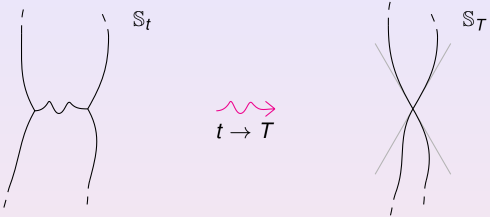
Theorem

If **M1** *is true, every* interior *vertex of such limit network either is a regular triple junction or it is a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* $120/60$ *degrees among them.*

This is exactly what we saw in the simulation when a single curve vanishes.

Analysis of singularities – Collapse with bounded curvature



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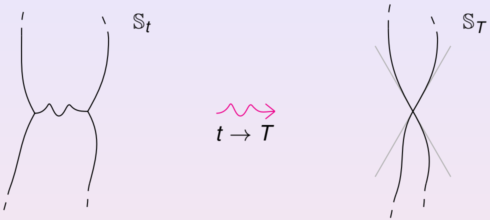

This analysis can be extended to the curves containing the fixed boundary points (*boundary curves*). If any of them vanishes, we get two curves forming an angle of 120 degrees (remember the example of the *spoon* network). In such case, the flow stops (we really “have to decide” whether/how to continue the flow).

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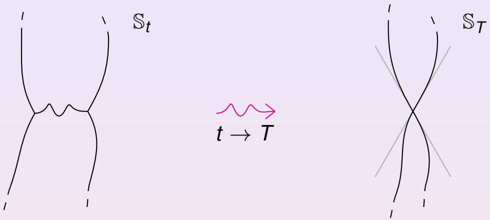

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As the limit network $\mathbb{S}_T$ contains at least one 4–junction, in order to restart the flow we need the short time existence theorem of Ilmanen, Neves and Schulze.

Analysis of singularities – Tree–like networks

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If **M1** *holds and the network is a* tree *(no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

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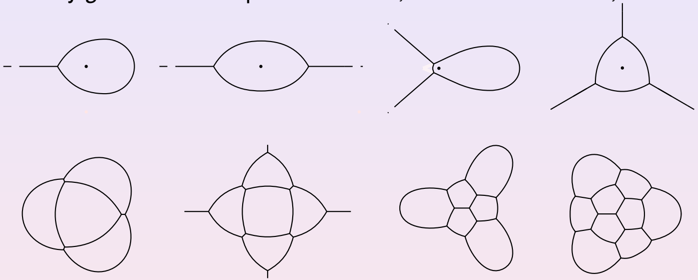
Also this corollary can be localized.

Analysis of singularities – Collapse of a region/Unbounded curvature

In the situation when the curvature is unbounded, that is, at least one region is collapsing, even if **M1** is true, using again the blow–up procedure we can obtain a very *general* blow–up limit shrinker, like the ones below, for instance:



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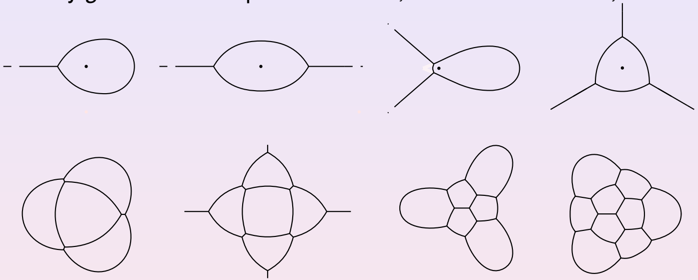

Then, the (local) structure (topology) of the evolving network plays an important role in the analysis/classification of the possible limit shrinkers (as the topology can possibly only “simplify” taking a limit).

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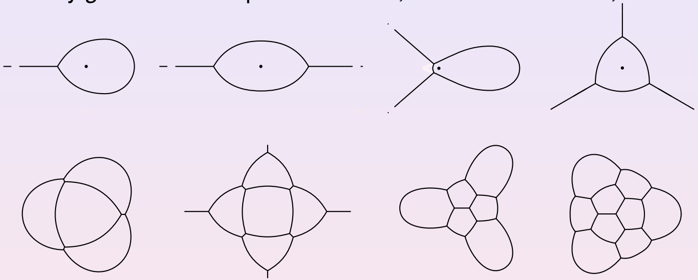

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In this case the blow–up limit is a shrinker *with regions*. These regions are the “memory” of the collapsing regions in the flow $\mathbb{S}_t$, as $t \to T$, and the unbounded halflines of the shrinker give the *limit tangents* of the *non–vanishing* curves of $\mathbb{S}_t$ arriving at the group of collapsing regions.

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If, as in the case of bounded curvature, the networks $\mathbb{S}_t$ converge to some limit $\mathbb{S}_T$, as $t \to T$, then the point of collapse will be a multi–point of $\mathbb{S}_T$ and the unbounded halflines of the blow–up limit shrinker give the tangents of the curves of $\mathbb{S}_T$ concurring at such point.

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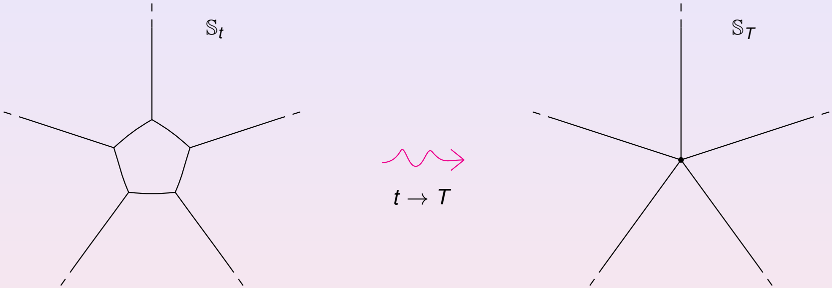
Assuming such uniqueness, one can prove that (possibly after reparametrization) the *vanishing* part of $\mathbb{S}_t$ collapses to a point and the *non–vanishing* part converges in C^1 to a limit network $\mathbb{S}_T$. The point of collapse is then a multi–point of $\mathbb{S}_T$ and the curves of $\mathbb{S}_T$ concurring at such point are of class C^∞ far from the multi–point.
Moreover, the curvature of each of such curves goes like $k = o(1/d)$, where d is the arclenght distance to the multi–point along the curve.

Analysis of singularities – Collapse of a region/Unbounded curvature



An example of a (homothetic) collapse of a (symmetric) pentagonal region of $\mathbb{S}_t$ (*five–ray star*).

Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network $\mathbb{S}_T$ satisfies the hypotheses of the INS Theorem:

Theorem (Ilmanen–Neves–Schulze, 2014)

For any initial network of non–intersecting curves there exists a (possibly non–unique) Brakke flow by curvature *in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*

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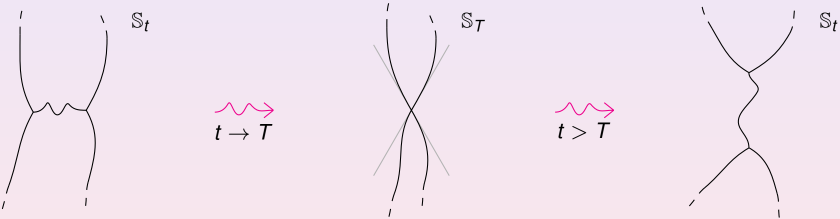
We anyway state the following conjecture:

Conjecture

The limit network $\mathbb{S}_T$ has always bounded curvature.

Restarting the flow after a singularity – The special case of bounded curvature/Standard transition

In the case of the collapse of an isolated curve with bounded curvature, the INS Theorem produces a curvature flow like we observed in the simulation.



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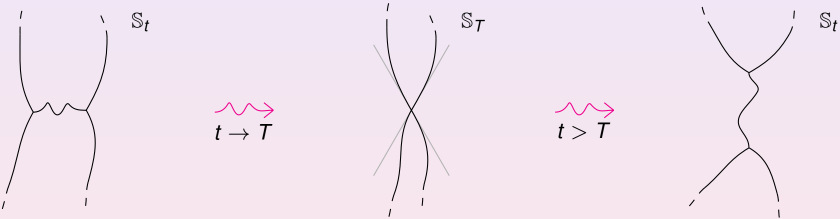

There is actually hope for uniqueness in this case, that we call *standard transition*.

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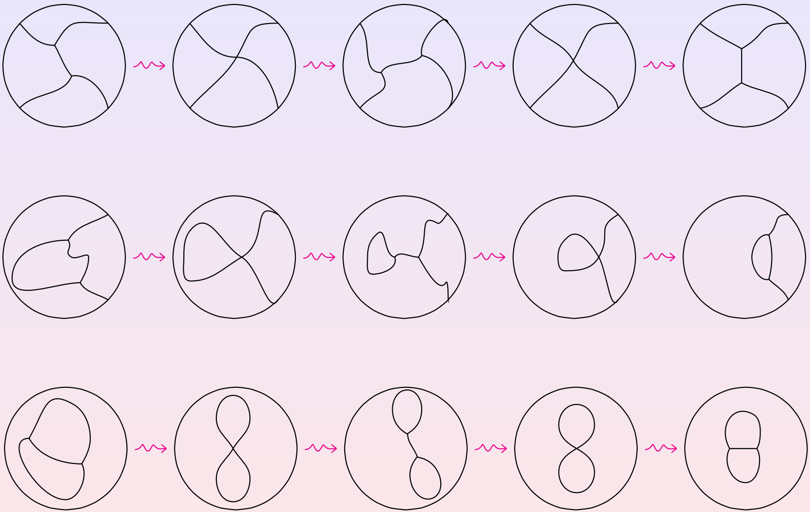
The limit of $\mathbb{S}_t$ as $t \to +\infty$, is unique (the full sequence of networks converges).

Possible “oscillation of shape” phenomenon



Possible infinite “oscillations” via standard transitions from a shape to another and back.

Open problems and research directions

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Theorem (CM, M. Novaga, A. Pluda – 2015)

- *If during the flow the triple–junctions stay uniformly far each other, then* **M1** *is true.*
- *If the initial network has at most two triple junctions, then* **M1** *is true.*

Analysis of singularity formation for some flows of networks with "few" triple junctions can then be made rigorous (under the previous "uniqueness of the limit network" conjecture in some cases).

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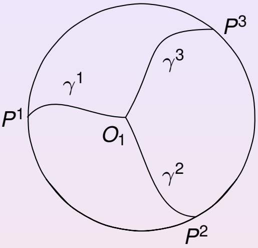
The Triod – A. Magni, CM, M. Novaga, V. Tortorelli



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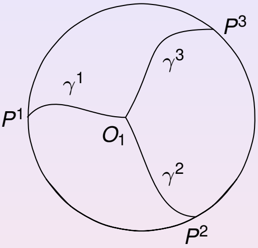
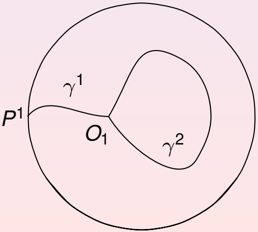
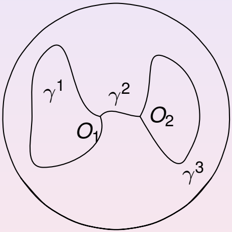


The Spoon – A. Pluda



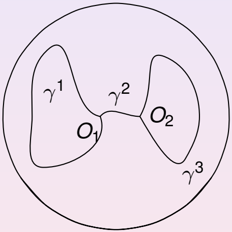
2 triple junctions – CM, M. Novaga, A. Pluda

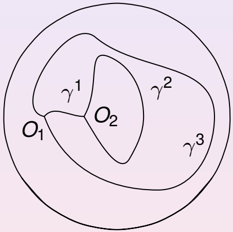
The Eyeglasses



2 triple junctions – CM, M. Novaga, A. Pluda

The Eyeglasses and... the Broken Eyeglasses





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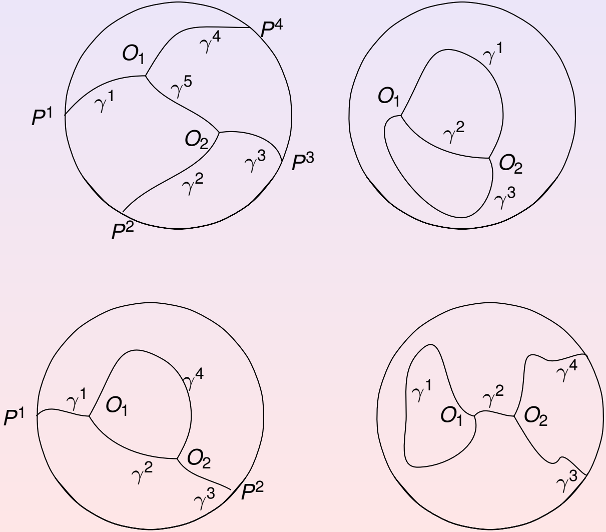
The “Steiner”, Theta, Lens and Island



Open problems and research directions

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Uniqueness issues (for initial regular and non–regular networks)

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Uniqueness of the blow–up limits

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Existence of Type II singularities/Bound from below on the blow–up rate of the curvature

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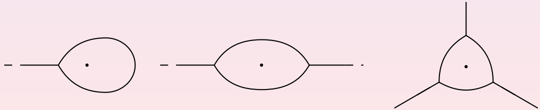

are the only dynamically stable shrinkers?

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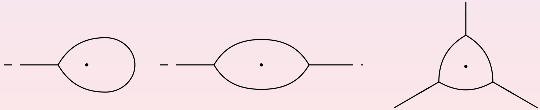

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- *Genericically, only regions with at most three edges can collapse*

Open problems and research directions – Higher dimensions

Study of the motion by mean curvature of 2–dimensional interfaces in $\mathbb{R}^3$ (*double–bubble*, for instance)

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Thanks for your attention